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MANIFOLDS WHICH ADMIT $\mathbf{R}^n$ ACTIONS

by G. CHATELET *and* H. ROSENBERG

INTRODUCTION

The purpose of this paper is to determine which n-manifolds admit smooth locally free actions of $\mathbf{R}^{n-1}$. We shall restrict ourselves to compact connected orientable manifolds V^n and locally free actions φ of $\mathbf{R}^{n-1}$ on V^n which are of class C^2 and tangent to ∂V^n, i.e. the components of ∂V^n are orbits of φ. For $n=3$, we know that V^3 admits such an $\mathbf{R}^2$ action if and only if $V^3=\mathbf{T}^2\times I$ or V^3 is a bundle over $\mathbf{S}^1$ with fibre $\mathbf{T}^2$ [7]. Moreover, the topological type of such $\mathbf{R}^2$ actions has been completely determined [8]. We recall that the rank of V^n is the largest integer k such that V^n admits a smooth locally free action of $\mathbf{R}^k$.

Now suppose that φ is a locally free action of $\mathbf{R}^{n-1}$ on V^n. We shall prove:

Theorem 1. — If ∂V^n *is not empty, then* V^n *is homeomorphic to* $\mathbf{T}^{n-1}\times I$ (here $\mathbf{T}^i$ denotes the torus of dimension i).

Theorem 2. — If ∂V^n *is empty and* φ *has at least one compact orbit, then* V^n *is a bundle over* $\mathbf{S}^1$ *with fibre* $\mathbf{T}^{n-1}$.

Theorem 3. — If ∂V^n *is empty and* φ *has no compact orbits then* V^n *is a bundle over a torus* $\mathbf{T}^k$ *with fibre a torus* $\mathbf{T}^{n-k}$.

Theorem 2 follows directly from Theorem 1 by cutting V^n along a compact orbit. Theorem 3 depends upon an observation of Novikov [4], and independently Joubert: suppose φ acts on V^n with no compact orbits. By Sacksteder [9], all the orbits of φ are $\mathbf{T}^{n-k}\times\mathbf{R}^{k-1}$ for some k. Choose linearly independent vector fields $X_1, \ldots, X_{n-1}$ tangent to the orbits of φ such that all the integral curves of $X_1, \ldots, X_{n-k}$ are periodic, of period one. Then $X_1, \ldots, X_{n-k}$ define a locally free action of $\mathbf{T}^{n-k}$ on V^n and the orbit space M is a smooth manifold of dimension k. Also M admits an action of $\mathbf{R}^{k-1}$ with all the orbits $\mathbf{R}^{k-1}$. It follows that M is homeomorphic to $\mathbf{T}^k$, which proves Theorem 3 ([5] and [3]). Consequently, our main result is Theorem 1. Here is how we proceed to prove Theorem 1: by inductive arguments similar to those used in [7], we restrict ourselves to actions φ with no compact orbits in the interior of V^n. We then remark that the foliation defined by the orbits of φ is almost without holonomy, i.e. the noncompact leaves have no holonomy. With this, we construct collar neighborhoods U_i of each component T_i of ∂V, such that $\partial U_i = T_i \cup T_i'$ with T_i' transverse to the foliation. We construct U_i so that some linear field Y (tangent to the orbits of φ) is transverse to each T_i'. We then prove the integral curves of Y go from T_i' to T_j' hence define a homeomorphism of V^n to $\mathbf{T}^{n-1} \times I$.

1. Some Preliminaries.

(1.1) *Let $\mathscr{F}$ be the foliation of* V *defined by the orbits of* φ. *Then each noncompact leaf of $\mathscr{F}$ has zero holonomy.*

Proof. — If T is a compact leaf of $\mathscr{F}$, then the germ of $\mathscr{F}$ in a neighborhood of T is without holonomy outside of T, provided T is an isolated compact leaf (page 13 of [8]). This is also true if T is an isolated compact leaf on one side in V and one considers the germ of $\mathscr{F}$ on this side. Now if φ has no compact orbits then $\mathscr{F}$ is without holonomy and we are done [9]. So suppose F is a noncompact leaf of $\mathscr{F}$ and $\mathscr{F}$ has compact leaves. Since $\mathscr{F}$ has no exceptional minimal sets [9], there is a compact leaf T of $\mathscr{F}$ such that T is in the closure of F. Let x be a point of F and $\alpha(x)$ a non zero element of $\pi_1(F, x)$. Let X be a vector field on V such that the integral curve of X through x is closed and homotopic to $\alpha(x)$, and all the integral curves of X on F are closed. X is easily constructed using the action φ (cf. [6]). Since T is in the closure of F, we know the integral curves of X on T are also closed. Now T is an isolated compact leaf at least on one side in V, the side where F intersects a transverse arc infinitely often. Let U be a neighborhood of T, on this side, such that all the leaves of $\mathscr{F}$ in U, except T, have zero holonomy. Then U contains closed integral curves of X which are on F, so such an integral curve C has zero holonomy. Since C is conjugate to $\alpha(x)$, it follows that $\alpha(x)$ has zero holonomy; thus F as well.

(1.2) *Suppose ∂V is not empty and φ has no compact orbits in the interior of* V. *Let* T *be a compact orbit of* φ; $T \subset \partial V$. *The leaves which contain* T *in their closure are homeomorphic to* $\mathbf{T}^k \times \mathbf{R}^{n-k-1}$ *where* $k =$ *the rank of the kernel of the holonomy map on* T.

Proof. — Let F be an open leaf whose closure contains T; $F \approx \mathbf{T}^j \times \mathbf{R}^{n-j-1}$. Suppose Z^k is the kernel of the holonomy homomorphism on T. Let $\mathbf{T}^k$ be a k-torus embedded in T which lifts onto nearby leaves by the holonomy. Since $\overline{F} \supset T$, we can lift $\mathbf{T}^k$ to a k-torus T_1 in F. Also $i_\# : \pi_1(T) \to \pi_1(V)$ is injective, where $i : T \hookrightarrow V$ (cf. [4]), hence $\pi_1(T_1)$ embeds in $\pi_1(F)$ and $k \leq j$.

Next we show $j \leq k$. Let $x \in F$ and $\alpha \in \pi_1(F, x)$, $\alpha \neq 0$. Let X be a vector field tangent to the orbits of φ, such that the integral curves of X on F are closed and the integral curve of X through x is homotopic to α. Since $\overline{F} \supset T$, all the integral curves of X on T are closed. Let C be an integral curve of X on T. We know that C lifts to a closed curve on F, so by (1.1), the holonomy of C is trivial; i.e. C is in the kernel of the holonomy homomorphism. Hence $j \leq k$.

2. The transverse torus and vector field.

Throughout this section, we suppose φ acts on V so that there are no compact orbits in the interior of V and T is a compact orbit in ∂V. Let k denote the rank of the kernel of the holonomy map associated to T; k varies between 0 and $n-2$. Let $Y_1, \ldots, Y_{n-1}$ be linearly independent commuting vector fields on V satisfying:

(i) they are tangent to the φ-orbits;

(ii) their integral curves are closed and of period one on T; and

(iii) the integral curves of $Y_1, \ldots, Y_k$ represent the kernel of the holonomy map on T.

We shall construct an $(n-1)$-torus $T' \subset \operatorname{Int} V$ such that $T \cup T'$ bound a trivial cobordism in V, *and* Y_{n-1} *is transverse to* T' *at each point.*

By (1.1), we know the orbits of $Y_{k+1}, \ldots, Y_{n-1}$ on T induce germs in Diffo($\mathbf{R}^+$) which are contractions or expansions, via the holonomy. Here Diffo($\mathbf{R}^+$) is the set of C^2-germs of diffeomorphisms of $\mathbf{R}^+$ to itself, which leave 0 fixed. After reversing the sign of Y_j if necessary, we shall assume the germs are all contractions, for $k+1 \leq j \leq n-1$.

Choose a metric on V and let U_ε be a geodesic collar neighborhood of T isometric to $\mathbf{T}^{n-1} \times [0, \varepsilon]$, with the obvious product metric. Clearly, if ε is small enough, the geodesics normal to T in U_ε will be transverse to the orbits of φ. Let f_x^i be the holonomy diffeomorphism associated to the Y_i orbit through x; f_x^i is the identity for $1 \leq i \leq k$ and a contraction for $k < i < n$.

Proposition **(2.1)**. — *There is an* $(n-1)$*-torus* T' *contained in* U_ε *such that* Y_{n-1} *is transverse to* T' *and* $T \cup T'$ *bound a trivial cobordism.*

In an earlier version of this paper we gave a proof of (2.1) which used calculus. Charles Pugh pointed out to us how one can use a theorem of W. Wilson on the existence of Liapounov functions for uniform stable attractors of vector fields [13]. We present this proof here and in an appendix we give our original proof.

We need some definitions before stating Wilson's theorem. Let X be a vector field on V and let A be a closed invariant subset of V (here V is a compact manifold). A is called a *uniform stable attractor* of X if the following conditions are satisfied:

a) there exists an increasing function δ sending $\mathbf{R}^+$ into itself such that

$$d(X(p, t), A) < \varepsilon$$

whenever $d(p, A) < \delta(\varepsilon)$ and $t \geq 0$;

b) there exists a neighborhood U of A such that $\omega(p)\subset A$ whenever $p\in U$ ($\omega(p)$ is the ω-limit set of p);

c) let D(A) be the set of p such that $\omega(p)\subset A$; D(A) is an open set, called the *basin of attraction* of A.

Wilson has proved [13] that if A is a uniform stable attractor for X then there exists a C^∞ Liapounov function, i.e.

a) there is a C^∞ function $f: D(A)\to \mathbf{R}^+$ with $f^{-1}(0)=A$; and

b) $X(f)(p)<0$ whenever $f(p)\neq 0$.

Hence f has no singularities outside A and all the level surfaces of f are diffeomorphic. Before proving (2.1) we need three lemmas.

Lemma **(2.2)**. *(Action box lemma.)*

There exists a unique mapping

$$F : J^{n-1}\times[0, \varepsilon] \to U_\varepsilon \subset V$$

(*where* $J=[-1, 2]$) *satisfying the following conditions*:

a) F *is a* C^2*-immersion*;

b) F *sends the horizontal plaques* $J^{n-1}\times\{z\}$ *into the leaves of* $\mathcal{F}$;

c) F *sends vertical arcs* $\{\Lambda\}\times[0, \varepsilon]$ *onto the geodesic arcs normal to* T;

d) F, *when restricted to* $J^{n-1}\times\{0\}$, *is the restriction of the natural covering map:* $\mathbf{R}^{n-1}\to T$ *induced by* φ, *which sends the i-direction line onto the* Y_i *circular orbit*;

e) *let* $x_0\in T$; *then* F *sends* $\{0\}\times[0, \varepsilon]$ *isometrically onto the geodesic arcs issued from* X_0, *normal to* T *and pointing inside* T.

Proof. — Define first F via e) and d). F obviously extends to $J^{n-1}\times[0, \varepsilon]$ using b) and c).

a) is clear, for geodesic arcs are normal to $\mathcal{F}$ in U_ε. Note that each Y_i orbit on T is covered three times by F.

Lemma **(2.3)**. *(Commuting contraction lemma.)*

If f_1 *and* f_2 *are commuting embeddings* $[0, \varepsilon]\to[0, \infty[$ *and* f_2 *is a contraction towards* 0, *then there exists a* K *so large that* $f_1 f_2^K$ *is a contraction to* 0.

Proof. — f_2 commuting with $f_1 f_2^K$, $f_1 f_2^K$ is an embedding without fixed point or is the identity (N. Koppel's Thesis). For sufficiently large k, $f_1 f_2^K$ is not the identity. Hence $f_1 f_2^K$ is a contraction or an expansion. For $f_1 f_2^K[0, \varepsilon]=f_2^K f_1[0, \varepsilon]$, and K may be chosen so large that $f_2^K f_1[0, \varepsilon]\subset\left[0, \frac{\varepsilon}{2}\right]$. $f_1 f_2^K$ is therefore a contraction.

3. Attraction Lemma.

There exists ε *and* $\delta>0$ *such that whenever* X *is a* C^1 *vector field on* $\mathbf{R}^{n-1}$ *and* $|X|_{C^0}<\delta$, *then* $Y=\Phi_*\left(\frac{\partial}{\partial\lambda_{n-1}}+X\right)$ *generates a flow having* T *as a uniform and stable attractor,* U_ε *being in the basin of attraction of* T.

— Look at the application F of Lemma (2.2) (action box lemma). If

$$Y=\Phi_*\left(\frac{\partial}{\partial\lambda_{n-1}}+X\right),$$

F^*Y is a C^1 vector field defined on $J^{n-1}\times[0,\varepsilon]$ (F is a C^2-immersion); F^*Y has no vertical component and may be chosen arbitrarily close to $\frac{\partial}{\partial\lambda_{n-1}}$ for a suitable choice of δ.

Let $I=[0,1]$, $A_0=I^{n-2}\times\{0\}\times[0,\varepsilon]$, $A_1=I^{n-2}\times\{1\}\times[0,\varepsilon]$ and $x\in A_0$. I being interior to J, choose δ such that the positive orbit of F^*Y through x crosses A_1 before reaching the boundary of $J^{n-1}\times[0,\varepsilon]$. Let $\bar{x}$ be the point of intersection of A_1 with the orbit. Via F, $\bar{x}$ is identified with a point $x_1\in A_0$ and hence may be written in the form $(\lambda'_1,\ldots,\lambda'_{n-1},0,z_1)$ where

$$z_1=f_{K+1}^{\varepsilon_{K+1}}\circ\ldots\circ f_{K+j}^{\varepsilon_{K+j}}\circ\ldots\circ f_{n-1}(z) \quad \text{if} \quad x=(\lambda_1,\ldots,\lambda_{n-1},0,z).$$

Recall that for $1\leq j\leq n-K+1$, f_{K+j} are the contracting holonomy diffeomorphisms associated to the circular Y_{K+j} orbits.

Using the contraction commuting lemma, we choose N such that

$$f_{K+1}^{-1}\circ\ldots\circ f_{n-2}^{-1}\circ f_{n-1}^{N}$$

is a contraction. For ε and δ small, we may build a sequence $(x,\bar{x},x_1,\ldots,x_{N-1},\bar{x}_{N-1},x_N)$ where the F^*Y orbit through x_i crosses A_1 at $\bar{x}_i$ and $\bar{x}_i$ being identified via F with x_{i+1} in A_0. So if $x=(\lambda_1,\ldots,\lambda_{n-1},0,z)$, then $x_N=(\lambda''_1,\ldots,\lambda''_{n-1},0,h(z))$ where

$$h(z)=\prod_j f_{K+j}^{\varepsilon_{K+j}}\circ f_{n-1}^{N}(z).$$

Thus we have shown that the vertical coordinate of any Y-orbit tends to 0 in a manner dominated by a fixed contraction $f_{K+1}^{-1}\circ\ldots\circ f_{n-1}^{N})$ as we proceed along the orbit in forward times, i.e. T is a uniformly stable attractor.

Let us prove now Proposition (2.1).

— The choice of the Y'_js on T allow us to write T as a trivial fibration $\Sigma\times\mathbf{S}_1$ where Σ is a manifold diffeomorphic to $\mathbf{T}^{n-2}$ and transversal to the circular orbits of Y_{n-1} which are the fibers of that fibration. Over these circles, consider the normal geodesic fibers of U_ε. This gives a two dimensional foliation of U_ε by cylinders. Call it $\mathscr{A}$; $\mathscr{A}$ is clearly transversal to $\mathscr{F}$.

Let $Y_{n-1}=X+Y$ where Y is tangent to $\mathscr{A}\cap\mathscr{F}$ and orthogonal to X; clearly $Y_{n-1}(x)-Y(x)=X(x)$ tends to 0 when $d(x, T)$ tends to 0. Due to the attraction lemma, Y admits T as a uniform stable attractor. Let $V_1=U_{\frac{\varepsilon}{3}}$, $V_2=U_{\frac{2\varepsilon}{3}}$ and let β be a bump function such that $\beta=1$ on V_1 and $\beta=0$ outside V_2. Let $Z=\beta Y+(1-\beta)Y_{n-1}$. It is easy to check that Z admits T as a uniform stable attractor and hence there exists a Liapounov fonction f for Z. For $\varepsilon>\varepsilon_0>\frac{2\varepsilon}{3}$, $Z=Y_{n-1}$ and $f^{-1}(\varepsilon_0)$ is transversal to Y_{n-1}. For $\frac{\varepsilon}{3}>\varepsilon_1>0$, $f^{-1}(\varepsilon_1)$ is transverse to Y; $f^{-1}(\varepsilon_1)$ is diffeomorphic to $f^{-1}(\varepsilon_0)$. It remains to prove $f^{-1}(\varepsilon_1)$ is a $(n-1)$-dimensional torus for $f^{-1}(\varepsilon_0)$ will be then a torus satisfying conditions of (2.1).

Y being transverse to $f^{-1}(\varepsilon_1)$, $f^{-1}(\varepsilon_1)$ is transverse to $\mathscr{A}$. Let $\mathscr{A}_x$ be the leaf of $\mathscr{A}$ through x; $\mathscr{A}_x\cap f^{-1}(\varepsilon_1)$ is a compact one-dimensional manifold and hence diffeomorphic to a circle. Writing $T=\Sigma\times\mathbf{S}_1$ and $x=(\lambda, s)$ here $\lambda\in\Sigma$ and $s\in\mathbf{S}_1$, one produces a family of embeddings of $\mathbf{S}_1$, $(\pi_\lambda)_{\lambda\in\Sigma}$ such that $\pi_\lambda(\mathbf{S}_1)=\mathscr{A}_x\cap f^{-1}(\varepsilon_1)$. We define now an application $\pi:\Sigma\times\mathbf{S}_1\to f^{-1}(\varepsilon_1)$ by $\pi(\lambda, s)=\pi_\lambda(s)$ which is clearly an embedding. Proposition (2.1) is thereby proved for Σ is diffeomorphic to $\mathbf{T}^{n-2}$.

Proof of Theorem 1. — We now assume ∂V is not empty and φ has no compact orbits in the interior of V. Let T, T′, and $Y_1, \dots, Y_{n-1}$ be as in section 2; so that Y_{n-1} is transverse to T′ and pointing into V along T′, i.e. Y_{n-1} points out of the tubular neighborhood of T. Let F be an orbit of φ which intersects T′ and let L be a connected component of $F\cap T'$.

Lemma **(3.1)**. — $\bigcup_{t\in\mathbf{R}} Y_{n-1}(t, L)=F$.

Proof. — We know F is diffeomorphic to $\mathbf{T}^k\times\mathbf{R}^{n-k-1}$ (in the leaf topology) and we have a covering map $\pi:\mathbf{R}^{n-1}\to F$ induced by φ. Since $Y_1, \dots, Y_{n-1}$ define the action φ, we can take $\pi^*(Y_{n-1})=\frac{\partial}{\partial x_{n-1}}$ where $(x_1, \dots, x_{n-1})$ denote the usual coordinates in $\mathbf{R}^{n-1}$. Let X denote $\frac{\partial}{\partial x_{n-1}}$, and let W be a connected component of $\pi^{-1}(L)$. It suffices to prove that each orbit of X starting at a point of the hyperplane $x_{n-1}=0$, intersects W, since this implies $\bigcup_t X(t, W)=\mathbf{R}^{n-1}$.

Now W is a closed submanifold of $\mathbf{R}^{n-1}$, of codimension one, and X is transverse to W, and makes an angle with W that is strictly bounded away from zero, since Y_{n-1} is transverse to T′. Clearly, the set of points of the hyperplane $x_{n-1}=0$, whose X orbits intersect W, is an open non empty set Ω. It suffices to show Ω is closed. Let $z\in\overline{\Omega}$, and $z_n\in\Omega$, satisfying: $\lim_{n\to\infty} z_n=z$ and for each n, there exists $t_n\in\mathbf{R}$, such that $X(t_n, z_n)\in W$. If some subsequence of (t_n) converges to a number t then we have $X(t, z)\in W$; hence we can suppose no subsequence converges. Let (s_n) be a subsequence of (t_n) such that $|s_n - s_{n+1}| \geq 1$ and $|z_n - z_{n+1}| < \frac{1}{n}$. Let $E(n)$ denote the line segment joining z_n to z_{n+1} and consider $(E(n) \times \mathbf{R}) \cap W$. This is a curve in W with endpoints $X(S_n, z_n)$ and $X(S_{n+1}, z_{n+1})$. There exists a point U_n on this curve where the tangent to the curve is parallel to the cord joining the endpoints. The angle this cord makes with X tends to zero as $n \to \infty$, which contradicts the fact that the angle between X and W is strictly positive.

Lemma **(3.2)**. — *Let* F, W, L, T *and* T′ *be as in* (3.1). *Then there exists a compact orbit* T_1 *of* φ *such that* $\overline{F} \supset T_1$ *and* $T_1 \neq T$.

Proof. — Let $W_0 = W$ and $W_n = X(n, W_0)$ for each positive integer n. By an argument analogous to that of (3.1), one sees that the distances $d(W_k, W_{k+s})$ tend to infinity as $s \to \infty$. Let $L_0 = L$ and $L_n = Y_{n-1}(n, L_0)$, so that $\lim_{s\to\infty} d(L_k, L_{k+s}) = \infty$, where the metric is that induced by π. We define $\Omega = \bigcap_n \overline{E}_n$, where E_n is the connected component of $F - L_n$ towards which Y_{n-1} points on L_n. Ω is an intersection of a nested family of compact sets, hence Ω is not empty and compact. We claim Ω is invariant under the φ action: clearly $\Omega = \{y \in V \mid \text{there exists } x_n \in E_n \text{ and } x_n \to y\}$. Let $F(y)$ be the orbit of φ by $y \in \Omega$ and let $y' \in F(y)$. Let $[y, y']$ denote a path in $F(y)$ joining y to y' and let $[x_n, x'_n]$ be the holonomy lifting of this path to the leaf of x_n. By construction we have $d(x_n, x'_n)$ bounded above by some number ℓ, independent of n. Since

$$d(L_n, L_{n+s}) \to \infty$$

as $s \to \infty$, we can choose a subsequence of (x'_n), call it (y_n), such that $y_n \in E_n$. Thus $y' \in \Omega$ and Ω is invariant. Thus Ω contains a φ-minimal set, which must be a compact orbit by Sacksteder's theorem. Since Y_{n-1} points away from T, this compact leaf $T_1 \subset \Omega$, is different from T.

(3.3) *Let* V^n *be of rank* $n-1$ *and let* φ *be an action of* $\mathbf{R}^{n-1}$ *on* V *such that the only compact orbits of* φ *are in* ∂V *and* ∂V *is not empty. Then* V *is homeomorphic to* $\mathbf{T}^{n-1} \times I$.

Proof. — We use the notation of (3.1) and (3.2). From these lemmas, it follows that the open leaves having T int heir closure are homeomorphic to the open leaves having T_1 in their closure, i.e. to $\mathbf{T}^k \times \mathbf{R}^{n-k-1}$, where k is the rank of the kernel of the holonomy map of T. Now since all the integral curves of $Y_1, \ldots, Y_k$ are closed in F, and $\overline{F} \supset T_1$, we know they are also closed in T_1; hence the k-tori in T_1 spanned by the orbits of $Y_1, \ldots, Y_k$ represents the kernel of the holonomy map of T_1. Now the orbits of $Y_{k+1}, \ldots, Y_{n-1}$ are not necessarily closed but we can choose vector fields (from lines through the origin in $\mathbf{R}^{n-1}$) $\widetilde{Y}_{k+1}, \ldots, \widetilde{Y}_{n-1}$, such that $Y_1, \ldots, Y_k, \widetilde{Y}_{k+1}, \ldots, \widetilde{Y}_{n-1}$ are linearly independent, commute, are tangent to the φ orbits, and all the integral curves of $\widetilde{Y}_{k+1}, \ldots, \widetilde{Y}_{n-1}$ in T_1 are closed. Clearly this can be done so that $\widetilde{Y}_{n-1}$ is

C^0-close to Y_{n-1}. We choose $\widetilde{Y}_{n-1}$ so close that $\widetilde{Y}_{n-1}$ is also transverse to T'. Now we go through the construction of a torus T'_1, bounding a collar neighborhood with T_1, such that $\widetilde{Y}_{n-1}$ is transverse to T'_1; this is (2.1). Letting Y denote $\widetilde{Y}_{n-1}$, we now have a linear vector field Y transverse to both tori T' and T'_1. We know the set of points A in T' whose Y-integral curve intersects T'_1 is an open non empty set. By the same reasoning, the complement of A in T' is open; hence $A = T'$. Now using the integral curves of Y, it is easy to construct a homeomorphism between V and $\mathbf{T}^{n-1} \times I$.

Proof of Theorem 1. — The proof follows from (3.3), and a reasoning identical to that on page 462 of [7].

Remarks 1. — A basic question remains unanswered: suppose φ is a locally free action of $\mathbf{R}^{n-1}$ on a closed manifold V^n, with no compact orbits. Then we know V^n fibres over a torus with fibre a torus, hence V^n fibres over $\mathbf{S}^1$ with fibre F (this also follows from [10]). Is F homeomorphic to $\mathbf{T}^{n-1}$?

2. Suppose V^n is a closed, orientable, bundle over $\mathbf{S}^1$ with fibre M. Then there exists a diffeomorphism $f: M \to M$ such that V is obtained from $M \times I$ by identifying points $(f(x), 1)$ with $(x, 0)$ for $x \in M$. We claim that if $f^*: H^1(M, \mathbf{R}) \to H^1(M, \mathbf{R})$ does not have one as an eigenvalue, then every locally free action of $\mathbf{R}^{n-1}$ on V has a compact orbit. To see this, first observe that f^* does not have 1 as an eigenvalue if and only if rank $H^1(V, \mathbf{R}) = 1$ [11]. Now suppose $\mathscr{F}$ is any foliation of V of codimension one, class C^2 and with no compact leaves. By [12], we can suppose L is a covering space of M for L a leaf of $\mathscr{F}$. We have an exact sequence of free abelian groups:

$$0 \to \pi_1(F)/\pi_1(L) \to \pi_1(V)/\pi_1(L) \to \frac{\pi_1(V)}{\pi_1(F)} \to 0.$$

Since $H^1(V, \mathbf{R}) \approx \mathbf{R}$, the last two groups are of rank one. Hence $\pi_1(L) = \pi_1(F)$ and L must be compact.

APPENDIX

Proof of (2.1)

Notation. — If X is a vector field on V, $t \mapsto X(t, x)$ will denote the integral curve of X passing through x at $t = 0$. For $A \subset V$, $X(t, A) = \{X(t, x) \mid x \in A\}$, and

$$X([a, b], A) = \bigcup_{a \leq t \leq b} X(t, A).$$

If $x \in T$, we define $\alpha_i(x) = Y_i([0, 1], x)$ for $i = 1, \ldots, n-1$, and $T_i(x)$ is the i-torus in T which is the orbit through x of the $\mathbf{R}^i$-action determined by $Y_1, \ldots, Y_i$. If $\bar{x}$ is on the normal arc through $x \in T$ and if the holonomy germs are defined on $\bar{x}$, then we denote by $\overline{T}_k(\bar{x})$ the lifting of $T_k(x)$ into the leaf of $\bar{x}$, given by the holonomy.

Let N be a unit vector field on V, normal to the orbits of φ and pointing into V along T (with respect to some metric on V). Let $U = N(I, T)$ where $I = [0, 1]$. We may suppose U is a tubular neighborhood of T in which the holonomy liftings of $\alpha_1(x), \ldots, \alpha_{n-1}(x)$ are defined, for $x \in T$. Let f_x^i be the holonomy diffeomorphism of $\alpha_{k+i}(x)$; $1 \leq i \leq n-k-1$.

Let $\pi : U \to T$ be the projection along N orbits. If $x \in T$ and $\bar{x} \in \pi^{-1}(x)$, let $\bar{\alpha}_i(\bar{x})$ denote the holonomy lifting of $\alpha_i(x)$ starting at $\bar{x}$; for $1 \leq i \leq k$, $\bar{\alpha}_i(\bar{x})$ is an embedded circle, and for $i > k$, $\bar{\alpha}_i(\bar{x})$ is diffeomorphic to I. For $x \in T$ and for all $\bar{x} \in \pi^{-1}(x)$, the $\bar{\alpha}_i(\bar{x})$ form a one dimensional foliation of U. Let C_i be a vector field in U, tangent to this foliation, and coinciding with Y_i on T.

We fix a base point $x_0 \in T$ and we let $\alpha_i = \alpha_i(x_0)$, $T_i = T_i(x_0)$, etc., and define $A_i = N(I, T_i)$.

Let $E_\ell(A_j)$ be the vector bundle of exterior products of order ℓ of vectors tangent to A_j. We identify $E_\ell(A_j)$ with $A_j \times \overset{\ell}{\wedge} \mathbf{R}^j$; so sections of $E_\ell(A_j)$ are functions from A_j to $\overset{\ell}{\wedge} \mathbf{R}^j$. We give these sections the canonical norm.

Let f be a function defined in a neighborhood of 0 such that $\lim_{x \to 0} f(x) = 0$. We write $f = \sigma_1(x)$ if

$$f(x) = ax + x\mathscr{E}(x),$$

with $a \neq 0$ and $\mathscr{E}(x) \to 0$ when $x \to 0$. Finally, we let $\beta_{k+j} = Y_{k+j} \wedge \ldots \wedge Y_{n-1}$.

Proposition (2.1). — *For each j, $1 \leq j \leq n-k-1$, there is a family of tori $G(k+j)$, satisfying*:

c_1) there is a neighborhood U_j of T_{k+j} and the $G(k+j)$'s are a foliation of U_j by tori of dimension $k+j$;

c_2) there is a section g_{k+j} of $E_{k+j}(A_{k+j})$ such that $g_{k+j}(x)$ represents the tangent space at x to $G(k+j)(x)$ and

$$(g_{k+j} \wedge \beta_{k+j})_p \neq 0$$

for all $p \in U_j - T_{k+j}$;

c_3) on T, $G_{k+s}(x) = T_{k+s}$.

Remark. — In particular c_2) implies

$$g_{n-1} \wedge Y_{n-1} \neq 0 \quad \text{in} \quad U_{n-1} - T.$$

Hence there exist $(n-1)$-tori, transverse to Y_{n-1}, as close to T as we wish.

Proof of (2.1). — We proceed by induction on j; first " cylinders " are constructed in $\mathbf{T}^j \times I$, $k+1 \leq j \leq n-2$, and then these cylinders are closed, to give tori, by the map F_j defined by the holonomy of α_j.

We start by constructing the foliation $G(k+1)$. Let $U_1 = \pi^{-1}(T_{k+1})$, and let (θ, z, λ) be coordinates for $\mathbf{T}^k \times I \times J$ where $\theta = (\theta_1, \ldots, \theta_k) \in T^k$, and $I = J = [0, 1]$. Let $F_1 : \mathbf{T}^k \times I \times J \to U_1$ be defined by:

$$F_1(0, 0, 0) = x_0$$
$$F_1(0, z, 0) = N(z, x_0)$$

$F_1(\theta, z, \lambda)$ is the endpoint of the holonomy lifting of the arc in T given by:

$$(t, Y(\lambda t, F_1(\theta, 0, 0)),$$

$0 \leq t \leq 1$, starting at $F_1(\theta, z, 0) = N(z, F_1(\theta, 0, 0))$.

Here we have identified $\mathbf{T}^k = \mathbf{R}^k / \mathbf{Z}^k$ with T_k by the linear diffeomorphism $(0, \ldots, \theta_i, \ldots, 0) \mapsto Y_i(\theta_i)(x_0)$.

By definition of F_1 we have:

— $DF_1\left(\dfrac{\partial}{\partial \theta_j}\right)$ is colinear with C_j for $1 \leq j \leq k$,

— F_1 sends the tori $\mathbf{T}^k \times \{z\} \times \{\lambda\}$ to the holonomy liftings of the tori $T_k(F_1(0, 0, \lambda))$ to the point $F_1(0, z, \lambda)$,

— $DF_1\left(\dfrac{\partial}{\partial \lambda}\right) = C_{k+1} = Y_{k+1}$ on T,

— F_1 send the segments $\{\theta\} \times I \times \{\lambda\}$ to the orbits of N starting at $F_1(\theta, 0, \lambda)$,

— the segments $\{\theta\} \times \{z\} \times J$ are sent to $\overline{\alpha_{k+1}}(F_1(\theta, z, 0))$,

— F_1 is a local diffeomorphism to U_1.

From these remarks, it is easy to see that the map $z \mapsto F_1(\theta, z, \lambda)$ (respectively $\lambda \mapsto F_1(\theta, z, \lambda)$) is a reparimetrization of the N-orbits (orbits of C_{k+1}). Hence there exist functions φ_1 and ψ_1, invertible in z and λ such that

$$DF_1\left(\frac{\partial}{\partial z}\right) = \frac{\partial \varphi_1}{\partial z} N$$

$$DF_1\left(\frac{\partial}{\partial \lambda}\right) = \frac{\partial \varphi_1}{\partial \lambda} C_{k+1}$$

(both φ_1 and ψ_1 have strictly positive derivatives on $\mathbf{T}^k \times I \times J$).

Now we construct a family of curves, $\gamma_1(\theta, z)$, in $\mathbf{T}^k \times I \times J$

$$\gamma_1(\theta, z) : \lambda \mapsto (Z_1(\theta, Z, \lambda), \lambda)$$

satisfying conditions *A)* and *B)*:

A) For fixed θ, z, $F_1(\gamma_1(\theta, z))$ is a closed curve in U_1, of class C^1. For θ and z in a neighborhood of $\mathbf{T}^k \times \{0\}$, the $F_1(\gamma_1(\theta, Z))$ form a one dimensional foliation of a neighborhood of T_{k+1} in A_{k+1}, *B)* Let $\Delta_1(\theta, z)=z-f_x^1(z)$, where $x=F_1(\theta, 0, 0)$, and let $\Delta_1(z)=\Delta_1(0, z)$. Then we require that:

$$\frac{\partial Z_1}{\partial \lambda}=\sigma_1(\Delta_1)$$

(here Δ_1 is the function $z\mapsto\Delta_1(z)$). The condition *B)* is not necessary to construct $G(k+1)$; however, it is necessary to insure the transversality relation *c_2)* when we construct $G(k+j)$, $j>1$.

Lemma **(2.2)**. — *There exists in* $\mathbf{T}^k\times I\times J$, *a family of curves* $\gamma_1(\theta, z)$, *satisfying conditions A) and B).*

Proof of **(2.2)**. — Let γ' be the tangent vector field to the γ_1 curves, with the λ-parametrization. Then condition *A)* can be written:

$$(1)\qquad (DF_1)_a\gamma'_a\wedge(DF_1)_b\gamma'_b=0$$

where $b=(\theta, f_x^1(z), 1)$, $x=F_1(\theta, 0, 0)$, and $a=(\theta, z, 0)$ (cf. figure 1).

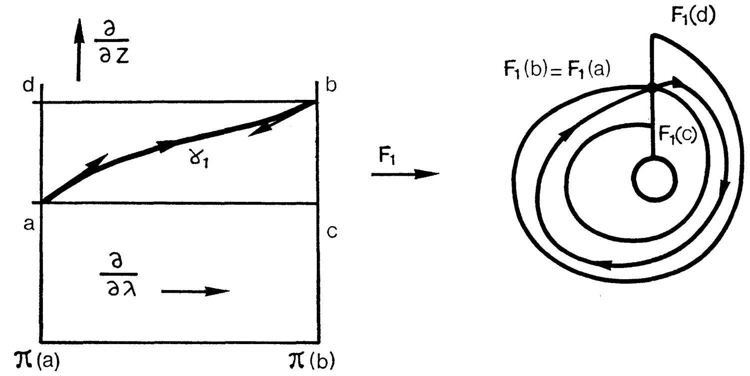

FIG. 1

An easy calculation shows that (1) can be written:

$$\left.\frac{\partial Z_1}{\partial \lambda}\right|_b=K(\theta, z)\left.\frac{\partial Z_1}{\partial \lambda}\right|_a$$

where K is a strictly positive function. Therefore, we can rewrite *A)* and *B)* as:

$$\left.\frac{\partial Z_1}{\partial \lambda}\right|_b=K\left.\frac{\partial Z_1}{\partial \lambda}\right|_a$$

$$\frac{\partial Z_1}{\partial \lambda}=\sigma_1(\Delta_1).$$

A tedious, but simple calculation, shows that the cubics (see fig. 1):

$$\lambda\to\left(\lambda, Z_1=\Delta_1(\theta, z)\left[\left(\frac{1+K}{1+K_0}-2\right)\lambda^3+\left(3-\frac{K+2}{K_0+1}\right)\lambda^2+\frac{\lambda}{1+K_0}\right]+f_x^1(z)\right),$$

satisfy these equations, where

$$K_0 = \sup K(\theta, z), \qquad (\theta, z) \in \mathbf{T}^k \times I \times \{1\}.$$

Now, one can write:

$$\frac{\partial Z_1}{\partial \lambda} = \Delta_1(\theta, z)\, g(\theta, z, \lambda),$$

where $g > 0$ on $\mathbf{T}^k \times I \times J$. Also $\frac{\partial Z_1}{\partial z} > 0$ on $\mathbf{T}^k \times [0, h] \times J$ for a suitable h, $0 < h \leq 1$. Hence the curves γ_1 form a foliation of $\mathbf{T}^k \times [0, h] \times J$. Their image by F_1 is a foliation (of class C^1) of a neighborhood $V_1 \subset U_1$, of T_{k+1} in A_{k+1}. This completes the proof of (2.2) (see fig. 2).

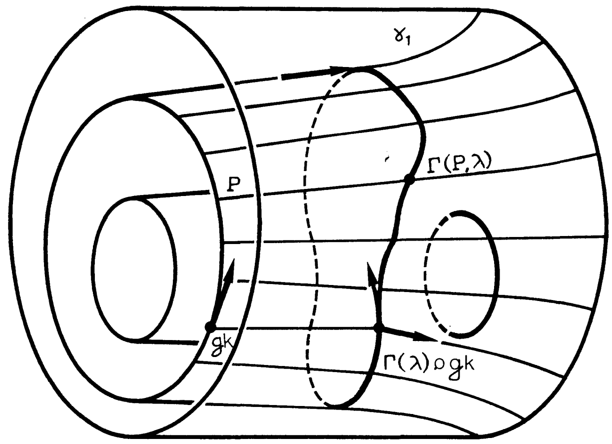

FIG. 2

We can now define $G(k+1)$. The submanifolds:

$$H(z) = \bigcup_{\theta \in \mathbf{T}^k} \gamma_1(\theta, z) \times \{\theta\}$$

are diffeomorphic to $\mathbf{T}^k \times [0, 1]$ and form a foliation of $\mathbf{T}^k \times [0, h] \times J$. Hence their image by F_1 is a foliation of V_1 by tori $G(k+1)$ (figure 2). We now check conditions $c_2)$ and $c_3)$.

We have a C^2 vector field γ' in $\mathbf{T}^k \times [0, h] \times J$; γ' is the tangent field to the γ_1 curves with the λ-parametrization. This field induces a natural action of $\mathbf{R}$ on the exterior products of vector fields, which we note by $\Gamma(\lambda)$: $\Gamma(\lambda)$ is the differential of the map induced by γ' of $\mathbf{T}^k \times [0, h] \times \{0\}$ to $\mathbf{T}^k \times [0, h] \times \{\lambda\}$. Then the tangent space to $H(z)$ at the point (θ, z, λ) is given by:

$$\Gamma(\lambda)\left(\frac{\partial}{\partial \theta_1} \wedge \ldots \wedge \frac{\partial}{\partial \theta_k}\right)_{(\theta, z, 0)} \wedge \gamma'_{(\theta, z, \lambda)}.$$

Now F_1 sends the tori $\mathbf{T}^k\times\{z\}\times\{0\}$ to the trivial holonomy liftings of the T_k; hence the tangent space to $G(k+1)$ at $p=F_1(\theta, z, \lambda)$ is given by:

$$(g_{k+1})_p=(DF_1\circ\Gamma(\lambda)\left(\frac{\partial}{\partial\theta_1}\wedge\ldots\wedge\frac{\partial}{\partial\theta_k}\right)\wedge DF_1\circ\gamma')_{(\theta, z, \lambda)}$$

$$\gamma'=\frac{\partial}{\partial\lambda}+\sigma_1(\Delta_1)\frac{\partial}{\partial Z}.$$

We recall that:

$$DF_1\left(\frac{\partial}{\partial Z}\right)=\frac{\partial\theta_1}{\partial Z}N, \qquad DF_1\left(\frac{\partial}{\partial\lambda}\right)=\frac{\partial\psi_1}{\partial\lambda}C_{k+1}.$$

Let $\tilde{\sigma}(\Delta_1)$ denote a function such that $\dfrac{\tilde{\sigma}(\Delta_1)}{\Delta_1}$ tends towards a limit a; then

$$\tilde{\sigma}(\Delta_1)=a\Delta_1+\sigma(\Delta_1),$$

(with a not necessarily different from 0 and $\sigma(\Delta_1)=\Delta_1\varepsilon(\Delta_1)$, $\varepsilon(\Delta_1)\rightarrow 0$ whenever $\Delta_1\rightarrow 0$). We take the tangent spaces to the trivial holonomy liftings of the T_k, to be given by a section of $E_k(A_k)$, equal to $Y_1\wedge\ldots\wedge Y_k$ on T_k.

Let $C^*_{k+1}(\lambda)$ denote the action induced by C_{k+1} on the vectors tangent to A_k (if Y is tangent to the φ-orbits, then so is $C^*_{k+1}(\lambda)(Y)$). Then we obtain for g_{k+1}:

$$(g_{k+1})_p=(C^*_{k+1}(\lambda)\circ(g_k)_{F_1(\theta, z, 0)}+\tilde{\sigma}(\Delta_1)\Omega_p)\wedge(C_{k+1}+\sigma_1(\Delta_1)N)_p,$$

where $p=F_1(\theta, z, \lambda)$ and Ω is a section of $E_{k+1}(A_{k+1})$ defined on V_1. We can rewrite this as:

$$C_{k+1}\wedge C^*_{k+1}(\lambda)\circ g_k+\sigma_1(\Delta_1)C^*_{k+1}(\lambda)\circ g_k\wedge N+\sigma(\Delta_1)C_{k+1}\wedge\Omega+\sigma(\Delta_1)N\wedge\Omega.$$

Now $\beta_{k+1}\wedge C_{k+1}\wedge C^*_{k+1}(\lambda)\circ g_k$ is zero, since it is a linear combination of exterior products of n vectors tangent to the φ-orbits. Hence:

$$g_{k+1}\wedge\beta_{k+1}=\sigma_1(\Delta_1)C^*_{k+1}(\lambda)\circ g_k\wedge N\wedge\beta_{k+1}+\sigma(\Delta_1)C_{k+1}\wedge\Omega\wedge\beta_{k+1}+\sigma(\Delta_1)N\wedge\Omega\wedge\beta_{k+1}.$$

We have:

$$C^*_{k+1}(\lambda)\circ g_k\wedge N\wedge\beta_{k+1}=Y_1\wedge\ldots\wedge Y_k\wedge N\wedge Y_{k+1}\wedge\ldots\wedge Y_{n-1}$$

on T_{k+1}. Hence for all points p in a neighborhood V_2 of T_{k+1}, we have:

$$|C^*_{k+1}(\lambda)g_k\wedge N\wedge\beta_{k+1}|_p>\alpha>0.$$

We want to show $(g_{k+1}\wedge\beta_{k+1})_p\neq 0$, for p in a suitable neighborhood of T_{k+1} in A_{k+1}. Dividing by $\sigma_1(\Delta_1)$ ($\neq 0$ if $z\neq 0$):

$$\frac{g_{k+1}\wedge\beta_{k+1}}{\sigma_1(\Delta_1)}=\rho_k+C_{k+1}\wedge\Omega'\wedge\beta_{k+1}+\mathscr{E}(\Delta_1)N\wedge\Omega\wedge\beta_{k+1},$$

where $|\rho_k|_p>\alpha>0$ in V_2, and Ω' is a bounded section on V_2-T_{k+1}, $\mathscr{E}(\Delta_1)\rightarrow 0$ as $\Delta_1\rightarrow 0$. Since $C_{k+1}=Y_{k+1}$ on T_{k+1}, the second term is less than $\alpha/3$ if p is in some neighborhood V_3 of T_{k+1}. Also $|\mathscr{E}(\Delta_1)||N\wedge\Omega\wedge\beta_{k+1}|<\alpha/3$ if p is in some neighborhood V_4 of T_{k+1}. Hence for $p\in V_2\cap V_3\cap V_4$,

$$\frac{|g_{k+1}\wedge\beta_{k+1}|}{\sigma_1(\Delta_1)}>\alpha/3>0,$$

which proves the theorem for $j=1$.

It is useful for the induction to write g_{k+1} in the form:

$$g_{k+1}=\alpha_{k+1}+\sigma(\Delta_1)\Omega''$$

where α_{k+1} is a section of $E_{k+1}(A_{k+1})$, defined in a neighborhood of T_{k+1} and equal to $Y_1\wedge\ldots\wedge Y_k\wedge Y_{k+1}$ on T_{k+1}.

Construction of $G(k+j+1)$. — Let $\Delta_{k+j}(s)=s-f^{k+j}(s)$, where $s\geq 0$ denotes the normal N-coordinate.

Fondamental little lemma:

$$\lim_{s\to 0}\frac{\Delta_{K+j}}{\Delta_{K+j+1}} \text{ exists.}$$

Following (1), one may find an homeomorphism $H:[0,\varepsilon]\to[0,\varepsilon']$ such that $H^{-1}f_{K+j}H=\lambda_{K+j}$ and $H^{-1}f_{K+j+1}H=\lambda_{K+j+1}$ where λ_{K+j} and λ_{K+j+1} are the homotheties the ratio of which are λ_{K+j} and λ_{K+j+1} (recall f_{K+j} and f_{K+j+1} are contractions). Define on $[0,\varepsilon]$ a metric δ such that $\delta(x,x')=|H^{-1}(x)-H^{-1}(x')|$ — this metric is topologically equivalent to the classical one — and $\delta(x,0)\to 0$ whenever $x\to 0$. We prove $\dfrac{\delta(x,f_{K+j}(x))}{\delta(x,f_{K+j+1}(x))}$ has a limit when $x\to 0$ (with respect to δ); we shall then be over. Then let $f=f_{K+j}$, $g=f_{K+j+1}$.

$$\frac{\delta(x,f(x))}{\delta(x,g(x))}=\frac{|H^{-1}(x)-H^{-1}f(x)|}{|H^{-1}(x)-H^{-1}g(x)|}=\frac{|H^{-1}(x)-\lambda_{K+j}\circ H^{-1}(x)|}{|H^{-1}(x)-\lambda_{K+j+1}\circ H^{-1}(x)|}$$

$$\rho=\frac{\delta(x,f(x))}{\delta(x,g(x)}=\frac{1-\lambda_{K+j}}{1-\lambda_{K+j+1}}\times H^{-1}(x)$$

when $x\to 0$, $\delta(x,0)\to 0$ and $\rho\to\dfrac{1-\lambda_{K+j}}{1-\lambda_{K+j+1}}$.

Our inductive hypothesis asserts the existence of the foliation $G(k+j)$ and a section g_{k+j} satisfying:

$$g_{k+j}=\alpha_{k+j}+\sigma(\Delta_{k+j})\Omega,$$

where α_{k+j} is a section of $E_{k+j}(A_{k+j})$, defined in a neighborhood U of T_{k+j} in A_{k+j}, which is a linear combination of vectors tangent to the φ-orbits, and equal to $Y_1\wedge\ldots\wedge Y_{k+j}$ on T_{k+j}. Ω is a section of $E_{k+j}(A_{k+j})$ defined in U. Henceforth, we work in U.

Since the $G(k+j)$ form a foliation of U transverse to the normals, we can construct, by the holonomy, a map F_{j+1} satisfying:

F_{j+1} sends $\mathbf{T}^{k+j}\times I\times J$ to U and is of maximal rank;

$F_{j+1}(0, z, 0)=N(z, x_0)$;

F_{j+1} sends the tori $\mathbf{T}^{k+j}\times\{z\}\times\{0\}$ to the tori $G(k+j)$ passing by $F_{j+1}(0, z, 0)$; restricted to $\mathbf{T}^{k+j}\times\{0\}\times J$, we have:

$$DF_{j+1}\left(\frac{\partial}{\partial\theta_\ell}\right)=Y_\ell, \qquad 1\leq \ell\leq k+j,$$

$$DF_{j+1}\left(\frac{\partial}{\partial\lambda}\right)=Y_{k+j+1};$$

$$DF_{j+1}\left(\frac{\partial}{\partial\theta_1}\wedge\ldots\wedge\frac{\partial}{\partial\theta_{k+j}}\right)=g_{k+j} \text{ in } A_{k+j};$$

F_{j+1} is the holonomy lifting, restricted to the plaques $\{\theta\}\times I\times J$, i.e. $(F=F_{j+1})$ $F(\theta, z, \lambda)$ is the endpoint of the holonomy lifting of the path $Y_{k+j+1}([0, \lambda], F(\theta, 0, 0))$ to the point $F(\theta, z, 0)$.

Exactly as the case $j=1$, we have a family of curves $\gamma_{j+1}(\theta, z)$ satisfying the conditions *A)* and *B)*, with F_1 replaced by $F=F_{j+1}$. This gives us a foliation of $\mathbf{T}^{k+j}\times I\times J$ by submanifolds diffeomorphic to $\mathbf{T}^{k+j}\times I$, and closing the cylinders by F we obtain a foliation by tori $G(k+j+1)$. We must verify $c_2)$.

Let $g=g_{k+j+1}$, $C=C_{k+j+1}$, $\Delta=\Delta_{k+j+1}$ and $C^*=C^*_{k+j+1}$. Then we have:

$$\begin{aligned}g&=(C+\sigma_1(\Delta)N)\wedge(C^*(\lambda)\circ g_{k+j}+\sigma(\Delta)\Omega')\\&=C\wedge C^*(\lambda)\circ g_{k+j}+\sigma(\Delta)N\wedge\Omega'+\sigma_1(\Delta)N\wedge C^*(\lambda)\circ g_{k+j}+\sigma(\Delta)C\wedge\Omega'.\end{aligned}$$

Now we write $g_{k+j}=\alpha_{k+j}+\sigma(\Delta_{k+j})\Omega$ (a section defined in A_{k+j}), to obtain (defined in A_{k+j+1}):

$$\begin{aligned}g=C\wedge C^*(\lambda)\circ\alpha_{k+j}+(C\wedge C^*(\lambda)\Omega)\,\sigma(\Delta_{k+j})+\sigma_1(\Delta)N\wedge C^*(\lambda)\circ\alpha_{k+j}\\+\sigma(\Delta)\,\sigma(\Delta_{k+j})\Omega+\sigma(\Delta)C\wedge\Omega'+\sigma(\Delta)N\wedge\Omega''.\end{aligned}$$

Notice that $C\wedge C^*(\lambda)\circ\alpha_{k+j}$ is a linear combination of products of order $k+j+1$ of vectors tangent to the leaves, and on T_{k+j+1}, it equals $Y_{k+j+1}\wedge Y_1\wedge\ldots\wedge Y_{k+j}$. Composing g with β_{k+j+1}:

— the term $C\wedge C^*(\lambda)\alpha_{k+j}\wedge\beta_{k+j+1}=0$ since it is a multiple of n vectors tangent to the φ orbits;

— $N\wedge C^*(\lambda)\circ\alpha_{k+j}=N\wedge Y_1\wedge\ldots\wedge Y_{k+j}$ on T_{k+j+1}.

Dividing by $\sigma_1(\Delta)$ we obtain:

$$\frac{g\wedge\beta_{k+j+1}}{\sigma_1(\Delta)}=N\wedge C^*(\lambda)\alpha_{k+j}+\frac{\sigma(\Delta_{k+j})}{\sigma_1(\Delta)}C\wedge C^*(\lambda)\Omega+\mathscr{E}(\Delta)\Omega''.$$

Now, as $s \to 0$, Δ_{k+j}/Δ is bounded hence $\sigma(\Delta_{k+j})/\sigma_1(\Delta)$ is bounded. Ω'' is a bounded section of $E_{k+j+1}(A_{k+j+1})$ and $\mathscr{E}(\Delta) \to 0$ as $\Delta \to 0$. Hence, in a small enough neighborhood of T_{k+j+1}, we have $g \wedge \beta_{k+j+1} \neq 0$, which completes the proof of (2.1).

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