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ASYMPTOTIC INVERSION OF CONVOLUTION OPERATORS

par Harold WIDOM (1)

1. Introduction

A beautiful and important theorem of G. Szegö describes the asymptotic behavior, for large N, of the Toeplitz determinants

$$\det(c_{j-k}), \qquad 0 \leq j,\ k \leq N$$

associated with sequences

$$c = \{c_k\}, \qquad -\infty < k < \infty.$$

If

$$\hat{c}(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}, \qquad \log \hat{c}(\theta) = \sum_{k=-\infty}^{\infty} s_k e^{ik\theta}$$

then Szegö's theorem asserts that under certain conditions the determinant, denoted by $D_N[c]$, satisfies the asymptotic relation

$$(1.1) \qquad \log D_N[c] = (N+1)s_0 + \sum_{k=1}^{\infty} k s_k s_{-k} + o(1), \qquad N \to \infty.$$

In the twenty years since the appearance of Szegö's paper [13] a host of mathematicians have been inspired to try their hands at going further, either to weaken the conditions needed to guarantee the validity of the formula or to find analogues in other situations. Some of these investigations have had important consequences apparently far removed from the original question.

There have been several different approaches. Szegö, who proved the result for $\hat{c}$ positive and having a derivative satisfying a Lipschitz condition, showed that the formula (1.1) became an identity for certain c and large enough N, and then used an approximation argument for more general c.

A method of Baxter [2] and Hirschman [8] makes use of an identity for $D_N[c]$ valid for all c which yields (1.1) for more general, and not necessarily real, $\hat{c}$. Devinatz [4] has refined these methods to obtain the asymptotic formula under the most general

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conditions to date. Golinskii and Ibragimov [6], using an identity related to that of Baxter and Hirschman, have shown essentially that if $\hat{c} \geq 0$ then (1.1) is true if it makes sense.

Kac [10] derived the formula by a completely different method which was based on a certain combinatorial identity. He was the first to obtain a continuous analogue, and exactly the same identity played a critical role. Although there is an inherent limitation (c must be close to the Kronecker δ in some sense) the method admits considerable generalization, in particular to variable coefficients [11] and to higher dimensions [14].

Hartwig and Fisher [7] started with the asymptotic inversion of the Toeplitz matrix

$$\mathbf{T}_N[c] = (c_{j-k}), \qquad 0 \leq j,\ k \leq N.$$

If $\mathbf{T}(\lambda)$ is an analytic family of matrices defined for λ in some domain of the complex plane, and each $\mathbf{T}(\lambda)$ is invertible, then we have the relation [5, p. 163]

$$\text{(1.2)} \qquad \frac{d}{d\lambda} \log \det \mathbf{T}(\lambda) = \operatorname{tr} \mathbf{T}'(\lambda) \mathbf{T}(\lambda)^{-1}.$$

Thus if the given sequence c can be embedded in an analytic family $c(\lambda)$ for which all the corresponding traces in (1.2) can be evaluated asymptotically, and if for some λ the corresponding Toeplitz determinant can be evaluated trivially, then

$$\log D_N[c]$$

can be found by integration with respect to λ. Hirschman [9], using a different kind of approximate inversion from that of Hartwig and Fisher, showed that this approach could also yield a continuous analogue of (1.1).

The success of this method depends on the quality of the approximation one uses for $\mathbf{T}_N[c]^{-1}$. It must be extremely accurate yet simple enough so that the computations are still manageable. We shall present here such an approximation which leads to a proof of (1.1) which is remarkably easy and quite elementary, with conditions on c weaker than hitherto required. The appropriate inversion formula is of such a form that extensions to other cases suggest themselves. Thus the continuous analogue will hardly be more difficult to obtain. (The main problem there is: exactly what is meant by determinant?) We shall also consider in this paper generalizations to higher dimensions and to the case of variable convolutions.

We recall certain facts from operator theory. The reader is referred to [5] for details. Given a compact operator $\mathbf{T}$ on Hilbert space one defines

$$\|\mathbf{T}\|_p = \Big(\sum_i e_i((\mathbf{T}^*\mathbf{T})^{1/2})^p\Big)^{1/p} \quad (0 < p < \infty)$$

where e_i denotes the i-th eigenvalue. The norm $\|\mathbf{T}\|_\infty$, whether or not $\mathbf{T}$ is compact, is the ordinary uniform (operator) norm of $\mathbf{T}$. One has the inequalities

$$\text{(1.3)} \qquad \|\mathbf{T}_1\mathbf{T}_2\|_p \leq \|\mathbf{T}_1\|_\infty \|\mathbf{T}_2\|_p, \qquad \|\mathbf{T}_1\mathbf{T}_2\|_1 \leq \|\mathbf{T}_1\|_2 \|\mathbf{T}_2\|_2.$$

If $\|\mathbf{T}\|_2<\infty$, then $\mathbf{T}$ is said to be of Hilbert-Schmidt type and $\|\mathbf{T}\|_2$ is called the Hilbert-Schmidt norm. In case $\mathbf{T}$ is an integral operator with kernel $K(x,y)$

$$\|\mathbf{T}\|_2^2=\iint |K(x,y)|^2 dx\, dy.$$

If $\|\mathbf{T}\|_1<\infty$, then $\mathbf{T}$ is said to be nuclear, or of trace class, and $\|\mathbf{T}\|_1$ is called the trace norm. If $\mathbf{T}$ is nuclear then its trace is defined as

$$\text{tr}\,\mathbf{T}=\sum_k (\mathbf{T}x_k,\, x_k) \tag{1.4}$$

where $\{x_k\}$ is an orthonormal basis for the Hilbert space. The series always converges and its sum is independent of the choice of basis. There is the inequality

$$|\text{tr}\,\mathbf{T}|\leq\|\mathbf{T}\|_1. \tag{1.5}$$

We shall use an analogue of the O, o notation for operators. If $\{\mathbf{T}_N\}$ is a family of operators depending on a parameter N and if $\psi(N)$ is a positive function then we write

$$\mathbf{T}_N=O_p(\psi(N)) \qquad (0<p<\infty)$$

if

$$\|\mathbf{T}_N\|_p=O(\psi(N))$$

in the usual sense. The notation

$$\mathbf{T}_N=o_p(\psi(N))$$

is defined similarly.

To invert a Toeplitz matrix $\mathbf{T}_N$ approximately it is not necessary actually to invert it. Given a candidate $\mathbf{U}_N$ for an approximation to $\mathbf{T}_N^{-1}$ one need only define a matrix $\mathbf{E}_N$ ($\mathbf{E}$ for error) by

$$\mathbf{T}_N\mathbf{U}_N=\mathbf{I}-\mathbf{E}_N \tag{1.6}$$

and prove that $\mathbf{E}_N$ is small in an appropriate sense. If

$$\|\mathbf{E}_N\|_\infty<1$$

then $\mathbf{T}_N$ must be invertible and

$$\mathbf{T}_N^{-1}=\mathbf{U}_N(\mathbf{I}-\mathbf{E}_N)^{-1}.$$

If $\mathbf{E}_N=o_p(1)$ for some p and $\|\mathbf{U}_N\|_\infty=O(1)$ then

$$\mathbf{T}_N^{-1}=\mathbf{U}_N+o_p(1).$$

This follows from the first inequality of (1.3) if one uses the Neumann series expansion of $(\mathbf{I}-\mathbf{E}_N)^{-1}$.

For the computation of traces of matrices involving $\mathbf{T}_N^{-1}$ it is of course desirable to show that $\mathbf{E}_N=o_1(1)$. This is not absolutely necessary though. For example $\mathbf{E}_N=o_2(1)$ implies

$$\mathbf{T}_N^{-1}=\mathbf{U}_N+\mathbf{U}_N\mathbf{E}_N+o_1(1).$$

The paper is divided as follows. The next section contains a heuristic derivation of an approximation $\mathbf{U}_N[c]$ for $\mathbf{T}_N[c]^{-1}$. In the following section it is shown that the resulting $\mathbf{E}_N$ defined by (1.6) is $o_1(1)$ (in fact, curiously enough, $o_{2/3}(1)$), and that the asymptotic formula (1.1) holds. The assumptions on c are that there be a determination of $\log \hat{c}$ which is bounded and has a bounded conjugate function, and whose Fourier coefficients s_k satisfy

$$\sum_{k=-\infty}^{\infty} |k|\,|s_k|^2 < \infty.$$

Section 4 treats the continuous analogue. Given a tempered distribution c on the real line with Fourier transform $\hat{c}$ which is a bounded function, the finite Wiener-Hopf operator $\mathbf{W}_R[c]$ on $L_2(0, R)$ is defined as follows. For $f \in L_2(0, R)$ take the Fourier transform of f extended to be zero outside the interval, multiply by $\hat{c}$, take the inverse Fourier transform, and restrict it to $[0, R]$. The result is $\mathbf{W}_R[c]f$. In an appropriate sense the operator is convolution by c on $L_2(0, R)$. With $\mathbf{U}_R[c]$ defined in analogy with the discrete case it is shown (with conditions on c analogous to those in the discrete case) that the corresponding error operator is $o_1(1)$. With a further assumption needed even to define the determinant (and there are two different ways of doing this) the analogue of (1.1) is derived.

Section 5 treats the case where the interval $[0, R]$ is replaced by $R\Omega$ where Ω is a bounded region in n-dimensional space. Because of the extra complications involved we make stronger assumptions on c than previously. In the analogue of (1.1) the first term on the right is a constant times R^n, the second term a constant times R^{n-1}, and the error $o(R^{n-1})$. (The formula, with considerably stronger assumptions on c, was obtained in [14].) It is interesting to note that the coefficient of R^{n-1} may be written as the integral over the unit sphere $\mathbf{S}^{n-1}$ of a function associated with c with respect to a measure associated with Ω. This measure is well-known to differential geometers. It is induced from surface measure on $\partial\Omega$ by the Gauss map

$$\partial\Omega \to \mathbf{S}^{n-1}$$

which takes any point on $\partial\Omega$ into the point of $\mathbf{S}^{n-1}$ representing the inner unit normal to $\partial\Omega$ at that point. Convex sets are determined up to translation by these measures [3, § 59], but not sets in general.

Finally certain variable convolutions are investigated. A convolution operator has kernel of the form $c(x-y)$. A variable convolution has kernel of the form

$$c(x, y, x-y).$$

These operators may be thought of as bearing the same relation to ordinary convolutions as variable coefficient linear differential operators do to constant coefficient operators. Indeed pseudodifferential operators are variable convolutions of a particular kind [12]. (Strictly speaking of course every kernel is the kernel of a variable convolution operator.

One should regard " variable convolution " as a way of thinking about certain operators rather than as a definition.) We shall be concerned with variable convolutions having kernels of the form

$$c\left(\frac{x}{R}, \frac{y}{R}, x-y\right) \qquad x, y \in R\Omega$$

and obtain for them analogues of the result of the preceding section.

Each of the last three sections is more complicated than, but in many ways similar to, the preceding. In each of these sections we shall give details when something new is involved, but details will be omitted if they would be essentially repetitions of previous arguments.

2. The approximate inverse

The matrix $\mathbf{T}_N[c]$ represents convolution by c on the finite (integer) interval [0, N]. If we confine attention to well into the interior of the interval then its boundary cannot play much of a role and the interval should be replaceable by the set of all integers. Thus if c has the convolution inverse h

$$c*h=h*c=\delta, \tag{2.1}$$

it seems reasonable to suppose that for j and k well away from both 0 and N the j, k entry of $\mathbf{T}_N[c]^{-1}$ is approximately h_{j-k}.

Next suppose we are well away from the right end-point N. Then we should be able to replace the interval by the set of nonnegative integers. Convolution on L_2 of the nonnegative integers is a (semi-infinite) Toeplitz operator, or discrete Wiener-Hopf operator, whose inversion is by now very well known. If $[r]$ denotes r-fold convolution, with the 0-fold convolution of any sequence taken to equal δ, the convolution exponential of a sequence c is

$$\operatorname{Exp} c=\sum_{r=0}^{\infty} c^{[r]}/r!.$$

We write " Exp " rather than " exp " to distinguish it from ordinary numerical exponentiation. Write

$$s=\operatorname{Log} c$$

if $c=\operatorname{Exp} s$. The sequences c_+ and c_- are defined by

$$c_+=c\chi_{[0,\,\infty)}, \qquad c_-=c\chi_{(-\infty,\,0)}$$

where χ denotes characteristic function and multiplication is pointwise. (That $k=0$ is put with c_+ is not important. It could have been put with c_- just as well or split between the two.) Finally, with h the convolution inverse of c as in (2.1) and $s=\operatorname{Log} c$, define h^+ and h^- by

$$h^+=\operatorname{Exp}(-s_+),\qquad h^-=\operatorname{Exp}(-s_-),$$

so that

$$h_k^+=h_{-k}^-=0,\qquad k<0, \tag{2.2}$$

and

$$h^+*h^-=h.$$

The inverse of the semi-infinite Toeplitz matrix

$$(c_{j-k})\qquad 0\leq j,\ k<\infty$$

has, under certain conditions, its j, k entry equal to

$$\sum_{m=0}^{\infty} h_{j-m}^+ h_{-k+m}^-. \tag{2.3}$$

(A proof of a similar inversion will be given in Lemma (5.1).)

This should be close to the j, k entry of $\mathbf{T}_N[c]^{-1}$ if j and k are well away from N. But if j and k are also well away from 0 we have already argued that this entry should be close to h_{j-k}. To reconcile these statements note that

$$h_k=\sum_{m=-\infty}^{\infty} h_m^+ h_{k-m}^-$$

or, more symmetrically

$$h_{j-k}=\sum_{m=-\infty}^{\infty} h_{j+m}^+ h_{-k-m}^-. \tag{2.4}$$

Therefore (2.3) equals

$$h_{j-k}-\sum_{m=1}^{\infty} h_{j+m}^+ h_{-k-m}^-$$

and the second term is indeed small if j or k is large and positive.

Similarly an approximation to the j, k entry of $\mathbf{T}_N[c]^{-1}$, valid for j and k well away from the left end-point 0, should be

$$h_{j-k}-\sum_{m=1}^{\infty} h_{-N+j-m}^- h_{N-k+m}^+.$$

If we put the last two expressions together we see that

$$h_{j-k}-\sum_{m=1}^{\infty} h_{j+m}^+ h_{-k-m}^- - \sum_{m=1}^{\infty} h_{-N+j-m}^- h_{N-k+m}^+$$

is close to each of the approximations in its range of validity. Hence we define

$$\mathbf{U}_N[c]=\Big(h_{j-k}-\sum_{m=1}^{\infty} h_{j+m}^+ h_{-k-m}^- - \sum_{m=1}^{\infty} h_{-N+j-m}^- h_{N-k+m}^+\Big)\qquad 0\leq j,\ k\leq N. \tag{2.5}$$

Note that the expression for $\mathbf{U}_N[c]$ consists of three terms, the first arising from the interior of [0, N] and the others from the two end-points.

3. Toeplitz matrices

The natural setting for our approach is a certain algebra $\mathscr{A}$ of sequences. We say that $c \in \mathscr{A}$ if

$$|||c|||^2 = \sum_{k=-\infty}^{\infty} |k|\,|c_k|^2 < \infty, \qquad \hat{c}(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta} \in L_\infty(0, 2\pi).$$

Use of the identity

$$|||c|||^2 = \frac{1}{4\pi^2}\iint \left| \frac{\hat{c}(\theta) - \hat{c}(\varphi)}{2 \sin \frac{1}{2}(\theta - \varphi)} \right|^2 d\theta\, d\varphi,$$

which follows easily from Parseval's identity, shows that

$$|||c_1 * c_2||| \leq |||c_1|||\, ||\hat{c}_2||_\infty + |||c_2|||\, ||\hat{c}_1||_\infty$$

from which it follows that $\mathscr{A}$ is a Banach algebra under convolution with norm

$$||c|| = |||c||| + ||\hat{c}||_\infty.$$

The relevance of the $||\hat{c}||_\infty$ part of $||c||$ is clear. It is equal to the norm of convolution by c on L_2 of the integers, and an upper bound for convolution by c on L_2 of any subset of the integers. These things also follow easily from Parseval's identity. The other part of $||c||$, namely $|||c|||$, is equal to the Hilbert-Schmidt norm of the direct sum of the semi-infinite matrices

$$(c_{j+k+1}) \qquad 0 \leq j,\ k < \infty,$$
$$(c_{-j-k-1}) \qquad 0 \leq j,\ k < \infty.$$

Given a sequence $c \in \mathscr{A}$ define the Cesàro mean $M_n c$ by

$$(M_n c)_k = \begin{cases} \left(1 - \dfrac{|k|}{n}\right) c_k, & |k| \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $(M_n c)\hat{}$ is just the n-th Cesàro mean of the Fourier series for $\hat{c}$. Although the sequence of Cesàro means does not converge to c in the norm of $\mathscr{A}$, it does have useful properties with respect to the norm $|||\ \ |||$, as the following lemmas show.

Lemma **(3.1)**. — *If* $|||c||| < \infty$, *then*

$$\lim_{n\to\infty} |||c - M_n c||| = 0.$$

Proof. — We have

$$|(M_n c)_k| \leq |c_k|$$

for all n and

$$\lim_{n\to\infty} (M_n c)_k = c_k$$

for each k. The conclusion therefore follows from the dominated convergence theorem.

Lemma **(3.2)**. — *Suppose* a_n *and* b_n *are two sequences of elements of* $\mathscr{A}$, *and* a *and* b *two elements of* $\mathscr{A}$, *such that as* $n \to \infty$

$$\hat{a}_n \to \hat{a}, \quad \hat{b}_n \to \hat{b} \quad \textit{boundedly almost everywhere,}$$

$$|||a_n - a||| \to 0, \quad |||b_n - b||| \to 0.$$

Then also

$$|||a_n * b_n - a * b||| \to 0.$$

Proof. — We have

$$a_n * b_n - a * b = (a_n - a) * (b_n - b) + (a_n - a) * b + a * (b_n - b).$$

It suffices therefore to prove the assertion in two special cases, the first where both of the limits are zero, and the second where one of the limits is zero and the other sequence is constant. In either case we write

$$\hat{a}_n(\theta)\hat{b}_n(\theta) - \hat{a}_n(\varphi)\hat{b}_n(\varphi) = (\hat{a}_n(\theta) - \hat{a}_n(\varphi))\hat{b}_n(\theta) + \hat{a}_n(\varphi)(\hat{b}_n(\theta) - \hat{b}_n(\varphi))$$

and we must show that the L_2 norm of this function of θ and φ, with respect to a certain measure, tends to zero.

In the first case, where $a = b = 0$, the norm of the first term on the right tends to zero because the norm of

$$\hat{a}_n(\theta) - \hat{a}_n(\varphi)$$

does and because the $\hat{b}_n(\theta)$ are uniformly bounded. The norm of the second term tends to zero for the same reason.

In the second case we may assume that $a = 0$ and b_n is independent of n. Then the norm of the first term on the right tends to zero for the same reason as before and the norm of the second term on the right tends to zero by the dominated convergence theorem.

Lemma **(3.3)**. — *Suppose* $c_n, c \in \mathscr{A}$ *and as* $n \to \infty$

$$\hat{c}_n \to \hat{c} \quad \textit{boundedly almost everywhere,}$$

$$|||c_n - c||| \to 0.$$

Then also

$$|||\operatorname{Exp} c_n - \operatorname{Exp} c||| \to 0.$$

Proof. — We have

$$|||\operatorname{Exp} c_n - \operatorname{Exp} c||| \leq \sum_{r=1}^{\infty} |||c_n^{[r]} - c^{[r]}|||/r!.$$

By the preceding lemma $|||c_n^{[r]} - c^{[r]}||| \to 0$ for each r, and we have the inequality

$$|||c_n^{[r]} - c^{[r]}||| \leq 2(\sup_m |||c_m|||)^r$$

for all n. It follows that the sum of the last series tends to zero as $n \to \infty$.

Next a lemma from operator theory.

Lemma **(3.4)**. — *Let* $\mathbf{T}_n$ *be bounded operators on Hilbert space satisfying*

$$\mathbf{T}_n \to \mathbf{T} \text{ strongly}$$

(that is, $\mathbf{T}_n x \to \mathbf{T} x$ *for each vector* x*). Let* $\mathbf{S}_n$ *be nuclear operators satisfying*

$$\|\mathbf{S}_n - \mathbf{S}\|_1 \to 0.$$

Then also

$$\|\mathbf{T}_n \mathbf{S}_n - \mathbf{T}\mathbf{S}\|_1 \to 0.$$

Proof. — We have

$$\|\mathbf{T}_n \mathbf{S}_n - \mathbf{T}\mathbf{S}\|_1 \leq \|\mathbf{T}_n(\mathbf{S}_n - \mathbf{S})\|_1 + \|(\mathbf{T}_n - \mathbf{T})\mathbf{S}\|_1.$$

Since $\mathbf{T}_n = O_\infty(1)$ (by the uniform boundedness principle) an application of the first inequality of (1.3) shows that the first term on the right tends to zero.

Now because $\mathbf{T}_n \to \mathbf{T}$ strongly and $\mathbf{S}$ is compact

$$\|(\mathbf{T}_n - \mathbf{T})\mathbf{S}\|_\infty \to 0.$$

This implies that the eigenvalues

$$e_i((\mathbf{S}^*(\mathbf{T}_n^* - \mathbf{T}^*)(\mathbf{T}_n - \mathbf{T})\mathbf{S})^{1/2})$$

tend to 0 as $n \to \infty$ for each i. Since

$$e_i((\mathbf{S}^*(\mathbf{T}_n^* - \mathbf{T}^*)(\mathbf{T}_n - \mathbf{T})\mathbf{S})^{1/2}) \leq 2 \sup_m \|\mathbf{T}_m\|_\infty e_i((\mathbf{S}^*\mathbf{S})^{1/2})$$

for all n and i, and since

$$\|\mathbf{S}\|_1 = \Sigma e_i((\mathbf{S}^*\mathbf{S})^{1/2}) < \infty$$

an application of the dominated convergence theorem shows that

$$\|(\mathbf{T}_n - \mathbf{T})\mathbf{S}\|_1 = \sum_i e_i((\mathbf{S}^*(\mathbf{T}_n^* - \mathbf{T}^*)(\mathbf{T}_n - \mathbf{T})\mathbf{S})^{1/2}) \to 0.$$

Now that these dull but useful lemmas are out of the way we proceed to show that $\mathbf{U}_N[c]$ as defined by (2.5) is a good approximation to $\mathbf{T}_N[c]^{-1}$. None of the lemmas, incidentally, is needed for this. They will be used only later in this section, where (1.1) is proved.

Theorem **(3.1)**. — *Suppose* $c = \operatorname{Exp} s$ *where* s_+ *and* s_- *belong to* $\mathscr{A}$. *Then* $\mathbf{T}_N[c]$ *is invertible for sufficiently large* N *and with* $\mathbf{U}_N[c]$ *defined by* (2.5) *we have, as* $N \to \infty$

$$\mathbf{T}_N[c]^{-1} = \mathbf{U}_N[c] + o_1(1).$$

Proof. — The j, k entry of $\mathbf{T}_N[c]\mathbf{U}_N[c]$ equals

$$\sum_{\ell=0}^{N} c_{j-\ell} h_{\ell-k} - \sum_{\ell=0}^{N} c_{j-\ell} \sum_{m=1}^{\infty} h^+_{\ell+m} h^-_{-k-m} - \sum_{\ell=0}^{N} c_{j-\ell} \sum_{m=1}^{\infty} h^-_{-N+\ell-m} h^+_{N-k+m}.$$

Since $c*h=\delta$ we may write

$$\sum_{\ell=0}^{N} c_{j-\ell} h_{\ell-k} = \delta_{j-k} - \sum_{\ell=-\infty}^{-1} c_{j-\ell} h_{\ell-k} - \sum_{\ell=N+1}^{\infty} c_{j-\ell} h_{\ell-k}$$

and by (2.2) and (2.4) this is

$$\delta_{j-k} - \sum_{\ell=-\infty}^{-1} c_{j-\ell} \sum_{m=1}^{\infty} h^+_{\ell+m} h^-_{-k-m} - \sum_{\ell=N+1}^{\infty} c_{j-\ell} \sum_{m=1}^{\infty} h^-_{-N+\ell-m} h^+_{N-k+m}.$$

Thus if

$$\mathbf{T}_N[c]\mathbf{U}_N[c] = \mathbf{I} - \mathbf{E}_N$$

then $\mathbf{E}_N$ has j, k entry

$$\sum_{\ell=-\infty}^{N} c_{j-\ell} \sum_{m=1}^{\infty} h^+_{\ell+m} h^-_{-k-m} + \sum_{\ell=0}^{\infty} c_{j-\ell} \sum_{m=1}^{\infty} h^-_{-N+\ell-m} h^+_{N-k+m}.$$

It follows from the definition of h^+ that

$$c*h^+ = \operatorname{Exp} s_-.$$

Consequently

$$\sum_{\ell=-\infty}^{\infty} c_{j-\ell} h^+_{\ell+m} = 0$$

if $j+m>0$ and so

$$\sum_{\ell=-\infty}^{\infty} c_{j-\ell} \sum_{m=1}^{M} h^+_{\ell+m} h^-_{-k-m} = 0$$

for each M. Now as $M\to\infty$ (with k fixed)

$$\{\sum_{m=1}^{\infty} h^+_{\ell+m} h^-_{-k-m}\}_{-\infty<\ell<\infty} \to \{\sum_{m=1}^{\infty} h^+_{\ell+m} h^-_{-k-m}\}_{-\infty<\ell<\infty}$$

in L_2 of the integers. Since convolution by c is continuous it follows that

$$\sum_{\ell=-\infty}^{\infty} c_{j-\ell} \sum_{m=1}^{\infty} h^+_{\ell+m} h^-_{-k-m} = 0.$$

Similarly

$$\sum_{\ell=-\infty}^{\infty} c_{j-\ell} \sum_{m=1}^{\infty} h^-_{-N+\ell-m} h^+_{N-k+m} = 0$$

and so the j, k entry of $\mathbf{E}_N$ is equal to

$$-\sum_{\ell=N+1}^{\infty} c_{j-\ell} \sum_{m=1}^{\infty} h^+_{\ell+m} h^-_{-k-m} - \sum_{\ell=-\infty}^{-1} c_{j-\ell} \sum_{m=1}^{\infty} h^-_{-N+\ell-m} h^+_{N-k+m}$$

$$= -\sum_{\ell=1}^{\infty} c_{j-\ell-N} \sum_{m=1}^{\infty} h^+_{N+\ell+m} h^-_{-k-m} - \sum_{\ell=1}^{\infty} c_{j+\ell} \sum_{m=1}^{\infty} h^-_{-N-\ell-m} h^+_{N-k+m}.$$

Now

$$\sum_{\ell=1}^{\infty} c_{j-\ell-N} \sum_{m=1}^{\infty} h^+_{N+\ell+m} h^-_{-k-m} \tag{3.1}$$

is the j, k entry of the product of three (infinite) matrices, the first having uniform norm at most $||\hat{c}||_\infty$, the second having Hilbert-Schmidt norm at most

$$\Big(\sum_{\ell, m=1}^{\infty} |h^{+}_{N+\ell+m}|^2\Big)^{1/2} = \Big(\sum_{k=1}^{\infty} k\,|h^{+}_{N+k+1}|^2\Big)^{1/2} \leq \Big(\sum_{k=N+2}^{\infty} k\,|h^{+}_{k}|^2\Big)^{1/2},$$

and the third having Hilbert-Schmidt norm at most

$$\Big(\sum_{m\geq 1, k\geq 0} |h^{-}_{-k-m}|^2\Big)^{1/2} = \Big(\sum_{k=1}^{\infty} k\,|h^{-}_{-k}|^2\Big)^{1/2}.$$

Since, by (1.3), the trace norm of the matrix represented by (3.1) is at most the product of these three quantities, and since

$$(3.2) \qquad \lim_{N\to\infty} \sum_{k=N+2}^{\infty} k\,|h^{+}_{k}|^2 = 0,$$

the matrix represented by (3.1) is $o_1(1)$. Similarly

$$(3.3) \qquad \sum_{\ell=1}^{\infty} c_{j+\ell} \sum_{m=1}^{\infty} h^{-}_{-N-\ell-m} h^{+}_{N-k+m}$$

represents a matrix whose trace norm is at most the product of $|||c|||$,

$$\Big(\sum_{k=N+2}^{\infty} k\,|h^{-}_{-k}|^2\Big)^{1/2},$$

and $||\hat{h}^{+}||_\infty$. Hence $\mathbf{E}_N = o_1(1)$ and, as was observed in the introduction, this implies the assertion of the theorem.

Remark 1. — We mentioned in the introduction that actually

$$\mathbf{E}_N = o_{2/3}(1).$$

This can be seen as follows. The index j belongs to $[0, N]$ so the trace norm of the matrix represented by (3.1) is unchanged if j is replaced by $N-j$ there. Then (3.1) becomes

$$\sum_{\ell=1}^{\infty} c_{-j-\ell} \sum_{m=1}^{\infty} h^{+}_{N+\ell+m} h^{-}_{-k-m}.$$

But in this form it is clear that we have a product of three Hilbert-Schmidt operators, two of which are $O_2(1)$ and the other $o_2(1)$, so the product is $o_{2/3}(1)$. Similarly the matrix represented by (3.3) is $o_{2/3}(1)$ and so $\mathbf{E}_N = o_{2/3}(1)$.

Remark 2. — Suppose we had an analytic family $c(\lambda)$ of sequences satisfying the hypothesis of the theorem. More precisely suppose

$$\lambda \mapsto (\operatorname{Log} c(\lambda))_{\pm}$$

are analytic from some open set in the complex plane to $\mathscr{A}$. Then the conclusions of the theorem hold uniformly for λ in any compact subset. For example to see that

(3.2) hold uniformly on compact subsets it is only necessary to notice that (with obvious notation)

$$\sum_{k=N}^{\infty} k\,|h_k^+(\lambda)|^2, \qquad N=1, 2, \ldots$$

is a nonincreasing sequence of continuous functions converging pointwise to zero. Such a sequence necessarily converges to zero uniformly on compact subsets, by Dini's theorem.

Theorem **(3.2)**. — *Under the assumptions of Theorem* (3.1) *the relation* (1.1) *holds.*

Proof. — Embed c in the analytic family

$$c(\lambda)=\operatorname{Exp}\lambda s$$

which clearly is of the type described in Remark 2. The relation (1.1) for $c(\lambda)$ is (with an obvious notation)

$$\textbf{(3.4)} \qquad \log D_N[c(\lambda)]=(N+1)s_0(\lambda)+\sum_{k=1}^{\infty} k s_k(\lambda)s_{-k}(\lambda)+o(1).$$

To prove this for all λ it suffices to check it for a single λ (and it is trivial for $\lambda=0$) and to prove that the relation obtained by formally differentiating it

$$\textbf{(3.5)} \qquad \frac{d}{d\lambda}\log D_N[c(\lambda)]=(N+1)s_0'(\lambda)+\sum_{k=1}^{\infty} k\frac{d}{d\lambda}[s_k(\lambda)s_{-k}(\lambda)]+o(1),$$

holds uniformly on compact sets. For then (3.4) follows by integration. Our proof of (3.5) will make virtually no use of the specific form of the family $c(\lambda)$.

By (1.2) the left side of (3.5) equals

$$\operatorname{tr} \mathbf{T}_N[c'(\lambda)]\mathbf{T}_N[c(\lambda)]^{-1}$$

and by Theorem (3.1) this is

$$\operatorname{tr} \mathbf{T}_N[c'(\lambda)]\mathbf{U}_N[c]+o(1).$$

By Remark 2 this holds uniformly on compact sets in the λ-plane.

A little computation gives

$$\operatorname{tr} \mathbf{T}_N[c']\mathbf{U}_N[c]=\sum_{k=-N}^{N}(N+1-|k|)c_k'h_{-k}-2\sum_{j,k=0}^{N} c_{k-j}'\sum_{m=1}^{\infty} h_{j+m}^+h_{-k-m}^-.$$

(Prime denotes differentiation with respect to λ. We no longer display the dependence of the various quantities on λ.) From the fact

$$\frac{d}{d\lambda}\operatorname{Log} c(\lambda)=c'(\lambda)*c(\lambda)^{-1}$$

one deduces

$$\textbf{(3.6)} \qquad s_k'=\sum_{j=-\infty}^{\infty} c_j'h_{k-j}$$

and so

$$(N+1)\sum_{k=-N}^{N} c'_k h_{-k}=(N+1)s'_0+(N+1)\sum_{|k|>N} c'_k h_{-k}.$$

But

$$(N+1)\sum_{|k|>N}|c'_k h_{-k}|\le\sum_{|k|>N}|k|\,|c'_k h_{-k}|$$

and so is $o(1)$. Therefore (3.5) is equivalent to the assertion

$$-\sum_{k=-N}^{N}|k|\,c'_k h_{-k}-2\sum_{j,k=0}^{N} c'_{k-j}\sum_{m=1}^{\infty} h^+_{j+m}h^-_{-k-m}=\sum_{k=1}^{\infty} k(s_k s_{-k})'+o(1).$$

As $N\to\infty$ the first sum on the left tends to the limit

$$\sum_{k=-\infty}^{\infty}|k|\,c'_k h_{-k}.$$

If

$$\mathbf{T}[c']=(c'_{j-k}),\qquad 0\le j,\ k<\infty,$$

$$\mathbf{H}=\Big(\sum_{m=1}^{\infty} h^+_{j+m}h^-_{-k-m}\Big)\qquad 0\le j,\ k<\infty,$$

and $\mathbf{P}_N$ denotes projection from L_2 of the nonnegative integers to L_2 of $[0, N]$, then the second sum on the left side is exactly

$$-2\ \mathrm{tr}\ \mathbf{P}_N\mathbf{T}[c']\mathbf{P}_N\mathbf{H}.$$

By Lemma (3.4) this converges to

$$-2\ \mathrm{tr}\ \mathbf{T}[c']\mathbf{H}$$

as $N\to\infty$. Thus we have shown that (3.5) is equivalent to the identity

$$\textbf{(3.7)}\qquad -\sum_{k=-\infty}^{\infty}|k|\,c'_k h_{-k}-2\ \mathrm{tr}\ \mathbf{T}[c']\mathbf{H}=\sum_{k=1}^{\infty} k(s_k s_{-k})'.$$

Now this identity can be proved, but it is a little messy and it turns out to suffice to prove a considerably simpler identity (actually a special case), as we shall now see. With M_n denoting n-th Cesàro mean as before, set

$$c^{(n)}(\lambda)=\mathrm{Exp}(M_n s(\lambda)),\qquad h^{(n)\pm}(\lambda)=\mathrm{Exp}(-M_n s(\lambda))_{\pm},$$

etc. It follows from Lemmas (3.1)-(3.3) that as $n\to\infty$

$$|||c^{(n)'}-c'|||\to 0,\qquad |||h^{(n)}-h|||\to 0,\qquad |||h^{(n)\pm}-h^{\pm}|||\to 0,$$

and of course

$$(c^{(n)'})^{\wedge}(\theta)\to\hat{c}'(\theta)$$

boundedly almost everywhere. This implies that the identity (3.7) would follow if the corresponding identity could be proved for each $c^{(n)}$. (One uses Lemma (3.4) to handle the second term of (3.7).) Moreover λ may be taken arbitrarily small since it is a matter of proving the identity of two analytic functions. But since (3.7) is equivalent to (3.4) it suffices to prove

$$\log D_N[c]=(N+1)s_0+\sum_{k=1}^{\infty} ks_k s_{-k}+o(1)$$

in case s_k is nonzero for only finitely many k and c is close to δ in any sense we choose, for example

$$|c_0-1|+\sum_{k\neq 0}|c_k|=\alpha<1.$$

Such a sequence c can be joined to the identity sequence δ by a simpler family than $\operatorname{Exp}(\lambda \operatorname{Log} c)$. In fact consider

$$c(\lambda)=\lambda c+(1-\lambda)\delta, \qquad |\lambda|<\alpha^{-1}.$$

It is easily seen that

$$\lambda \mapsto (\operatorname{Log} c(\lambda))_{\pm}$$

are analytic from the disc $|\lambda|<\alpha^{-1}$ to $\mathscr{A}$ and so it is now a question of proving (3.5) for this family. If $c(\lambda)$ is replaced by $\lambda^{-1}c(\lambda)$ then each side of (3.5) has $(N+1)\lambda^{-1}$ subtracted from it. Therefore it suffices to prove (3.5) for the family

$$c+\lambda^{-1}(1-\lambda)\delta, \qquad |\lambda|<\alpha^{-1},$$

or equivalently for the family

$$c+\lambda\delta, \qquad |\lambda+1|>\alpha.$$

Since, as we have seen, (3.5) is equivalent to (3.7), it is sufficient to prove the latter. But for the family $c+\lambda\delta$ it reads

$$-2\operatorname{tr}\mathbf{H}=\sum_{k=1}^{\infty} k(s_k s_{-k})',$$

or

$$-2\sum_{k=0}^{\infty} kh_k^+ h_{-k}^- = \sum_{k=1}^{\infty} k(s_k s_{-k})'. \tag{3.8}$$

This is proved as follows. The definition of h^+ implies that

$$\sum_{k=0}^{\infty} h_k^+ z^k=\exp\{-\sum_{k=0}^{\infty} s_k z^k\}.$$

Differentiating with respect to z and equating coefficients of like powers of z give

$$kh_k^+=-\sum_{j=1}^{\infty} js_j h_{k-j}^+.$$

Therefore

$$\sum_{k=0}^{\infty} kh_k^+ h_{-k}^- = -\sum_{j=1}^{\infty} js_j \sum_{k=0}^{\infty} h_{k-j}^+ h_{-k}^-.$$

(There is no need to justify the interchange of the j and k summations since the summation over j is finite.) By (2.4)

$$\sum_{k=0}^{\infty} h_{k-j}^{+} h_{-k}^{-} = h_{-j}$$

and for our family $c+\lambda\delta$ this is, by (3.6), just s'_{-j}. Hence

$$\sum_{k=0}^{\infty} k h_k^{+} h_{-k}^{-} = -\sum_{j=1}^{\infty} j s_j s'_{-j}.$$

Similarly

$$\sum_{k=0}^{\infty} k h_k^{+} h_{-k}^{-} = -\sum_{j=1}^{\infty} j s_{-j} s'_j$$

and (3.8) is established. This completes the proof of the theorem.

Remark. — Perhaps we should mention explicitly why the Cesàro means were introduced. To prove (3.4), with an arbitrary s, for the family

$$c(\lambda) = \operatorname{Exp} \lambda s$$

it sufficed to prove it, as we saw, for λ small. Thus, just as in the proof of the theorem, it suffices to prove (3.4) for the family

$$c+\lambda\delta, \qquad |\lambda|>1$$

where it is assumed that $||c||<1$. One certainly has the analytic family

$$s(\lambda) = \delta \log \lambda + \sum_{r=1}^{\infty} (-1)^{r+1} \lambda^{-r} c^{[r]}/r$$

from the exterior of the unit circle, cut say along $(1, \infty)$, to $\mathscr{A}$. The difficulty is that there is no guarantee that $s(\lambda)_{\pm}$ belong to $\mathscr{A}$ also. If we knew that these $s(\lambda)_{\pm}$ belonged to $\mathscr{A}$ then Theorem (3.1) could be applied and Lemmas (3.1)-(3.3) (and also, it turns out, Lemma (3.4)) could have been dispensed with and the entire proof shortened considerably. An assumption that guarantees that $s(\lambda)_{\pm}$ belong to $\mathscr{A}$ is, for our original s,

$$\sum_{k=-\infty}^{\infty} |s_k| < \infty.$$

This is in addition, of course, to the assumption $|||s|||<\infty$. It was under just these assumptions that (1.1) was proved in [8].

4. Finite Wiener-Hopf operators

As in the discrete case we first introduce an algebra, which we again call $\mathscr{A}$ since there seems to be no possibility of confusion. A tempered distribution c on the real line belongs to $\mathscr{A}$ if its Fourier transform $\hat{c}$ is a bounded function and if, on the complement of $\{0\}$, c is equal to a function belonging to L_2 with weight function $|x|$. If we call this function c_0 and write

$$|||c||| = |||c_0||| = \left(\int_{-\infty}^{\infty} |x|\, |c_0(x)|^2\, dx\right)^{1/2}$$

then the norm on A is

$$||c|| = ||\hat{c}||_\infty + |||c|||.$$

Since

$$\hat{c}_0(\xi) - \hat{c}_0(\xi+\eta) = \int c_0(x)(1 - e^{ix\eta}) e^{ix\xi}\, dx \tag{4.1}$$

(integrals are taken over the entire real line unless indicated otherwise) we see that for each η

$$\hat{c}_0(\xi) - \hat{c}_0(\xi+\eta)$$

is an L_2 function of ξ. Now the most general distribution supported on $\{0\}$ is a finite linear combination of derivatives of the Dirac distribution δ. Thus $\hat{c}$ equals $\hat{c}_0$ plus a polynomial. But $\hat{c}$ is bounded and

$$\hat{c}_0(\xi) - \hat{c}_0(\xi+\eta)$$

is in L_2 and it is a simple exercise to deduce that the polynomial must be constant. Thus c equals c_0 plus a constant times δ. This implies, using (4.1) and Parseval's identity

$$|||c|||^2 = \frac{1}{\pi^2}\iint \left|\frac{\hat{c}(\xi) - \hat{c}(\eta)}{\xi - \eta}\right|^2 d\xi\, d\eta.$$

From this we see that $\mathscr{A}$ is a Banach algebra under convolution.

Given $c \in \mathscr{A}$ with associated function c_0 we define c_+ and c_- by

$$c_- = c_0 \chi_{(-\infty, 0)}, \qquad c_+ = c - c_-.$$

Clearly, if $\hat{c}_+$ or $\hat{c}_-$ is a bounded function, then c_+ and c_- belong to $\mathscr{A}$. But, just as in the discrete case, this may or may not occur.

We can now define the approximate inverse $\mathbf{U}_R[c]$ for the finite Wiener-Hopf operator $\mathbf{W}_R[c]$ defined in the introduction. We assume that

$$c = \operatorname{Exp} s$$

where s and s_+ (and so also s_-) belong to $\mathscr{A}$. As before this is the convolution exponential, not the pointwise exponential, and we write also

$$s = \operatorname{Log} c.$$

Define

$$h^+ = \operatorname{Exp}(-s_+), \qquad h^- = \operatorname{Exp}(-s_-), \qquad h = h^+ * h^-$$

so that in particular

$$c * h = \delta.$$

It will be convenient to talk about kernels of the form

$$\alpha\, \delta(x-y) + K(x,y)$$

where α is a constant, δ the Dirac distribution, and K a kernel in the classical sense. It is clear what is meant by the integral operator with such a kernel.

We define $\mathbf{U}_R[c]$ to be the integral operator on $L_2(0, R)$ with kernel

$$h(x-y)-\int_0^R h^+(x+z)h^-(-y-z)\,dz-\int_0^R h^-(-R+x-z)h^+(R-y+z)\,dz. \tag{4.2}$$

We can, and do, restrict x and y to the open interval $(0, R)$. The integrands in the two integrals are then ordinary functions on the ranges of integration. (Perhaps we ought to have inserted subscripts to denote the functions corresponding to the distributions, but this hardly seems necessary.)

Theorem **(4.1)**. — *Assume* $c = \operatorname{Exp} s$ *where* s *and* s_+ *belong to* $\mathscr{A}$. *Then* $\mathbf{W}_R[c]$ *is invertible for* R *sufficiently large and with* $\mathbf{U}_R[c]$ *defined by* (4.2) *we have, as* $R\to\infty$,

$$\mathbf{W}_R[c]^{-1} = \mathbf{U}_R[c] + o_1(1).$$

The proof of this is entirely analogous to that of Theorem (3.1) and so need not be given. Only one point might be mentioned. The proof establishes directly only the right invertibility of $\mathbf{W}_R[c]$. Left invertibility is obtained by considering the adjoint.

The method of the preceding section will also give a continuous analogue to Theorem (3.2) once a definition of determinant has been agreed upon. From the point of view of operator theory it is most natural to define

$$\det \mathbf{T} = \prod_i e_i(\mathbf{T}) \tag{4.3}$$

if $\mathbf{T}-\mathbf{I}$ is a nuclear operator. Here $e_i(\mathbf{T})$ are the eigenvalues of T arranged in any order. This product necessarily converges. Moreover if $\mathbf{T}(\lambda)$ is an analytic family of nuclear operators then [5, p. 163]

$$\frac{d}{d\lambda}\log\det(\mathbf{I}+\mathbf{T}(\lambda)) = \operatorname{tr}\mathbf{T}'(\lambda)(\mathbf{I}+\mathbf{T}(\lambda))^{-1} \tag{4.4}$$

as long as -1 is not an eigenvalue of $\mathbf{T}(\lambda)$.

One point that requires care is that the trace of a nuclear integral operator with kernel $K(x,y)$ is not necessarily given by the formula

$$\int K(x, x)\,dx. \tag{4.5}$$

This does hold if K is continuous and there are formulas for the trace similar to this in the general case. By the method of [5, § 10] one can prove the following.

Let φ *be any function on the real line whose Fourier transform* $\hat\varphi$ *satisfies*

$$\hat\varphi \geq 0, \quad \hat\varphi \in L_1 \cap L_\infty, \quad \lim_{\xi\to 0}\hat\varphi(\xi) = 1. \tag{4.6}$$

Then if K *is the kernel of a nuclear operator on* $L_2(a, b)$ *the trace of the operator is equal to*

$$\lim_{\varepsilon\to 0}\int_a^b\int_a^b \varepsilon^{-1}\varphi(\varepsilon^{-1}(x-y))K(x,y)\,dx\,dy.$$

From this it is easy to deduce formulas for the traces of nuclear finite Wiener-Hopf operators.

Let φ *be any function satisfying* (4.6) *and in addition*

$$\int |x|\,|\varphi(x)|^2\,dx<\infty. \tag{4.7}$$

Suppose $c\in L_2(-\infty,\infty)$ *is such that* $\mathbf{W}_R[c]$ *is a nuclear operator. Then*

$$\operatorname{tr} \mathbf{W}_R[c]=R\lim_{\varepsilon\to 0}\int \varepsilon^{-1}\varphi(\varepsilon^{-1}x)c(x)\,dx. \tag{4.8}$$

For the trace is

$$\lim_{\varepsilon\to 0}\int_0^R\int_0^R \varepsilon^{-1}\varphi(\varepsilon^{-1}(x-y))c(x-y)\,dx\,dy$$

$$\lim_{\varepsilon\to 0}\int_{-R}^R (R-|x|)\varepsilon^{-1}\varphi(\varepsilon^{-1}x)c(x)\,dx.$$

Schwarz's inequality gives

$$\left|\int_{|x|>R} R\varepsilon^{-1}\varphi(\varepsilon^{-1}x)c(x)\,dx\right|\le \|c\|_2 R\varepsilon^{-1/2}\left(\int_{|x|>\varepsilon^{-1}R}|\varphi(x)|^2\,dx\right)^{1/2}$$

$$\le \|c\|_2 R^{1/2}\left(\int_{|x|>\varepsilon^{-1}R}|x|\,|\varphi(x)|^2\,dx\right)^{1/2}$$

and this tends to zero as $\varepsilon\to 0$. Moreover

$$\left|\int_{-R}^R |x|\varepsilon^{-1}\varphi(\varepsilon^{-1}x)c(x)\,dx\right|\le \|c\|_2\left(\int_{-\varepsilon^{-1}R}^{\varepsilon^{-1}R}\varepsilon|x|^2\,|\varphi(x)|^2\,dx\right)^{1/2}.$$

For each ε the last integrand is at most

$$R|x|\,|\varphi(x)|^2\in L_1(-\infty,\infty)$$

so the integral tends to zero by the dominated convergence theorem. The asserted formula follows.

Note that if c is continuous at 0 the trace is simply $Rc(0)$.

Suppose now that c satisfies the hypothesis of Theorem (4.1) and in addition that $\mathbf{W}_R[c]$ differs from $\mathbf{I}$ by a nuclear operator, so that its determinant may be defined by (4.3). Since a nuclear operator is necessarily Hilbert-Schmidt, $c-\delta$ must be a locally square integrable function, and therefore also globally square integrable since $|||c|||$ is finite. This implies that

$$\hat{s}=\log \hat{c}\in L_2\cap L_\infty.$$

Since

$$\hat{c}-1=\sum_{r=1}^{\infty}\hat{s}^r/r!$$

$\hat{c}-1$ equals $\hat{s}$ plus a function belonging to $L_1(-\infty,\infty)$. But $\hat{a}\in L_1$ implies $\mathbf{W}_R[a]$ is nuclear. (In case $\hat{a}\ge 0$ this follows from Mercer's theorem since $\mathbf{W}_R[a]$ is positive semi-definite and is an integral operator with continuous kernel; the most general function of L_1 is a linear combination of four nonnegative L_1 functions.) Thus the assumption that $\mathbf{W}_R[c]-\mathbf{I}$ is nuclear is equivalent to the assumption that $\mathbf{W}_R[s]$ is nuclear.

Theorem **(4.2)**. — a) *Assume that c satisfies the hypothesis of Theorem* (4.1) *and that* $\mathbf{W}_{\mathrm{R}}[s]$ *is nuclear for each* $\mathrm{R}>0$. *Then with determinant defined by* (4.3), *and with φ any function satisfying the conditions* (4.6) *and* (4.7) *we have as* $\mathrm{R}\to\infty$

$$\log \det \mathrm{W}_{\mathrm{R}}[c]=\mathrm{R}\lim_{\varepsilon\to 0}\int \varepsilon^{-1}\varphi(\varepsilon^{-1}x)\,s(x)\,dx+\int_0^\infty xs(x)\,s(-x)\,dx+o(1).$$

Proof. — Carrying through the argument of Theorem (3.1) requires the computation of

$$\operatorname{tr}\mathbf{W}_{\mathrm{R}}[c']\,\mathbf{U}_{\mathrm{R}}[c]$$

and since the trace is not necessarily given by the formula (4.5) some care must be exercised.

As $\mathbf{U}_{\mathrm{R}}[c]$ is defined as the sum of three operators so also is the product a sum of three operators. Two of these, corresponding to the integrals in (4.2), will have continuous kernels (arising from what are essentially convolutions of L_2 functions) and their traces may therefore be computed by (4.5). The third operator is

$$\mathbf{W}_{\mathrm{R}}[c']\,\mathbf{W}_{\mathrm{R}}[h].$$

Now

$$\mathbf{W}_{\mathrm{R}}[c'*h]-\mathbf{W}_{\mathrm{R}}[c']\,\mathbf{W}_{\mathrm{R}}[h]$$

has kernel

$$\int_0^\infty c'(x+z)\,h(-z-y)\,dz+\int_0^\infty c'(x-\mathrm{R}-z)\,h(z+\mathrm{R}-y)\,dz,\qquad 0<x,y<\mathrm{R}.$$

This function is continuous since c' and h belong to L_2 of the complement of $\{0\}$, and is the kernel of a nuclear operator since it represents the sum of products of Hilbert-Schmidt operators. Therefore (4.5) may be used to compute

$$\operatorname{tr}\,(\mathbf{W}_{\mathrm{R}}[c']\,\mathbf{W}_{\mathrm{R}}[h]-\mathbf{W}_{\mathrm{R}}[c'*h]).$$

Moreover

$$\operatorname{tr}\mathbf{W}_{\mathrm{R}}[c'*h]=\operatorname{tr}\mathbf{W}_{\mathrm{R}}[s']=\mathrm{R}\lim_{\varepsilon\to 0}\int \varepsilon^{-1}\varphi(\varepsilon^{-1}x)\,s'(x)\,dx.$$

With these points kept in mind there is no difficulty reducing the proof to the verification of the continuous analogue of identity (3.7). In the discrete case it was shown that it sufficed to check this identity for the simple family

$$c(\lambda)=c+\lambda\delta$$

and with extremely nice c. Since

$$\det \mathbf{W}_{\mathrm{R}}[c+\lambda\delta]$$

is not defined unless $\lambda=0$ there is a problem. Of course (3.7) and its continuous analogue could be proved for the family

$$c(\lambda)=\operatorname{Exp}\lambda s$$

but the computations are a little unpleasant and best avoided. The family $c+\lambda\delta$ should be used if at all possible, and one can accomplish this by a simple device.

Extend the definitions of determinant and trace to operators of the form

$$\alpha\mathbf{I}+\mathbf{T}$$

with α a nonzero constant and $\mathbf{T}$ nuclear, by defining

$$\det(\alpha\mathbf{I}+\mathbf{T})=\det(\mathbf{I}+\alpha^{-1}\mathbf{T}),\qquad \operatorname{tr}(\alpha\mathbf{I}+\mathbf{T})=\operatorname{tr}\mathbf{T}.$$

These definitions look ridiculous but they make perfectly good sense for infinite dimensional Hilbert spaces. Moreover the analogue of (4.4) for analytic families of the form

$$\alpha(\lambda)\mathbf{I}+\mathbf{T}(\lambda)$$

holds if these definitions are used. The formula follows easily from the corresponding formula for the family

$$\mathbf{I}+\alpha(\lambda)^{-1}\mathbf{T}(\lambda).$$

Having established this, one sees that the formula

$$\frac{d}{d\lambda}\log\det\mathbf{W}_{\mathrm{R}}[c]=\frac{d}{d\lambda}\operatorname{tr}\mathbf{W}_{\mathrm{R}}[s]+\frac{d}{d\lambda}\int_{0+}^{\infty}xs(x)\,s(-x)\,dx+o(1)$$

for the family $c(\lambda)=c+\lambda\delta$ is equivalent to the continuous analogue of (3.8), which is proved without difficulty.

The other definition of determinant is the classical Fredholm one. If T is the integral operator with kernel $\mathrm{K}(x,y)$ then one defines

$$\det(\mathrm{I}+\mathrm{T})=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\int\cdots\int\det(\mathrm{K}(x_i,x_j))\,dx_1\ldots dx_r. \tag{4.9}$$

This makes sense if

$$\iint|\mathrm{K}(x,y)|^2\,dx\,dy<\infty,\qquad \int|\mathrm{K}(x,x)|\,dx<\infty,$$

and (4.4) holds if $(\mathbf{I}+\mathbf{T}(\lambda))^{-1}$ is thought of in terms of resolvent kernels in the usual way and (4.5) is used as the definition of the trace. Of course the determinant as defined by (4.9) is not really a function of $\mathbf{T}$ since two kernels may give rise to the same operator but to different right hand sides of (4.8). Nevertheless we retain the notation $\det(\mathbf{I}+\mathbf{T})$.

To apply this definition to $\mathbf{W}_{\mathrm{R}}[c]$ we must assume as before that $s\in\mathrm{L}_2$ in order to guarantee that $\mathbf{W}_{\mathrm{R}}[c]$ differs from $\mathbf{I}$ by a Hilbert-Schmidt operator. But it is also necessary to define the kernel of $\mathbf{W}_{\mathrm{R}}[c]-\mathbf{I}$, namely

$$c(x-y)-\delta(x-y),$$

almost everywhere on the diagonal $x=y$. Now

$$c-\delta=\sum_{r=1}^{\infty}s^{[r]}/r! \tag{4.10}$$

and since $\hat{s}\in\mathrm{L}_2\cap\mathrm{L}_\infty$ the convolutions $s^{[r]}$ for $r\geq 2$ are everywhere defined and even continuous. Therefore what remains is to define $s(0)$ somehow and this is used to assign a value to $c-\delta$ at o by (4.10). How $s(0)$ is defined is irrelevant, but once it is done that value is used to define $\det \mathbf{W}_R[c]$ by (4.9).

Theorem **(4.2)**. — b) *Assume that c satisfies the hypothesis of Theorem* (4.1) *and that in addition* $s \in L_2$. *Then with determinant defined by* (4.9) *we have as* $R \to \infty$

$$\log \det \mathbf{W}_R[c] = Rs(0) + \int_0^\infty xs(x)\, s(-x)\, dx + o(1).$$

Proof. — We omit the details as usual. Use is made of the fact that for nuclear operators with continuous kernels the two definitions of trace coincide, so one may use inequality (1.5) to show that the error term $o_1(1)$ in the statement of Theorem (4.1) contributes $o(1)$ to the traces, as defined by (4.5), that arise.

5. Higher dimensional convolutions

Throughout this section and the next Ω will denote a compact set in n-dimensional Euclidean space E^n whose boundary is of class C^1. More exactly for every point of $\partial\Omega$ there is a neighborhood N of the point in E^n and a C^1 diffeomorphism σ of N onto an open ball of E^n such that $\sigma(N \cap \Omega)$ is the part of this ball to one side of a hyperplane through its center. Roughly speaking, at each point of its boundary Ω looks like a half-space. This suggests that the approximate inversion of convolution operators on Ω, or on $R\Omega$ for large R, may be effected in terms of the inversion of Wiener-Hopf operators on half-spaces. That is exactly what happens.

As mentioned in the introduction we make stronger assumptions on our kernels than in the one-dimensional case. We consider the algebra $\mathscr{B}$ of distributions c on R^n which are of the form

$$\alpha\delta + c_0$$

where α is a constant, δ the Dirac distribution, and c_0 a function on E^n satisfying

$$\int |c_0(x)|\, dx < \infty, \qquad \int |x|\, |c_0(x)|^2\, dx < \infty.$$

(Integrals are taken over E^n unless otherwise indicated.) It is easy to see that $\mathscr{B}$ is a Banach algebra under convolution with norm

$$\|c\| = |\alpha| + \int |c_0(x)|\, dx + \left(\int |x|\, |c_0(x)|^2\, dx\right)^{1/2}.$$

This algebra $\mathscr{B}$ is easier to work with than the n-dimensional analogue of the algebra $\mathscr{A}$ of the last section because $\mathscr{B}$ is closed under the taking of absolute values, and also for the reason mentioned at the end of the third section. The results we obtain might very well hold for the n-dimensional analogue of $\mathscr{A}$ but the proofs would have to be considerably more sophisticated.

The operator $\mathbf{W}_{\mathrm{R}}[c]$ on $\mathrm{L}_2(\mathrm{R}\Omega)$ is defined as convolution by c. For $c\in\mathscr{B}$ the operator is a scalar multiple of $\mathbf{I}$ plus a compact operator.

Our basic assumption will be that $c=\operatorname{Exp} s$ with $s\in\mathscr{B}$. In dimension one this is equivalent to

$$c\in\mathscr{B},$$

$\hat{c}$ bounded away from 0, and

$$\underset{-\infty<\xi<\infty}{\Delta} \arg \hat{c}(\xi)=0.$$

In higher dimensions the last condition is unnecessary. These things follow from the Arens-Calderón extension of the Wiener-Lévy theorem [1]. The simplification in higher dimensions occurs because

$$\lim_{|\xi|\to\infty} \hat{c}(\xi)$$

exists and the one-point compactification of E^n is simply connected, so that if $\hat{c}$ is bounded away from zero it has a continuous logarithm.

Unlike in dimension one there is now a (convolution) factorization of c, or an additive decomposition of $s=\operatorname{Log} c$, corresponding to each direction. Given $c\in\mathscr{B}$ and a unit vector v define (with the dot denoting inner product)

$$c_{+(v)}=c\chi_{\{x:\,x.v\geq 0\}}, \qquad c_{-(v)}=c\chi_{\{x:\,x.v<0\}}.$$

Just as before the δ part of c is arbitrarily put with c_+ rather than c_-. Note though that if $c\in\mathscr{B}$ then c_+ and c_- necessarily belong to $\mathscr{B}$.

Given c with $s=\operatorname{Log} c\in\mathscr{B}$ we define

$$h_v^+=\operatorname{Exp}(-s_{+(v)}), \qquad h_v^-=\operatorname{Exp}(-s_{-(v)}), \qquad h=\operatorname{Exp}(-s).$$

The inversion of Wiener-Hopf operators on half-spaces is given by the following lemma, which is well-known.

Lemma (5.1). — *Assume* $s=\operatorname{Log} c\in\mathscr{B}$ *and let* v *be a unit vector in* E^n. *Then the integral operator on* L_2 *of the half-space*

$$\{x:\ x.v\geq 0\}$$

with kernel $c(x-y)$ *has inverse the integral operator with kernel*

$$h(x-y)-\int_{z.v>0} h_v^+(x+t)\,h_v^-(-y-t)\,dt.$$

Proof. — We have

$$\int c(x-z)\,dz\int_{t.v>0} h_v^+(z+t)\,h_v^-(-y-t)\,dt=\int_{t.v>0} h_v^-(-y-t)\,dt\int c(x-z)\,h_v^+(z+t)\,dz$$
$$=\int_{t.v>0} h_v^-(-y-t)\,dt\int c(x+t-z)\,h_v^+(z)\,dz.$$

It follows from the definition of h_v^+ that the convolution $c * h_v^+(u)$ vanishes unless $u.v \geq 0$. Since $x.v \geq 0$ and $t.v > 0$ the inner integral vanishes. Hence

$$\int_{x.v \geq 0} c(x-z)\, dt \int_{t.v>0} h_v^+(z+t)\, h_v^-(-y-t)\, dt$$
$$= -\int_{x.v<0} c(x-z)\, dz \int_{t.v>0} h_v^+(z+t)\, h_v^-(-y-t)\, dt.$$

Since $h_v^+(u)$ vanishes unless $u.v \geq 0$ this may also be written

$$-\int_{x.v<0} c(x-z)\, dz \int h_v^+(z+t)\, h_v^-(-y-t)\, dt = -\int_{x.v<0} c(x-z)\, h(z-y)\, dz.$$

Therefore

$$\int_{x.v \geq 0} c(x-z) \Big(h(z-y) - \int_{t.v>0} h_v^+(z+t)\, h_v^-(y+t)\, dt \Big)\, dz$$
$$= \int_{x.v \geq 0} c(x-z)\, h(z-y)\, dz + \int_{x.v<0} c(x-z)\, h(z-y)\, dz$$
$$= \int c(x-z)\, h(z-y)\, dz = \delta(x-y).$$

This shows that the asserted inverse is actually a right inverse. That it is also a left inverse may be seen by considering adjoints.

The approximate inversion of $\mathbf{W}_R[c]$ is not as neat in higher dimensions as in one. The reason seems to be that in one dimension there are only two boundary points and they are far from each other so their interference is negligible. In higher dimensions there are many boundary points very close to each other and their interference is substantial. The obvious way of defining an approximate inverse in analogy with (4.2) would be to replace the sum (of integrals) on the right side, one term for each boundary point, by an integral over $\partial\Omega$. Unfortunately this is essentially wrong and we use a less natural definition, one that does not reduce to (4.2) for $n=1$ although it differs from it by only $o_1(1)$.

Let $y \mapsto \bar{y}$ be any measurable mapping from Ω to $\partial\Omega$ satisfying for some $\gamma > 0$

(5.1) $$(y-\bar{y}) . v(\bar{y}) \geq \gamma |y-\bar{y}|$$

(5.2) $$\lim_{y \to \partial\Omega} |y-\bar{y}| = 0.$$

Here $v(\bar{y})$ is the inner unit normal to $\partial\Omega$ at $\bar{y}$, and the dot denotes inner product. This induces in an obvious way a mapping, also denoted by $y \mapsto \bar{y}$, from $R\Omega$ to $\partial(R\Omega)$.

We define $\mathbf{U}_R[c]$ to be the integral operator on $L_2(R\Omega)$ with kernel

$$h(x-y) - \int_{t.v(\bar{y})>0} h_{v(\bar{y})}^+(x-\bar{y}+t)\, h_{v(\bar{y})}^-(-y+\bar{y}-t)\, dt.$$

This depends of course on the mapping $y \mapsto \bar{y}$, but exactly which mapping we take will not matter, as long as (5.1) and (5.2) are satisfied.

Before we see in what sense this is a good approximation to $\mathbf{W}_R[c]^{-1}$, we consider some consequences of the assumption that $\partial\Omega$ be of class C^1. Let σ be a C^1-diffeomorphism of an open set N in E^n to a neighborhood of o in E^n such that $\sigma(N \cap \Omega)$ consists of those points of $\sigma(N)$ whose first coordinate σ^1 is nonnegative. For

$$z \in N \cap \partial\Omega, \qquad x \in N \cap \Omega$$

we have, since $\sigma^1(z) = 0$

$$\sigma^1(x) = (x-z) \,.\, \text{grad}\ \sigma^1(z) + o(|x-z|)$$

as $|x-z| \to 0$, and this holds uniformly on compact subsets of N. It follows that given $\varepsilon > 0$ there is an r such that, if

$$B(z, r) = \{x : |x-z| \leq r\},$$

then

$$\Omega \cap B(z, r) \subset \{x : (x-z) \,.\, v(z) \geq -\varepsilon|x-z|\},$$
$$\Omega \supset \{x : (x-z) \,.\, v(z) \geq \varepsilon|x-z|\} \cap B(z, r).$$

As before, $v(z)$ denotes the inner unit normal at z.

For $z \in \partial\Omega$ define $D(z)$ to be the symmetric difference between Ω and the half-plane

$$\{x : (x-z) \,.\, v(z) \geq 0\}$$

which locally approximates Ω at z. Then we deduce that

$$D(z) \cap B(z, r) \subset \{x : |(x-z) \,.\, v(z)| < \varepsilon|x-z|\}$$

for sufficiently small r.

It follows that for our function $y \to \bar{y}$ from Ω to $\partial\Omega$, if $x \in D(\bar{y})$ and is sufficiently close to $\bar{y}$ then x must be much further from y than from $\partial\Omega$. More exactly, if we define (with d denoting distance)

$$\beta(\rho) = \inf_{x, y}\{|x-y| : x \in D(\bar{y}),\ d(x, \partial\Omega) \geq \rho\} \tag{5.3}$$

then $\beta(\rho) > 0$ and

$$\lim_{\rho \to 0} \beta(\rho)/\rho = \infty. \tag{5.4}$$

(The proof of this is very simple and is omitted.) This will be used to show that the error committed in certain computations is small if Ω is replaced by its approximating half-spaces.

Lemma **(5.2)**. — *For any* $c \in \mathscr{B}$ *the integral operator from* $L_2(\Omega)$ *to* $L_2(E^n)$ *with kernel*

$$R^n c(R(x-y))\chi_{D(\bar{y})}(x)$$

is $o_\infty(1)$ *and* $o_2(R^{(n-1)/2})$.

Proof. — Note first that since $y \notin D(\bar{y})$ the δ part of c does not contribute to the operator and so may be assumed to be zero.

The arguments for the two conclusions are quite different. First we show the operator is $o_\infty(1)$ and all that will be needed for this is that $c \in L_1(E^n)$. Since the uniform norm of convolution by $|c|$ on $L_2(E^n)$ is equal to $\|c\|_1$, and since the bounded functions with compact support are dense in $L_1(E^n)$, it suffices to prove the desired conclusion for c the characteristic function of some ball $B(0, r)$. Instead of the operator we may consider its adjoint, so what is to be shown is that

$$\sup \left\| R^n \int_{D(\bar{y}) \cap B(y, r/R)} f(x)\, dx \right\|_2 \Big/ \|f\|_2$$

is $o(1)$ as $R \to \infty$. The supremum is taken over all $f \in L_2(E^n)$ and the first norm is taken with respect to $y \in \Omega$.

There is a finite collection of open sets $N_0, N_1, \ldots$ in E^n which cover Ω such that the closure of N_0 is contained in the interior of Ω and each N_i with $i \geq 1$ is one of the coordinate neighborhoods for $\partial\Omega$ described at the beginning of the section. We can find compact subsets K_i of these N_i $(i \geq 0)$ whose union contains an open set containing Ω. If f is supported in the complement of this union then the integral

$$R^n \int_{D(\bar{y}) \cap B(y, r/R)} f(x)\, dx \tag{5.5}$$

will vanish, for sufficiently large R, for all $y \in \Omega$. The same holds if f is supported in K_0 since

$$\beta(d(K_0, \partial\Omega)) > 0,$$

where β was defined by (5.3). Thus we may assume that f is supported on one of the K_i with $i \geq 1$. We shall simply write K, N for K_i, N_i.

For sufficiently large R the integral will be nonzero only if y belongs to a slightly larger compact set K' of N. Thus we may confine attention to $x \in K$ and $y \in K'$. By means of the diffeomorphism

$$\sigma : N \to E^n,$$

with $\sigma(N \cap \Omega)$ consisting of those points of $\sigma(N)$ with $\sigma^1 \geq 0$, the integral (5.5) may be written as an integral with respect to the variable $\xi = \sigma(x)$. The integral with respect to y of the square of (5.5), which must be evaluated to determine the L_2 norm of this integral, may be written as an integral with respect to the variable $\eta = \sigma(y)$. The integral

$$\int |f(x)|^2\, dx$$

determining the norm of f may also be written as an integral with respect to ξ. The Jacobians arising in these transformed integrals are bounded and bounded away from zero since x and y are restricted to the compact subset K' of N.

The region of integration in (5.5) is contained in the set

$$\{x : d(x, \partial\Omega) \leq \beta^{-1}(|x-y|),\ |x-y| \leq r/R\}$$

where β^{-1} is the inverse of β, defined at points of ambiguity to be continuous on the right. Upon applying σ the region of integration becomes contained in

$$\{\xi : |\xi^1| \leq A\beta^{-1}(|\xi-\eta|),\ |\xi-\eta| \leq A/R\}$$

where A is some constant. It follows from (5.4) that

$$\lim_{\rho\to 0} \beta^{-1}(\rho)/\rho = 0 \tag{5.6}$$

and so for any $\varepsilon > 0$ this set is contained in

$$\Delta(\eta) = \{\xi : |\xi^1| \leq \varepsilon/R,\ |\xi - \eta| \leq A/R\}$$

for R sufficiently large.

We shall show that

$$\sup_f \left\| R^n \int_{\Delta(\eta)} f(\xi)\, d\xi \right\|_2 \Big/ \|f\|_2 \tag{5.7}$$

is at most a constant times $\varepsilon^{1/2}$ for large enough R, and this will give the desired conclusion. Here f runs over $L_2(E^n)$ and the first norm is taken with respect to $\eta \in E^n$.

For a point $\xi = (\xi^1, \xi^2, \ldots, \xi^n)$ we write $\tilde{\xi} = (0, \xi^2, \ldots, \xi^n)$. Set

$$g(\tilde{\xi}) = \left(\int_{-\infty}^{\infty} |f(\xi^1, \xi^2, \ldots, \xi^n)|^2\, d\xi^1\right)^{1/2}.$$

Estimating the integral with respect to ξ^1 in (5.7) by Schwarz's inequality gives

$$\left| R^n \int_{\Delta(\eta)} f(\xi)\, d\xi \right| \leq R^n (2\varepsilon/R)^{1/2} \int_{|\tilde{\xi} - \tilde{\eta}| \leq A/R} g(\tilde{\xi})\, d\tilde{\xi}.$$

Since $\Delta(\eta)$ is empty unless $|\eta^1| \leq (A + \varepsilon)/R$ and since the integral on the right side of the last inequality is independent of η^1 we see that the L_2 norm with respect to η^1 of the left side of the inequality is at most a constant times

$$R^{n-1} \varepsilon^{1/2} \int_{|\tilde{\xi} - \tilde{\eta}| \leq A/R} g(\tilde{\xi})\, d\tilde{\xi}. \tag{5.8}$$

But this is exactly the value at $\tilde{\eta}$ of the $(n-1)$-dimensional convolution of $g(\tilde{\xi})$ with

$$R^{n-1} \varepsilon^{1/2} \chi_{B(0,A)}(R\tilde{\xi}),$$

a function in $L_1(E^{n-1})$ with L_1 norm a constant times $\varepsilon^{1/2}$. Hence (5.8) has L_2 norm, with respect to $\tilde{\eta} \in E^{n-1}$, at most a constant times

$$\varepsilon^{1/2} \left(\int_{E^{n-1}} g(\tilde{\xi})^2\, d\tilde{\xi}\right)^{1/2} = \varepsilon^{1/2} \left(\int_{E^n} |f(\xi)|^2\, d\xi\right)^{1/2}.$$

Thus (5.7) is at most a constant times $\varepsilon^{1/2}$ for R sufficiently large, $\varepsilon > 0$ was arbitrary, and the first assertion of the lemma is established.

To establish the second part we make a few preliminary observations. First, the integral

$$\int |x|\, |c(x)|^2\, dx$$

is equal to

$$\int_0^\infty \rho\, |dS_c(\rho)|$$

where

$$S_c(\rho) = \int_{|x| > \rho} |c(x)|^2\, dx.$$

From the fact that

$$\int |x| \, |c(x)|^2 \, dx < \infty$$

it follows easily that

(5.9) $$\lim_{\rho \to \infty} \rho S_c(\rho) = 0, \qquad \lim_{\rho \to 0} \rho S_c(\rho) = 0.$$

Second, if

(5.10) $$\Omega_\rho = \{x \in E^n : d(x, \partial\Omega) \leq \rho\}, \qquad \mu(\rho) = \mathrm{vol}(\Omega_\rho)$$

then as $\rho \to 0$

(5.11) $$\mu(\rho) = O(\rho).$$

This follows easily from the fact that $\partial\Omega$ is of class C^1.

The Hilbert-Schmidt norm of the operator in the statement of the lemma is the square root of

(5.12) $$\int_\Omega \int_{E^n} R^{2n} |c(R(x-y))|^2 \chi_{D(\bar{y})}(x) \, dx \, dy.$$

Let A be so large that $\Omega \subset B(0, A/2)$. Then integration with respect to x over the complement of $B(0, A)$ gives, for each $y \in \Omega$, at most

$$\int_{|x| \geq A/2} R^{2n} |c(Rx)|^2 \, dx = R^n S_c(AR/2) = o(R^{n-1})$$

as $R \to \infty$, by (5.9). Integration with respect to $y \in \Omega$ just multiplies this estimate by the volume of Ω.

Consider now the part of (5.12) where $x \in B(0, A)$. The integrand vanishes unless $x \in D(\bar{y})$ and so we must have

$$|x - y| \geq \beta(d(x, \partial\Omega)).$$

Therefore, since also $x \in \Omega_A$, integration first with respect to y shows that this part of (5.12) is at most

$$\int_{\Omega_A} R^n S_c(R\beta(\mathrm{dist}(x, \partial\Omega)) \, dx = \int_0^A R^n S_c(R\beta(\rho)) \, d\mu(\rho).$$

Since $\beta(\rho)$ is at least a constant times ρ, integration by parts, (5.9), and (5.11) show that this is at most a constant times

$$R^n \int_0^A \rho \, |dS_c(R\beta(\rho))| + o(R^{n-1}) = R^n \int_0^{\beta(A)} \beta^{-1}(\rho) \, |dS_c(R\rho)| + o(R^{n-1}).$$

Since $\beta^{-1}(\rho)$ is at most a constant times ρ, integration over $\rho \geq P/R$ contributes at most a constant times

$$R^n \int_{P/R}^\infty \rho \, |dS_c(R\rho)| = R^{n-1} \int_P^\infty \rho \, |dS_c(\rho)|$$

which is at most εR^{n-1} if P is chosen large enough. For each P, if R is big enough, integration over $\rho \leq P/R$ will contribute, by (5.6), at most

$$\varepsilon R^n \int_0^{P/R} \rho \, |dS_c(R\rho)| \leq \varepsilon R^{n-1} \int_0^\infty \rho \, |dS_c(\rho)|.$$

Thus this other part of (5.12) is also $o(R^{n-1})$ and the second part of the lemma is established.

The domains $D(z)$ were defined for Ω and $z\in\partial\Omega$. Analogously we define the domains $D_R(z)$ defined for $z\in\partial(R\Omega)$. We write $\mathbf{F}_R[c]$ for the integral operator from $L_2(R\Omega)$ to $L_2(E^n)$ with kernel

$$c(x-y)\chi_{D_R(\bar{y})}(x).$$

This is unitarily equivalent to the operator of Lemma (5.2) and so

$$\textbf{(5.13)} \qquad \mathbf{F}_R[c]=o_\infty(1), \qquad \mathbf{F}_R[c]=o_2(R^{(n-1)/2}).$$

Recall that we defined $\mathbf{U}_R[c]$ to be the operator with kernel

$$\textbf{(5.14)} \qquad U_R(x,y)=h(x-y)-\int_{t.v(\bar{y})>0} h^+_{v(\bar{y})}(x-\bar{y}+t)h^-_{v(\bar{y})}(-y+\bar{y}-t)\,dt.$$

The final lemma gives a rough but useful bound on $U_R(x,y)$.

Lemma **(5.3)**. — *For some* $c_0\in\mathscr{B}$ *we have*

$$|U_R(x,y)|\leq c_0(x-y)$$

for all R.

Proof. — Since $\mathscr{B}$ is closed under convolution and absolute value and since

$$0\leq|c_1|\leq c_2, \qquad c_2\in B$$

imply $c_1\in\mathscr{B}$, it suffices to show that for some $c_0\in\mathscr{B}$ we have

$$|h_v^+|\leq c_0, \qquad |h_v^-|\leq c_0$$

for all unit vectors v. But since

$$|s_{\pm(v)}|\leq|s|$$

we have

$$|h_v^\pm|\leq\sum_{r=0}^{\infty}|s|^{[r]}/r!$$

which belongs to $\mathscr{B}$.

Theorem **(5.1)**. — *Assume* $c=\operatorname{Exp} s$ *where* $s\in\mathscr{B}$. *Then* $\mathbf{W}_R[c]$ *is invertible for sufficiently large* R, *and if* $\mathbf{U}_R[c]$ *is defined by* (5.14) *we have as* $R\to\infty$

$$\mathbf{W}_R[c]^{-1}=\mathbf{U}_R[c]+o_\infty(1), \qquad \mathbf{W}_R[c]^{-1}=\mathbf{U}_R[c]+o_2(R^{(n-1)/2}).$$

Proof. — The operator $\mathbf{W}_R[c]\mathbf{U}_R[c]$ has kernel

$$\int_{R\Omega}c(x-z)U_R(z,y)\,dz=\int_{(z-\bar{y}).v(\bar{y})\geq 0}c(x-z)U_R(z,y)\,dz$$
$$-\int c(x-z)U_R(z,y)(\chi_{\{z:(z-\bar{y}).v(\bar{y})\geq 0\}}-\chi_{R\Omega}(z))\,dz.$$

By Lemma (5.1) the first term on the right side is $\delta(x-y)$. Thus if $\mathbf{E}_R$ is the operator on $L_2(R\Omega)$ whose kernel is the last integral we have

$$\textbf{(5.15)} \qquad \mathbf{W}_R[c]\mathbf{U}_R[c]=\mathbf{I}-\mathbf{E}_R.$$

The absolute value of the kernel of $\mathbf{E}_R$ is at most that of $\mathbf{F}_R[c_0]$ left multiplied by convolution by $|c|$ as an operator from $L_2(E^n)$ to $L_2(R\Omega)$; here c_0 is as given by Lemma (5.3). It follows from (5.13) therefore that

$$\mathbf{(5.16)}\qquad \mathbf{E}_R = o_\infty(1), \quad \mathbf{E}_R = o_2(R^{(n-1)/2}).$$

In particular $\mathbf{W}_R[c]$ is right invertible for sufficiently large R and its left invertibility follows upon considering adjoints. Since $\mathbf{U}_R = o_\infty(1)$, (5.15) and (5.16) give

$$\mathbf{(5.17)}\qquad \mathbf{W}_R[c]^{-1} = \mathbf{U}_R[c] + \mathbf{U}_R[c]\mathbf{E}_R + o_1(R^{n-1})$$

and the theorem follows.

The theorem as stated is not strong enough to enable us to deduce formulas for the traces with error $o(R^{n-1})$. We shall use rather (5.17) which contains more information. Recall that $\mathbf{E}_R$ has kernel

$$\int c(x-z)U_R(z,y)\left(\chi_{\{z:(z-\bar{y}).v(\bar{y})\geq 0\}} - \chi_{R\Omega}(z)\right) dz.$$

We shall see that as far as traces go the term $\mathbf{U}_R[c]\mathbf{E}_R$ can contribute at most $o(R^{n-1})$.

If φ is any function satisfying conditions (4.6) and (4.7) and $\mathbf{T}$ is a nuclear operator on L_2 of a subset of E^n with kernel K then

$$\mathbf{(5.18)}\qquad \operatorname{tr} \mathbf{T} = \lim_{\varepsilon\to 0} \iint \varepsilon^{-n}\varphi(\varepsilon^{-1}(x-y))K(x,y)\, dx\, dy.$$

If a nuclear operator $\mathbf{T}$ is written as a sum of operators each of whose nuclearity is dubious, the right side of (5.18) could be used to estimate the "traces" of these summands. The sum of these estimates is then an estimate on tr $\mathbf{T}$ itself. This suggests that, whether $\mathbf{T}$ is nuclear or not, it is useful to consider

$$\limsup_{\varepsilon\to 0} \left|\iint \varepsilon^{-n}\varphi(\varepsilon^{-1}(x-y))K(x,y)\, dx\, dy\right|.$$

We call this the "estimated trace" of T. We shall assume that, in addition to satisfying (4.6) and (4.7), the function φ is nonnegative. This implies that if $\mathbf{T}_1$, $\mathbf{T}_2$ have kernels K_1, K_2 satisfying

$$|K_1(x,y)| \leq K_2(x,y)$$

then the estimated trace of $\mathbf{T}_1$ is at most that of $\mathbf{T}_2$. If $\mathbf{T}$ is nuclear then the estimated trace of $\mathbf{T}$ is, of course, equal to the absolute value of tr $\mathbf{T}$.

In the next lemma $\mathbf{P}_R$ denotes the projection operator from $L_2(E^n)$ to $L_2(R\Omega)$.

Lemma **(5.4)**. — *If* $c_1, c_2 \in \mathscr{B}$ *then the estimated trace of*

$$\mathbf{W}_R[c_1]\mathbf{P}_R\mathbf{F}_R[c_2]$$

is $o(R^{n-1})$ *as* $R\to\infty$.

Proof. — As mentioned in the proof of Lemma (5.2) we may assume c_2 has no δ summand. To take care of the δ summand in c_1 we shall show first that the estimated trace of

$$\mathbf{P}_R\mathbf{F}_R[c_2]$$

is zero. For this we may assume $R=1$. If we define

$$S_\varepsilon(\rho)=\int_{|x|>\rho}\varepsilon^{-n}\varphi(\varepsilon^{-1}x)|c_2(x)|\,dx$$

then integration first with respect to x gives

$$\left|\int_\Omega\int_{D(\bar{y})}\varepsilon^{-n}\varphi(\varepsilon^{-1}(x-y))c_2(x-y)\,dx\,dy\right|\leq\int_\Omega S_\varepsilon(\beta(d(y,\,\partial\Omega)))\,dy\leq\int_0^A S_\varepsilon(\beta(\rho))\,d\mu(\rho)$$

where $\Omega\subset B(0,A)$. As in the last part of the proof of Lemma (5.2), integration by parts shows this is at most a constant times

$$S_\varepsilon(\beta(A))+\int_0^\infty S_\varepsilon(\rho)\,d\rho=S_\varepsilon(\beta(A))+\int\varepsilon^{-n}|x|\,\varphi(\varepsilon^{-1}x)\,|c_2(x)|\,dx.$$

For any subset B of E^n Schwarz's inequality shows that

$$\left(\int_B\varepsilon^{-n}|x|\,\varphi(\varepsilon^{-1}x)\,|c_2(x)|\,dx\right)^2\leq\left(\int_B\varepsilon^{-2n}|x|\,\varphi(\varepsilon^{-1}x)^2\,dx\right)\left(\int_B|x|\,|c_2(x)|^2\,dx\right)$$
$$=\left(\int_{\varepsilon^{-1}B}|x|\,\varphi(x)^2\,dx\right)\left(\int_B|x|\,|c_2(x)|^2\,dx\right).$$

Each of these last integrals has a bound independent of B and ε. If we first choose B to be the ball $B(0,\delta)$ the second integral will be arbitrarily small if δ is small enough. Having fixed δ and then taking B to be the complement of the same ball, the first integral is $o(1)$ as $\varepsilon\to 0$. Thus

$$\int\varepsilon^{-n}|x|\,\varphi(\varepsilon^{-1}x)\,|c_2(x)|\,dx=o(1).$$

An application of Schwarz's inequality also gives

$$S_\varepsilon(\beta(A))=o(1)$$

and so the first part of the lemma is established.

To prove the second part we may assume c_1 has no δ summand, and note that the trace of the product of Hilbert-Schmidt operators with kernels $K_i(x,y)$ $(i=1,2)$ is always given by

$$\iint K_1(y,\,x)K_2(x,y)\,dx\,dy.$$

Therefore

$$|\operatorname{tr}\mathbf{W}_R[c_1]\mathbf{P}_R\mathbf{F}_R[c_2]|\leq\int_{R\Omega}\int_{R\Omega}|c_1(y-x)c_2(x-y)|\,.\,\chi_{D_R(\bar{y})}(x)\,dx\,dy$$
$$\leq\int_{R\Omega}\int_{R\Omega}(|c_1(y-x)|^2+|c_2(x-y)|^2)\,\chi_{D_R(\bar{y})}(x)\,dx\,dy$$

and this is $o(R^{n-1})$ by the second part of (5.13).

Lemma **(5.5).** — *For any* $c_1\in\mathscr{B}$ *the estimated trace of*

$$\mathbf{W}_R[c_1]\mathbf{U}_R[c]\mathbf{E}_R$$

is $o(R^{n-1})$ *as* $R\to\infty$.

Proof. — If we write

$$c = \alpha_c \delta + \tilde{c}, \qquad c_0 = \alpha_{c_0} \delta + \tilde{c}_0, \qquad c_1 = \alpha_{c_1} \delta + \tilde{c}_1$$

where α_c, α_{c_0}, α_{c_1} are constants and $\tilde{c}$, $\tilde{c}_0$, $\tilde{c}_1$ functions in $L_1(E^n)$ (this notation is different from that used at the beginning of the preceding section) then the kernel of $\mathbf{E}_R$ has absolute value at most

$$|\alpha_c|\,|\tilde{c}_0(x-y)|\,\chi_{D_R(\bar{y})}(x) + \int |\tilde{c}(x-z)\tilde{c}_0(z-y)\,(\chi_{\{z:(z-\bar{y}).v(\bar{y})\geq 0\}} - \chi_{R\Omega}(z))|\,dz.$$

Here c_0 is as in Lemma (5.3). Moreover the absolute value of the kernel of

$$\mathbf{W}_R[c_1]\mathbf{U}_R[c]$$

is at most

$$|\alpha_{c_1}\alpha_{c_0}|\,\delta + |\alpha_{c_1}|\,|\tilde{c}_0(x-y)| + |\alpha_{c_0}|\,|\tilde{c}_1(x-y)| + |\tilde{c}_1 * \tilde{c}_0|(x-y).$$

It follows that the absolute value of the kernel of

$$\mathbf{W}_R[c_1]\mathbf{U}_R[c]\mathbf{E}_R$$

is at most that of

$$|\alpha_c\alpha_{c_1}\alpha_{c_0}|\,\mathbf{F}_R[|\tilde{c}_0|] + \mathbf{W}_R[c_2]\mathbf{P}_R\mathbf{F}_R[|\tilde{c}_0|]$$

where c_2 is a nonnegative member of $\mathscr{B}$ without a δ summand. The conclusion follows from Lemma (5.4).

The preceding lemma takes care of the term $\mathbf{U}_R[c]E_R$ appearing in (5.17). Unfortunately there are two more lemmas to go.

Lemma **(5.6).** — *If* $c_1, c_2 \in \mathscr{B}$ *then*

$$\mathbf{W}_R[c_1]\mathbf{W}_R[c_2] - \mathbf{W}_R[c_1 * c_2]$$

is a nuclear operator with trace $O(R^{n-1})$.

Proof. — The kernel of the operator is

$$\int_{(R\Omega)^c} c_1(x-z)c_2(z-y)\,dz \tag{5.19}$$

where the superscript c denotes complement. This is the resultant of two kernels and the first part of the lemma will follow if we can show each of them is Hilbert-Schmidt. To show for example that

$$\int_{R\Omega}\int_{(R\Omega)^c} |c_1(x-y)|^2\,dy\,dx \tag{5.20}$$

is finite, we integrate first with respect to y. The integral is seen to be at most

$$\int_{R\Omega} S_{c_1}(d(x, \partial(R\Omega)))\,dx \tag{5.21}$$

where, as before

$$S_c(\rho) = \int_{|x|>\rho} |c(x)|^2\,dx.$$

We estimate the integral (5.21) just as we estimated similar integrals before. If

$$\Omega \subset B(0, A)$$

then (5.21) is at most

$$R^n \int_{|x| \leq A} S_{c_1}(R\, d(x, \partial\Omega))\, dx = R^n \int_0^A S_{c_1}(R\rho)\, d\mu(\rho) = R^n \int_0^A \mu(\rho)\, |dS_{c_1}(R\rho)| + O(R^{n-1})$$

and since $\mu(\rho)$ is at most a constant times ρ the first term on the right side is at most a constant times

$$R^n \int_0^\infty \rho\, |dS_{c_1}(R\rho)| = R^{n-1} \int_0^\infty \rho\, |dS_{c_1}(\rho)|.$$

Thus (5.21) is not only finite, it is even $O(R^{n-1})$.

Since the kernel (5.19) is the resultant of two Hilbert-Schmidt kernels, the trace of the operator it represents is given by

$$\int_{R\Omega} \int_{(R\Omega)^c} c_1(x-z)\, c_2(z-x)\, dz\, dx.$$

This has absolute value at most

$$\int_{R\Omega} \int_{(R\Omega)^c} [|c_1(x-z)|^2 + |c_2(x-z)|^2]\, dz\, dx,$$

the sum of two integrals each of which we have already seen to be $O(R^{n-1})$.

In the next lemma we write $\mathbf{H}_R$ for the operator on $L_2(R\Omega)$ with kernel

$$H_R(x, y) = \int_{t.v(\bar{y})>0} h^+_{v(\bar{y})}(x - \bar{y} + t)\, h^-_{v(\bar{y})}(-y + \bar{y} - t)\, dt$$

so that

$$\mathbf{U}_R[c] = \mathbf{W}_R[h] - \mathbf{H}_R.$$

Lemma **(5.7)**. — *Assume* $s = \operatorname{Log} c \in \mathscr{B} \cap L_2$. *Then for any* $c_1 \in \mathscr{B} \cap L_2$

$$\mathbf{W}_R[c_1]\mathbf{H}_R$$

is nuclear and its trace is $O(R^{n-1})$ *as* $R \to \infty$.

Proof. — Let us look carefully at the integral representing $H_R(x, y)$. The second factor of the integrand cannot have vanishing argument over the range of integration, by (5.1). However the first factor may have vanishing argument for some t. If so, then

$$x \in R\Omega, \qquad (x - \bar{y}).v(\bar{y}) < 0$$

so $x \in D_R(\bar{y})$. Thus if α_v denotes the coefficient of the δ summand of h^+_v then these δ summands contribute

(5.22) $$\alpha_{v(\bar{y})}\, h^-_{v(\bar{y})}(y - x)\, \chi_{D_R(\bar{y}) \cap R\Omega}(x)$$

to $\mathbf{H}_R$. The method of proof of Lemma (5.3) shows that there is a $k \in \mathscr{B}$ such that

$$|h^+_v(x)| \leq k(x), \qquad |h^-_v(x)| \leq k(x)$$

and $\tilde{k}=k-\delta$ is a function in $L_2(E^n)$. Therefore (5.22) has absolute value at most a constant times

$$\tilde{k}(y-x)\chi_{D_R(\bar{y})\cap R\Omega}(x)$$

and so by Lemma (5.4) the product with $\mathbf{W}_R[c_1]$ on the left is nuclear and has trace $o(R^{n-1})$.

The remaining part of $H_R(x,y)$ has absolute value at most

$$\int_{t.v(\bar{y})>0}\tilde{k}(x-y+t)\,\tilde{k}(-y+\bar{y}-t)\,dt \tag{5.23}$$

which is on the one hand at most

$$\tilde{k}^{[2]}(x-y)$$

and on the other at most, by Schwarz's inequality

$$S_{\tilde{k}}(\max((x-\bar{y}).v(\bar{y}),\,0))^{1/2}S_{\tilde{k}}((y-\bar{y}).v(\bar{y}))^{1/2}.$$

The first bound shows that (5.23) is bounded uniformly in R, since $\tilde{k}\in L_2$. In particular this part of $\mathbf{H}_R$ is Hilbert-Schmidt and so when left multiplied by $\mathbf{W}_R[c_1]$, gives a nuclear operator. A bound on the absolute value of the trace of this product is

$$\int_{R\Omega}\int_{R\Omega}|c_1(y-x)|\,dx\,dy\int_{t.v(\bar{y})>0}\tilde{k}(x-\bar{y}+t)\,\tilde{k}(-y+\bar{y}-t)\,dt. \tag{5.24}$$

First let us confine attention to the set where

$$(x-\bar{y}).v(\bar{y})\leq\tfrac{1}{2}\gamma|x-\bar{y}|$$

where γ is as in (5.1). This implies that

$$|x-y|\geq\gamma_1\,d(x,\,\partial(R\Omega)) \tag{5.25}$$

for some constant $\gamma_1>0$. Therefore the contribution of this set to the integral (5.24) is at most

$$\iint|c_1(y-x)|\,\tilde{k}^{[2]}(x-y)\,dx\,dy$$

integrated over that part of $R\Omega\times R\Omega$ for which (5.25) holds. If

$$S(\rho)=\int_{|y|>\rho}|c_1(y)|\,\tilde{k}^{[2]}(-y)\,dy$$

then integration with respect to y first shows that the last double integral is at most

$$\int_{R\Omega}S(\gamma_1 d(x,\,\partial(R\Omega)))\,dx=O(R^{n-1})$$

by the same sort of argument used in the proof of Lemma (5.6).

There remains the contribution to (5.24) of the set where

$$(x-\bar{y}).v(\bar{y})\geq\tfrac{1}{2}\gamma|x-\bar{y}|.$$

On this set we have

$$(x-\bar{y}).v(\bar{y})\geq\tfrac{1}{2}\gamma\,d(x,\,\partial(R\Omega)),\qquad(y-\bar{y}).v(\bar{y})\geq\tfrac{1}{2}\gamma\,d(y,\,\partial(R\Omega)),$$

so its contribution is at most

$$\int_{R\Omega}\int_{R\Omega}|c_1(y-x)|\,S_{\tilde{k}}(\tfrac{1}{2}\gamma\, d(x,\ \partial(R\Omega)))^{1/2}S_{\tilde{k}}(\tfrac{1}{2}\gamma\, d(y,\ \partial(R\Omega)))^{1/2}\,dx\,dy.$$

Since $\|\mathbf{W}_R[|c_1|]\|_\infty \leq \|c_1\|_1$ this is at most

$$\int_{\Omega R} S_{\tilde{k}}(\tfrac{1}{2}\gamma\, d(x,\ \partial(R\Omega)))\,dx = O(R^{n-1})$$

as before. The lemma is established.

We are ready to give the higher dimensional analogue of (1.1) or, more precisely, of Theorem (4.2), *a)*. As in that theorem the assumption that $\mathbf{W}_R[c]$ differ from $\mathbf{I}$ by a nuclear operator is equivalent to the assumption that $\mathbf{W}_R[s]$ be nuclear, and this implies in particular that s, $c-\delta$, $h-\delta$ are functions in $L_2(E^n)$. In the statement of the theorem the integral over $\partial\Omega$ is taken with respect to surface area.

Theorem **(5.2)**. — *Assume that* $s = \operatorname{Log} c \in \mathscr{B}$ *and that* $\mathbf{W}_R[s]$ *is nuclear for each* $R>0$. *Then with determinant defined by* (4.3), *and with* φ *any function satisfying the conditions* (4.6) *and* (4.7), *we have as* $R\to\infty$

$$\log\det \mathbf{W}_R[c] = R^n \operatorname{vol}(\Omega)\lim_{\varepsilon\to 0}\int \varepsilon^{-n}\varphi(\varepsilon^{-1}x)s(x)\,dx + \tfrac{1}{2}R^{n-1}\int_{\partial\Omega}dz\int_{t.v(z)>0} t.v(z)s(t)s(-t)\,dt + o(R^{n-1}).$$

Proof. — The first term on the right is just the n-dimensional analogue of the right side of (4.8) and so is nothing but

$$\operatorname{tr}\mathbf{W}_R[s].$$

The assertion of the theorem is therefore equivalent to

$$\log\det \mathbf{W}_R[c] = \operatorname{tr}\mathbf{W}_R[s] + \tfrac{1}{2}R^{n-1}\int_{\partial\Omega}dz\int_{t.v(z)>0} t.v(z)s(t)s(-t)\,dt + o(R^{n-1}). \tag{5.26}$$

As before we embed c in the analytic family

$$c(\lambda) = \operatorname{Exp}\lambda s$$

but we proceed a little differently now. We shall show first that

$$\log\det \mathbf{W}_R[c(\lambda)] = \operatorname{tr}\mathbf{W}_R[\lambda s] + O(R^{n-1}) \tag{5.27}$$

uniformly for λ belonging to any compact set. Since this holds trivially for $\lambda=0$ it suffices to show

$$\frac{d}{d\lambda}\log\det \mathbf{W}_R[c(\lambda)] = \operatorname{tr}\mathbf{W}_R[s] + O(R^{n-1}) \tag{5.28}$$

with the prescribed uniformity. The left side is

$$\operatorname{tr}\mathbf{W}_R[c'(\lambda)]\mathbf{W}_R[c(\lambda)]^{-1}.$$

Since $c'(\lambda) * c(\lambda) = h(\lambda)$, (5.28) follows from (5.17) and Lemmas (5.5-5.7). The uniformity follows from the uniformity of the conclusions of those lemmas, which is easily verified.

Thus (5.27) holds, and this implies that to prove (5.26) for the entire family $c(\lambda)$ it suffices to prove it for small λ. In particular it suffices to prove (5.26) if c is such that, with $c = \delta + \widetilde{c}$

$$\int |\widetilde{c}(x)|\, dx < 1.$$

We now use the same trick as in the proof of Theorem (4.2), *a)* and extend the definitions of determinant and trace as was done there. Then c may be embedded in the family

(5.29) $$c(\lambda) = \widetilde{c} + \lambda\,\delta$$

with λ in some neighborhood of the interval $(1, \infty)$. Clearly

$$s(\lambda) = \delta \log \lambda + \sum_{r=1}^{\infty} (-1)^{r+1} \lambda^{-r} \widetilde{c}^{[r]}/r$$

(with $\log 1 = 0$) is analytic from some neighborhood of $[1, \infty]$ to $\mathscr{B}$ and it suffices to prove (5.26) for this family. Since both sides vanish at $\lambda = \infty$ it suffices to prove the differentiated relation, which for the family (5.29) is

$$\operatorname{tr} \mathbf{W}_{\mathrm{R}}[c]^{-1} = \operatorname{tr} \mathbf{W}_{\mathrm{R}}[h] + \tfrac{1}{2} \mathrm{R}^{n-1} \int_{\partial\Omega} dz \int_{t.v(z)>0} t.v(z)\,(s(t)s(-t))'\, dt + o(\mathrm{R}^{n-1}).$$

(The prime denotes differentiation with respect to λ.) Note that this need only be proved for each λ, since uniformity on closed sets follows from this plus the uniformity of

$$\log \det \mathbf{W}_{\mathrm{R}}[c] = \operatorname{tr} \mathbf{W}_{\mathrm{R}}[h] + O(\mathrm{R}^{n-1})$$

on closed sets.

From (5.17) and Lemma (5.5) we see that what must be proved is that

$$-\tfrac{1}{2} \mathrm{R}^{n-1} \int_{\partial\Omega} dz \int_{t.v(z)>0} t.v(z)\,(s(t)s(-t))'\, dt$$

differs from the trace of $\mathrm{H_R}$, as defined by (5.18), by $o(\mathrm{R}^{n-1})$. As we saw at the beginning of the proof of Lemma (5.7) the contribution to $\mathrm{H_R}(x, y)$ of the δ summands of the various $h_v^{\pm}$ has absolute value at most

$$\widetilde{k}(y - x)\, \chi_{\mathrm{D_R}(\overline{y}) \cap \mathrm{R}\Omega}(x)$$

with $\widetilde{k} \in \mathscr{B}$, and by Lemma (5.4) this contributes $o(\mathrm{R}^{n-1})$. (Indeed, the proof of that lemma shows that the contribution is actually zero.)

Thus we may ignore the δ summands of the $h_v^{\pm}$. What remains of $\mathrm{H_R}(x, y)$ is a function which, although not necessarily continuous in x and y, is continuous in x uniformly for $y \in \mathrm{R}\Omega$. This follows easily from the fact that the $h_v^{\pm}$ with δ summands removed are all bounded by a single function belonging to $\mathrm{L_2(E}^n)$. Continuity in x uniformly in y is enough to guarantee that (5.18) is equal, in our case, to

$$\int_{\mathrm{R}\Omega} \mathrm{H_R}(y, y)\, dy.$$

Thus what must be verified is

(5.30) $$\int_{R\Omega} H_R(y,y)\,dy = -\tfrac{1}{2}R^{n-1}\int_{\partial\Omega} dz \int_{t.v(z)>0} t.v(z)(s(t)s(-t))'\,dt + o(R^{n-1}).$$

The idea is this. We must integrate

$$H_R(y,y) = \int_{t.v(\bar{y})>0} h^+_{v(\bar{y})}(y-\bar{y}+t)h^-_{v(\bar{y})}(-y+\bar{y}-t)\,dt$$

with respect to y over $R\Omega$. We integrate first over those y for which $\bar{y}=z$. If these y were exactly the points running along the inner normal from z we would obtain

$$\int_{t.v(z)>0} t.v(z)h^+_{v(z)}(t)h^-_{v(z)}(-t)\,dt.$$

Then integration over $\partial(R\Omega)$ would give

$$R^{n-1}\int_{\partial\Omega} dz \int_{t.v(z)>0} t.v(z)h^+_z(t)h^-_z(-t)\,dt,$$

which turns out to be equal to the first term on the right side of (5.30).

Of course the difficulty is that $R\Omega$ is not the Cartesian product of its boundary with its normals. What we are going to do is cover most of a neighborhood of $\partial(R\Omega)$, which we shall see is all that counts in evaluating the integral on the left side of (5.30), by finitely many disjoint sets, each contained in one of the coordinate neighborhoods, such that after applying the coordinate mapping into a half-space the set corresponds to a Cartesian product and the mapping $y\mapsto\bar{y}$ corresponds to the orthogonal projection onto the hyperplane bounding the half-space. Note that there was flexibility in defining the mapping $y\mapsto\bar{y}$. It had only to satisfy the two conditions (5.1) and (5.2).

We shall use here the notation

$$Q_\delta = \{x\in Q:\ d(x,\partial Q)\le\delta\},\qquad Q^\delta = Q - Q_\delta$$

for an arbitrary set Q. We observe first that for any $\delta>0$

$$\int_{R\Omega^\delta} H_R(y,y)\,dy = o(R^{n-1}).$$

For with $\tilde{k}$ and $S_{\tilde{k}}$ as at the beginning of the proof of Lemma (5.7)

(5.31) $$|H_R(y,y)| \le S_{\tilde{k}}(\gamma\, d(y,\partial(R\Omega))),$$

and so

$$\int_{R\Omega^\delta} |H_R(y,y)|\,dy \le R^n\int_{\Omega^\delta} S_{\tilde{k}}(\gamma\, d(y,\partial\Omega))\,dy$$

which is at most a constant times

$$R^n\int_\delta^\infty \rho\,|dS_{\tilde{k}}(\gamma R\rho)| = \gamma^{-1}R^{n-1}\int_{\gamma R\delta}\rho\,|dS_{\tilde{k}}(\rho)| = o(R^{n-1}).$$

Thus we need only consider the integral of $H_R(y,y)$ over $R\Omega_\delta$. The δ-neighborhood of $\partial\Omega$ in E^n may be covered by finitely many coordinate neighborhoods N_i. We can find disjoint sets B_i such that

(i) the union of the B_i is $\partial\Omega$,

(ii) ∂B_i in $\partial\Omega$ has $(n-1)$-dimensional measure 0,

(iii) the closed δ-neighborhood of B_i in E^n is contained in N_i.

Finally, if π_i on $\sigma_i(N_i \cap \Omega)$ (where σ_i are the coordinate mappings which take the $N_i \cap \Omega$ into half-spaces) denotes projection onto the boundary half-plane, let

$$\Omega_i = \{y \in \Omega : d(y, (B_i)^{2\delta}) \leq \delta, \ \pi_i \sigma_i(y) \in \sigma_i((B_i)^{2\delta})\}.$$

Note that since the B_i are disjoint the distance between any two different $(B_i)^{2\delta}$ is at least 4δ, and this implies that the Ω_i are disjoint. Moreover

$$\bigcup_i \Omega_i \subset \Omega_\delta$$

and

$$\Omega_\delta - \bigcup_i \Omega_i \subset \bigcup_i \{y \in \Omega : d(y, \partial B_i) \leq A\delta\}$$

where A is some positive constant.

We shall show first that the contribution of

$$\{y \in R\Omega : d(y, \partial(RB_i)) \leq AR\delta\}$$

to the left side of (5.30) is at most an arbitrarily small constant times R^{n-1} if δ is small enough. In fact by (5.31) this contribution is at most

$$\int_{d(y,\partial(RB_i)) \leq AR\delta} S_{\tilde{k}}(\gamma\, d(y, \partial R\Omega))\, dy = R^n \int_{d(y,\partial B_i) \leq A\delta} S_{\tilde{k}}(\gamma R\, d(y, \partial\Omega))\, dy.$$

If

$$\mu_i(\rho) = \mathrm{vol}\{y : d(y, \partial\Omega) \leq \rho, \ d(y, \partial B_i) \leq A\delta\}$$

then for sufficiently small δ we shall have

$$\mu_i(\rho) \leq \varepsilon\rho$$

for all ρ. (Here ε is an arbitrary but fixed positive number.) This is seen by applying the coordinate mapping σ_i and using property (ii) of the B_i. This implies that

$$R^n \int_{d(y,\partial B_i) \leq A\delta} S_{\tilde{k}}(\gamma R\, d(y, \partial\Omega))\, dy$$

is at most a constant times

$$\varepsilon R^n \int_0^\infty \rho\, |dS_{\tilde{k}}(\gamma R\rho)| = \varepsilon\gamma^{-1} R^{n-1} \int_0^\infty \rho\, |dS_{\tilde{k}}(\rho)|.$$

Thus if δ is sufficiently small then the left side of (5.30) differs from

$$\sum_i \int_{R\Omega_i} H_R(y, y)\, dy$$

by an arbitrarily small multiple of R^{n-1}. We now specify that

$$\bar{y} = \sigma_i^{-1}(\pi_i \sigma_i(y)), \quad y \in \Omega_i.$$

We define $\bar{y}$ arbitrarily in the complement of $\bigcup_i \Omega_i$ except that, of course, (5.1) and (5.2) must be satisfied.

To evaluate the integral

$$\int_{R\Omega_i} H_R(y,y)\,dy \tag{5.32}$$

we use on Ω_i not the coordinate function σ_i but a modification τ_i defined by

$$\tau_i(y)=((y-\bar{y}).v(\bar{y}),\ \sigma_i^2(y),\ \ldots,\ \sigma_i^n(y)).$$

From the way $\bar{y}$ was defined the Jacobian determinant

$$J_i=\left|\frac{\partial\tau_i}{\partial y}\right|$$

is bounded and bounded away from zero, and if we set

$$H_i(\eta)=H_R(\tau_i^{-1}(\eta),\ \tau_i^{-1}(\eta))$$

then (5.32) equals

$$R^n\int\ldots\int H_i(R\eta^1,\eta^2,\ldots,\eta^n)J(\eta^1,\ldots,\eta^n)^{-1}\,d\eta^1,\ldots,d\eta^n. \tag{5.33}$$

The integral with respect to $\eta^2, \ldots, \eta^n$ is taken over $\tau_i((B_i)^{2\delta})$ and for each $\eta^2, \ldots, \eta^n$ in this set the integral over η^1 is taken from zero to a quantity which is bounded and bounded away from zero.

Now given any functions $f\in L_1(0,\infty)$ and $g\in L_\infty(0,\infty)$ such that

$$\lim_{\eta\to 0} g(\eta)=0,$$

we have

$$\lim_{R\to\infty} R\int_0^\infty f(R\eta)g(\eta)\,d\eta=0. \tag{5.34}$$

This is very easy. It follows that with an error induced in (5.33) of at most $o(R^{n-1})$ the Jacobian may be replaced by its boundary function

$$J(0,\eta^2,\ldots,\eta^n).$$

Therefore (5.33) equals

$$\begin{aligned}R^n\int\ldots\int H_i(R\eta^1,\eta^2,\ldots,\eta^n)J(0,\eta^2,\ldots,\eta^n)\,d\eta^1,\ldots,d\eta^n+o(R^{n-1})\\ =R^{n-1}\int\ldots\int\int_0^\infty H_i(u,\eta^2,\ldots,\eta^n)J(0,\eta^2,\ldots,\eta^n)\,du\,d\eta^2,\ldots,d\eta^n+o(R^{n-1}),\end{aligned}$$

where the integration with respect to $\eta^2, \ldots, \eta^n$ is over $\tau_i((B_i)^{2\delta})$ as before. But the first term on the right is exactly equal to

$$R^{n-1}\int_{(B_i)^{2\delta}} dz\int_{t.v(z)>0} t.v(z)h^+_{v(z)}(t)h^-_{v(z)}(-t)\,dt.$$

Since the difference

$$\int_{\partial\Omega}\ldots dz-\sum_i\int_{(B_i)^{2\delta}}\ldots dz$$

can be made arbitrarily small by choosing δ sufficiently small, this shows that

$$\int_{R\Omega} H_R(y,y)\,dy = R^{n-1}\int_{\partial\Omega} dz \int_{t.v(z)>0} t.v(z) h^+_{v(z)}(t) h^-_{v(z)}(-t)\,dt + o(R^{n-1}).$$

To prove (5.30) now it suffices to show that

$$\int_{t.v>0} t.vh^+_v(t)h^-_v(-t)\,dt = -\tfrac{1}{2}\int_{t.v>0} t.v(s(t)s(-t))'\,dt$$

for each unit vector v. But this is proved just as the analogous identity (3.8), as the reader should have no difficulty verifying.

It was mentioned in the introduction that the coefficient of R^{n-1} in the asymptotic formula, namely

$$\int_{\partial\Omega} dz \int_{t.v(z)>0} t.v(z)s(t)s(-t)\,dt, \tag{5.35}$$

may be written as an integral over the $(n-1)$-sphere S^{n-1} rather than as an integral over $\partial\Omega$. If G denotes the Gauss map

$$\partial\Omega \to S^{n-1}$$

which takes any $z\in\partial\Omega$ to the point of S^{n-1} corresponding to the inner unit normal vector $v(z)$ then we can define the measure ν_Ω on S^{n-1} by defining

$$\nu_\Omega(A) = \text{surface measure of } G^{-1}(A) \text{ on } \partial\Omega.$$

Then the integral (5.35) is equal to

$$\int_{S^{n-1}} d\nu_\Omega(v) \int_{t.v>0} t.vs(t)s(-t)\,dt.$$

This way of writing the coefficient makes a little clearer the individual contributions of Ω and s to the asymptotic formula.

6. Variable convolution operators

We consider first operators on $L_2(R\Omega)$ with kernels of the form

$$c\left(\frac{x}{R}, x-y\right).$$

At the end of the section we shall indicate what can be done with the more general form

$$c\left(\frac{x}{R}, \frac{y}{R}, x-y\right).$$

The setting will be the algebra, which we call $\mathscr{B}(\Omega)$, of C^1 functions from Ω to the algebra $\mathscr{B}$ of the last section. More exactly, $c\in\mathscr{B}(\Omega)$ if c is a function from Ω to $\mathscr{B}$ for which there is a continuous function, denoted by

$$\text{grad}\, c,$$

from Ω to n-vectors with components in $\mathscr{B}$, satisfying

$$\text{(6.1)} \qquad \|c(x)-c(y)-(x-y)\,.\,\text{grad}\, c(y)\|_{\mathscr{B}}=o(|x-y|)$$

for $|x-y|\to 0$. The subscript indicates that the norm taken in (6.1) is the norm in $\mathscr{B}$.

In the set of nonnegative elements of $\mathscr{B}$ with its natural partial ordering, bounded subsets have least upper bounds. We define, for $c\in\mathscr{B}(\Omega)$

$$\text{(6.2)} \qquad \bar{c}=\underset{x\in\Omega}{\ell.u.b.}\,|c(x)|, \qquad \bar{\bar{c}}=\underset{x\in\Omega}{\ell.u.b.}\,|\text{grad}\, c(x)|.$$

Here $|\text{grad}\, c(x)|$ is taken to mean the sum of the norms of the components of grad $c(x)$. The norm in $\mathscr{B}(\Omega)$ is given by

$$\|c\|_{\mathscr{B}(\Omega)}=\|\bar{c}\|_{\mathscr{B}}+\|c\|_{\mathscr{B}}.$$

The set $\mathscr{B}(\Omega)$ is a Banach algebra under convolution defined by

$$(c_1 * c_2)\,(x)=c_1(x) * c_2(x).$$

It is convenient to associate with each $c\in\mathscr{B}(\Omega)$ a function of two variables $c(x, t)$ defined everywhere on $\Omega\times E^n$; given $x\in\Omega$ let $c(x, t)$ be an everywhere defined function of t (more exactly a constant times $\delta(t)$ plus an everywhere defined function of t) representing the element $c(x)$ of $\mathscr{B}$. Clearly we have for all x

$$\text{(6.3)} \qquad |c(x, t)|\leq\bar{c}(t), \qquad |\text{grad}\, c(x, t)|\leq\bar{\bar{c}}(t)$$

for almost every t (and for $t=0$ which corresponds to the δ summands of the distributions $c(x)$). The gradient, of course, is taken with respect to the variable x.

For any $c\in\mathscr{B}(\Omega)$ we denote by $\mathbf{W}_R[c]$ the integral operator on $L_2(R\Omega)$ with kernel

$$c\left(\frac{x}{R}, x-y\right).$$

Note that with the notation of the last section

$$c\left(\frac{x}{R}, t\right)=\alpha\left(\frac{x}{R}\right)\delta(t)+\tilde{c}\left(\frac{x}{R}, t\right)$$

where $\tilde{c}$ is an ordinary function of two variables. Correspondingly $\mathbf{W}_R[c]$ is multiplication by $\alpha(x/R)$ plus the integral operator with kernel

$$\tilde{c}\left(\frac{x}{R}, x-y\right)$$

in the classical sense. It follows that

$$\|\mathbf{W}_R[c]\|\leq\|\alpha\|_\infty+\int_{t\neq 0}\bar{c}(t)\,dt\leq\|c\|_{\mathscr{B}(\Omega)}.$$

In the special case $c(x)=f(x)c$ with $c\in\mathscr{B}$ the operator on $L_2(E^n)$ with kernel

$$f(x)\,c(x-y)$$

has inverse exactly equal to the operator with kernel

$$f(y)^{-1}h(x-y)$$

where h is the inverse of c in $\mathscr{B}$. This suggests that in defining the approximate inverse $\mathbf{U}_{\mathrm{R}}[c]$ of $\mathbf{W}_{\mathrm{R}}[c]$ in general, one should replace the first term of (5.14) by

$$h\left(\frac{y}{\mathrm{R}}, x-y\right)$$

where h is the inverse of c in $\mathscr{B}(\Omega)$.

In analogy with the last section our basic assumption will be that

$$c = \operatorname{Exp} s, \qquad s \in \mathscr{B}(\Omega). \tag{6.4}$$

As with $\mathscr{B}$, this is equivalent to

$$c \in \mathscr{B}(\Omega)$$

$$\hat{c} \text{ bounded away from } 0 \text{ on } \Omega \times \mathrm{E}^n$$

(the circumflex denotes Fourier Transform with respect to the second variable) plus, in dimension one

$$\underset{-\infty<\xi<\infty}{\Delta} \arg \hat{c}(x, \xi) = 0 \quad \text{for each } x.$$

This follows from the Arens-Calderón form of the Wiener-Lévy theorem and the fact that the most general homomorphism of $\mathscr{B}(\Omega)$ onto the complex numbers is of the form

$$c \to \hat{c}(x, \xi)$$

for some $(x, \xi) \in \Omega \times \mathrm{E}^n$. The proof of this is not hard and is left as an exercise for the interested reader.

If (6.4) holds we define the elements h and $h_v^{\pm}$ of $\mathscr{B}(\Omega)$ in analogy with the last section. The mapping $y \mapsto \bar{y}$, just as before, is to satisfy (5.1) and (5.2). Then we define $\mathbf{U}_{\mathrm{R}}[c]$ to be the operator on $\mathrm{L}_2(\mathrm{R}\Omega)$ with kernel

$$h\left(\frac{y}{\mathrm{R}}, x-y\right) - \int_{t.v(\bar{y})>0} h_{v(\bar{y})}^{+}\left(\frac{y}{\mathrm{R}}, x-\bar{y}+t\right) h_{v(\bar{y})}^{-}\left(\frac{y}{\mathrm{R}}, -y+\bar{y}-t\right) dt. \tag{6.5}$$

When carrying out the procedure of the last section we encounter new operators which arise because $c(x, t)$ is not independent of x. Given $c_1, c_2 \in \mathscr{B}(\Omega)$ we write

$$\mathbf{V}_{\mathrm{R}}[c_1, c_2]$$

for the integral operator on $\mathrm{L}_2(\mathrm{R}\Omega)$ with kernel

$$\int_{\mathrm{R}\Omega} \left(c_1\left(\frac{x}{\mathrm{R}}, x-z\right) - c_1\left(\frac{y}{\mathrm{R}}, x-z\right)\right) c_2\left(\frac{y}{\mathrm{R}}, z-y\right) dz$$

and $\overline{\mathbf{V}}_{\mathrm{R}}[c_1, c_2]$ for the operator with kernel given by a similar formula but with absolute value signs around the integrand. Although one such operator will make a significant contribution to the asymptotic formula for the determinant, these operators are all reasonably small in certain senses.

Lemma **(6.1)**. — *If* $c_1, c_2 \in \mathscr{B}(\Omega)$ *then as* $R \to \infty$

$$\overline{\mathbf{V}}_R[c_1, c_2] = o_\infty(1), \qquad \overline{\mathbf{V}}_R[c_1, c_2] = o_2(R^{(n-1)/2}).$$

If moreover $c_3 \in \mathscr{B}$ *then the estimated trace of*

$$\mathbf{W}_R[c_3]\,\overline{\mathbf{V}}_R[c_1, c_2]$$

is $O(R^{n-1})$.

Proof. — It follows from (6.1) and (6.3) that

$$\left| c_1\left(\frac{x}{R}, x-z\right) - c_2\left(\frac{y}{R}, x-z\right)\right| \leq AR^{-1}|x-y|\,\bar{\bar{c}}_1(x-z)$$

for some constant A. Hence the kernel of $\overline{\mathbf{V}}_R[c_1, c_2]$ is at most

$$R^{-1}|x-y|\,c_4(x-y) \tag{6.6}$$

where $c_4 = A\bar{\bar{c}}_1 * \bar{c}_2$. (Note that the δ summand of c_4 plays no role here.) We may write this as

$$R^{-1}|x-y|\,c_4(x-y)\,\chi_{\{(x,y):|x-y|\leq \varepsilon R\}} + R^{-1}|x-y|\,c_4(x-y)\,\chi_{\{(x,y):|x-y|\geq \varepsilon R\}}.$$

The first of these summands is the kernel of convolution with a function whose L_1 norm is at most $\varepsilon\|c_4\|_1$. Since $|x-y|$ is at most a constant times R for $x, y \in R\Omega$, the second summand is at most a constant times the kernel of convolution with a function whose L_1 norm is

$$\int_{|x|\geq \varepsilon R} c_4(x)\,dx = o(1)$$

as $R \to \infty$. Hence $\overline{\mathbf{V}}_R[c_1, c_2] = o_\infty(1)$.

We use the same decomposition to estimate the Hilbert-Schmidt norm. The square of the first summand is

$$R^{-2}|x-y|^2 c_4(x-y)^2\,\chi_{\{(x,y):|x-y|\leq \varepsilon R\}} \leq \varepsilon R^{-1}|x-y|\,c_4(x-y)^2.$$

This must be integrated over $R\Omega \times R\Omega$. Integration with respect to x gives at most

$$\varepsilon R^{-1}\int |x|\,c_4(x)^2\,dx.$$

Subsequent integration with respect to y gives at most a constant times εR^{n-1}. Similarly the square of the second summand is at most a constant times

$$R^{-1}|x-y|\,c_4(x-y)^2\,\chi_{\{(x,y):|x-y|\geq \varepsilon R\}}$$

and the integral of this over $R\Omega \times R\Omega$ is at most

$$R^{n-1}\,\mathrm{vol}\,\Omega\int_{|x|\geq \varepsilon R}|x|\,c_4(x)^2\,dx$$

which is $o(R^{n-1})$ as $R \to \infty$. Thus we have shown that the square of the Hilbert-Schmidt norm of $\overline{\mathbf{V}}_R[c_1, c_2]$ is $o(R^{n-1})$.

To prove the last part of the lemma, note first that the estimate (6.6) implies that $\overline{\mathbf{V}}_{\mathrm{R}}[c_1, c_2]$ has estimated trace zero. Hence we may assume c_3 has no δ summand. The estimated trace in question is at most the estimated trace of the operator with kernel

$$\int_{\mathrm{R}\Omega} \mathrm{R}^{-1} |c_3(x-z)|\,|z-y|\,c_4(z-y)\,dz.$$

This is the resultant of two nonnegative Hilbert-Schmidt kernels. The estimated trace is therefore equal to the absolute value of the trace, which is

$$\int_{\mathrm{R}\Omega}\int_{\mathrm{R}\Omega} \mathrm{R}^{-1}|x-y|\,|c_3(y-x)|\,c_4(x-y)\,dx\,dy$$
$$\leq \int_{\mathrm{R}\Omega}\int_{\mathrm{R}\Omega} \mathrm{R}^{-1}|x-y|\,(|c_3(y-x)|^2+c_4(x-y)^2)\,dx\,dy$$

and this is $O(\mathrm{R}^{n-1})$.

Theorem **(6.1)**. — *Assume* $c=\operatorname{Exp} s$ *with* $s\in\mathscr{B}(\Omega)$, *and define* $\mathbf{U}_{\mathrm{R}}[c]$ *by* (6.5). *Then* $\mathbf{W}_{\mathrm{R}}[c]$ *is invertible for sufficiently large* R *and we have, as* $\mathrm{R}\to\infty$

$$\mathbf{W}_{\mathrm{R}}[c]^{-1}=\mathbf{U}_{\mathrm{R}}[c]+o_\infty(1), \qquad \mathbf{W}_{\mathrm{R}}[c]^{-1}=\mathbf{U}_{\mathrm{R}}[c]+o_2(\mathrm{R}^{(n-1)/2}).$$

Moreover for any $c_1\in\mathscr{B}(\Omega)$ *the operator*

$$\mathbf{W}_{\mathrm{R}}[c_1]\,(\mathbf{W}_{\mathrm{R}}[c]^{-1}-\mathbf{U}_{\mathrm{R}}[c]+\mathbf{U}_{\mathrm{R}}[c]\mathbf{V}_{\mathrm{R}}[c,h])$$

has estimated trace $o(\mathrm{R}^{n-1})$.

Proof. — If $\mathrm{U}_{\mathrm{R}}(x,y)$ denotes the kernel of $\mathbf{U}_{\mathrm{R}}[c]$ then

$$\mathbf{W}_{\mathrm{R}}[c]\mathbf{U}_{\mathrm{R}}[c]$$

has kernel

$$\int_{\mathrm{R}\Omega} c\left(\frac{\mathrm{X}}{\mathrm{R}}, x-z\right)\mathrm{U}_{\mathrm{R}}(z-y)\,dz=\int_{\mathrm{R}\Omega} c\left(\frac{y}{\mathrm{R}}, x-z\right)\mathrm{U}_{\mathrm{R}}(z-y)\,dz$$
$$+\int_{\mathrm{R}\Omega}\left(c\left(\frac{x}{\mathrm{R}}, x-z\right)-c\left(\frac{y}{\mathrm{R}}, x-z\right)\right)\mathrm{U}_{\mathrm{R}}(z,y)\,dz.$$

By Theorem (5.1), or more exactly its proof, the first term in the right is the kernel of an operator

$$\mathbf{I}+\mathbf{E}^1_{\mathrm{R}}$$

where

$$\mathbf{E}^1_{\mathrm{R}}=o_\infty(1), \qquad \mathbf{E}^1_{\mathrm{R}}=o_2(\mathrm{R}^{(n-1)/2})$$

and for each $c_2\in\mathrm{B}$ the product $\mathbf{W}_{\mathrm{R}}[c_2]\mathbf{E}^1_{\mathrm{R}}$ has estimated trace $o(\mathrm{R}^{n-1})$.

The second integral on the right side is the kernel of

$$\mathbf{V}_{\mathrm{R}}[c,h]+\mathbf{E}^2_{\mathrm{R}}.$$

Here $\mathbf{E}^2_{\mathrm{R}}$ has kernel of the form

$$\mathbf{E}^2_{\mathrm{R}}(x,y)=\int_{\mathrm{R}\Omega}\left(c\left(\frac{x}{\mathrm{R}}, x-z\right)-c\left(\frac{y}{\mathrm{R}}, x-z\right)\right)\mathrm{H}_{\mathrm{R}}(z,y)\,dz$$

where H_R has the following properties. First

$$|H_R(x,y)| \leq k(x-y)$$

for some $k \in \mathscr{B}$ without a δ summand. Second

$$\int_{R\Omega}\int_{R\Omega} |H_R(x,y)|^2\,dx\,dy = O(R^{n-1}). \tag{6.7}$$

These things may be checked by referring to the proof of Lemma (5.7).

Thus

$$\mathbf{W}_R[c]\mathbf{U}_R[c] = \mathbf{I} + \mathbf{V}_R[c, h] + \mathbf{E}_R^1 + \mathbf{E}_R^2.$$

By Lemma (6.1)

$$\mathbf{V}_R[c, h] = o_\infty(1), \qquad \mathbf{E}_R^2 = o_\infty(1).$$

(Note that the kernel of $\mathbf{E}_R^2$ is at most that of $\overline{\mathbf{V}}_R[c, k]$.) Hence in particular $\mathbf{W}_R[c]$ is right invertible.

To see why this implies $\mathbf{W}_R[c]$ is also left invertible, write

$$c(x, t) = \alpha(x)\delta(t) + \tilde{c}(x, t)$$

where $\tilde{c}$ has no δ summand. Our hypothesis implies that $\alpha(x)$ is nonzero on Ω; in fact $\alpha(x)$ is the exponential of the coefficient of the δ summand of s. Therefore $\mathbf{W}_R[c]$ consists of an invertible operator, multiplication by $\alpha(x)$, right multiplied by the operator

$$\mathbf{W}_R[\delta + \alpha^{-1}\tilde{c}]$$

which differs from $\mathbf{I}$ by a compact operator. Hence, by the Fredholm alternative, left and right invertibility for $\mathbf{W}_R[c]$ are completely equivalent.

Hence $\mathbf{W}_R[c]$ is invertible for sufficiently large R and

$$\mathbf{W}_R[c]^{-1} = \mathbf{U}_R[c](\mathbf{I} - \mathbf{V}_R[c, h] - \mathbf{E}_R^1 - \mathbf{E}_R^2) + o_1(R^{n-1}).$$

Since

$$|U_R(x,y)| \leq c_0(x,y)$$

for some $c_0 \in B$, what remains to be verified is that for each $c_2 \in B$

$$\mathbf{W}_R[c_2]\mathbf{E}_R^1, \qquad \mathbf{W}_R[c_2]\mathbf{E}_R^2$$

have estimated traces equal to $O(R^{n-1})$. That the first does has already been mentioned. As for the second, it follows from (6.7) that the kernel of $\mathbf{E}_R^2$ has absolute value at most

$$R^{-1}|x-y|$$

times the kernel of an operator which is $o_2(R^{(n-1)/2})$. It follows that $\mathbf{E}_R^2$ itself has estimated trace zero. One may assume therefore that c_2 has no δ summand, and Schwarz's inequality shows that the estimated trace of

$$\mathbf{W}_R[c_2]\mathbf{E}_R^2$$

is at most $O(R^{(n-1)/2})$ times

$$R^{-1}\left(\int_{R\Omega}\int_{R\Omega} |x-y|^2\,|c_2(x-y)|^2\,dx\,dy\right)^{1/2}.$$

If Ω is contained in the ball $B(0, A)$ then this is at most a constant times

$$R^{\frac{n}{2}-1}\left(\int_{|x|\leq 2AR} |x|^2 |c_2(x)|^2 dx\right)^{1/2} = o(R^{(n-1)/2}).$$

Theorem **(6.2)**. — *Assume that* $s = \text{Log}\, c \in \mathscr{B}(\Omega)$ *and that* $\mathbf{W}_R[s]$ *is nuclear for each* $R > 0$. *Then with determinant defined by* (4.3), *and with* φ *any function satisfying the conditions* (4.6) *and* (4.7), *we have as* $R \to \infty$

$$\log\det \mathbf{W}_R[c] = R^n \lim_{\varepsilon\to 0}\int_\Omega dx \int \varepsilon^{-n}\varphi(\varepsilon^{-1}t)s(x, t)\,dt$$
$$+ \tfrac{1}{2}R^{n-1}\int_{\partial\Omega} dz \int_{t.v(z)>0} t.v(z)s(z, t)s(z, -t)\,dt$$
$$- \tfrac{1}{2}R^{n-1}\int_\Omega dy \int_{E^n} s(y, -x)x.\text{grad}\, s(y, x)\,dx + o(R^{n-1}).$$

Proof. — The first term on the right side is, by (5.18)

$$\text{tr}\, \mathbf{W}_R[s].$$

As in the proof of Theorem (5.2) we first embed c in the family

$$c(\lambda) = \text{Exp}\,\lambda s$$

and show that

$$\text{tr}\, \mathbf{W}_R[c'(\lambda)]\mathbf{W}_R[c(\lambda)]^{-1} = \text{tr}\, \mathbf{W}_R[s] + o(R^{n-1})$$

uniformly for λ in any compact set. The only difference with the corresponding point in the proof of Theorem (5.2) is that now we must show

$$\text{tr}\, \mathbf{W}_R[c']\mathbf{U}_R[c]\mathbf{V}_R[c, h] = O(R^{n-1}).$$

This follows, however, from the last part of Lemma (6.1).

Thus once again it suffices to consider the family

$$c(\lambda) = c + \lambda\,\delta$$

and to compute, for this family

$$\frac{d}{d\lambda}\det\log \mathbf{W}_R[c] = \text{tr}\, \mathbf{W}_R[c]^{-1}.$$

By the preceding theorem this is

$$\text{tr}\, \mathbf{U}_R[c] - \text{tr}\, \mathbf{U}_R[c]\mathbf{V}_R[c, h] + O(R^{n-1}).$$

The first term, the trace of $\mathbf{U}_R[c]$, is

$$\textbf{(6.8)}\quad \text{tr}\, \mathbf{W}_R[h] - \int_{R\Omega} dy \int_{t.v(\bar{y})>0} h^+_{v(\bar{y})}\left(\frac{y}{R}, x - \bar{y} + t\right)h^-_{v(\bar{y})}\left(\frac{y}{R}, -y + \bar{y} - t\right)dt$$
$$= \text{tr}\, W_R[h] - R^{n-1}\int_{\partial\Omega} dz \int_{t.v(z)>0} t.v(z)h^+_{v(z)}(z, t)h^-_{v(z)}(z, -t)\,dt + o(R^{n-1}).$$

The proof of this is like the corresponding part of the proof of Theorem (5.2). One uses an argument based on (5.34) to show that if in the integral on the left side of (6.8) the quantity $R^{-1}y$ is replaced by $R^{-1}\bar{y}$ the error incurred is $o(R^{n-1})$. We omit the details.

We are left with

$$\operatorname{tr} \mathbf{U}_R[c]\mathbf{V}_R[c, h].$$

Now $\mathbf{U}_R[c]$ is the sum of $\mathbf{W}_R[h]$ and another operator whose contribution to the trace of the product can only be $o(R^{n-1})$, by (6.7) and Lemma (6.1). Therefore what we must compute is

(6.9) $\operatorname{tr} \mathbf{W}_R[h]\mathbf{V}_R[c, h]$

$$= \int_{R\Omega}\int_{R\Omega} h\left(\frac{x}{R}, y-x\right) dx\, dy \int_{R\Omega} \left(c\left(\frac{x}{R}, x-z\right) - c\left(\frac{y}{R}, x-z\right)\right) h\left(\frac{y}{R}, z-y\right) dz.$$

First, observe that we may replace

$$h\left(\frac{x}{R}, y-x\right)$$

in this integral by

$$h\left(\frac{y}{R}, y-x\right)$$

with error $o(R^{n-1})$. The reason is that the difference between these functions is the kernel of the operator $\mathbf{V}_R[h, \delta]$ and

$$\operatorname{tr} \mathbf{V}_R[h, \delta]\mathbf{V}_R[c, h] = o(R^{n-1})$$

by Lemma (6.1).

Next, let us see what error is incurred in the integral (6.9) if the z integration is taken over all E^n. If $\bar{h}$, $\bar{\bar{c}}$ are the least upper bounds for h and grad c as in (6.2) the error has absolute value at most

$$\int_{R\Omega}\int_{R\Omega} R^{-1}|x-y|\, \bar{h}(y-x)\, dx\, dy \int_{(R\Omega)^c} \bar{\bar{c}}(x-z)\bar{h}(z-y)\, dz.$$

The proof of Lemma (5.6) shows that the inner integral, as a function on $R\Omega \times R\Omega$, has L_2 norm $O(R^{n-1})$. Therefore the triple integral is at most $O(R^{(n-1)/2})$ times

$$R^{-1}\left\{\int_{R\Omega}\int_{R\Omega} |x-y|^2\, \bar{h}(y-x)\, dx\, dy\right\}^{1/2} = O(R^{(n-1)/2})$$

just as at the end of the proof of Theorem (6.1).

Thus we have shown that

$$\operatorname{tr} \mathbf{U}_R[c]\mathbf{V}_R[c, h]$$

is equal to

$$\int_{R\Omega}\int_{R\Omega} h\left(\frac{y}{R}, y-x\right) dx\, dy \int_{E^n} \left(c\left(\frac{x}{R}, x-z\right) - c\left(\frac{y}{R}, x-z\right)\right) h\left(\frac{y}{R}, z-y\right) dz$$

plus $o(\mathrm{R}^{n-1})$. We leave it to the reader to check that the error incurred upon replacing

$$c\left(\frac{x}{\mathrm{R}}, x-z\right)-c\left(\frac{y}{\mathrm{R}}, x-z\right)$$

by

$$\mathrm{R}^{-1}(x-y)\,.\,\mathrm{grad}\, c\left(\frac{y}{\mathrm{R}}, x-z\right)$$

is also $o(\mathrm{R}^{n-1})$. Since

$$\mathrm{grad}\, s = h * \mathrm{grad}\, c$$

(this is just the old formula $(\log f)' = f'/f$ suitably interpreted) we conclude that

$$\mathrm{tr}\, \mathbf{U}_{\mathrm{R}}[c]\mathbf{V}_{\mathrm{R}}[c, h] = \int_{\mathrm{R}\Omega}\int_{\mathrm{R}\Omega} \mathrm{R}^{-1} h\left(\frac{y}{\mathrm{R}}, y-x\right)(x-y)\,.\,\mathrm{grad}\, s\left(\frac{y}{\mathrm{R}}, x-y\right) dx\, dy + o(\mathrm{R}^{n-1}).$$

Again the error incurred upon integration with respect to x over E^n rather than $\mathrm{R}\Omega$ is $o(\mathrm{R}^{n-1})$ and so

$$\mathrm{tr}\, \mathbf{U}_{\mathrm{R}}[c]\mathbf{V}_{\mathrm{R}}[c, h] = \mathrm{R}^{n-1}\int_{\Omega} dy \int h(y, -x)x\,.\,\mathrm{grad}\, s(y, x)\, dx + o(\mathrm{R}^{n-1}).$$

Recall that we are dealing with the family

(6.10) $$c(\lambda) = c + \lambda\,\delta$$

and are trying to verify that the trace of $\mathbf{W}_{\mathrm{R}}[c]^{-1}$ differs by $o(\mathrm{R}^{n-1})$ from the sum of the derivatives with respect to λ of the three terms appearing on the right side of the formula given in the statement of the theorem. The trace of $\mathbf{U}_{\mathrm{R}}[c]$ is given by (6.8). That the two terms on the right side of (6.8) are the derivatives of the first two terms of the formula is no different from the corresponding fact in the last section.

There remains the verification of the identity

$$2\int h(y, -x)x\,.\,\mathrm{grad}\, s(y, x)\, dt = \frac{d}{d\lambda}\int s(y, -x)x\,.\,\mathrm{grad}\, s(y, x)\, dx$$

for each y. The right side is (prime denotes differentiation with respect to λ)

$$\int s'(y, -x)x\,.\,\mathrm{grad}\, s(y, x)\, dx + \int s(y, -x)x\,.\,\mathrm{grad}\, s'(y, x)\, dx$$

$$= \int s'(y, -x)x\,.\,\mathrm{grad}\, s(y, x)\, dx - \int s(y, x)x\,.\,\mathrm{grad}\, s'(y, -x)\, dx$$

$$= 2\int s'(y, -x)x\,.\,\mathrm{grad}\, s(y, x)\, dx - \int x\,.\,\mathrm{grad}[s(y, x)s'(y, -x)]\, dx.$$

The first term is exactly

$$2\int h(y, -x)x\,.\,\mathrm{grad}\, s(y, x)\, dx$$

since $h' = s$ for the family (6.10). We shall shall show that the second term is zero.

It suffices to show that for any $s \in \mathscr{B}$ without a δ summand, and with

$$h = \operatorname{Exp}(-s),$$

one has

$$\int x s(x) h(-x)\, dx = 0.$$

It is simplest to assume first that

$$\int (1+|x|)|s(x)|\, dx < \infty. \tag{6.11}$$

Then also

$$\int_{x \neq 0} (1+|x|)\, |h(x)|\, dx < \infty \tag{6.12}$$

since L_1 with weight function $1+|x|$ is a Banach algebra under convolution. If $h = \tilde{h} + \delta$ then taking Fourier transforms gives

$$1 + \int \tilde{h}(x) e^{ix.\xi}\, dx = \exp\left(-\int s(x) e^{ix.\xi}\, dx\right).$$

Because of (6.11) we can differentiate (that is, take the gradient of) both sides with respect to ξ to deduce

$$\int x\tilde{h}(x) e^{ix.\xi}\, dx = -\int x s(x) e^{ix.\xi}\, dx \exp\left(-\int s(x) e^{ix.\xi}\, dx\right) = -\int e^{ix.\xi}\, dx \int y s(y) \tilde{h}(x-y)\, dy.$$

Therefore

$$x\tilde{h}(x) = \int y s(y) \tilde{h}(x-y)\, dy$$

almost everywhere. If s is a continuous function with compact support then both sides are continuous, the equality holds everywhere, and at $x=0$ it gives

$$0 = \int y s(y) \tilde{h}(-y)\, dy = \int y s(y) h(-y)\, dy.$$

For general $s \in \mathscr{B}$ we use the facts that the continuous functions with compact support are dense in $\mathscr{B}$ and that

$$(c_1, c_2) \to \int x c_1(x) c_2(x)\, dx$$

is $|||\ \ |||$-continuous. This completes the proof of the theorem.

We shall now indicate how one might obtain asymptotic results for more general operators $\mathbf{W}_R[c]$ with kernels of the form

$$c\left(\frac{x}{R}, \frac{y}{R}, x-y\right).$$

If one defines (supposing for the moment that c is defined for all values of its second variable)

$$c^0(x, t) = c\left(x, x - \frac{t}{R}, t\right)$$

then

$$c^0\left(\frac{x}{R}, x-y\right)=c\left(\frac{x}{R}, \frac{y}{R}, x-y\right).$$

Therefore an asymptotic formula for

$$\log \det \mathbf{W}_R[c]$$

should be obtainable form Theorem (6.2) applied to the function c^0 (which, admittedly, depends on R).

To a first approximation $c^0(x, t)$ equals

$$c(x, x, t)-R^{-1}t.\mathrm{grad}_2 c(x, x, t)$$

where the subscript refeis to the fact that the gradient is taken with respect to the second variable. Substituting this into the formula given by the theorem gives, after a little manipulation

$$\begin{aligned}\log \det \mathbf{W}_R[c]=R^n \lim_{\varepsilon\to 0}\int_\Omega dx\int \varepsilon^{-n}\varphi(\varepsilon^{-1}t)s(x, x, t)\,dt \\ +\tfrac{1}{2}R^{n-1}\int_{\partial\Omega} dz\int_{t.v(z)>0} t.v(z)s(z, z, t)s(z, z, -t)\,dt \\ +\tfrac{1}{2}R^{n-1}\int_\Omega dy\int_{E^n} s(y, y, -x)x.(\mathrm{grad}_2 s(y, y, x)-\mathrm{grad}_1 s(y, y, x))\,dx \\ +o(R^{n-1}).\end{aligned}$$

Here of course $c(x, y, t)$ is, for fixed x and y, supposed equal to

$$\operatorname{Exp} s(x, y, t).$$

Note the asymmetry in the first two variables. This should not be unexpected since the form of the kernel also displays this asymmetry. If c is symmetric in its first two variables though, then so is s and the third term of the formula vanishes.

Although what we just described is the simplest way of formally deriving the asymptotic formula from what has already been done, it may not be the simplest way to go about proving it. Perhaps better would be to prove an analogue of Theorem (6.1) for the general case, where instead of the term

$$h\left(\frac{y}{R}, x-y\right)$$

appearing in (6.5) one uses

$$h\left(\frac{y}{R}, \frac{x}{R}, x-y\right),$$

and continuing as before. The computations would certainly be quite lengthy and the whole thing might not be worth the effort. There can be no doubt that under appropriate conditions the last asymptotic formula is correct.

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