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RIEMANN-ROCH FOR SINGULAR VARIETIES

by PAUL BAUM, WILLIAM FULTON and ROBERT MACPHERSON (1)

CONTENTS

0. Introduction.

(0.1) Grothendieck's version of the Riemann-Roch theorem for non-singular projective varieties [Borel-Serre] is expressed by saying that the mapping $\zeta \mapsto ch(\zeta) \smile Td(X)$ from K^0X to $H^\bullet X$ is a natural transformation of covariant functors. Here K^0X denotes

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the Grothendieck group of algebraic vector bundles on X, $H^{\bullet}X$ is a suitable cohomology theory, ch is the Chern character, and Td(X) is the Todd class of the tangent bundle to X; K^0 and $H^{\bullet}$ are naturally contravariant functors, but for non-singular varieties they can be made covariant.

A Riemann-Roch theorem for singular varieties in terms of K^0 and $H^{\bullet}$ can be formulated only for those maps $f: X \to Y$ for which Gysin homomorphisms

$$f_*: K^0X \to K^0Y \quad \text{and} \quad f_*: H^{\bullet}X \to H^{\bullet}Y$$

are available. Such a theorem can be proved when f is a complete intersection morphism, and the cohomology is

1) $H^{\bullet}X = \mathrm{Gr}^{\bullet}(X)_{\mathbf{Q}}$ = the associated graded ring to the λ-filtration of $K^0X_{\mathbf{Q}}$ [SGA 6], **or**

2) $H^{\bullet}X = A^{\bullet}X_{\mathbf{Q}}$ = the Chow cohomology ring (Chapter IV, § 3; [App., § 3]), or

3) $H^{\bullet}X = H^{\bullet}(X; \mathbf{Q})$ = singular cohomology (Chapter IV, §§ 3, 4).

With such a theorem, however, one obtains a Hirzebruch Riemann-Roch formula for the Euler characteristic of a vector-bundle on X only if X itself is a local complete intersection in projective space.

Our Riemann-Roch theorem for projective varieties (which may be singular) is formulated in terms of naturally covariant functors from the category of projective varieties to the category of abelian groups. We construct a natural transformation τ from K_0 to $H_{\bullet}$. Here K_0X is the Grothendieck group of coherent algebraic sheaves on X, and $H_{\bullet}X$ is a suitable homology group. In the classical case, when the ground field is $\mathbf{C}$, $H_{\bullet}X$ may be $H_{\bullet}(X; \mathbf{Q})$ = singular homology with rational coefficients. For varieties over any field we may take $H_{\bullet}X$ to be the Chow group $A_{\bullet}X_{\mathbf{Q}}$ of cycles modulo rational equivalence, with rational coefficients [App., § 1]. Each of these homology theories has a corresponding cohomology theory $H^{\bullet}$ with a cap product $H^{\bullet} \otimes H_{\bullet} \xrightarrow{\frown} H_{\bullet}$; each variety has a fundamental class [X] in $H_{\bullet}X$.

Riemann-Roch theorem. — There is a unique natural transformation $\tau: K_0 \to H_{\bullet}$ *such that:*

1) *For any* X *the diagram*

$$\begin{tikzcd} K^0X \otimes K_0X \arrow[r, "\otimes"] \arrow[d, "ch \otimes \tau"] & K_0X \arrow[d, "\tau"] \\ H^{\bullet}X \otimes H_{\bullet}X \arrow[r, "\frown"] & H_{\bullet}X \end{tikzcd}$$

is commutative.

2) *If* X *is non-singular, and* $\mathcal{O}_X$ *is the structure sheaf on* X, *then*

$$\tau(\mathcal{O}_X) = \mathrm{Td}(X) \frown [X].$$

For each projective variety X, $\tau : K_0X \to H_\bullet X$ is a homomorphism of abelian groups. The naturality of τ means, as usual, that if $f : X \to Y$ is a morphism, then the following diagram commutes:

$$\begin{tikzcd} K_0X \arrow[r, "\tau"] \arrow[d, "f_*"'] & H_\bullet X \arrow[d, "f_*"] \\ K_0Y \arrow[r, "\tau"] & H_\bullet Y \end{tikzcd}$$

(If an element η in K_0X is represented by a sheaf $\mathscr{F}$, then $f_*\eta$ in K_0Y is represented by $f_!\mathscr{F} = \sum_i (-1)^i R^i f_* \mathscr{F}$.)

We call $\tau(\mathcal{O}_X)$ the *homology Todd class* of X, and denote it $\tau(X)$. Let $\varepsilon : H_\bullet X \to \mathbf{Q}$ be the map induced by mapping X to a point. Then $\varepsilon(\tau(X)) = \chi(X, \mathcal{O}_X)$ is the *arithmetic genus* of X.

Corollary. — *If* E *is an algebraic vector bundle on a projective variety* X, *then*

$$\chi(X, E) = \varepsilon(\mathrm{ch}(E) \frown \tau(X)).$$

In particular, for fixed X, $\chi(X, E)$ depends only on the Chern classes of E. Of course, if X is non-singular, the corollary becomes Hirzebruch's formula

$$\chi(X, E) = (\mathrm{ch}\, E \smile \mathrm{Td}\, X)[X].$$

The uniqueness assertion in the Riemann-Roch theorem can be strengthened considerably (Chapter III, § 2):

Uniqueness theorem. — *The* τ *of the Riemann-Roch theorem is the only additive natural transformation from* K_0 *to* $H_\bullet$ *satisfying either of the following conditions*:

1) τ *is compatible with the Chern character, as in* 1) *of the Riemann-Roch theorem, and if* X *is a point,* $\tau(\mathcal{O}_X) = 1 \in \mathbf{Q} = H_\bullet X$.

2) *If* X *is a projective space, the top-dimensional cycle in* $\tau(\mathcal{O}_X)$ *is* [X].

Neither condition mentions the Todd class of a bundle; condition 2) does not even mention Chern classes. This theorem holds over an arbitrary field when $H_\bullet X = A_\bullet X_{\mathbf{Q}}$, as well as in the classical case when $H_\bullet X = H_\bullet(X; \mathbf{Q})$.

We can also deduce from our Riemann-Roch theorem (Chapter III, § 1) a result known previously only for non-singular varieties [SGA 6; XIV, § 4]. Let $\mathrm{Gr}_\bullet X$ be the graded group associated to the filtration of K_0X by dimension of support. Assigning to each subvariety of X its structure sheaf induces a homomorphism $\varphi : A_\bullet X \to \mathrm{Gr}_\bullet X$.

Theorem. — *The mapping* φ *is an isomorphism modulo torsion*:

$$A_\bullet X_{\mathbf{Q}} \xrightarrow{\cong} \mathrm{Gr}_\bullet X_{\mathbf{Q}}.$$

(0.2) For morphisms which are complete intersections, our theory lifts to cohomology (Chapter IV, § 3). This allows us to recover the "cohomology Riemann-Roch theorem" of [SGA 6], for quasi-projective schemes, with values in $A^{\bullet}_{\mathbf{Q}} \simeq \mathrm{Gr}^{\bullet}_{\mathbf{Q}}$.

For a complete intersection morphism $f: X \to Y$ of complex varieties we construct Gysin "wrong-way" homomorphisms

$$f_*: H^{\bullet}(X; \mathbf{Z}) \to H^{\bullet}(Y; \mathbf{Z}) \quad \text{and} \quad f^*: H_{\bullet}(Y; \mathbf{Z}) \to H_{\bullet}(X; \mathbf{Z})$$

(Chapter IV, § 4). The problem of constructing such maps was raised by Grothendieck [SGA 6; XIV]. This allows us to prove a cohomology Riemann-Roch theorem without denominators for a local complete intersection $X \subset Y$ of singular complex varieties (Chapter IV, § 5), as well as extend the Riemann-Roch theorem of [SGA 6] to the singular cohomology theory.

When $X \subset Y$ are smooth, in any characteristic, our methods also give a Riemann-Roch theorem without denominators for the Chow theory; this was conjectured by Grothendieck, and proved using other methods by Jouanolou [*Inventiones Math.*, 11 (1970), pp. 15-26].

For morphisms $f: X \to Y$ which are complete intersections, there are formulas relating the Todd classes of X and Y (Chapter IV, § 1 and § 3). In particular, if X is a local complete intersection in a non-singular variety, its Todd class $\tau(X) = \mathrm{td}(T_X) \frown [X]$, where T_X is the virtual tangent bundle (Chapter IV, § 1).

For general singular varieties, however, the Todd class may not be the cap product of any cohomology class with the fundamental class (Chapter IV, § 6). One method of attack is to find a map $\pi: \widetilde{X} \to X$ which resolves the singularities of X. Then $\mathcal{O}_X - \pi_! \mathcal{O}_{\widetilde{X}} = \sum_i n_i \mathcal{O}_{V_i}$ in $K_0 X$, where the V_i are irreducible subvarieties of the singular locus of X. So

$$\tau(X) - \pi_* \tau(\widetilde{X}) = \sum_i n_i \varphi_{i*}(\tau(V_i))$$

where φ_i is the inclusion of V_i in X. If one can find $\widetilde{X}$, and calculate V_i and n_i, one may reduce the problem to a lower-dimensional case. In this paper we make no use of resolution of singularities (except in an unrelated way for surfaces in Chapter II).

(0.3) The way the homology Todd class generalizes the arithmetic genus is quite analogous to the way the homology Chern class generalizes the topological Euler characteristic [M 2]. (In fact our work on Riemann-Roch began with our trying to find an analogy with this theory of Chern classes.) However, a basic property of the arithmetic genus is that it is constant in a (flat) family of varieties, while the topological Euler characteristic can vary, so one cannot expect the sort of relation between them as one has in the non-singular case (cf. Chapter IV, § 6).

We generalize this property of the arithmetic genus as follows (Chapter IV, § 2).

Theorem. — If $X \to C$ *is a flat family parametrized by a non-singular curve* C, *then the Todd class of the general fibre specializes to the Todd class of the special fibre.*

Similarly the formula giving the arithmetic genus of a Cartesian product $X \times Y$ as the arithmetic genus of X times the arithmetic genus of Y generalizes to the fact that $\tau(X \times Y) = \tau(X) \times \tau(Y)$ (Chapter III, § 3).

(0.4) We give two proofs of the Riemann-Roch theorem. Both proceed by imbedding X in a non-singular variety M. Since a coherent sheaf on X can be resolved by locally free sheaves on M, we are led to consider complexes $E_\bullet$ of vector bundles on M which are exact off X.

For such a complex $E_\bullet$ its Chern character $\sum_i (-1)^i \operatorname{ch} E_i \frown [M] \in H_\bullet M$ restricts to zero in $H_\bullet(M-X)$, so it should come from an element in $H_\bullet X$. From our point of view, an essential step in proving Riemann-Roch is to construct such a "localized class" $\operatorname{ch}_X^M E_\bullet$ in $H_\bullet X$.

Another essential step is to compare an imbedding of non-singular varieties $M \subset P$ with the imbedding of M as the zero-section of the normal bundle. This problem was overcome in [B-S, SGA 6] by blowing up P along M to reduce to the case of a hypersurface, and in [A-H 2] by using a local diffeomorphism (with a suitable complex analytic property) between the two imbeddings. Here we use a different approach which we believe is simpler. We find a family of imbeddings which deforms the given imbedding algebraically into the imbedding as the zero-section of the normal bundle (Chapter I, § 5). Our construction of this deformation uses a simplified form of the "Grassmannian graph construction" (cf. § 0.7) which is vital to our general proof of Riemann-Roch.

(0.5) Chapter I contains the first proof, valid for complex varieties, with values in singular homology with rational coefficients. The class $\operatorname{ch}_X^M E_\bullet$ is constructed using the "difference bundle" of Atiyah and Hirzebruch [A-H 1], and its basic properties are proved in §§ 1, 2. More properties are deduced from those in § 3, and §§ 4, 5, 6 contain the construction of τ and the proof of Riemann-Roch.

(0.6) In Chapter II we construct the localized class $\operatorname{ch}_X^M E_\bullet$ in the Chow group $A_\bullet X_{\mathbf{Q}}$ for any closed subvariety (or subscheme) X of a quasi-projective variety M over an arbitrary field, and a complex of bundles $E_\bullet$ on M, exact off X. This greater generality allows us to study local complete intersections, and also extends the Riemann-Roch theorem to all quasi-projective varieties and proper morphisms. Once the localized class $\operatorname{ch}_X^M E_\bullet$ is constructed, the proof of Riemann-Roch proceeds as in Chapter I, §§ 3-6.

Note that our theorem gives a Riemann-Roch theorem in any homology theory $H_\bullet$ for which there is a natural transformation $A_\bullet \to H_\bullet$, where $A_\bullet$ is the Chow theory. In the classical case this gives another proof for singular homology.

(0.7) We say a few words about the basic Grassmannian graph construction [M 1] for a vector-bundle map $\varphi : E \to F$ of bundles on a complex variety M. The graph of φ at each point $p \in M$ is a subspace of $E_p \oplus F_p$, so we have a section of a Grassmann bundle $G = \operatorname{Grass}_e(E \oplus F)$ over M, with $e = \operatorname{rank} E$. For each complex number λ, we can apply this to $\lambda\varphi$, and get a section s_λ of G over M. This family of imbeddings can be completed at $\lambda = \infty$ to get a rational equivalence. The cycle obtained at infinity contains a great deal of information about where and how φ becomes singular. Riemann-Roch is only one of the applications of this construction.

(0.8) In the classical case the Riemann-Roch map $\tau: K_0X \to H_.(X; \mathbf{Q})$ factors through topological homology K-theory $K_0^{top}(X)$ with integer coefficients. In fact the construction becomes more natural in this context (cf. [A-H 2] for the non-singular case). The Todd class $\tau(\mathcal{O}_X) \in K_0^{top}X$ becomes an orientation class for X in topological K-theory.

If one regards Riemann-Roch as a translation from algebraic geometry to topology, the K-theory version is the most natural and precise way to formulate it. On the other hand, factoring through the Chow group shows that the Todd class is an algebraic cycle which is well-defined up to rational equivalence (over $\mathbf{Q}$). The relations between these theories are made clearer by the commutative diagram

$$\begin{array}{ccc} K_0 & \longrightarrow & K_0^{top} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle ch} \\ A_{.\mathbf{Q}} & \longrightarrow & H_.(\ ;\mathbf{Q}) \end{array}$$

where the maps out of K_0 are the maps we construct in our Riemann-Roch theorems, the right vertical map is the homology Chern character, and the lower horizontal map takes an algebraic cycle to its homology class. This should be thought of as "dual" to the diagram

$$\begin{array}{ccc} K^0 & \longrightarrow & K^0_{top} \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ A^{\cdot}_{\mathbf{Q}} & \longrightarrow & H^{\cdot}(\ ;\mathbf{Q}) \end{array}$$

where the horizontal maps are the natural maps from algebraic objects to topological ones.

All four of these pairs of natural transformations are compatible, as in 1) of our Riemann-Roch theorem. The horizontal maps translate algebraic geometry to topology. The top maps are with integer coefficients, and the bottom maps are induced by maps with integer coefficients. All the vertical maps become isomorphisms over $\mathbf{Q}$ (provided we take just the even part of the homology and cohomology) (Chapter IV, § 1 and [App., 3.3]).

We will give the K-theory version of Riemann-Roch in another paper.

(0.9) The methods of this paper extend to give a Lefschetz fixed point theorem for singular varieties which specializes to [P. Donovan, The Lefschetz-Riemann-Roch

Formula, *Bull. Soc. Math. France*, 97 (1969), pp. 257-273] in the non-singular case. We also obtain explicit contributions to the Lefschetz number at isolated (possibly singular) fixed points. For an automorphism of finite order, this extends the Atiyah-Bott formula ([M. F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes, I, *Annals of Math.*, 86 (1967), pp. 374-407], [M. F. Atiyah and G. B. Segal, The index of elliptic operators: II, *Annals of Math.*, 87 (1968), pp. 531-545]) to singular varieties. This will be the subject of another paper.

It also appears that this Riemann-Roch map is just the zero-th part of Riemann-Roch maps $K'_i X \to K_i^{\mathrm{top}} X$, where $K'_i X$ is the higher K-group of Quillen [Higher algebraic K-theory, Algebraic K-theory I, *Springer Lecture Notes in Mathematics*, 341 (1973)]. For non-singular varieties this question is not difficult; for singular varieties we have a proposed definition of these maps. We plan to report on this later.

(0.10) *Notation*:

If X is a subspace of Y, and $i : X \to Y$ is the imbedding, and $x \in H_\bullet X$, $y \in H^\bullet Y$, we write $y \frown x$ instead of $i^* y \frown x$, for any of our homology-cohomology theories.

If E is a vector bundle on a space X, we write P(E) for the bundle over X whose fibre over a point in X is the set of lines in E over that point, as in [G], not [EGA]; similarly for Grassmann-bundles. We often use the same letter to denote an algebraic vector bundle and the associated locally free sheaf, saying " the bundle E ", or " the sheaf E " to distinguish the concepts when necessary. We write $\check{E}$ for the dual bundle (or, sheaf).

The Todd class of a bundle E is denoted td(E). If M is non-singular, we write

$$\mathrm{Td}(M) = \mathrm{td}(T_M)$$

for the Todd class of its tangent bundle T_M.

(0.11) An outline of our Riemann-Roch theorem, using differential-geometric methods, appears in [Baum]. The main results were also announced at Arcata [F], where a preliminary version of this paper was distributed.

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Chapter I

RIEMANN-ROCH BY DIFFERENCE-BUNDLE

In this chapter we use singular homology and cohomology with rational coefficients; we write $H_{\bullet}X$ for $H_{\bullet}(X; \mathbf{Q})$ and $H^{\bullet}(A, B)$ for $H^{\bullet}(A, B; \mathbf{Q})$. The Grothendieck group of topological vector bundles on a compact space X will be denoted $K^0_{\mathrm{top}}(X)$. When X has a base point the reduced group will be denoted by $\widetilde{K}^0_{\mathrm{top}}(X)$.

1. The Localized Class $\mathrm{ch}^M_X E_{\bullet}$ by Difference-Bundle.

Let X be a compact complex analytic subspace of a complex manifold M. Define

$$K^0(M, M-X) = \varprojlim \widetilde{K}^0(M/C)$$

where the limit is over all closed subsets C of $M-X$.

Atiyah and Hirzebruch have shown [A-H 1] how to construct an element $d(E_{\bullet})$ in $K^0(M, M-X)$ from a complex $E_{\bullet}$:

$$0 \to E_r \xrightarrow{d_r} E_{r-1} \to \dots \to E_0 \to 0$$

of topological vector-bundles on M which is exact off X. We recall their construction.

Let $F_i = \mathrm{Ker}(d_i)$ and choose splitting isomorphisms $E_i \cong F_i \oplus F_{i-1}$ on $M-X$. This gives isomorphisms

$$E_{ev} = \sum_k E_{2k} \cong \sum_i F_i$$
$$E_{odd} = \sum_k E_{2k+1} \cong \sum_i F_i.$$

Composing the first with the inverse of the second gives an isomorphism $\sigma : E_{ev} \xrightarrow{\cong} E_{odd}$ on $M-X$. Choose an isomorphism of $E_{odd} \oplus F$ with a trivial bundle ε^N, for a suitable bundle F on M. Then

$$E_{ev} \oplus F \xrightarrow{\sigma \oplus 1} E_{odd} \oplus F \cong \varepsilon^N$$

trivializes $E_{ev} \oplus F$ on $M-X$. Therefore $E_{ev} \oplus F$ defines a compatible collection of bundles on M/C, C closed in $M-X$, and so $E_{ev} \oplus F - \varepsilon^N$ determines the desired element $d(E_{\bullet})$ in the limit group $K^0(M, M-X)$.

If we note that $H^{\bullet}(M, M-X) = \varprojlim \widetilde{H}^{\bullet}(M/C)$, the Chern character gives a mapping

$$\mathrm{ch} : K^0(M, M-X) \to H^{\bullet}(M, M-X).$$

The Lefschetz duality isomorphism $\widetilde{\mathrm{H}}^{\bullet}(\mathrm{M}/\mathrm{C}) \cong \mathrm{H}_{\bullet}(\mathrm{M}-\mathrm{C})$ for C a neighborhood retract (cf. [Spanier, *Algebraic Topology*, McGraw-Hill (1966), p. 297]) passes to the limit to give an isomorphism

$$\mathrm{L} : \mathrm{H}^{\bullet}(\mathrm{M}, \mathrm{M}-\mathrm{X}) \xrightarrow{\approx} \mathrm{H}_{\bullet}\mathrm{X}.$$

We then have $\mathrm{K}^0(\mathrm{M}, \mathrm{M}-\mathrm{X}) \xrightarrow{\mathrm{ch}} \mathrm{H}^{\bullet}(\mathrm{M}, \mathrm{M}-\mathrm{X}) \xrightarrow{\mathrm{L}} \mathrm{H}_{\bullet}\mathrm{X}.$

Define $\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_{\bullet} = \mathrm{L}(\mathrm{ch}(d(\mathrm{E}_{\bullet})))$.

2. Basic Properties of $\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_{\bullet}$.

We list six fundamental properties of this construction. Except for a variation in (2.5), X, M and $\mathrm{E}_{\bullet}$ will be as in § 1.

Property **(2.1)** (Localization).

(a) If $\mathrm{X} \subset \mathrm{Y} \subset \mathrm{M}$, where Y is another compact analytic subspace of M, and j denotes the imbedding of X in Y, then

$$j_{*}\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_{\bullet} = \mathrm{ch}_{\mathrm{Y}}^{\mathrm{M}}\mathrm{E}_{\bullet}.$$

(b) If i is the imbedding of X in M, then

$$i_{*}\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_{\bullet} = \mathrm{ch}\ \mathrm{E}_{\bullet} \frown [\mathrm{M}] = \sum_{i} (-1)^{i}\, \mathrm{ch}\ \mathrm{E}_{i} \frown [\mathrm{M}].$$

Property **(2.2)** (Additivity). — If $\mathrm{E}_{\bullet}$ is a direct sum of two complexes $\mathrm{E}'_{\bullet}$ and $\mathrm{E}''_{\bullet}$, then

$$\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_{\bullet} = \mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}'_{\bullet} + \mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}''_{\bullet}.$$

Property **(2.3)** (Module). — If F is a vector-bundle on M, then

$$\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}(\mathrm{F} \otimes \mathrm{E}_{\bullet}) = \mathrm{ch}\ \mathrm{F} \frown \mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_{\bullet}.$$

Property **(2.4)** (Excision). — If $\mathrm{X} \subset \mathrm{U} \subset \mathrm{M}$, with U open in M, then

$$\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_{\bullet} = \mathrm{ch}_{\mathrm{X}}^{\mathrm{U}}(\mathrm{E}_{\bullet} \mid \mathrm{U}).$$

Property **(2.5)** (Homotopy). — Let $\mathrm{X} \subset \mathrm{M}$ as in § 1. Let C be a connected complex manifold, D a complex manifold, $\pi : \mathrm{D} \to \mathrm{C}$ a smooth (1) mapping, and $i : \mathrm{M} \times \mathrm{C} \to \mathrm{D}$ a closed imbedding so that

$$\begin{tikzcd}
\mathrm{M} \times \mathrm{C} \arrow[rr, hook, "i"] \arrow[dr, "p"'] & & \mathrm{D} \arrow[dl, "\pi"] \\
& \mathrm{C} &
\end{tikzcd}$$

(1) In this context "smooth" means a holomorphic mapping such that for each $p \in \mathrm{M}$ the induced map of tangent spaces $\mathrm{T}_p\mathrm{M} \to \mathrm{T}_{\pi(p)}\mathrm{C}$ is surjective. For general algebraic varieties we refer to [EGA IV, 17.5].

commutes, where p is the projection. Let $E_\bullet$ be a complex of bundles on D, exact off $X \times C$. Then for each $t \in C$, $E_\bullet$ induces a complex $E_{\bullet t}$ on $D_t = \pi^{-1}(t)$ exact off $X_t = X \times \{t\} = X$, and the resulting class $\mathrm{ch}_X^{D_t}(E_{\bullet t})$ in $H_\bullet X$ is independent of t.

Property **(2.6)** (Pull-back). — Let $p : P \to M$ be a smooth, proper mapping, and let $Q = p^{-1}(X)$, $q : Q \to X$ the restriction to X. Then $p^*(E_\bullet)$ is a complex on P exact off Q, and

$$q^*(\mathrm{ch}_X^M E_\bullet) = \mathrm{ch}_Q^P(p^* E_\bullet)$$

where $q^* : H_\bullet X \to H_\bullet Q$ is the homology Gysin map.

(When $H_\bullet$ is singular homology, we define the homology Gysin map

$$q^* : H_\bullet X \to H_\bullet Q,$$

for simplicity, by requiring commutativity in the diagram

$$\begin{tikzcd} H^\bullet(M, M-X) \arrow[r, "p^*"] \arrow[d, "L"', "\wr"] & H^\bullet(P, P-Q) \arrow[d, "L"', "\wr"] \\ H_\bullet X \arrow[r, "q^*"] & H_\bullet Q \end{tikzcd}$$

If X is non-singular, this agrees with the map obtained by using Poincaré duality.)

The first four properties are easy consequences of the definition. For the homotopy, we may replace C by a compact disk. Then by standard techniques of extending $\mathscr{C}^\infty$ vector fields, the product structure on $M \times C$ extends to a neighborhood U of $M \times C$ in D, $U = U_0 \times C$. Let i_t inject U_0 as $U_0 \times t$ and let $[U_0]_t$ be the Borel-Moore homology orientation of U_0 given by the complex structure on U_0 induced by i_t. If we apply the construction of § 1 to $X \times C \subset D$ and $E_\bullet$, then $\mathrm{ch}(d(E_\bullet))$ maps to $\mathrm{ch}_X^{D_t}(E_{\bullet t})$ by the composite

$$H^\bullet(U, U - X \times C) \xrightarrow{i_t^*} H^\bullet(U_0, U_0 - X) \xrightarrow{\frown [U_0]_t} H_\bullet(X).$$

But these are equal since the i_t are homotopic and the $[U_0]_t$ are determined by homotopic complex structures.

Property (2.6) follows from the fact that $d(p^* E_\bullet) = p^*(d(E_\bullet))$ in $K^0(P, P-Q)$, and the above description of the homology Gysin map.

3. More Properties of $\mathrm{ch}_X^M E_\bullet$.

We prove several more facts about this construction. Although some of these could be proved directly and easily from the definition—the reader is invited to do so—we prefer to show how they can be derived from the basic Properties (2.1-2.6).

When we construct a localized class algebraically in Chapter II which satisfies Properties (2.1-2.6), we will then be able to conclude that it satisfies all the other properties of this section, and that Riemann-Roch is true for the Chow theory.

Proposition **(3.1)**. — *Let* $0\to E'_\bullet \xrightarrow{\alpha} E_\bullet \xrightarrow{\beta} E''_\bullet \to 0$ *be an exact sequence of complexes on* M, *each exact off* X. *Then*

$$\mathrm{ch}_X^M E_\bullet = \mathrm{ch}_X^M E'_\bullet + \mathrm{ch}_X^M E''_\bullet.$$

Proof. — We deform the exact sequence into the split exact sequence. Let $p: M\times \mathbf{C}\to M$ be the projection, and define a surjection of complexes on $M\times\mathbf{C}$

$$h: p^*E_\bullet \oplus p^*E''_\bullet \to p^*E''_\bullet$$

by $h(e, e'') = \beta(e) - te''$ if e and e'' are in fibres over a point $(m, t)\in M\times\mathbf{C}$, $t\in\mathbf{C}$. Let $\widetilde{E}_\bullet$ be the kernel of h. Then $\widetilde{E}_\bullet$ is exact off $X\times\mathbf{C}$, and $\widetilde{E}_\bullet$ restricts to $E'_\bullet\oplus E''_\bullet$ at $t=0$, and to $E_\bullet$ at $t=1$, so the result follows from Properties (2.5) and (2.2).

Lemma **(3.2)**. — *Let* $F_\bullet$ *be the complex obtained by shifting* $E_\bullet$ *one place to the left*: $F_i = E_{i-1}$ *(with corresponding boundaries). Then*

$$\mathrm{ch}_X^M F_\bullet = -\mathrm{ch}_X^M E_\bullet.$$

Proof. — Construct the « algebraic mapping cylinder » $G_\bullet$, where

$$G_i = F_i \oplus E_i = E_{i-1}\oplus E_i, \quad \text{and} \quad d_i(f, e) = (df, de + (-1)^i f).$$

Then $G_\bullet$ is exact on all of M, so $\mathrm{ch}_X^M G_\bullet = 0$ (Property (2.1) for $\emptyset\subset X\subset M$). Since there is an exact sequence

$$0\to E_\bullet \to G_\bullet \to F_\bullet \to 0$$

we can conclude by Proposition (3.1).

Proposition **(3.3)**. — *Let* $E_\bullet$ *be a complex of bundles on* M, *exact off* X, *and let* $F_\bullet$ *be any complex of bundles on* M. *Then* $F_\bullet\otimes E_\bullet$ *is exact off* X, *and*

$$\mathrm{ch}_X^M(F_\bullet\otimes E_\bullet) = \mathrm{ch}(F_\bullet) \frown \mathrm{ch}_X^M E_\bullet.$$

Proof. — If the boundary maps in $F_\bullet$ are all zero this follows from the lemma and Properties (2.2) and (2.3). For the general case let $p: M\times\mathbf{C}\to M$, and consider the complex $\widetilde{F}_\bullet\otimes p^*E_\bullet$ on $M\times\mathbf{C}$, where $\widetilde{F}_i = p^*F_i$ but the boundary maps of $\widetilde{F}_\bullet$ over a point $(m, t)\in M\times\mathbf{C}$ are t times the boundary maps of $F_\bullet$. This gives a homotopy between the zero-boundary case and the general case.

Proposition **(3.4)**. — *Let* $E_\bullet$ *be a complex of bundles on* M *exact off* X, *and let* $\pi: N\to M$ *be a vector bundle over* M, *with* M *regarded as a subspace of* N *by the zero-section. Let* $\Lambda^\bullet\pi^*\check{N}$ *be the Koszul-Thom complex on* N (cf. [A-H 2, Prop. (2.5)]). *Then* $\Lambda^\bullet\pi^*\check{N}\otimes\pi^*E_\bullet$ *is exact on* $N-X$, *and*

$$\mathrm{ch}_X^N(\Lambda^\bullet\pi^*\check{N}\otimes\pi^*E_\bullet) = \mathrm{td}(N)^{-1}\frown \mathrm{ch}_X^M(E_\bullet).$$

Proof. — The exactness on $N-X$ results from the fact that a tensor product of complexes is exact where either of the complexes is exact.

Imbed N in its projective completion $P=P(N\oplus 1)$ (cf. [G, § 5]), let $p: P\to M$ be the projection, and let $q: Q=p^{-1}(X)=P((N\oplus 1)\,|\,X)\to X$ be the restriction over X.

On P we have an exact sequence

$$0\to H\to p^*(\check{N}\oplus 1)\to \mathcal{O}_P(1)\to 0.$$

Since $p^*(\check{N}\oplus 1)=p^*(\check{N})\oplus 1$, projection on the second factor gives a homomorphism of sheaves

$$H\to \mathcal{O}_P$$

which is surjective off M. Such a homomorphism from a locally free sheaf H to the trivial sheaf $\mathcal{O}_P$ gives rise to a Koszul complex $\wedge^{\bullet}H$ on P, exact off M. This complex restricts to $\wedge^{\bullet}\pi^*\check{N}$ on N. By the excision Property (2.4)

$$\mathrm{ch}_X^N(\wedge^{\bullet}\pi^*\check{N}\otimes\pi^*E_{\bullet})=\mathrm{ch}_X^P(\wedge^{\bullet}H\otimes p^*E_{\bullet}).$$

Let $s: X\to Q$ be the zero section. Then

$$s_*(\mathrm{ch}_X^P(\wedge^{\bullet}H\otimes p^*E_{\bullet}))=\mathrm{ch}_Q^P(\wedge^{\bullet}H\otimes p^*E_{\bullet})$$

by the localization Property (2.1). But $p^*E_{\bullet}$ is exact off Q, so by Proposition (3.3)

$$\mathrm{ch}_Q^P(\wedge^{\bullet}H\otimes p^*E_{\bullet})=\mathrm{ch}(\wedge^{\bullet}H)\frown\mathrm{ch}_Q^P(p^*E_{\bullet}).$$

Now $\mathrm{ch}_Q^P(p^*E_{\bullet})=q^*\mathrm{ch}_X^M(E_{\bullet})$ by the pull-back Property (2.6), and $q_*s_*=\text{identity}$. Therefore (cf. [App., § (3.1)])

$$q_*(\mathrm{ch}(\wedge^{\bullet}H)\frown q^*\mathrm{ch}_X^M E_{\bullet})=p_*(\mathrm{ch}(\wedge^{\bullet}H))\frown\mathrm{ch}_X^M E_{\bullet}.$$

Putting all this together, we are reduced to proving the formal identity

$$p_*(\mathrm{ch}\wedge^{\bullet}H)=\mathrm{td}(N)^{-1}$$

or, by the projection formula,

$$p_*(\mathrm{ch}\wedge^{\bullet}H\smile p^*\,\mathrm{td}(N))=1.$$

We use the basic identity [B-S; Lemma 18]

$$\mathrm{ch}\wedge^{\bullet}H=c_e(\check{H})\,\mathrm{td}(\check{H})^{-1}$$

where $e=\mathrm{rank}\,H=\mathrm{rank}\,N$. From the exact sequence defining H we see that

$$p^*\,\mathrm{td}(N)=\mathrm{td}(p^*N\oplus 1)=\mathrm{td}(\check{H})\,\mathrm{td}(\mathcal{O}(-1)).$$

Therefore $\mathrm{ch}(\wedge^{\bullet}H)\,.\,p^*\,\mathrm{td}(N)=c_e(\check{H})\,.\,\mathrm{td}(\mathcal{O}(-1))$, so we are reduced to showing that

$$p_*(c_e(\check{H})\,\mathrm{td}(\mathcal{O}(-1)))=1.$$

Let $z=c_1(\mathcal{O}(1))$. Since

$$0=c_{e+1}(\check{N}\oplus 1)=c_e(\check{H})\,.\,c_1(\mathcal{O}(-1))=-zc_e(\check{H}),$$

and $\mathrm{td}(\mathcal{O}(-1))-1$ is a multiple of z, we are reduced to showing

$$p_*(c_e(\check{\mathrm{H}}))=1.$$

Finally, since

$$c(\check{\mathrm{H}})=p^*c(\mathrm{N})/c(\mathcal{O}(-1)),$$

$$c_e(\check{\mathrm{H}})=\sum_{i=0}^{e}p^*c_i(\mathrm{N})z^{e-i}, \quad \text{so} \quad p_*c_e(\check{\mathrm{H}})=p_*z^e=1.$$

4. Coherent Sheaves.

Let X be a projective variety, and imbed X in a non-singular quasi-projective variety M. If $\mathscr{F}$ is a coherent sheaf on X, let $\mathrm{E}_\bullet$ be a complex of vector bundles on M that resolves $\mathscr{F}$, and define

$$\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathscr{F}=\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_\bullet.$$

Proposition **(4.1)**. — $\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathscr{F}$ *does not depend on the resolution* $\mathrm{E}_\bullet$.

Proof. — Since two resolutions are dominated by a third [B-S; Lemma 13], if $\mathrm{E}''_\bullet$ is another we may assume there is an exact sequence $0\to\mathrm{E}'_\bullet\to\mathrm{E}_\bullet\to\mathrm{E}''_\bullet\to 0$, where $\mathrm{E}'_\bullet$ is exact on all of M. Then $\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}_\bullet=\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}''_\bullet+\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}'_\bullet=\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathrm{E}''_\bullet$ by Proposition (3.1) and Property (2.1).

Since an exact sequence of sheaves can be resolved by an exact sequence of bundles [B-S; Proof of Lemma 12], we likewise deduce the following fact:

Proposition **(4.2)**. — *If* $0\to\mathscr{F}'\to\mathscr{F}\to\mathscr{F}''\to 0$ *is an exact sequence of sheaves on* X, *then*

$$\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathscr{F}=\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathscr{F}'+\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}\mathscr{F}''.$$

Therefore $\mathrm{ch}_{\mathrm{X}}^{\mathrm{M}}$ defines a homomorphism from $\mathrm{K}_0\mathrm{X}$ to $\mathrm{H}_\bullet\mathrm{X}$. We can see from Proposition (3.4) how this homomorphism depends on the imbedding, at least in a special case.

5. Deformation to the Normal Bundle.

Proposition **(5.1)**. — *Let* $\mathrm{M}\subset\mathrm{P}$ *be an imbedding of non-singular quasi-projective varieties, and let* N *be the normal bundle. Then there is a non-singular variety* D, *an imbedding* $\mathrm{M}\times\mathbf{C}\subset\mathrm{D}$, *and a smooth morphism* $\pi:\mathrm{D}\to\mathbf{C}$ *which restricts to the projection* $\mathrm{M}\times\mathbf{C}\to\mathbf{C}$ *on* $\mathrm{M}\times\mathbf{C}$:

$$\begin{tikzcd} \mathrm{M}\times\mathbf{C} \arrow[rr, hook] \arrow[dr] & & \mathrm{D} \arrow[dl, "\pi"] \\ & \mathbf{C} & \end{tikzcd}$$

For each $t\in\mathbf{C}$ *we get an imbedding*

$$\mathrm{M}=\mathrm{M}\times\{t\}\subset\pi^{-1}(t)=\mathrm{D}_t$$

with the following properties:

1) *For* $t \neq 0$, *the imbedding* $M \subset D_t$ *is isomorphic to the given imbedding of* M *in* P.

2) *For* $t = 0$, *the imbedding* $M \subset D_0$ *is isomorphic to the imbedding of* M *as the zero-section of* N.

Proof [1]. — Imbed P as a locally closed subvariety of a projective space $\mathbf{P}^N$, and choose homogeneous polynomials $F_1, \ldots, F_r$ (in $N+1$ variables), with $\deg F_i = d_i$, which define M (scheme-theoretically) in P. Let E be the bundle over P whose sheaf of sections is $\mathcal{O}_P(d_1) \oplus \ldots \oplus \mathcal{O}_P(d_r)$, and let $s : P \to E$ be the section determined by $(F_1, \ldots, F_r)$. The fact that $F_1, \ldots, F_r$ define M scheme-theoretically means that $(F_1, \ldots, F_r)$ maps the sheaf $\check{E} = \bigoplus_i \mathcal{O}(-d_i)$ onto the ideal-sheaf $\mathcal{I}$ of M in P. Restricting to M gives $\check{E} | M \to \mathcal{I}/\mathcal{I}^2 \to 0$. This is dual to an imbedding of the bundle N in $E | M$.

Throughout the proof we regard $M \subset P \subset E$ by means of the zero-section of E; thus $M = s^{-1}(P)$ as a scheme.

Let $\mathbf{C}^* = \mathbf{C} - \{0\}$, and consider the imbedding

$$P \times \mathbf{C}^* \hookrightarrow E \times \mathbf{C}$$

by the map $(p, t) \to \left(\frac{1}{t} s(p), t\right)$. Let D be the closure of $P \times \mathbf{C}^*$ in $E \times \mathbf{C}$, $\pi : D \to \mathbf{C}$ the projection.

We first notice that the product imbedding $M \times \mathbf{C} \subset E \times \mathbf{C}$ imbeds $M \times \mathbf{C}$ in D, since s is the zero-section on M.

If $t \neq 0$, $D_t = \frac{1}{t} s(P) \times \{t\}$, and the imbedding $M \subset \frac{1}{t} s(P)$ is isomorphic to the imbedding of M in P, proving (1).

To check (2) and smoothness, we study the situation locally on P. We assume P is an affine subvariety of $\{(x_0, \ldots, x_N) \in \mathbf{P}^N | x_0 \neq 0\}$, so the ideal of M is generated by $f_i = F_i(1, x_1, \ldots, x_n)$ in the coordinate ring of P. Shrinking P if necessary, and renumbering the f_i, we may assume $f_1, \ldots, f_k$ define M in P, and $f_i = \sum_{j=1}^{k} a_{ij} f_j$ for $i > k$; k is the codimension of M in P, and a_{ij} are regular functions on P. Since $\mathcal{O}(1)$ is canonically trivial on $\{(x_0, \ldots, x_N) | x_0 \neq 0\}$, E is trivial over P; let $y_1, \ldots, y_r$ be fibre coordinates for E. We claim that in $E \times \mathbf{C} = P \times \mathbf{C}^r \times \mathbf{C}$ the equations for D are

$$t y_i = f_i \qquad i = 1, \ldots, k$$

$$y_i = \sum_{j=1}^{k} a_{ij} y_j \qquad i = k+1, \ldots, r.$$

[1] *Note added in proof*. S. Kleiman and I. Vainsencher have pointed out that this construction may be done intrinsically, as in [M. Gerstenhaber, On the deformation of rings and algebras: II, *Annals of Math.*, **84** (1966), 1-19].

To see this let D' be the subscheme of $E \times \mathbf{C}$ defined by these equations. The Jacobian criterion shows $D' \to \mathbf{C}$ is smooth, with fibres of the same dimension as P. It is clear that $D'_t = D_t$ for $t \neq 0$. And D'_0 is defined by the equations

$$f_i = 0 \qquad i = 1, \ldots, k$$

$$y_i = \sum_{j=1}^{k} a_{ij} y_j \qquad i = k+1, \ldots, r.$$

But these equations define the normal bundle N in $E|M$.

Since $D' \to \mathbf{C}$ is smooth and all the fibres are connected, D' is non-singular and irreducible; since D' agrees with D where $t \neq 0$, $D' = D$. This finishes the proof.

Remark. — Even if P is projective (complete), the variety D is not proper over $\mathbf{C}$. If one takes the closure $\overline{D}$ of D in $P(E \oplus 1) \times \mathbf{C}$ the fibre $\overline{D}_0$ has two components $P(N \oplus 1)$ and $\hat{P} = P$ blown up along M, which meet transversally along $P(N)$ (see Chapter IV, § 3).

Lemma (5.2). — *With* M, P, D, $M \times \mathbf{C} \subset D$ *as in Proposition* (5.1), *let* $p : M \times \mathbf{C} \to M$ *be the projection. Let* $\mathscr{F}$ *be a coherent sheaf on* M, *and let* $E_{\bullet}$ *be a resolution of* $p^*\mathscr{F}$ *by vector bundles on* D. *Then for all* $t \in \mathbf{C}$, $E_{\bullet,t}$ *is a resolution of* $\mathscr{F}$ *by vector bundles on* D_t.

Proof. — Let $\pi : D \to \mathbf{C}$ be the projection. The natural resolution of $\mathcal{O}_{D_t}$ by locally free sheaves is

$$0 \to \mathcal{O}_D \xrightarrow{\pi - t} \mathcal{O}_D \to \mathcal{O}_{D_t} \to 0.$$

Since $\pi - t$ is not a zero divisor on $p^*\mathscr{F} = \mathscr{F} \otimes_{\mathcal{O}_M} \mathcal{O}_{M \times \mathbf{C}}$, tensoring the above sequence with $p^*\mathscr{F}$ shows that $\mathrm{Tor}_i^{\mathcal{O}_D}(p^*\mathscr{F}, \mathcal{O}_{D_t}) = 0$ for $i > 0$. Since $\mathrm{Tor}_i^{\mathcal{O}_D}(p^*\mathscr{F}, \mathcal{O}_{D_t})$ is the i-th homology of $E_{\bullet,t} = E_{\bullet} \otimes_{\mathcal{O}_D} \mathcal{O}_{D_t}$, this proves the lemma.

Proposition (5.3). — *Let* $X \subset M$, $M \subset P$ *be closed subvarieties, with* M *and* P *non-singular. Let* N *be the normal bundle of* M *in* P. *Then for any coherent sheaf* $\mathscr{F}$ *on* X

$$\mathrm{ch}_X^P \mathscr{F} = \mathrm{td}(N)^{-1} \frown \mathrm{ch}_X^M \mathscr{F}.$$

Proof. — Take $M \times \mathbf{C} \subset D$ as in Proposition (5.1), and a resolution $E_{\bullet}$ of $p^*\mathscr{F}$ as in Lemma (5.2). Then the homotopy Property (2.5) reduces it to the case where M is embedded as the zero section of N. And this case is covered by Proposition (3.4), since if $E_{\bullet}$ resolves $\mathscr{F}$ on M, $\Lambda^{\bullet}\pi^*\check{N} \otimes \pi^* E_{\bullet}$ resolves $\mathscr{F}$ on N.

6. Construction of τ and Proof of Riemann-Roch.

Fix a projective variety X. For any imbedding of X in a non-singular quasi-projective variety M, and sheaf $\mathscr{F}$ on X, define

$$\tau^M(\mathscr{F}) = \mathrm{Td}(M) \frown \mathrm{ch}_X^M(\mathscr{F})$$

where Td(M) is the Todd class of the tangent bundle to M. By Proposition (4.2), τ^M defines a homomorphism from K_0X to $H_\bullet X$. We will show that $\tau=\tau^M$ is independent of the imbedding and satisfies the conditions of the Riemann-Roch theorem (§ 0.1). We do this in several small steps.

(1) If $X\subset Y\subset M$, and j is the imbedding of X in Y, the diagram

$$\begin{array}{ccc} K_0X & \xrightarrow{\tau^M} & H_\bullet X \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ K_0Y & \xrightarrow{\tau^M} & H_\bullet Y \end{array}$$

commutes. This follows from Property (2.1).

(2) If $X\subset M\subset P$, with M and P non-singular, then $\tau^M=\tau^P$. This follows from Proposition (5.3) and the identity

$$\mathrm{Td}(P)\smile \mathrm{td}(N)^{-1}=\mathrm{Td}(M) \quad \text{in } H^\bullet M.$$

(3) If $p: P\to \mathrm{pt.}$ maps a projective space to a point, then the diagram

$$\begin{array}{ccc} K_0P & \xrightarrow{\tau^P} & H_\bullet P \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ K_0(\mathrm{pt.}) & \xrightarrow{\tau^{\mathrm{pt.}}} & H_\bullet(\mathrm{pt.}) \end{array}$$

commutes. This is an easy formal calculation, since K_0P is generated by powers of the hyperplane bundle [B-S; Prop. 10].

(4) If F is an algebraic vector-bundle on M, and $\mathscr{F}$ is a sheaf on X, $X\subset M$ as above, then

$$\tau^M(F\otimes\mathscr{F})=\mathrm{ch}\, F\frown\tau^M(\mathscr{F}).$$

This follows from the module property (2.3), since if $E_\bullet$ resolves $\mathscr{F}$ on M, then $F\otimes E_\bullet$ resolves $F\otimes\mathscr{F}$.

(5) If $X\subset M$ as above, and P is a projective space, then the diagram

$$\begin{array}{ccc} K_0X\otimes K_0P & \xrightarrow{\tau^M\otimes\tau^P} & H_\bullet X\otimes H_\bullet P \\ \downarrow & & \downarrow \\ K_0(X\times P) & \xrightarrow{\tau^{M\times P}} & H_\bullet(X\times P) \end{array}$$

commutes, where the vertical arrows are Künneth maps. For $K_0P = K^0P$ is generated by vector bundles, so by (4) we are reduced to showing

$$\tau^{M\times P}(q^*\mathscr{F}) = \tau^M\mathscr{F} \times (\mathrm{Td}\,P \frown [P])$$

where $\mathscr{F}$ is a sheaf on X, and $q : X\times P \to X$ is the projection. But this follows from the pull-back property (2.6) applied to $p : M\times P \to M$, and the fact that

$$\mathrm{Td}(M\times P) = \mathrm{Td}\,M \times \mathrm{Td}\,P.$$

(6) If $X \subset M$, and P is a projective space, so $X\times P \subset M\times P$ by the product, then the diagram

$$\begin{tikzcd} K_0(X\times P) \arrow[r, "\tau^{M\times P}"] \arrow[d, "p_*"'] & H_\bullet(X\times P) \arrow[d, "p_*"] \\ K_0X \arrow[r, "\tau^M"] & H_\bullet X \end{tikzcd}$$

commutes. Here p is the projection. We can see this by fitting a " cube " over this square, whose top square is

$$\begin{tikzcd} K_0X\otimes K_0P \arrow[r, "\tau^M\otimes\tau^P"] \arrow[d, "1\otimes p_*"] & H_\bullet X\otimes H_\bullet P \arrow[d, "1\otimes p_*"] \\ K_0X\otimes K_0(\mathrm{pt.}) \arrow[r, "\tau^M\otimes\tau^{\mathrm{pt.}}"] & H_\bullet X\otimes H_\bullet(\mathrm{pt.}) \end{tikzcd}$$

and the maps to the bottom square are all Künneth maps. The top commutes by (3), two sides commute by (5), and the other two commute by natural properties of the Künneth maps. Since $K_0X\otimes K_0P \to K_0(X\times P)$ is surjective [B-S; Prop. 9], the bottom square must commute.

(7) Let $X\subset P$, $Y\subset Q$ be imbeddings of varieties X and Y in projective spaces P and Q. Let $f : X\to Y$ be a morphism, and regard

$$X \subset X\times Y \subset P\times Y \subset P\times Q$$

by means of the mapping $x \mapsto (x, f(x))$. Then the diagram

$$\begin{tikzcd} K_0X \arrow[r, "\tau^{P\times Q}"] \arrow[d, "f_*"'] & H_\bullet X \arrow[d, "f_*"] \\ K_0Y \arrow[r, "\tau^Q"] & H_\bullet Y \end{tikzcd}$$

commutes. For this diagram is obtained by fitting together the diagrams

$$\begin{array}{ccc} K_0X & \xrightarrow{\tau^{P\times Q}} & H_{\bullet}X \\ \downarrow & & \downarrow \\ K_0(P\times Y) & \xrightarrow{\tau^{P\times Q}} & H_{\bullet}(P\times Y) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ K_0Y & \xrightarrow{\tau^{Q}} & H_{\bullet}Y \end{array}$$

and the top of this commutes by (1), and the bottom by (6).

(8) The mapping $\tau=\tau^{M}$ is independent of the imbedding. For by (2) we need only consider imbeddings in projective spaces. And if $X\subset P$, $X\subset Q$ were two such imbeddings, apply (7) to the identity map on X to conclude that $\tau^{P\times Q}=\tau^{Q}$, and by symmetry $\tau^{P}=\tau^{P\times Q}=\tau^{Q}$.

(9) The mapping τ is natural. For if $f: X\rightarrow Y$ is an imbedding, just imbed Y in a non-singular M and use (1). If f is a projection $P\times Y\rightarrow Y$, it follows from (7). A general f is a composite of two such mappings, as in (7).

(10) The mapping τ gives the right formula on a non-singular variety X. This follows from (2) above, with $X=M\subset P$.

(11) The module property follows from (4) and the fact that a vector-bundle on any quasi-projective variety is the restriction of an algebraic vector-bundle on some non-singular M containing X [App., § (3.2)].

Remark. — If one assumes all the results of [A-H 2], this proof of Riemann-Roch may be shortened considerably. The construction of τ and proof of naturality is as given in this section, but using only imbeddings in projective spaces. The fact that τ gives the right answer for non-singular varieties is the content of [A-H 2; § 3].

Chapter II

RIEMANN-ROCH BY GRASSMANNIAN-GRAPH

In this chapter we work in the category of quasi-projective schemes over an algebraically closed field k of arbitrary characteristic. In fact k need not be algebraically closed. We leave to the reader interested in that case the verification that all the cycles constructed are rational over the ground-field. The reader in the opposite camp may read " variety " wherever we write " scheme ".

For such a scheme X, we let $A_{\bullet}X$ be the Chow group of cycles modulo rational equivalence, graded according to dimension. This " Chow homology theory " is discussed in the appendix [App.], where a " cohomology " theory $A^{\bullet}$ is constructed to go with this, with the usual formal properties—cap products, projection formulae, Poincaré duality for non-singular varieties, Gysin homomorphisms, Chern classes, etc.

Write $H_{\bullet}X = A_{\bullet}X_{\mathbf{Q}} = A_{\bullet}X \otimes \mathbf{Q}$, $H^{\bullet}X = A^{\bullet}X_{\mathbf{Q}}$. There is the Chern character $\mathrm{ch} : K^0 \to H^{\bullet}$ [App., § (3.3)]. We will prove:

Theorem. — There is a unique natural transformation $\tau : K_0 \to H_{\bullet}$ *of covariant functors from the category of quasi-projective schemes and proper mappings to the category of abelian groups satisfying:*

(1) *For any* X *the diagram*

$$\begin{tikzcd}
K^0X \otimes K_0X \arrow[r, "\otimes"] \arrow[d, "\mathrm{ch}\otimes\tau"] & K_0X \arrow[d, "\tau"] \\
H^{\bullet}X \otimes H_{\bullet}X \arrow[r, "\frown"] & H_{\bullet}X
\end{tikzcd}$$

is commutative.

(2) *If* X *is non-singular*

$$\tau(\mathcal{O}_X) = \mathrm{Td}(X) \frown [X].$$

(3) *If* U *is an open subscheme of* X, *the diagram*

$$\begin{array}{ccc} K_0X & \xrightarrow{\tau} & H_\bullet X \\ \downarrow & & \downarrow \\ K_0U & \xrightarrow{\tau} & H_\bullet U \end{array}$$

is commutative, where the vertical maps are restrictions [App., § (1.9)]. (Chapters III and IV contain more properties of the map τ.)

To prove this we will construct localized classes satisfying properties analogous to (and more general than) those in Chapter I, § 2. (The construction gives an alternate approach to the case with singular homology; for non-compact varieties Borel-Moore homology [*Michigan Math. J.*, 7 (1960), pp. 137-159] should be used.)

In this chapter $\mathbf{A}^n$ and $\mathbf{P}^n$ denote affine and projective space over k.

1. The Localized class $\mathrm{ch}_X^M E_\bullet$ by Grassmannian Graph.

Let X be a closed subscheme of an irreducible variety M. It is not necessary to assume M is smooth over k, but the smooth case will suffice for the Riemann-Roch theorem and most applications. (In fact the construction goes through with little change even if M is not irreducible or reduced, but for simplicity here we take M to be a variety.)

For each complex $E_\bullet$ of bundles on M, exact off X, we will construct a class $\mathrm{ch}_X^M E_\bullet$ in $H_\bullet X$ by using the Grassmannian graph construction. The notation of this section will be used throughout the rest of Chapter II.

Suppose our complex is

$$0 \longrightarrow E_r \xrightarrow{d_r} E_{r-1} \xrightarrow{d_{r-1}} \dots \xrightarrow{d_1} E_0 \xrightarrow{d_0} E_{-1} = 0.$$

Let e_i be the rank of E_i, and let $G_i = \mathrm{Grass}_{e_i}(E_i \oplus E_{i-1})$ be the Grassmann bundle (over M) of e_i-dimensional planes in $E_i \oplus E_{i-1}$. Let ξ_i be the tautological bundle on G_i; it is the subbundle of $E_i \oplus E_{i-1}$ (pulled back to G_i) whose fibre over a point in G_i is the subspace represented by that point.

Let $G = G_r \times_M G_{r-1} \times \dots \times_M G_0$, $\pi : G \to M$ the projection. The bundles ξ_i pull back to bundles on G, still denoted ξ_i, and we take

$$\xi = \xi_0 - \xi_1 + \xi_2 - \dots + (-1)^r \xi_r$$

to be the " virtual tautological bundle " on G.

Any bundle map $\varphi : E_i \to E_{i-1}$ determines a section $s(\varphi)$ of G_i over M; the value of $s(\varphi)$ at $m \in M$ is the graph of φ in the fibre over m. Thus

$$s(\varphi)(m) = \{(v, \varphi(v)) \mid v \in (E_i)_m\} \in G_i.$$

For each $\lambda \in k$ we obtain a section $s_\lambda : M \to G$ by taking the section $s(\lambda d_i)$ in the factor G_i, where $d_i : E_i \to E_{i-1}$ is the boundary map in the complex $E_\bullet$.

Regard $\mathbf{A}^1 \subset \mathbf{P}^1$ by $\lambda \mapsto (1 : \lambda)$ as usual, so $\mathbf{P}^1 = \mathbf{A}^1 \cup \{\infty\}$, $\infty = (0 : 1)$. The mapping $(m, \lambda) \mapsto (s_\lambda(m), (1 : \lambda))$ gives an imbedding

$$\mathrm{M} \times \mathbf{A}^1 \to \mathrm{G} \times \mathbf{P}^1.$$

Let n be the dimension of M. Let W be the closure of $\mathrm{M} \times \mathbf{A}^1$ in $\mathrm{G} \times \mathbf{P}^1$ under this imbedding. Let Z_∞ be the n-cycle cut out by W at ∞; i.e. let $\varphi : \mathrm{W} \to \mathbf{P}^1$ be the projection, and let $\mathrm{Z}_\infty \times \{\infty\} = \varphi^*([\infty]) = \mathrm{W}_{\bullet\varphi}[\infty]$ ([S; V], [App., § 2]). If M is non-singular, $\mathrm{Z}_\infty \times \{\infty\}$ is the intersection-cycle of W and $\mathrm{G} \times \{\infty\}$.

Lemma (**1.1**). — *The cycle* Z_∞ *has a unique decomposition* $\mathrm{Z}_\infty = \mathrm{Z} + [\mathrm{M}_*]$, *where*

(1) M_* *is an irreducible variety.*

(2) π *maps* M_* *birationally onto* M, *isomorphically off* X.

(3) π *maps the cycle* Z *into* X.

Proof. — Since the construction of Z_∞ restricts naturally to open subsets of M, we may reduce to the case where $\mathrm{E}_\bullet$ is exact on all of M. We show in this case how to extend the imbedding $\mathrm{M} \times \mathbf{A}^1 \to \mathrm{G} \times \mathbf{P}^1$ to an imbedding $\mathrm{M} \times \mathbf{P}^1 \subset \mathrm{G} \times \mathbf{P}^1$, from which it will follow that $\mathrm{Z}_\infty = [\mathrm{M}_*] \cong [\mathrm{M}]$.

Now $\mathrm{Ker}(d_i)$ is a subbundle of E_i. We imbed $\mathrm{M} \times \mathbf{P}^1$ in $\mathrm{G} \times \mathbf{P}^1$ by assigning to a point $(m, (\lambda_0 : \lambda_1))$ in $\mathrm{M} \times \mathbf{P}^1$ the point $(\mathrm{H}, (\lambda_0 : \lambda_1))$ where H is the subspace of $(\mathrm{E}_i)_m \oplus (\mathrm{Ker}\, d_{i-1})_m$ defined by the equations

$$\lambda_0 z_{i-1} = \lambda_1 d_i e_i$$

where $z_{i-1} \in (\mathrm{Ker}\, d_{i-1})_m$, $e_i \in (\mathrm{E}_i)_m$. If $\lambda_0 \neq 0$, this gives the same subspace of

$$(\mathrm{E}_i)_m \oplus (\mathrm{E}_{i-1})_m$$

as

$$s\left(\frac{\lambda_1}{\lambda_0} d_i\right),$$

but if $\lambda_0 = 0$ we get the subspace $(\mathrm{Ker}\, d_i)_m \oplus (\mathrm{Ker}\, d_{i-1})_m$, still of the right dimension. One checks that this imbeds $\mathrm{M} \times \mathbf{P}^1$ in $\mathrm{G} \times \mathbf{P}^1$, and so concludes the proof.

The cycle Z determines a class in $\mathrm{H}_\bullet(\pi^{-1}\mathrm{X})$, which may also be denoted Z. Then $\mathrm{ch}\,\xi \frown \mathrm{Z} \in \mathrm{H}_\bullet(\pi^{-1}\mathrm{X})$, and we define

$$\mathrm{ch}_\mathrm{X}^\mathrm{M} \mathrm{E}_\bullet = \pi_*(\mathrm{ch}\,\xi \frown \mathrm{Z}) \quad \text{in } \mathrm{H}_\bullet \mathrm{X}.$$

In Chapter IV, § 3 all these cycles and classes are determined explicitly in the case where X is a local complete intersection in M and $\mathrm{E}_\bullet$ resolves a locally free sheaf on X.

2. Basic Properties of $\mathrm{ch}_\mathrm{X}^\mathrm{M} \mathrm{E}_\bullet$.

We prove stronger versions of the properties stated in Chapter I.

Property (**2.1**) (Localization). — (a) *If* $\mathrm{X} \subset \mathrm{Y} \subset \mathrm{M}$, *where* Y *is another subscheme of* M, *and* j *denotes the imbedding of* X *in* Y, *then*

$$j_* \mathrm{ch}_\mathrm{X}^\mathrm{M} \mathrm{E}_\bullet = \mathrm{ch}_\mathrm{Y}^\mathrm{M} \mathrm{E}_\bullet.$$

(b) *If* i *is the imbedding of* X *in* M, *then*
$$i_* \operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}_\bullet = \operatorname{ch} \mathrm{E}_\bullet \frown [\mathrm{M}].$$

Proof. — (a) is clear from the construction. We prove (b). Let $\mathrm{Z}_\lambda = s_\lambda(\mathrm{M}) \subset \mathrm{G}$. Then W gives a rational equivalence between Z_0 and Z_∞. So $\operatorname{ch} \xi \frown \mathrm{Z}_\infty = \operatorname{ch} \xi \frown \mathrm{Z}_0$ in $\mathrm{H}_\bullet \mathrm{G}$. When $\lambda = 0$, λd_i is the zero map, so ξ restricts to $\sum\limits_i (-1)^i \mathrm{E}_i$ on $\mathrm{Z}_0 \cong [\mathrm{M}]$. So $\pi_*(\operatorname{ch} \xi \frown \mathrm{Z}_\infty) = \operatorname{ch} \mathrm{E}_\bullet \frown [\mathrm{M}]$ in $\mathrm{H}_\bullet \mathrm{M}$.

To finish the proof we must show that $\pi_*(\operatorname{ch} \xi \frown [\mathrm{M}_*]) = 0$. In fact we will show that ξ restricts to zero on M_*.

Let k_i be the rank of $\operatorname{Ker}(d_i)$ on $\mathrm{M} - \mathrm{X}$, where it is a bundle. Define
$$\mathrm{G}_* = \operatorname{Grass}_{k_r} \mathrm{E}_r \times_{\mathrm{M}} \ldots \times_{\mathrm{M}} \operatorname{Grass}_{k_0} \mathrm{E}_0.$$
There is a closed imbedding $\mathrm{G}_* \subset \mathrm{G}$ of bundles over M which assigns to the collection of subspaces S_i of E_i the collection of subspaces $\mathrm{S}_i \oplus \mathrm{S}_{i-1}$ of $\mathrm{E}_i \oplus \mathrm{E}_{i-1}$. Then the virtual tautological bundle ξ restricts to zero on G_*.

There is a section
$$s : \mathrm{M} - \mathrm{X} \to \mathrm{G}_*$$
which assigns to a point m in $\mathrm{M} - \mathrm{X}$ the collection of subspaces $(\operatorname{Ker} d_i)_m$ of $(\mathrm{E}_i)_m$. If we look at the proof of Lemma (1.1), and consider how G_* is imbedded in G, we see that $s(\mathrm{M} - \mathrm{X})$ agrees with M_* over $\mathrm{M} - \mathrm{X}$. Since G_* is closed in G, M_* (being the closure of $s(\mathrm{M} - \mathrm{X})$) must be contained in G_*, so $\xi \mid \mathrm{M}_* = 0$, as desired.

Remark. — Although the construction of $\operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}_\bullet$ is rather delicate, the above proof shows one fortunate way in which it is not. With Z_∞ as in § 1, we may take *any* cycle $\mathrm{M}'_* \subset \mathrm{G}_*$ such that Z_∞ and M'_* agree over $\mathrm{M} - \mathrm{X}$. Then if we set $\mathrm{Z}' = \mathrm{Z}_\infty - \mathrm{M}'_*$, $\operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}_\bullet = \pi_*(\operatorname{ch} \xi \frown \mathrm{Z}')$. This fact will be crucial in the proof of the homotopy property.

Property **(2.2)** (Additivity). — *If* $\mathrm{E}_\bullet$ *is a direct sum of two complexes* $\mathrm{E}'_\bullet$ *and* $\mathrm{E}''_\bullet$, *then*
$$\operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}_\bullet = \operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}'_\bullet + \operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}''_\bullet.$$

Proof. — We denote by one or two primes the spaces, bundles, cycles, and mappings constructed for $\mathrm{E}'_\bullet$ and $\mathrm{E}''_\bullet$ as in § 1. The natural imbedding $\mathrm{G}'_i \times_{\mathrm{M}} \mathrm{G}''_i \subset \mathrm{G}_i$ gives a closed imbedding $\mathrm{G}' \times_{\mathrm{M}} \mathrm{G}'' \subset \mathrm{G}$ under which ξ restricts to $\widetilde{\xi}' \oplus \widetilde{\xi}''$, where $\widetilde{\xi}'$ is the pull-back of ξ' to $\mathrm{G}' \times_{\mathrm{M}} \mathrm{G}''$, and similarly for ξ''. Since the imbedding of $\mathrm{M} \times \mathbf{A}^1$ in $\mathrm{G} \times \mathbf{P}^1$ maps it into $\mathrm{G}' \times_{\mathrm{M}} \mathrm{G}'' \times \mathbf{P}^1$, we may regard W as a cycle on $\mathrm{G}' \times_{\mathrm{M}} \mathrm{G}'' \times \mathbf{P}^1$. Let $p' : \mathrm{G}' \times_{\mathrm{M}} \mathrm{G}'' \times \mathbf{P}^1 \to \mathrm{G}' \times \mathbf{P}^1$ be the projection. Since p' is the identity on $\mathrm{M} \times \mathbf{A}^1$, $p'_*[\mathrm{W}] = [\mathrm{W}']$ as cycles. Since the push-forward of a rational equivalence is a rational equivalence [App., § 1.8], $p'_* \mathrm{Z}_\infty = \mathrm{Z}'_\infty$. Also $p'_*[\mathrm{M}_*] = [\mathrm{M}'_*]$, since $\mathrm{M}_* \to \mathrm{M}'_* \to \mathrm{M}$ is birational. So $p'_* \mathrm{Z} = \mathrm{Z}'$, and likewise $p''_* \mathrm{Z} = \mathrm{Z}''$. Therefore
$$\begin{aligned}
\operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}_\bullet &= \pi_*(\operatorname{ch}(\widetilde{\xi}'' \oplus \widetilde{\xi}'') \frown \mathrm{Z}) \\
&= \pi_*(\operatorname{ch} \widetilde{\xi}' \frown \mathrm{Z}) + \pi_*(\operatorname{ch} \widetilde{\xi}'' \frown \mathrm{Z}) \\
&= \pi'_* p'_*(\operatorname{ch} \widetilde{\xi}' \frown \mathrm{Z}) + \pi''_* p''_*(\operatorname{ch} \widetilde{\xi}'' \frown \mathrm{Z}) \\
&= \pi'_*(\operatorname{ch} \xi' \frown \mathrm{Z}') + \pi''_*(\operatorname{ch} \xi'' \frown \mathrm{Z}) \\
&= \operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}'_\bullet + \operatorname{ch}^{\mathrm{M}}_{\mathrm{X}} \mathrm{E}''_\bullet.
\end{aligned}$$

Property **(2.3)** (Module). — *If* F *is a vector-bundle on* M, *then*

$$\mathrm{ch}_X^M(F\otimes E_\bullet)=\mathrm{ch}\,F\frown \mathrm{ch}_X^M E_\bullet.$$

Proof. — Let $f=\mathrm{rank}\,F$, and let

$$\widetilde{G}_i=\mathrm{Grass}_{fe_i}((F\otimes E_i)\oplus(F\otimes E_{i-1})),\qquad \widetilde{G}=\widetilde{G}_r\times_M\ldots\times\widetilde{G}_0.$$

There is a natural imbedding of G in $\widetilde{G}$ which maps a subspace S_i of $E_i\oplus E_{i-1}$ to the subspace $F\otimes S_i$ of $(F\otimes E_i)\oplus(F\otimes E_{i-1})$. The virtual tautological bundle $\widetilde{\xi}$ on $\widetilde{G}$ restricts to $\pi^*F\otimes\xi$ on G. In the imbedding of $M\times\mathbf{A}^1$ in $\widetilde{G}\times\mathbf{P}^1$ used in constructing $\mathrm{ch}_X^M(F\otimes E_\bullet)$, we see that

$$M\times\mathbf{A}^1\subset G\times\mathbf{P}^1\overset{j}{\subset}\widetilde{G}\times\mathbf{P}^1.$$

It follows that the cycle j_*Z is the same as the corresponding cycle constructed for $F\otimes E_\bullet$, so

$$\begin{aligned}\mathrm{ch}_X^M(F\otimes E_\bullet)&=\widetilde{\pi}_*(\mathrm{ch}\,\widetilde{\xi}\frown j_*Z)\\&=\pi_*((\mathrm{ch}\,\pi^*F\smile\mathrm{ch}\,\xi)\frown Z)\\&=\mathrm{ch}\,F\frown\pi_*(\mathrm{ch}\,\xi\frown Z)\\&=\mathrm{ch}\,F\frown\mathrm{ch}_X^M E_\bullet.\end{aligned}$$

Property **(2.4)** (Excision). — *Let* M_0 *be an open subscheme of* M, $X_0=X\cap M_0$. *Then* $\mathrm{ch}_X^M E_\bullet$ *restricts to* $\mathrm{ch}_{X_0}^{M_0}(E_\bullet\,|\,M_0)$ *under the restriction* $H_\bullet X\to H_\bullet X_0$.

Proof. — This follows from the fact that the entire construction restricts to M_0. It is also a special case of property (2.6) below.

Property **(2.5)** (Homotopy). — *Let* C *be a smooth (geometrically) connected curve over* k. *Suppose* X *is a closed subscheme of* M, *and* $f:M\to C$ *is a flat morphism whose restriction* g *to* X *is also flat. Let* $E_\bullet$ *be a complex of bundles on* M, *exact off* X. *For each* $t\in C$ *we get an imbedding of the fibres* $X_t\subset M_t$, *and a complex* $E_{\bullet t}$ *on* M_t *exact off* X_t. *If* $i_t:X_t\to X$ *is the inclusion, then*

$$\mathrm{ch}_{X_t}^{M_t}E_{\bullet t}=i_t^*\,\mathrm{ch}_X^M E_\bullet$$

where $i_t^*:H_\bullet X\to H_\bullet X_t$ *is the Gysin homomorphism* [App., § 4].

Remark. — In the language of specialization [App., § 4.4], this implies that the localized class $\mathrm{ch}_{X_t}^{M_t}E_{\bullet t}$ for the general fibre X_t specializes to the localized class $\mathrm{ch}_{X_s}^{M_s}E_{\bullet s}$ for the special fibre X_s.

Corollary. — *If* $X=Y\times C$ *in the above,* g *is the projection to* C, *and* C *is a rational curve, then all the classes* $\mathrm{ch}_Y^{M_t}E_{\bullet t}$ *are equal in* $H_\bullet Y$.

Proofs. — The corollary follows since all the maps

$$i_t^*:H_\bullet(Y\times C)\to H_\bullet(Y\times\{t\})=H_\bullet Y$$

are the same if C is rational [App., § 4.3].

To prove the homotopy property, let $\pi : \mathrm{G} \to \mathrm{M}$, ξ, $\mathrm{W} \subset \mathrm{G} \times \mathbf{P}^1$ be as constructed in § 1 for $\mathrm{E}_\bullet$ on M. Examples show that the projection $\mathrm{W} \to \mathrm{C} \times \mathbf{P}^1$ may not be equidimensional (i.e. some fibres may have bigger dimension than the generic fibre), so W does not determine a family of cycles parametrized by $\mathrm{C} \times \mathbf{P}^1$. We will overcome this difficulty by blowing up $\mathrm{C} \times \mathbf{P}^1$ so that W becomes equidimensional (cf. claim below).

Let $\rho : \mathrm{V} \to \mathrm{C} \times \mathbf{P}^1$ be a birational, proper morphism from a non-singular surface V onto $\mathrm{C} \times \mathbf{P}^1$ which is an isomorphism over $\mathrm{C} \times \mathbf{A}^1$. For such V, and any subvariety S of V, and any scheme T over C, we denote by

$$\mathrm{T}_{\mathrm{S}} = \mathrm{T} \times_{\mathrm{C}} \mathrm{S}$$

the fibre product, where S maps to C by the composite $\mathrm{S} \subset \mathrm{V} \xrightarrow{\rho} \mathrm{C} \times \mathbf{P}^1 \to \mathrm{C}$. A similar subscript is used for morphisms between schemes over C. Note that if a point $v \in \mathrm{V}$ maps to a point $t \in \mathrm{C}$, then $\mathrm{T}_v = \mathrm{T}_t$ is the fibre of T over $t \in \mathrm{C}$. The following diagram may clarify the situation.

$$\begin{tikzcd}
\mathrm{G}_{\mathrm{S}} \arrow[r] \arrow[d] & \mathrm{G}_{\mathrm{V}} \arrow[r] \arrow[d, "\pi_{\mathrm{V}}"'] \arrow[dd, bend left, "p"] & \mathrm{G} \times \mathbf{P}^1 \arrow[r] \arrow[d] & \mathrm{G} \arrow[d, "\pi"] \\
\mathrm{M}_{\mathrm{S}} \arrow[r] \arrow[d] & \mathrm{M}_{\mathrm{V}} \arrow[r] \arrow[d] & \mathrm{M} \times \mathbf{P}^1 \arrow[r] \arrow[d] & \mathrm{M} \arrow[d] \\
\mathrm{S} \arrow[r] & \mathrm{V} \arrow[r, "\rho"'] & \mathrm{C} \times \mathbf{P}^1 \arrow[r] & \mathrm{C}
\end{tikzcd}$$

If $\mathrm{S} = \mathrm{V}$, then G_{V} maps birationally onto $\mathrm{G} \times \mathbf{P}^1$, under which an open subscheme of G_{V} becomes identified with $\mathrm{G} \times \mathbf{A}^1$. Thus for example the imbedding $\mathrm{M} \times \mathbf{A}^1 \subset \mathrm{G} \times \mathbf{A}^1$ of the Grassmannian-graph construction may be regarded as an imbedding $\mathrm{M} \times \mathbf{A}^1 \subset \mathrm{G}_{\mathrm{V}}$.

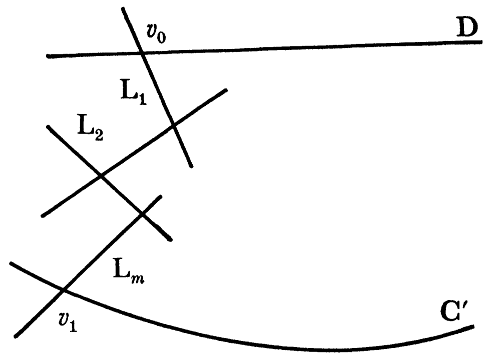
Claim. — There is a proper birational $\rho : \mathrm{V} \to \mathrm{C} \times \mathbf{P}^1$ from a non-singular surface V onto $\mathrm{C} \times \mathbf{P}^1$ which is an isomorphism over $\mathrm{C} \times \mathbf{A}^1$, so that if $\widetilde{\mathrm{W}}$ is the closure of $\mathrm{M} \times \mathbf{A}^1$ in G_{V}, then the morphism $\varphi : \widetilde{\mathrm{W}} \to \mathrm{V}$ induced by the projection $p : \mathrm{G}_{\mathrm{V}} \to \mathrm{V}$ is equidimensional.

Before discussing the claim, we show how it can be used to conclude the proof. Let M_* be the subvariety of G constructed in Lemma (1.1). Then $\mathrm{M}_{*\mathrm{V}} \to \mathrm{V}$ is equidimensional, since it pulls back from $\mathrm{M}_* \to \mathrm{C}$. Set

$$z = [\widetilde{\mathrm{W}}] - [\mathrm{M}_{*\mathrm{V}}],$$

an $(n+1)$-cycle on G_{V} ($n = \dim \mathrm{M}$).

Fix $t \in \mathrm{C}$, let D be the non-singular rational curve on V which maps isomorphically by ρ to $\{t\} \times \mathbf{P}^1$, and let v_0 be the point on D that maps to $\{t\} \times \{\infty\}$. Let C′ be the non-singular curve on V that maps isomorphically by ρ to $\mathrm{C} \times \{\infty\}$, and let v_1 be the point on C′ which maps to $\{t\} \times \{\infty\}$. Since ρ is a birational proper morphism between non-singular surfaces, $\rho^{-1}(\{t\} \times \{\infty\})$ is a connected collection of rational curves $L_1, \ldots, L_m$ which meet transversally. (Cf. [Zariski, Introduction to the problem of minimal models in the theory of algebraic surfaces, *Mathematical Society of Japan*, 1958].)



The idea of the proof is as follows. If we restrict z first to D, and then to v_0, we obtain the cycle needed to calculate the localized class for $E_{\bullet,t}$. By the equidimensionality assumption, using Serre's intersection theory, this restriction can be done directly from V to v_0. Similarly, restricting z to C′, and then to v_1, gives the cycle for $i_t^* \mathrm{ch}_X^M E$. Travelling from v_0 to v_1 along the lines L_i will give the required rational equivalence between them.

Since V is non-singular, for any cycle w on G_V whose components are all equidimensional over V, and any cycle η on V, the intersection cycle $w\bullet_p\eta$ on G_V is defined ([S; V], [App., § 2]).

Now $[\widetilde{W}\bullet_p D]$ is the "W-cycle" used in computing the localized class of $E_{\bullet,t}$, since it agrees with the desired cycle over $\{t\}\times\mathbf{A}^1$. Therefore [1] $[\widetilde{W}]\bullet_p[v_0]$ is the Z_∞-cycle used for this construction. Since $[\widetilde{W}]\bullet_p[v_0]$ and $[M_{*V}]\bullet_p[v_0]$ agree over $M_t - X_t$, and $M_{*V}\subset G_{*V}$, we may use the remark in § 2.1 to deduce:

(1) $\mathrm{ch}_{X_t}^{M_t} E_{\bullet,t} = \pi_{v_0*}(\mathrm{ch}\,\xi \frown (z\bullet_p[v_0]))$ in $H_\bullet X_t$ (where we identify $G_{v_0}=G_t$, $X_{v_0}=X_t$).
Similarly, with $C'\cong C$, $X_{C'}=X$, we get

(2) $\mathrm{ch}_X^M E_\bullet = \pi_{C'*}(\mathrm{ch}\,\xi \frown (z\bullet_p[C']))$ in $H_\bullet X$.

Consider the fibre square

$$\begin{tikzcd}
\pi^{-1}(X_{v_0}) \arrow[r, "j_{v_0}"] \arrow[d, "\pi_{v_0}"] & \pi_D^{-1}(X_D) \arrow[d, "\pi_D"] & \subset & G_D \arrow[d] \\
X_{v_0} \arrow[r, "i_{v_0}"] & X_D & \subset & M_D
\end{tikzcd}$$

Then $j_{v_0}^*(z\bullet_p[D]) = z\bullet_p[v_0]$, so

$$\pi_{v_0*}(\mathrm{ch}\,\xi \frown (z\bullet_p[v_0])) = \pi_{v_0*} j_{v_0}^*(\mathrm{ch}\,\xi \frown (z\bullet_p[D])) = i_{v_0}^* \pi_{D*}(\mathrm{ch}\,\xi \frown (z\bullet_p[D]))$$

---

[1] If v is a point on a non-singular curve S on V, then $w\bullet_p[v]=(w\bullet_p[S])\ {}_{p_S}[v]$ (cf. [App., § 2.2, Lemma 4]).

[App., § 4.2]. Therefore from (1) we get

(3) $$\mathrm{ch}_{X_t}^{M_t} E_{\bullet,t} = i_{v_0}^* \pi_{D*}(\mathrm{ch}\, \xi \frown (z \bullet_p [D])).$$

The same argument, using $v_0 \in L_1$ in place of $v_0 \in D$ shows that the right-hand side of (1) is also equal to the Gysin pull-back of $\pi_{L_1 *}(\mathrm{ch}\, \xi \frown (z \bullet_p [L_1]))$ under the imbedding $X_{v_0} = X_t \subset X_{L_1} = X_t \times L_1$. Now if we let v vary in L_1, these Gysin pull-backs will not vary [App., § 4.3]. We move similarly through the curves $L_2, \ldots, L_m$, until we arrive at the equation

(4) $$\mathrm{ch}_{X_t}^{M_t} E_{\bullet,t} = \pi_{v_1 *}(\mathrm{ch}\, \xi \frown (z \bullet_p [v_1])).$$

And the same argument applied to $v_1 \in C'$ shows that the right-hand side of (4) is equal to the Gysin pull-back of $\pi_{C'*}(\mathrm{ch}\, \xi \frown (z \bullet_p [C']))$ under the imbedding i_t of $X_t = X_{v_1}$ in $X = X_{C'}$. By (2) this completes the proof.

The claim is a consequence of Grothendieck's construction of the Hilbert schemes. This construction gives us a birational morphism $\rho_1 : V_1 \to C \times \mathbf{P}^1$, isomorphic over $C \times \mathbf{A}^1$, and a subscheme $\widetilde{W}_1$ of G_V which extends $M \times \mathbf{A}^1$ and is flat over V_1. (See [R; Chapter 4, § 2] for a discussion of this as well as generalizations to the non-projective case.) If $V \to V_1$ is taken to resolve the singularities of V_1, then the composite

$$V \to V_1 \to C \times \mathbf{P}^1$$

will satisfy the conditions of the claim.

Property **(2.6)** (Pull-back). — *Let* $p : P \to M$ *be a flat morphism, and let* $Q = p^{-1}(X)$, $q : Q \to X$ *the restriction to* X. *Then* $p^* E_\bullet$ *is a complex on* P *exact off* Q, *and*

$$q^*(\mathrm{ch}_X^M E_\bullet) = \mathrm{ch}_Q^P(p^* E_\bullet)$$

where $q^* : H_\bullet X \to H_\bullet Q$ *is the Gysin map* [App., § 1.9].

Proof. — We claim that the entire construction for $p^* E_\bullet$ on P is obtained by pulling back the construction for $E_\bullet$ on M. Denote the corresponding spaces for $\widetilde{E}_\bullet = p^* E_\bullet$ by $\widetilde{G}$, etc. We have a fibre square

$$\begin{array}{ccc} \widetilde{G} = G \times_M P & \xrightarrow{\widetilde{\pi}} & P \\ \big\downarrow{\scriptstyle \widetilde{p}} & & \big\downarrow{\scriptstyle p} \\ G & \xrightarrow[\pi]{} & M \end{array}$$

$\widetilde{\xi} = \widetilde{p}^* \xi$, $\widetilde{W} = \widetilde{p}^{-1} W$, so $\widetilde{Z}_\infty = p^* \widetilde{Z}_\infty$ since rational equivalence pulls back [App., § 1.9]. Also $\widetilde{M}_* = \widetilde{p}^* M_*$, so $\widetilde{Z} = \widetilde{q}^* Z$, where $\widetilde{q} : \widetilde{\pi}^{-1}(Q) \to \pi^{-1}(X)$. Therefore

$$\begin{aligned} \mathrm{ch}_Q^P \widetilde{E}_\bullet &= \widetilde{\pi}_*(\mathrm{ch}(\widetilde{p}^* \xi) \frown \widetilde{q}^* Z) \\ &= \widetilde{\pi}_*(\widetilde{q}^*(\mathrm{ch}\, \xi \frown Z)) \\ &= q^* \pi_*(\mathrm{ch}\, \xi \frown Z) \\ &= q^* \mathrm{ch}_X^M E_\bullet \end{aligned}$$

where we have used [App., § 3.1].

3. Proof of Riemann-Roch.

Since §§ 3-6 of Chapter I used only these six properties of the localized class (together with formal properties of homology and cohomology), we see that the Riemann-Roch theorem as stated at the beginning of this chapter is true. The additional condition (3) on restricting to open subschemes follows immediately from the strengthened form of the excision Property (2.4).

Chapter III

UNIQUENESS AND GRADED K

1. The Chow Groups and Graded K-Groups.

Let X be a quasi-projective scheme over a field. Consider the filtration on K_0X by dimension of support [SGA 6]. $\mathrm{Filt}_k K_0X$ is generated by classes of sheaves whose support has dimension $\leq k$, or by the structure sheaves of subvarieties of dimension $\leq k$. The associated graded groups $\mathrm{Gr}_k X$ define a theory closely related to the Chow groups $A_k X$. If we assign to a subvariety Y of X the class of its structure sheaf $\mathcal{O}_Y$ in K_0X, we obtain [App., § 1.9] a natural surjective transformation

$$A_\bullet \xrightarrow{\varphi} \mathrm{Gr}_\bullet$$

of functors from the category of quasi-projective schemes and proper morphisms to the category of graded abelian groups. Even if X is non-singular, φ may not be an isomorphism [SGA 6; XIV, 4.7]. Grothendieck showed in the non-singular case that φ is an isomorphism modulo torsion [*ibid.*, 4.2]. Our Riemann-Roch theorem enables us to extend this to the singular case, with a somewhat simpler proof.

Theorem. — For all quasi-projective schemes X *over a field:*

(a) φ *induces an isomorphism* $A_\bullet X_{\mathbf{Q}} \xrightarrow{\sim} \mathrm{Gr}_\bullet X_{\mathbf{Q}}$.

(b) *The Riemann-Roch map* τ *induces an isomorphism*

$$K_0 X_{\mathbf{Q}} \xrightarrow{\sim} A_\bullet X_{\mathbf{Q}}.$$

Proof. — We show that the associated graded map to the map in (b) gives the inverse to the map in (a). If Y is a subvariety of X, $i: Y \to X$ the imbedding, and we regard $\mathcal{O}_Y$ as a sheaf on X, then $\tau(\mathcal{O}_Y) = i_*\tau(Y)$ is contained in $i_*(A_\bullet Y_{\mathbf{Q}})$, by naturality of Riemann-Roch. Therefore τ maps $\mathrm{Filt}_k K_0 X$ into

$$\mathrm{Filt}_k(A_\bullet X_{\mathbf{Q}}) = \sum_{j \leq k} A_j X_{\mathbf{Q}}.$$

Thus τ induces a mapping $\mathrm{Gr}_\bullet X \to A_\bullet X_{\mathbf{Q}}$ of associated graded groups. Both (a) and (b) will follow if we show that the composite

$$A_\bullet X_{\mathbf{Q}} \xrightarrow{\varphi} \mathrm{Gr}_\bullet X_{\mathbf{Q}} \to A_\bullet X_{\mathbf{Q}}$$

is the identity, and this is an immediate consequence of the following lemma, applied to irreducible subvarieties of X.

Lemma. — *If* X *is an irreducible variety, then the top dimensional cycle in* $\tau(X)$ *is* $[X]$.

Proof. — This follows by restricting to the non-singular part X_0 of X, where it is clear by (2) of the Riemann-Roch Theorem. Or one may let $\overline{X}$ be a projective closure of X, and apply naturality to a finite map $\overline{X} \to \mathbf{P}^n$ to reduce it to $\mathbf{P}^n$.

2. Uniqueness Theorems.

We consider only projective varieties over a field. (If τ is determined on these, it is determined on all quasi-projective varieties by condition (3) of the theorem, and on schemes by applying naturality to injections of irreducible subvarieties.) $A_{\bullet}X_{\mathbf{Q}}$ is the Chow group with rational coefficients.

In our first uniqueness theorem no mention is made of Todd classes or Chern classes of bundles. We see that the Todd class, and the Riemann-Roch formula for a non-singular variety, are completely determined if we want any kind of natural theorem. The Todd class does, however, naturally enter into the arguments at several points (see Chapter I, Proposition 3.4 and Chapter IV, Proposition 1.3). For an explicit differential-forms approach to the inevitability of the Todd class see [Baum].

Theorem. — *There is only one additive natural transformation* $\tau : K_0 \to A_{\bullet\mathbf{Q}}$ *with the property that if* P *is a projective space, the top dimensional cycle in* $\tau(\mathcal{O}_P)$ *is* $[P]$.

Proof. — Let $\tau_{\mathbf{Q}} : K_{0\mathbf{Q}} \to A_{\bullet\mathbf{Q}}$ be the map induced by τ.

We have constructed one such τ. Suppose τ' were another. Then by § 1, we get a natural transformation

$$\alpha = \tau'_{\mathbf{Q}} \circ \tau_{\mathbf{Q}}^{-1} : A_{\bullet\mathbf{Q}} \to A_{\bullet\mathbf{Q}}$$

which takes $[P]$ to $[P]+$lower terms, for P a projective space. But the only such natural transformation is the identity [App., § 5].

Remark. — If $\mathscr{F}$ is a sheaf on an irreducible variety X, then the top-dimensional cycle in $\tau(\mathscr{F})$ is $\operatorname{rank}(\mathscr{F}).[X]$. Of course, this property also determines τ uniquely.

If we include compatibility with the Chern character in our conditions for τ, then it only needs to be normalized on a point.

Corollary. — *There is a unique additive natural transformation* $\tau : K_0 \to A_{\bullet\mathbf{Q}}$ *satisfying*

(1) *If* E *is a vector bundle on* X, *then* $\tau(E) = \operatorname{ch} E \frown \tau(\mathcal{O}_X)$.

(2) *If* X *is a point, then* $\tau(\mathcal{O}_X) = 1$ *in* $\mathbf{Q} = A_{\bullet}X_{\mathbf{Q}}$.

Proof. — We must show $\tau(\mathcal{O}_{\mathbf{P}^n}) = [\mathbf{P}^n] +$ lower terms. If p is a point in $\mathbf{P}^n$, the Riemann-Roch theorem for the imbedding $i : \{p\} \to \mathbf{P}^n$ gives $\operatorname{ch}(i_*\mathcal{O}_{\{p\}}) \frown [\mathbf{P}^n] = [p]$.

Since $i_*\mathcal{O}_{\{p\}} \in K_0\mathbf{P}^n$, by (1) we must have $\tau(i_*\mathcal{O}_{\{p\}}) = \operatorname{ch}(i_*\mathcal{O}_{\{p\}}) \frown \tau(\mathcal{O}_{\mathbf{P}^n})$. By naturality and (2), $\tau(i_*\mathcal{O}_{\{p\}}) = i_*[p]$. These two equations imply that $\tau(\mathcal{O}_{\mathbf{P}^n}) = [\mathbf{P}^n] +$ lower terms.

Remark. — The theorem and corollary also hold for complex varieties with values in singular homology with rational coefficients. As in the proof of the theorem, we get a natural transformation

$$\alpha : A_{\bullet\mathbf{Q}} \to H_{\bullet}(\ ;\mathbf{Q})$$

such that $\alpha[P]=[P]+$lower terms for P a projective space. And the only such natural transformation is the one induced by the usual cycle map $A_{\bullet} \to H_{\bullet}(\ ;\mathbf{Z})$ [App., § 5].

3. Cartesian Products.

Theorem. — Let X, Y *be quasi-projective schemes. Then the diagram*

$$\begin{array}{ccc} K_0X \otimes K_0Y & \xrightarrow{\tau\otimes\tau} & A_{\bullet}X_{\mathbf{Q}} \otimes A_{\bullet}Y_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ K_0(X\times Y) & \xrightarrow{\tau} & A_{\bullet}(X\times Y)_{\mathbf{Q}} \end{array}$$

commutes; the vertical maps are the usual Künneth maps.

Corollary. — For any quasi-projective schemes X, Y

$$\tau(X\times Y)=\tau(X)\times\tau(Y).$$

Proof. — By § 1, the horizontal maps are isomorphisms when tensored with $\mathbf{Q}$. Consider the mapping

$$\theta : A_{\bullet}X_{\mathbf{Q}} \otimes A_{\bullet}Y_{\mathbf{Q}} \to A_{\bullet}(X\times Y)_{\mathbf{Q}}$$

obtained by going around the diagram $(\otimes\mathbf{Q})$ from upper right to upper left to lower left to lower right. This θ is an additive natural transformation of functors from pairs (X, Y) of quasi-projective schemes and morphisms to abelian groups. We must show θ is the usual Künneth product.

Since θ is compatible with restriction to open subschemes, we may restrict attention to projective schemes. Note also that $\theta([X]\otimes[Y])=[X\times Y]+$lower terms for varieties X, Y (§ 1, Lemma). It is not difficult, following Landman's proof for single spaces [App., § 5] to show that there is only one such natural transformation θ.

Chapter IV

THE TODD CLASS AND GYSIN MAPS

For a quasi-projective scheme X, let $\tau(X)=\tau(\mathcal{O}_X)$ be its Todd class. Write $\tau(X)=\sum_i \tau_i(X)$, $\tau_i(X)\in A_i(X)_{\mathbf{Q}}$.

1. Mappings.

If $f: X \to Y$ is a morphism, it is natural to compare the Todd classes of X and Y in terms of properties of f. This section contains four facts of this type. All of these are special cases of a conjectured formula, which will be stated in § 3. From part (3) of the Riemann-Roch Theorem in Chapter II we obtain the following fact:

Proposition **(1.1)**. — *If* X *is an open subscheme of* Y, *then the Todd class of* Y *restricts to the Todd class of* X.

This determines $\tau_k(X)$ for all k bigger than the dimension of the singularities of X. For example, if X is a projective normal surface, then $\deg \tau_0 X = \chi(X, \mathcal{O}_X)$, $\tau_1(X)=-K/2$ where K is a canonical divisor on X, and $\tau_2(X)=[X]$.

Corollary. — *Let* $f: X \to Y$ *be a birational proper morphism, and let* Z *be closed in* Y *such that* f *maps* $X - f^{-1}(Z)$ *isomorphically onto* $Y - Z$. *Then* $f_*\tau_k X = \tau_k Y$ *for all* $k > \dim Z$.

Proof. — In fact, $f_*\tau X$ and τY agree in $A_\bullet(Y-Z)$, and $A_k(Y) \to A_k(Y-Z)$ is an isomorphism for $k > \dim Z$ [App., § 1.9].

Proposition **(1.2)**. — *Let* $g: M \to N$ *be a smooth morphism of non-singular varieties,* Y *a closed subvariety of* N, $X = g^{-1}(Y)$, $f: X \to Y$ *the restriction of* g *to* X. *Then*

$$\tau(X) = \operatorname{td}(T_f) \frown f^*\tau(Y),$$

where T_f *is the relative tangent bundle of* f.

Proof. — From property (2.6) of Chapter II, we deduce $f^*\mathrm{ch}_Y^N \mathcal{O}_Y = \mathrm{ch}_X^M \mathcal{O}_X$. Then

$$\begin{aligned}\tau(X) &= \operatorname{td}(T_M) \frown \mathrm{ch}_X^M \mathcal{O}_X = \operatorname{td}(T_f) . g^*(\operatorname{td} T_N) \frown f^*\mathrm{ch}_Y^N \mathcal{O}_Y \\ &= \operatorname{td}(T_f) \frown f^*(\operatorname{td} T_N \frown \mathrm{ch}_Y^N \mathcal{O}_Y) = \operatorname{td}(T_f) \frown f^*\tau(Y).\end{aligned}$$

This applies for example if $X = P(E)$ is a projectivized vector-bundle over Y, giving the Todd class of X in terms of the Todd class of Y and the Chern classes of E.

Proposition **(1.3)** (Adjunction formula). — *Let* X *be an effective Cartier divisor on* Y, $i : X \to Y$ *the inclusion. Let* $x = c_1(\mathcal{O}(X)) \in A^1 Y$ *be the class determined by* X. *Then*

$$i_* \tau(X) = (1 - e^{-x}) \frown \tau(Y) \quad \textit{in } A_\bullet Y_{\mathbf{Q}}.$$

Proof. — From the exact sequence

$$0 \to \mathcal{O}(-X) \to \mathcal{O} \to i_* \mathcal{O}_X \to 0$$

we see that $\mathrm{ch}(i_* \mathcal{O}_X) = 1 - e^{-x}$. Therefore

$$i_* \tau(X) = \tau(i_* \mathcal{O}_X) = (1 - e^{-x}) \frown \tau(Y).$$

Proposition **(1.4)**. — *Let* X *be a local complete intersection in a non-singular variety* Y, $i : X \to Y$ *the inclusion,* N *the normal bundle,* T_Y *the tangent bundle to* Y. *Let*

$$T_X = i^* T_Y - N \in K^0 X$$

be the virtual tangent bundle of X. *Then* $\tau(X) = \mathrm{td}(T_X) \frown [X]$.

Proof. — To prove this it is enough to show $\mathrm{ch}_X^Y \mathcal{O}_X = \mathrm{td}(N)^{-1} \frown [X]$. This follows from Proposition (5.3) of Chapter I (with $X = M$, $\mathcal{F} = \mathcal{O}_X$, $Y = P$). Note that the non-singularity of M was not used in Chapter I, § 5. In fact, the results of Chapter I, § 5 hold for any local complete intersection $X \subset Y$. In § 3 we will discuss this case in more detail.

Remark. — The virtual tangent bundle is independent of the imbedding in Y [SGA 6; VIII].

2. Families.

Let C be a smooth (geometrically), connected curve, and let $f : X \to C$ be a flat morphism. (If X is an irreducible variety, flatness means only that f does not map X to a point.)

Theorem. — *For each (closed) point* $t \in C$, *let* $i_t : X_t \to X$ *be the inclusion of the fibre* $f^{-1}(t)$ *in* X. *Then*

$$\tau(X_t) = i_t^* \tau(X)$$

where $i_t^* : A_\bullet X_{\mathbf{Q}} \to A_\bullet X_{t\mathbf{Q}}$ *is the Gysin map* [App., § 4].

In particular, the Todd class of the general fibre specializes to the Todd class of the special fibres [App., § 4.4].

Proof. — Factor f into an imbedding $X \to P \times C$, where P is smooth, followed by the projection to C. Let $E_\bullet$ resolve $\mathcal{O}_X$ on $P \times C$. Then, for all $t \in C$, $E_{\bullet t}$ resolves $\mathcal{O}_{X_t}$ on $P_t = P \times \{t\}$. Therefore by the homotopy property (2.5) of Chapter II

$$\mathrm{ch}_{X_t}^{P_t} \mathcal{O}_{X_t} = i_t^* \mathrm{ch}_X^{P \times C} \mathcal{O}_X.$$

Since $i_t^* \mathrm{Td}(P \times C) = \mathrm{Td}(P_t)$, the theorem follows.

It follows that if $z \in A^k X$, the numerical function $\deg(z \frown \tau_k(X_t))$ is a constant function of t.

3. Local Complete Intersections.

Let $i : X \to Y$ imbed a scheme X as a local complete intersection in a quasi-projective scheme Y, with normal bundle N. Let F be a vector bundle on X, and $E_\bullet \to i_*(F)$ a resolution by vector bundles on Y.

We will compute explicitly all the cycles and bundles involved in the Grassmannian graph construction (Chapter II, § 1). This will show how in this case $\mathrm{ch}_X^Y E_\bullet$ lifts canonically to a " cohomology " class $\widetilde{\mathrm{ch}}_X^Y E_\bullet$. From this we will be able to prove some " cohomology " Riemann-Roch theorems (cf. [SGA 6]) for quasi-projective schemes.

Here we take $H_\bullet X = A_\bullet X_{\mathbf{Q}} = \mathrm{Gr}_\bullet X_{\mathbf{Q}}$, and $H^\bullet X = A^\bullet X_{\mathbf{Q}} = \mathrm{Gr}^\bullet X_{\mathbf{Q}}$ (cf. [App., § 3]); or, for complex varieties, $H_\bullet X = H_\bullet(X; \mathbf{Q})$, $H^\bullet X = H^\bullet(X; \mathbf{Q})$, ordinary singular homology and cohomology.

Let $\pi : G \to Y$, $\xi, \varphi : W \to \mathbf{P}^1$ be as in the construction of Chapter II, § 1 for the complex $E_\bullet$ on $Y = M$. In this section, however, we let Z_λ be the scheme-theoretic fibre $\varphi^{-1}(\lambda)$; we regard Z_λ as a Cartier divisor on W, instead of a Weil divisor (cycle). (If Y is not reduced, the scheme W is not defined by its underlying set; the local equations for W will appear in the proof of the following proposition.)

Proposition. — (1) *The Cartier divisor* Z_∞ *has a unique decomposition* $Z_\infty = Z + Y_*$ *where* Z *and* Y_* *are Cartier divisors on* W, π *maps* Y_* *birationally onto* Y (Y_* *is the blow-up of* Y *along* X), *and* $\pi(Z) = X$.

(2) *There is a commutative diagram*

$$\begin{array}{ccc} P = P(N \oplus 1) & \xrightarrow{j} & G \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{i} & Y \end{array}$$

where j *maps* P *isomorphically onto* Z, *and* $j^*\xi = \sum_i (-1)^i \Lambda^i H \otimes p^* F$ *in* $K^0 P$, *with* H *as in the proof of Proposition* (3.4) *in Chapter I.*

(3) $Z \times_W Y_*$ *is a Cartier divisor on* Z *and* Y_*; W *is a local complete intersection in* $G \times \mathbf{P}^1$.

(4) $\mathrm{ch}_X^Y E_\bullet = \widetilde{\mathrm{ch}}_X^Y E_\bullet \frown [X]$, *where* $\widetilde{\mathrm{ch}}_X^Y E_\bullet = p_*(\mathrm{ch}(\Lambda^\bullet H \otimes p^* F)) = \mathrm{td}(N)^{-1} \smile \mathrm{ch}(F)$ *and* $p_* = H^\bullet P \to H^\bullet X$ *is the Gysin map* (cf. § 4 and [App., § 3.4]).

Proof. — We first construct the map j of (2). The restriction $E_\bullet | X$ of $E_\bullet$ to X is a complex whose homology sheaves $\mathscr{H}_i = \mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, F)$ are canonically isomorphic to $\Lambda^i \check{N} \otimes F$ ([B-S, § 15], [SGA 6; VII]). The inclusion $H \subset p^* \check{N} \oplus 1$ of bundles on $P = P(N \oplus 1)$ gives rise to an inclusion

$$\Lambda^i H \subset \Lambda^i(p^* \check{N} \oplus 1) = \Lambda^i p^* \check{N} \oplus \Lambda^{i-1} p^* \check{N}.$$

Tensoring this with $p^*\mathrm{F}$ gives

$$\Lambda^i \mathrm{H} \otimes p^* \mathrm{F} \subset p^* \mathscr{H}_i \oplus p^* \mathscr{H}_{i-1}.$$

By the universal property of Grassmannians, this induces a morphism

$$\mathrm{P}(\mathrm{N} \oplus 1) \to \bigtimes_{i=0}^{e} \mathrm{Grass}_{\binom{e}{i}f}(\mathscr{H}_i \oplus \mathscr{H}_{i-1})$$

where $e = \operatorname{rank} \mathrm{N}$, $f = \operatorname{rank} \mathrm{F}$.

Let $\mathscr{K}_i = \mathrm{Ker}(d_i \otimes \mathscr{O}_\mathrm{X})$, $\mathscr{B}_i = \mathrm{Im}(d_i \otimes \mathscr{O}_\mathrm{X})$. Since the $\mathscr{H}_i$ are locally free on X, so are the $\mathscr{K}_i$ and $\mathscr{B}_i$, and the surjections $\mathscr{K}_i \to \mathscr{H}_i$ give an imbedding

$$\bigtimes_i \mathrm{Grass}_{\binom{e}{i}f}(\mathscr{H}_i \oplus \mathscr{H}_{i-1}) \to \bigtimes_i \mathrm{Grass}_{e_i}(\mathscr{K}_i \oplus \mathscr{K}_{i-1}).$$

(Note that the tautological bundles in the i-th factor differ by $\mathscr{B}_i \oplus \mathscr{B}_{i-1}$.)

The imbedding $\mathscr{K}_i \subset \mathrm{E}_i | \mathrm{X}$ gives

$$\bigtimes_i \mathrm{Grass}_{e_i}(\mathscr{K}_i \oplus \mathscr{K}_{i-1}) \subset \bigtimes_i \mathrm{Grass}_{e_i}(\mathrm{E}_i | \mathrm{X} \oplus \mathrm{E}_{i-1} | \mathrm{X}) = \mathrm{G} | \mathrm{X}.$$

The composition of these maps is the morphism $j : \mathrm{P} = \mathrm{P}(\mathrm{N} \oplus 1) \to \mathrm{G} | \mathrm{X}$. By construction $j^*\xi = \sum_i (-1)^i \Lambda^i \mathrm{H} \otimes p^* \mathrm{F}$ in $\mathrm{K}^0\mathrm{P}$. (Note that the extra factors $\mathscr{B}_i \oplus \mathscr{B}_{i-1}$ cancel when we take the alternating sum on $\mathrm{P}(\mathrm{N} \oplus 1)$.)

The other assertions in (1)-(3) are local on Y.

We assume that Y is affine and small enough so F and N extend to (trivial) bundles $\widetilde{\mathrm{F}}$ and $\widetilde{\mathrm{N}}$ on Y, and that there is a section $s : \mathrm{Y} \to \widetilde{\mathrm{N}}$ whose zeros define X scheme-theoretically. (In terms of coordinates for $\widetilde{\mathrm{N}}$, s is given by a regular sequence of functions defining X.) Let $\Lambda^\bullet \widetilde{\mathrm{N}}^\vee$ be the Koszul complex defined by the section s. By the local uniqueness of resolutions (cf. [S; IV, App. I]) we may assume $\mathrm{E}_\bullet = \mathrm{E}'_\bullet \oplus \mathrm{E}''_\bullet$, where $\mathrm{E}'_\bullet = \Lambda^\bullet \widetilde{\mathrm{N}}^\vee \otimes \widetilde{\mathrm{F}}$, and $\mathrm{E}''_\bullet$ is exact on all of Y.

We first define a morphism

$$\tilde{j} : \mathrm{P}(\widetilde{\mathrm{N}} \oplus 1) \times \mathbf{P}^1 \to \mathrm{G} \times \mathbf{P}^1$$

which restricts to j over $\mathrm{X} \times \{\infty\}$. Corresponding to the decomposition $\mathrm{E}_\bullet = \mathrm{E}'_\bullet \oplus \mathrm{E}''_\bullet$ we have an inclusion $\mathrm{G}' \times \mathrm{G}'' \subset \mathrm{G}$, where

$$\mathrm{G}' = \bigtimes_i \mathrm{Grass}_{\binom{e}{i}f}((\Lambda^i \widetilde{\mathrm{N}}^\vee \otimes \widetilde{\mathrm{F}}) \oplus (\Lambda^{i-1} \widetilde{\mathrm{N}}^\vee \otimes \widetilde{\mathrm{F}}))$$

$$\mathrm{G}'' = \bigtimes_i \mathrm{Grass}_{e_i - \binom{e}{i}f}(\mathrm{E}''_i \oplus \mathrm{E}''_{i-1})$$

and $\tilde{j}$ will factor through $\mathrm{G}' \times \mathrm{G}'' \times \mathbf{P}^1$. Thus $\tilde{j}$ will be determined by constructing two mappings

$$\tilde{j}_1 = \mathrm{P}(\widetilde{\mathrm{N}} \oplus 1) \to \mathrm{G}'$$

$$\tilde{j}_2 = \mathrm{P}(\widetilde{\mathrm{N}} \oplus 1) \times \mathbf{P}^1 \to \mathrm{G}'' \times \mathbf{P}^1.$$

Then $\tilde{j}(x, y) = \tilde{j}_1(x) \times \tilde{j}_2(x, y)$.

The first morphism

$$\widetilde{j}_1 : \mathrm{P}(\widetilde{\mathrm{N}}\oplus 1) \to \mathrm{G}'$$

comes from the "Koszul complex" $\wedge^{\bullet}\widetilde{\mathrm{H}}\otimes p^*\widetilde{\mathrm{F}}$ on $\mathrm{P}(\widetilde{\mathrm{N}}\oplus 1)$, where $\widetilde{\mathrm{H}}$ is defined by the exact sequence

$$0 \to \widetilde{\mathrm{H}} \to p^*(\widetilde{\mathrm{N}}^{\vee}\oplus 1) \to \mathcal{O}(1) \to 0$$

(cf. the construction of j). The second mapping $\widetilde{j}_2$ factors:

$$\mathrm{P}(\widetilde{\mathrm{N}}\oplus 1)\times\mathbf{P}^1 \xrightarrow{p\times 1} \mathrm{Y}\times\mathbf{P}^1 \longrightarrow \mathrm{G}''\times\mathbf{P}^1$$

where the second is the map constructed in the proof of Chapter II, Lemma (1.1), for a complex $\mathrm{E}''_{\bullet}$ exact on all of Y.

If we define finally

$$\mathrm{Y}\times\mathbf{A}^1 \to \widetilde{\mathrm{N}}\times\mathbf{P}^1$$

by $(y, \lambda) \to (\lambda s(y), (1;\lambda))$, the composite

$$\mathrm{Y}\times\mathbf{A}^1 \to \widetilde{\mathrm{N}}\times\mathbf{P}^1 \subset \mathrm{P}(\widetilde{\mathrm{N}}\oplus 1)\times\mathbf{P}^1 \xrightarrow{j} \mathrm{G}\times\mathbf{P}^1$$

is exactly the morphism constructed in the Grassmannian-graph construction for E on Y.

It follows that W is the closure of $\mathrm{Y}\times\mathbf{A}^1$ in $\mathrm{P}(\widetilde{\mathrm{N}}\oplus 1)\times\mathbf{P}^1$. We have studied this closure in Chapter I, § 5 (here $\mathrm{Y}=\mathrm{M}$, $\lambda = 1/t$). If we choose coordinates $y_1, \ldots, y_e$ which trivialize $\widetilde{\mathrm{N}}$, so $y_0, \ldots, y_e$ are homogeneous coordinates for the fibre of $\mathrm{P}(\widetilde{\mathrm{N}}\oplus 1)$, and $s(x) = \sum_i f_i(x) y_i$, then local equations for W in $\mathrm{P}(\widetilde{\mathrm{N}}\oplus 1)\times\mathbf{P}^1 = \mathrm{Y}\times\mathbf{P}^e\times\mathbf{P}^1$ are

$$\lambda_0 y_i = \lambda_1 f_i(x) y_0 \qquad i = 1, \ldots, e$$
$$y_i f_j(x) = y_j f_i(x) \qquad i, j = 1, \ldots, e.$$

Then Z_∞ is defined by adding the equation $\lambda_0 = 0$, which is the sum of the two divisors $\mathrm{Z} = \mathrm{X}\times\mathbf{P}^e$ and $\mathrm{Y}_*\subset\mathrm{Y}\times\mathbf{P}^{e-1}$ defined by the equations $y_i f_j = y_j f_i$, i.e. Y_* is the blow-up of Y along X. And $\mathrm{Z}\times_{\mathrm{W}}\mathrm{Y}_* = \mathrm{X}\times\mathbf{P}^{e-1}$. The remaining assertions of (1)-(3) can be verified by looking at the local equations; we leave this to the reader.

The assertion (4) follows from the identification of Z and $\xi | \mathrm{Z}$ in (3), and the formal fact that $p_*(\mathrm{ch}\wedge^{\bullet}\mathrm{H}) = \mathrm{td}(\mathrm{N})^{-1}$ in $\mathrm{H}^{\bullet}\mathrm{X}$, which was proved in Chapter I, Proposition (3.4).

Let $f : \mathrm{X}\to\mathrm{Y}$ be a projective complete intersection morphism of quasi-projective schemes. This means [SGA 6; VIII] that f factors

$$\mathrm{X} \xrightarrow{i} \mathrm{Y}\times\mathrm{P} \xrightarrow{p} \mathrm{Y},$$

where P is a projective space, i imbeds X as a local complete intersection in Y, and p is the projection. If N is the normal bundle of the imbedding i, and T_p is the relative tangent bundle of p, then the "virtual tangent bundle of f"

$$\mathrm{T}_f = i^*\mathrm{T}_p - \mathrm{N} \quad \text{in } \mathrm{K}^0\mathrm{X}$$

is independent of the factorization [SGA 6; VIII, Cor. 2.5]. (Our T_f is dual to that in SGA 6.)

Corollary 1 (Berthelot, Grothendieck, Illusie *et al.*). — *If* $f : X \to Y$ *is a complete intersection morphism as above, and* $x \in K^0X$, *then*

$$\mathrm{ch}(f_* x) = f_*(\mathrm{ch}(x) \,.\, \mathrm{td}(T_f))$$

in $H^\bullet Y$, *where* $f_* : H^\bullet X \to H^\bullet Y$ *is the Gysin map* (cf. § 4 and [App., § 3.3]).

Proof. — The case of a projection is quite formal (cf. [B-S], [SGA 6]), so we confine ourselves to the case where $f = i$ is an imbedding. We may assume x is the class of a bundle F on X. Since $\mathrm{ch}(i_* F) = \mathrm{ch}(E_\bullet)$, where $E_\bullet$ is as at the beginning of this section, we are reduced by (4) of the proposition to showing

$$i_*(\widetilde{\mathrm{ch}}{}^Y_X E_\bullet) = \mathrm{ch}\, E_\bullet.$$

This is a cohomology version of our localization property (2.1) (b) of Chapter II. We prove it as follows. In the notation of the proposition

$$i_* \widetilde{\mathrm{ch}}{}^Y_X E_\bullet = i_* p_*(\mathrm{ch}\, j^* \xi) = \pi_* j_* \mathrm{ch}(j^* \xi).$$

Let $j_\lambda : Z_\lambda \to G$ be the inclusion. Since πj_0 is an isomorphism of Z_0 with Y, under which $j_0^* \xi$ corresponds to $\sum_i (-1)^i E_i$, we get $\mathrm{ch}\, E_\bullet = \pi_* j_{0*}(\mathrm{ch}(j_0^* \xi))$. So it suffices to show that

$$j_* \mathrm{ch}(j^* \xi) = j_{0*} \mathrm{ch}(j_0^* \xi) \quad \text{in } H^\bullet G.$$

We claim first that

$$(1) \qquad j_{0*}(1) = j_{\infty *}(1) \quad \text{in } H^\bullet G.$$

It is enough to show that all Z_λ define the same cohomology class in $H^\bullet W$, since j_λ factors: $Z_\lambda \to W \to G \times \mathbf{P}^1 \to G$. In the Chow theory $H^\bullet = \mathrm{Gr}^\bullet_{\mathbf{Q}}$ this follows from the fact that the Z_λ are all linearly equivalent Cartier divisors on W. For the singular theory see § 4, Proposition (4.2) *c)*.

Let k be the inclusion of Y_* in G. We claim secondly that

$$(2) \qquad j_{\infty *}(1) = j_*(1) + k_*(1) \quad \text{in } H^\bullet G.$$

In the Chow theory this follows from the exact sequence

$$0 \to \mathcal{O}_{Z_\infty} \to \mathcal{O}_Z \oplus \mathcal{O}_{Y_*} \to \mathcal{O}_{Z \times_W Y_*} \to 0$$

and the fact that the Gysin maps are determined by the corresponding sheaves; note that $Z \times_W Y_*$ is a local complete intersection of lower dimension, so it does not contribute [SGA 6; VII, 4.6]. For the singular case see § 4, Proposition (4.2) *e)*.

Since $j_* \mathrm{ch}(j^* \xi) = \mathrm{ch}\, \xi . j_*(1)$, and similarly for $j_{\infty *}$ and k_*, we deduce

$$j_{\infty *} \mathrm{ch}(j_\infty^* \xi) = j_* \mathrm{ch}(j^* \xi) + k_*(\mathrm{ch}\, k^* \xi);$$

but $k^* \xi = 0$ in $K^0 Y_*$ (cf. proof of property (2.1) in Chapter II) which concludes the proof.

Corollary 2. — *Let* $f : X \to Y$ *be a complete intersection morphism as above. Then*

$$f_* \tau(X) = f_*(\mathrm{td}(T_f)) \frown \tau(Y).$$

Proof. — Set $x=1$ in Corollary 1, and cap both sides with $\tau(Y)$ to get $\mathrm{ch}(f_*1)\frown\tau(Y)=f_*(\mathrm{td}(T_f))\frown\tau(Y)$. By the module property and naturality

$$\mathrm{ch}(f_*1)\frown\tau(Y)=\tau(f_*1)=f_*\tau(X),$$

as desired.

This contains Proposition 3 of § 1 as a special case. When one has a Gysin map $f^*: H_\bullet Y\to H_\bullet X$ for a complete intersection morphism $f: X\to Y$, one expects the stronger

Conjecture. — $\tau(X)=\mathrm{td}(T_f)\frown f^*\tau(Y)$.

We proved some cases of this in § 1; see also Chapter III, § 3.

In the singular homology theory for complex varieties we will construct such Gysin maps in the next section, but the conjectured formula has not been proved in this context ([1]).

4. Gysin Maps in the Classical Case.

Let $f: X\to Y$ be a proper complete intersection morphism of possibly singular quasi-projective schemes over the complex numbers. In this section we define a cohomology push-forward map $f_*: H^\bullet(X;\mathbf{Z})\to H^\bullet(Y;\mathbf{Z})$ which generalizes what in various cases has been called the Gysin homomorphism, the Umkehrhomomorphism, or integration over the fiber. We also define a dual homology pull-back

$$f^*:\ H_\bullet(Y;\mathbf{Z})\to H_\bullet(X;\mathbf{Z}).$$

The definitions and proofs apply to any pair of extraordinary cohomology and homology theories in which a complex vector bundle E has a canonical orientation (or Thom class in $H^\bullet(E, E-\{0\})$, where $\{0\}$ is the zero section). For example topological K-theory provides such a pair [B-F-M].

The main tool is an appropriate definition of a generalized Thom class

$$U_{XY}\in H^n(Y, Y-X)$$

where X is included in Y as a local complete intersection and $\dim Y=\dim X+n$. Note that the pair (Y, X) will not in general be locally homeomorphic to $(A\times\mathbf{R}^{2n}, A\times 0)$ for any A. Even when it is, U_{XY} may not be the classical Thom class if X is not reduced.

Let $X\subset Y$ be a local complete intersection. Choose an algebraic section $s: Y\to E$ of a vector bundle E over Y such that $X=s^{-1}(\{0\})$ as a scheme. This can be done similarly to the construction of Chapter I, § 5. Then as in Chapter I, § 5 the normal bundle N to X in Y sits naturally in the restriction of E to X. Choose a classical neighborhood V of X in Y and choose a topological complex vector bundle C over V

([1]) *Note added in proof.* — J.-L. Verdier has constructed these Gysin maps for the Chow homology and proved the conjecture in general [Séminaire Bourbaki, n° 464, Feb. 1975].

contained in E such that C restricted to X is a complement to N in E. This can be done by an argument using Urysohn's lemma. Let Q be the quotient topological vector bundle E/C over V. Note that Q identifies canonically with N over X. Let $\bar{s}: V \to Q$ be s followed by the quotient map.

Lemma (4.1). — *Shrinking* V *to a smaller neighborhood of* X *if necessary,* $\bar{s}$ *maps the pair* $(V, V-X)$ *to the pair* $(Q, Q-\{0\})$.

Proof. — We may work locally in Y. Locally as in Chapter IV, § 3, N extends (algebraically) to a subbundle $\widetilde{N}$ of E so that s maps V to $\widetilde{N}$. Since being a complement is an open condition, C is a complement to $\widetilde{N}$ in E on a possibly smaller neighborhood V of X. Then over V, the quotient map $q: \widetilde{N} \to Q$ is an isomorphism of topological vector bundles. Since s takes $V-X$ to $\widetilde{N}-\{0\}$, $\bar{s}$ takes $V-X$ to $Q-\{0\}$.

Definition. — The *generalized Thom class* $U_{XY} \in H^n(V, V-X)$ (which is $H^n(Y, Y-X)$ by excision) is given by

$$U_{XY} = \bar{s}^*(U_Q)$$

where $U_Q \in H^n(Q, Q-\{0\})$ is the Thom class determined by the complex structure on the vector bundle Q.

The pullback of U_{XY} to Y will be $\{X\}$, the cohomology class " carried by " X, or $i_* 1$ where i is the inclusion of X into Y.

We will sometimes use the subscript XY on objects $(E, V, C, Q, s, \bar{s})$ relating to the construction of U_{XY}. In particular V_{XY} denotes an arbitrarily small classical neighborhood of X in Y.

Proposition (4.2):

a) U_{XY} *is independent of the choices.*

b) *For* $X \subset Y \subset Z$, *if* $r: V_{XZ} \to V_{XY}$ *is a retraction and* $j: V_{XZ} \to V_{YZ}$ *is an inclusion, then*

$$U_{XZ} = j^* U_{YZ} \smile r^* U_{XY}.$$

c) *If*

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook] \arrow[d, "\pi|\widetilde{X}"] & \widetilde{Y} \arrow[d, "\pi"] \\ X \arrow[r, hook] & Y \end{tikzcd}$$

is a fiber square such that π *is flat and the inclusions are local complete intersections, then*

$$\pi^* U_{XY} = U_{\widetilde{X}\widetilde{Y}}.$$

d) *If* M *is non-singular and* $g': X \subset Y = X \times M$ *is the graph of* $g: X \to M$ *and* $h: V_{XY} \to g^{-1}TM$ *is a tubular neighborhood homeomorphism sending* $g'(X)$ *to the zero section, then*

$$h^* U_{g^{-1}TM} = U_{XY}$$

where $U_{g^{-1}TM}$ *is the classical Thom class.*

e) *The generalized Thom class of the sum of two Cartier divisors is the sum of their Thom classes. In particular, if* X *and* Y *are of codimension one in* W *and have no component in common, then*

$$U_{X\cup Y,W} = \varphi^* U_{X,W} + \psi^* U_{Y,W}$$

where φ *and* ψ *are the evident inclusion of pairs.*

Proof:

a) Let E', s', c', be different choices and let $\eta : Q' \to Q$ be a topological isomorphism extending the identification of Q with N with Q′ over X. (Here as always, shrink V when necessary.) We show that $t(\bar{s})+(1-t)\eta\bar{s}'$ maps $(V, V-X)$ to $(Q, Q-\{0\})$ and thus provides a homotopy from one situation to the other. Working locally as in the proof of the lemma above, we have the diagram

$$\begin{tikzcd} Q & & Q' \arrow[ll, "\eta"'] \\ & \tilde{N} \arrow[ul, "q"] \arrow[ur, "q'"'] & \\ & V \arrow[u, shift left, "s'"'] \arrow[u, shift right, "s"] & \end{tikzcd}$$

and we must show that

$$t.s+(1-t)(q^{-1}\eta q').s' \neq 0$$

off X. Introduce a norm $\|\ \|$ on $\tilde{N}$. Since $s-s'$ is given by functions in the square of the ideal of X, for any $\varepsilon>0$, V can be shrunk so that $\|s-s'\|<\varepsilon\|s'\|$. Since $q^{-1}\eta q$ is the identity on X, we can also have $\|q^{-1}\eta q' s'-s'\|<\varepsilon\|s'\|$. With $\varepsilon\leq 1/2$ the result follows.

b) By adroit choices, we can arrange things so that over V_{XZ} we have the following commutative diagram with an exact sequence of topological vector bundles accross the top

$$\begin{tikzcd} 0 \arrow[r] & Q_{XY} \arrow[r] & Q_{XZ} \arrow[r] & Q_{YZ} \arrow[r] & 0 \\ & Y \arrow[u, "\bar{s}_{XY}"] & Z \arrow[u, "\bar{s}_{XZ}"] \arrow[ur, "\bar{s}_{YZ}"'] & & \end{tikzcd}$$

Now our equality for the triple of spaces (Z, Y, X) can be pulled back from the corresponding known equality for the triple $(Q_{XZ}, Q_{XY}, \{0\})$.

c) We can make choices so that $Q_{\tilde{X}\tilde{Y}}=\pi^{-1}Q_{XY}$ and the following diagram commutes

$$\begin{tikzcd} Q_{\tilde{X}\tilde{Y}} \arrow[r] & Q_{XY} \\ V_{\tilde{X}\tilde{Y}} \arrow[u, "s_{\tilde{X}\tilde{Y}}"] \arrow[r, "\pi"] & V_{XY} \arrow[u, "s_{XY}"] \end{tikzcd}$$

d) Since all classes in $H^n(Y, Y-X)$ are multiples of $h^*U_{f^{-1}TM}$ by the Thom isomorphism, we can use *c)* to reduce to the case where X is a zero dimensional scheme. Here one checks that the following diagram can be made to commute:

$$\begin{tikzcd}
Q \arrow[r, equal] & V \times T_{f(x)}Y \arrow[d] \\
V_{XY} \arrow[u, "s"] \arrow[r, "h"] & T_{f(x)}Y
\end{tikzcd}$$

e) Since we are dealing with divisors, we have the global algebraic commutative diagram

$$\begin{tikzcd}
Q_{XW} & Q_{XW} \oplus Q_{YW} \arrow[l, "p_X"'] \arrow[r, "p_Y"] \arrow[d, "t"] & Q_{YW} \\
 & Q_{XW} \otimes Q_{YW} & \\
 & W \arrow[uul, "s_{XW}"] \arrow[uu, bend left, "s"] \arrow[u, "s_{X\cup Y, W}"'] \arrow[uur, "s_{YW}"'] &
\end{tikzcd}$$

where t takes $x \oplus y$ to $x \otimes y$. Our equality is then the pullback by s of the relation

$$p_X^* U_{Q_{XW}} + p_Y^* U_{Q_{YW}} = t^* U_{Q_{XW} \otimes Q_{YW}}$$

in $$H^\bullet(Q_{XW} \oplus Q_{YW}, Q_{XW} \oplus Q_{YW} - t^{-1}\{0\}).$$

But this relation is true because both sides agree when restricted to

$$H^\bullet(Q_{XW} \oplus Q_{YW} - \{0\}; Q_{XW} \oplus Q_{YW} - t^{-1}\{0\})$$

and this implies that they are equal by the long exact sequence of the triple

$$(Q_{XW} \oplus Q_{YW}, Q_{XW} \oplus Q_{YW} - \{0\}, Q_{XW} \oplus Q_{YW} - t^{-1}\{0\}).$$

If $f : X \to Y$ is an arbitrary proper complete intersection morphism, i.e. f lifts to an inclusion as a local complete intersection in $Y \times M$ for some smooth M, construct the following diagram:

$$\begin{tikzcd}
 & V \arrow[dl, "r"'] \arrow[d, phantom, "\cap"] & W \arrow[l, "r'"', "\subset"] \arrow[d, phantom, "\cap"] \\
X \arrow[r, phantom, "\subset"] \arrow[dr, "f"'] & Y\times M \arrow[r, phantom, "\subset"] \arrow[d] & Y\times D \arrow[dl, "p"] \\
 & Y &
\end{tikzcd}$$

Here V is a neighborhood of X in $Y \times M$ that retracts by r onto X. (For example V could be a regular neighborhood with respect to a triangulation of the pair $(Y \times M, X)$.) D is a disk of dimension at least $4 \dim M+4$ in which M is differentiably embedded; W is a tubular neighborhood of V in $Y \times D$; r is the retraction. Orient D and the fiber of r' so that the orientations add in the natural decomposition

$$T_m D = T_m M \oplus T_m r'^{-1}(m) \quad \text{for} \quad m \in M;$$

let U_D and U_W be the corresponding Thom classes.

Definition. — The *cohomology Gysin homomorphism*

$$f_* : H^{\bullet}(X; \mathbf{Z}) \to H^{\bullet}(Y; \mathbf{Z})$$

is the composition

$$\begin{tikzcd}
H^{\bullet}(X) \arrow[r, "(r\circ r')^*"] & H^{\bullet}(W) \arrow[r, "(r'^* U_{X,Y\times M} \smile U_W)\smile"] & H^{\bullet}(W, W-X) \arrow[d, equal, "\text{excision}"] \\
H^{\bullet}(Y) \arrow[r, "U_D \smile p^*", "\cong"'] & H^{\bullet}(Y\times D, Y\times \partial D) & H^{\bullet}(Y\times D, (Y\times D)-X). \arrow[l, "\text{inclusion}^*"']
\end{tikzcd}$$

Two special cases of this map are more classically known. If Y is non-singular and X is reduced then this is the Umkehrhomomorphism $f_*(c) = \text{Poincaré Dual} f_*(c \frown [x])$. If f is a fibration, then this is integration over the fiber [Borel and Hirzebruch, Characteristic Classes and Homogeneous Spaces, I, *Am. J. Math.*, 80 (1958), p. 482].

Proposition (4.3). — *The homomorphism f_* is independent of the choices involved.*

Proof. — The homomorphism is independent: of the imbedding of M in D since for D this large all embeddings are isotopic; of U_W since to change it would produce a cancelling change in U_D; of the map $r \circ r'$ since all such are homotopy inverses to the inclusion of X. It remains to show independence of the factorization of f through

$Y \times M$. Since any two such factorizations are dominated by the product, we reduce to the special case

$$\begin{tikzcd} & Y \times M \arrow[d] \\ X \arrow[ur, equal] \arrow[r, hook] & Y \end{tikzcd}$$

By applying the fact that the Thom class of a direct sum vector bundle is the product of the Thom classes pulled up, we reduce to showing the following fact. Let

$$r : V_{X, Y\times M} \to V_{X, X\times M}$$

be a retraction and $p : V_{X, Y\times M} \to V_{X,Y}$ be the projection and $h : V_{X, X\times M} \to g^{-1}TM$ be as in Proposition (4.2) *d)*. Then

$$U_{X, Y\times M} = (h \circ r)^* U_{g^{-1}TM} \smile p^* U_{XY}.$$

But this follows easily from Proposition (4.2) *b)*, *c)* and *d)*.

Proposition **(4.4)**:

a) *The Gysin homomorphism is functorial,* $(f \circ g)_* = f_* g_*$.

b) *The projection formula holds*

$$f_*(x \smile f^* y) = f_* x \smile y.$$

Using Proposition (4.2), the proof is entirely parallel to that for the classical Umkehrhomomorphism as in [Dyer, *Cohomology Theories*, Benjamin, 1969, p. 47].

Definition. — The *homology Gysin homomorphism*

$$f^* : H_\bullet(Y; \mathbf{Z}) \to H_\bullet(X; \mathbf{Z})$$

where $H_\bullet$ is homology with closed support (Borel-Moore homology) is the composition

$$\begin{tikzcd} H_\bullet(X) & H_\bullet(W) \arrow[l, "(r \circ r')_*"'] & & H_\bullet(W, W-X) \arrow[ll, "(r'^* U_{X, Y\times M} \smile U_W) \frown"'] \arrow[d, equal, "\text{excision}"] \\ H_\bullet(Y) \arrow[r, "p_* U_0 \frown", "\cong"'] & H_\bullet(Y\times D, Y\times \partial D) \arrow[rr, "\text{inclusion}_*"] & & H_\bullet(Y\times D, (Y\times D) - X). \end{tikzcd}$$

It has similarly proved independence of choices and functoriality.

5. Riemann-Roch Without Denominators.

In this section we work in either of the following contexts:

(1) Complex quasi-projective schemes; $H^\bullet$ denotes singular cohomology with integer coefficients.

(2) Smooth quasi-projective varieties over an arbitrary field; $H^{\bullet}$ denotes the Chow ring with integer coefficients.

If N is a vector-bundle of rank d on X, let $P=P(N\oplus 1)$, $p:P\to X$ be the projective completion, and let

$$0\to H\to p^{*}(\check{N}\oplus 1)\to \mathcal{O}(1)\to 0$$

be the universal exact sequence on P.

For any bundle F of rank f on X let $P(F, N)=p_{*}(c(\Lambda^{\bullet}H\otimes p^{*}F))$ in $H^{\bullet}X$. (For a complex $E_{\bullet}$, its Chern class $c(E_{\bullet})$ is $\prod_i c(E_i)^{(-1)^i}$). The calculation of $P(F, N)$ is purely formal. The component $P_q(F, N)=p_{*}(c_q(\Lambda^{\bullet}H\otimes p^{*}F))$ in $H^{q-d}X$ may be written

$$P_q(F, N)=P_q(f, c_1(F), \ldots, c_{q-d}(F); c_1(N), \ldots, c_{q-d}(N))$$

where $P_q(T_0, \ldots, T_{q-d}; U_1, \ldots, U_{q-d})$ is a universal polynomial with integer coefficients. This may be extended to any $F\in K^0X$ with $f=\varepsilon(F)$.

Theorem. — Let $i:X\to Y$ *imbed* X *as a local complete intersection in* Y, *with normal bundle* N *of rank d. Then for* $F\in K^0X$

$$c_q(i_{*}F)=i_{*}(P_q(F, N)) \quad \text{in } H^qY$$

where $i_{*}=H^{q-d}X\to H^qY$ *is the Gysin map.*

Proof. — We may assume F is a bundle. Let $E_{\bullet}$ be a resolution of $i_{*}F$ by bundles on Y, and let

$$\begin{array}{ccc} P & \xrightarrow{j} & G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[i]{} & Y \end{array}$$

be the diagram constructed in § 3, Proposition (2), for $E_{\bullet}$ on Y. Then $c(i_{*}F)=c(E_{\bullet})$, and $i_{*}P(F, N)=\pi_{*}j_{*}c(\Lambda^{\bullet}H\otimes p^{*}F)=\pi_{*}j_{*}c(j^{*}\xi)$. Then the proof proceeds exactly as in the corollary in § 3, replacing " ch " by " c ".

Remark. — A formal calculation shows that $P_d(1, N)=(-1)^{d-1}(d-1)!\in H^0X$. It follows that $c_d(i_{*}\mathcal{O}_X)=(-1)^{d-1}(d-1)!\,i_{*}(1)$ in H^dY. In the classical case, even for X a point on a three-dimensional Y, this was unknown before [SGA 6; XIV, § 6].

6. Examples.

(1) We first give an example to show that the Todd class is not always in the image of the " Poincaré duality " mapping $H^{\bullet}X\to H_{\bullet}X$ given by $a\mapsto a\frown[X]$. We construct a three-dimensional normal variety X with one singular point, such that $\tau_2(X)\in H_4(X; \mathbf{Q})$ is not in $H^2(X; \mathbf{Q})\frown[X]$.

Let C_1, C_2 be non-singular projective curves of genus 1, 0 respectively, and let L_1, L_2 be negative line bundles on C_1, C_2 of degrees $-d_1$, $-d_2$. Let $S=C_1\times C_2$, $L=L_1\otimes L_2$ (a negative line bundle on S), $P=P(L\oplus 1)$ the projective completion of L, $f: P\to S$ the projection. Regard $L\subset P$ as usual, and $S\subset L$ by the zero section. By Grauert's criterion (cf. [EGA II, 8.9.1]) we may form the variety $X=P/S$ obtained by blowing S down to a (singular) point; let $\pi: P\to X$ be the collapsing map.

Let $z=c_1\mathcal{O}_P(1)\in H^2(P)$. The standard formula $H^{\bullet}P=H^{\bullet}S\oplus H^{\bullet}S.z$, and the split exact homology and cohomology sequences of the pair (P, S) allow us to compute the homology and cohomology of X. In particular z gives a basis for H^2X, and

$$T_1=\pi_*(f^{-1}(C_1\times\{\text{pt.}\})) \qquad \text{and} \qquad T_2=\pi_*f^{-1}(\{\text{pt.}\}\times C_2)$$

give a basis for H_4X. The relation $[S]^{\text{dual}}=z+f^*c_1(L)$ in H^2P [G; § 5, Lemma 3] implies that $z\frown[X]=d_2T_1+d_1T_2$.

From the standard formula for the tangent bundle to a projectivized bundle we see that $c(T_P)=c(f^*(L\oplus 1)\otimes\mathcal{O}(1)).f^*c(T_s)$, i.e. $c(T_P)=(1-d_2T_1-d_1T_2)(1+z)(1+2T_1)$ Since $\tau_2X=\pi_*(\tau_2P)$ (Cor. to Proposition 1.1), we deduce that

$$\begin{aligned}\tau_2X&=\tfrac{1}{2}(-d_2T_1-d_1T_2+2z\frown[X]+2T_1)\\&=\tfrac{1}{2}z\frown[X]+T_1,\end{aligned}$$

which is not in $H^2(X;\mathbf{Q})\frown[X]=\mathbf{Q}.(z\frown[X])$.

(2) In the above example $\tau_2X=\frac{1}{2}c_2X$, where c_2X is the homology Chern class of X [M 2], since the singularities of X have dimension <2; but such a relation cannot be expected in general. To see this, fix a curve C of genus $g>2$, and an integer d between g and $2g$. For each line bundle L on C of degree $-d$, let X_L be obtained by blowing C down to a point in $P(L\oplus 1)$. Then the arithmetic genus

$$\tau_0(X_2)=g+\dim H^0(C, L^{\vee}),$$

which varies with L, but the Chern classes depend only on the degree of L.

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