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AN ALGEBRAIC CONSTRUCTION OF THE GENERIC SINGULARITIES OF BOARDMAN-THOM

by K. R. MOUNT [1] *and* O. E. VILLAMAYOR

Abstract. — In this paper we study some of the functorial properties of the infinite jet space in order to give a coordinate free algebraic definition of the generic singularities of Boardman-Thom. More precisely, suppose that k is a commutative ring with an identity and suppose that A is a commutative ring with an identity which is a k-algebra. An A-k-Lie algebra L is a k-Lie algebra with a k-Lie algebra map φ from L to the algebra of k-derivations of A to itself such that for $d, d' \in L$ and $a, a' \in A$, then

$$[ad, a'd'] = a(\varphi(d)a')d' - a'(\varphi(d')a)d + aa'[d, d'].$$

There is a universal enveloping algebra for such Lie algebras which we denote by E(L). Denote by ***L-alg*** the category of A-algebras B which have L and hence E(L) acting as left operators such that for $a \in A$, $d \in L$, $(da)1_B = d(a.1_B)$. If F is the forgetful functor from ***L-alg*** to the category of A-algebras, we show that F has a left adjoint $J(L, \cdot)$ which is the natural algebraic translation of the infinite jet space.

In the third section of this paper we construct a theory of singularities for a derivation from a ring to a module and then we apply this construction to J(L, C) where C is an A-algebra. These singularities are subschemas with defining sheaf of ideals given by Fitting invariants of appropriately chosen modules when A and B are polynomial rings over a field k and $C = A \otimes_k B$; these are the generic singularities of Boardman-Thom.

Finally we show that, under some rather general conditions on the structure of C as an A-algebra, the generic singularities are regular immersions in the sense of Berthelot.

§ 1

The usual construction of the infinite jet space used for the generic singularities of Boardman-Thom (see [2]) uses coordinates in the domain and range of the functions and then one appeals to a coordinate patching process to construct the required space. Because the discussions of singularities between schemas is the natural analogous algebraic

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subject of study it seems reasonable to seek a more functorial construction of the infinite jet space in order to free the construction as much as possible from coordinates. In this paper we carry out such a construction and in the course of the discussion it becomes clear that at the same time it is possible to isolate the effect of the Lie algebra of derivations of the domain variety.

More precisely, suppose that A is a commutative ring with an identity which is also a k-algebra where k is a commutative ring with 1. The collection of derivations from A to itself which annihilate k form a k-Lie algebra which is also an A-module. If d, d' are two of these derivations, and if $a, a' \in A$, then

$$[ad, a'd'] = ad(a')d' - a'd'(a)d + aa'[d, d'].$$

We begin this paper by studying k-Lie algebras L which are left A-modules and which are such that there is an A-linear map φ from L to the k-derivations of A to itself such that φ is a k-Lie algebra map and such that

$$[ad, a'd'] = a(\varphi(d)a')d' - a'(\varphi(d')a)d + aa'[d, d'].$$

We call these A-k-Lie algebras. These Lie algebras have naturally associated to them associative rings E(L) which are both left A-modules and universal enveloping algebras for L. Further if B is an A-algebra with an A-k-Lie algebra acting as left operators we have $(da)1_B = d(a.1_B)$. Each such B has the structure of a left E(L)-module in a natural way. There is a functor F from ***L-alg*** to the category of A-algebras which forgets the E(L) and L structure. The main construction of this paper is to show that the functor F has a left adjoint which we denote by $J(L, \cdot)$. If A and B are regular rings which are algebras over field k of characteristic zero, suppose $R = (A \otimes_k B)_{\mathfrak{P}}$ where $\mathfrak{P}$ is a prime of $A \otimes_k B$. Assume $\mathfrak{p} = \mathfrak{P} \cap A$ and assume that R has maximal ideal $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ where $x_1, \ldots, x_n$ generate $\mathfrak{p}.A_{\mathfrak{p}}$ and $y_1, \ldots, y_m$ generate $(\mathfrak{P} \cap B)B_{(\mathfrak{P} \cap B)}$. If we assume L is the Lie algebra of k-derivations from $A_{\mathfrak{p}}$ to itself and if this Lie algebra is a finitely generated $A_{\mathfrak{p}}$-module, then J(L, R) is a polynomial ring over B on indeterminates $z(\alpha, j)$ where $\alpha = (\alpha(1), \ldots, \alpha(n))$ is a sequence of nonnegative integers and $1 \leq j \leq m$. Furthermore the Lie algebra L acts on the $z(\alpha, j)$ by

$$\frac{\partial}{\partial x_t} z(\alpha, j) = z((\alpha(1), \ldots, \alpha(t)+1, \ldots, \alpha(n)), j).$$

Also J(L, B) is localizing in multiplicative sets in B. These remarks imply that $J(L, A \otimes_k B)$ satisfies the conditions required for the coordinate rings for the infinite jets of maps from $\mathrm{Spec}_k(A)$ to $\mathrm{Spec}_k(B)$, if we assume that L is the module of k-derivations from A to A, and if we suppose that L is a finitely generated A-module.

In the third section of this paper we have constructed a fairly simple theory of iterated singularities for a derivation Δ from a k-algebra R to a finitely generated R-module M. Given a sequence of integers $i(1) \geq i(2) \geq \ldots \geq i(r)$ we have defined a subschema $\Sigma(i(1), \ldots, i(r)) \subset \mathrm{Spec}(R)$ by forming Fitting invariants of the modules M,

$A/\mathfrak{f}_{i(1)-1}(M)\otimes_R M/\Delta\mathfrak{f}_{i(1)-1}(M)$, ..., etc. In the case that $R=J(L, B)$, there is a module over R and a derivation from R to this module whose singularities are generic in the sense of the Boardman-Thom singularities.

In the last section of this paper we show that under some rather general conditions on the structure of B as an A-algebra (conditions satisfied by $A\otimes_k B$) the generic singularities are actually regular immersions in the sense of Berthelot [1] (see the discussion following Theorem 4.3). We propose this as the algebraic analogue of the Boardman theory.

§ 2

As in [13] we shall say that a commutative ring with 1 is a *scalar* ring, and a nonscalar ring is a ring not necessarily satisfying these conditions. Unless otherwise specified in the discussion, when we say *ring* we shall mean a *scalar ring*.

We shall use the following notation. If A is a ring which is an algebra over a ring k, we shall denote by $D(A/k)$ the module of k differentials of A and by $L(A/k)$ the A-module $\mathrm{Hom}_A(D(A/k), A)$ with Lie product $[d_1, d_2]=d_1.d_2-d_2.d_1$, where

$$d_1, d_2\in \mathrm{Hom}_A(D(A/k), A).$$

It will be convenient to denote the module $\mathrm{Hom}_A(D(A/k), A)$ by $D^1(A/k)$.

Lemma **(2.1)**. — *$L(A/k)$ is a Lie algebra over k which is a left A-module. Further if $a, b\in A$ and $d_1, d_2\in L(A/k)$, then*

$$[ad_1, bd_2]=ad_1(b)d_2-bd_2(a)d_1+ab[d_1, d_2].$$

Proof. — The first remark is very well known. As for the second, this is a straightforward computation.

Definition **(2.2)**. — Suppose A and k are rings such that A is a k-algebra. A k-Lie algebra L is said to be an *A-k-Lie algebra* if the following conditions are satisfied:

(i) L is a left A-module.

(ii) There exists a k-Lie algebra homomorphism $\varphi : L\to L(A/k)$ which is A-linear. (We shall call φ the A-*structure map* of L).

(iii) If $a, b\in A$ and $x, y\in L$, then

$$[ax, by]=a(\varphi(x)b)y-b(\varphi(y)a)x+ab[x,y].$$

When there is no fear of confusion we shall denote $\varphi(d)a$ by da.

Definition **(2.3)**. — Suppose A, B and k are rings, where B is an A-algebra and A is a k-algebra. If L is an A-k-Lie algebra, then B is said to be an L-*algebra* if there exists an A linear map $\theta : L\to L(B/k)$ which is a k-Lie algebra map such that if $\varphi : L\to L(A/k)$ is the A-structure map of L and if $a\in A$, $d\in L$, then

$$(\varphi(d)a).1_B=\theta(d)(a.1_B).$$

The map θ will be called the L-*structure map* of B.

Definition (**2.4**). — Suppose that A and B are k-algebras, with B an A-algebra. Suppose that L and L^1 are respectively an A-k-Lie algebra and a B-k-Lie algebra with structure maps φ and φ'. Assume $\lambda : A \to B$ is the A-algebra structure map for B. A k-Lie algebra map $\theta : L \to L^1$ is said to be an *A-k-map* if for each $x \in L$ and each $a \in A$, $(\varphi'.(\theta(x)))(\lambda a) = \lambda(\varphi(x)a)$. If B and B^1 are L-algebras, a map $\alpha : B \to B^1$ which is an A-algebra map is said to be an *L-algebra map* if for each $b \in B$, $x\alpha(b) = \alpha(xb)$ for each $x \in L$.

Lemma (**2.5**). — *Suppose that k and A are rings and assume that A is a k-algebra. Assume that M is an A-module and assume that there exists an A-linear map $\theta : M \to L(A/k)$ and a k-bilinear product* [,] *from M to M such that for some set of A-generators $\{m_j\}$, $(j \in J)$, of M we have*

(i) $$[m_i, [m_j, m_k]] + [m_j, [m_k, m_i]] + [m_k, [m_i, m_j]] = 0.$$

(ii) $$[m_i, m_i] = 0 \quad \textit{and} \quad [m_i, m_j] = -[m_j, m_i].$$

(iii) $$[am_i, bm_j] = a(\theta(m_i)(b))m_j - b(\theta(m_j)(a))m_i + ab[m_i, m_j].$$

(iv) $$\theta[m_i, m_j] = [\theta(m_i), \theta(m_j)].$$

Then M has the structure of an A-k-Lie algebra with bracket [,].

Proof. — We need first to show that M is a k-Lie algebra. Thus if $\sum_i x_i m_i$, $\sum_j y_j m_j$ and $\sum_k z_k m_k$ are elements of M, where $x_i, y_j, z_k \in A$, then

$$[\sum_i x_i m_i, [\sum_j y_j m_j, \sum_k z_k m_k]] = \sum_{i,j,k} [x_i m_i, [y_j m_j, z_k m_k]].$$

In order to prove the Jacobi identity we need only show that

$$\sum_{i,j,k} ([x_i m_i, [y_j m_j, z_k m_k]] + [y_j m_j, [z_k m_k, x_i m_i]] + [z_k m_k, [x_i m_i, y_j m_j]]) = 0,$$

and thus it will suffice to show that for each i, j and k

$$T_{ijk} = [x_i m_i, [y_j m_j, z_k m_k]] + [y_j m_j, [z_k m_k, x_i m_i]] + [z_k m_k, [x_i m_i, y_j m_j]] = 0.$$

Set $x_i = a$, $y_j = a'$, $z_k = a''$, $m_i = m$, $m_j = m'$, $m_k = m''$. Then

$$\begin{aligned}[am, [a'm', a''m'']] = {} & a''(\theta(m'')a')(\theta(m')a)m - a'a''(\theta[m', m'']a)m \\ & - a'(\theta(m')a'')(\theta(m'')a)m - a(\theta(m)(a''\theta(m'')a'))m' \\ & + a(\theta(m)(a'\theta(m')a''))m'' + aa'(\theta(m')a'')[m, m''] \\ & - aa''(\theta(m'')a')[m, m'] + a(\theta(m)(a'a''))[m', m''] + aa'a''[m, [m', m'']].\end{aligned}$$

Similarly for $[a'm', [a''m'', am]]$ and $[a''m'', [am, a'm']]$. Thus if we set $\theta(m)(x) = m(x)$, then the coefficient of m in T_{ijk} is

$$\begin{aligned}& a''m''(a')m'(a) - a'a''([m', m''](a)) - a'm'(a'')m''(a) + a'm'(a'')m''(a) \\ & \qquad + a'a''m'(m''(a)) - a''m''(a')m'(a) - a''a'm''(m'(a)) \\ & \qquad = -a'a''([m', m''](a)) + a''a'(m'(m''(a)) - m''(m'(a))) = 0.\end{aligned}$$

The coefficient of $[m, m']$ in T_{ijk} is

$$-aa''m''(a')-a'a''m''(a)+a'a''m''(a)+a''am''(a')=0.$$

A simple symmetry argument shows that therefore

$$T_{ijk}=x_i y_j z_k([m_i, [m_j, m_k]]+[m_j, [m_k, m_i]]+[m_k, [m_i, m_j]])=0.$$

To complete the proof that M is a k-Lie algebra it will suffice to show that $0=[\sum_i x_i m_i, \sum_j x_j m_j]$. But

$$\begin{aligned}[\sum_i x_i m_i, \sum x_j m_j] &= \sum_{i,j}[x_i m_i, x_j m_j] \\ &= \sum_{i,j}(x_i(\theta(m_i)x_j)m_j - x_j(\theta(m_j)x_i)m_i + x_i x_j[m_i, m_j]) \\ &= \sum_k(\sum_j(x_j(\theta(m_j)x_k)-x_j(\theta(m_j)x_k)))m_k + \sum_{i<j}(x_i x_j - x_i x_j)[m_i, m_j] = 0.\end{aligned}$$

Finally, we must show that M with the given bracket has the structure of an A-k-algebra. Thus suppose m and m' are elements of M. Then $m=\sum_i a_i m_i$ and $m'=\sum_j a'_j m_j$ for some a'_i, $a_j \in A$. Then

$$\begin{aligned}\theta[m, m'] &= \theta[\sum_i a_i m_i, \sum_j a'_j m_j] = \theta(\sum_{i,j}[a_i m_i, a'_j m_j]) \\ &= \theta(\sum_{i,j} a_i(\theta(m_i)a'_j)m_j - a'_j(\theta(m_j)a_i)m_i + a_i a'_j[m_i, m_j]) \\ &= \sum_{i,j} a_i(\theta(m_i)a'_j)\theta(m_j) - a'_j(\theta(m_j)a_i)\theta(m_i) + a_i a'_j[\theta(m_i), \theta(m_j)] \\ &= \sum_{i,j}[a_i\theta(m_i), a'_j\theta(m_j)] = [\sum_i a_i\theta(m_i), \sum_j a'_j\theta(m_j)] = [\theta(m), \theta(m')].\end{aligned}$$

Finally

$$\begin{aligned}[am, bm'] &= [\sum_i aa_i m_i, \sum_j ba'_j m'_j] = \sum_{i,j}[aa_i m_i, ba'_j m'_j] \\ &= \sum_{i,j}(aa_i\theta(m_i)(ba'_j)m'_j - ba'_j\theta(m'_j)(aa_i)m_i + aba_i a'_j[m_i, m'_j]) \\ &= \sum_{i,j}(aa_i b(\theta(m_i)a'_j)m'_j + aa_i a'_j(\theta(m_i)b)m'_j - ba'_j a(\theta(m'_j)a_i)m_i \\ &\qquad - ba'_j a_i(\theta(m'_j)a)m_i + aba_i a'_j[m_i, m'_j]) \\ &= a\sum_{i,j} a_i(\theta(m_i)b)a'_j m'_j - b\sum_{i,j} a'_j(\theta(m'_j)a)a_i m_i + ab\sum_{i,j} a_i a'_j[m_i, m'_j] \\ &\qquad + \sum_{i,j} aa_i b(\theta(m_i)a'_j)m'_j - \sum_{i,j} ba'_j a(\theta(m'_j)a_i)m_i \\ &= a(\theta(\sum_i a_i m_i)b)(\sum_j a'_j m'_j) - b(\theta(\sum_j a'_j m'_j)a)(\sum_i a_i m_i) \\ &\qquad + ab(\sum_{i,j}(a_i(\theta(m_i)a'_j)m'_j - a'_j(\theta(m'_j)a_i)m_i + a_i a'_j[m_i, m'_j])) \\ &= a(\theta(m)b)m' - b(\theta(m')a)m + ab\sum_{i,j}[a_i m_i, a'_j m'_j] \\ &= a(\theta(m)b)m' - b(\theta(m')a)m + ab[m, m'].\end{aligned}$$

This completes the proof.

Theorem **(2.6)**. — *Suppose that* A, *k and* B *are rings such that* A *is a k-algebra and such that* B *is an* A-*algebra. Suppose that* L *is an* A-*k*-*Lie algebra and assume that* $\varphi : L \to L(A/k)$ *and* $\psi : L \to L(B/k)$ *are an* A-*structure map for* L *and an* L-*structure map for* B. *Then* $B\otimes_A L$ *has the structure of a* B-*k*-*Lie algebra such that*

$$[b\otimes d, b'\otimes d'] = b(\psi(d)b')\otimes_A d' - b'(\psi(d')b)\otimes_A d + bb'\otimes_A [d, d'].$$

Further the $(B\otimes_A L)$-*structure map to* $L(B/k)$ *is the map* $\gamma = \mathrm{Id}_B\otimes_A \psi$.

Proof. — We shall first show that $B\otimes_k L$ has the structure of a k-Lie algebra.

Suppose that $b\times d\in B\times L$. We define $[b\times d, \cdot]' : B\times L \to B\otimes_k L$ by

$$[b\times d, \beta\times\delta]' = b(\psi(d)\beta)\otimes_k \delta - \beta(\psi(\delta)b)\otimes_k d + b\beta\otimes_k [d, \delta].$$

It is easily seen that $[b\times d, \cdot]'$ is k-bilinear, thus there exists a k-linear map $[b\times d, \cdot]''$ from $B\otimes_k L$ to $B\otimes_k L$ such that $[b\times d, \beta\otimes_k\delta]'' = [b\times d, \beta\times\delta]'$. Suppose

$$m = \sum_i \beta_i\otimes_k\delta_i \in B\otimes_k L.$$

We can define a map $[\cdot, m]^\# : B\times L \to B\otimes_k L$ by $[b\times d, m]^\# = [b\times d, m]''$. For each m, $[\cdot, m]^\#$ is k-bilinear. Thus if $u, v\in k$ and $d, d'\in L$

$$\begin{aligned}
[b\times(ud+vd'), m]^\# &= [b\times(ud+vd'), \textstyle\sum_i \beta_i\otimes\delta_i]'' \\
&= \textstyle\sum_i [b\times(ud+vd'), \beta_i\times\delta_i]' \\
&= \textstyle\sum_i (b(\psi(ud+vd'))\beta_i\otimes_k\delta_i - \beta_i(\psi(\delta_i)b)\otimes_k(ud+vd') + b\beta_i\otimes_k[ud+vd', \delta_i]) \\
&= \textstyle\sum_i (ub(\psi(d)\beta_i)\otimes_k\delta_i + bv(\psi(d')\beta_i)\otimes_k\delta_i - u\beta_i(\psi(\delta_i)b)\otimes_k d - v\beta_i(\psi(\delta_i)b)\otimes_k d' \\
&\qquad + ub\beta_i\otimes_k[d, \delta_i] + vb\beta_i\otimes_k[d', \delta_i]) \\
&= \textstyle\sum_i (ub(\psi(d)\beta_i)\otimes_k\delta_i - u\beta_i(\psi(\delta_i)b)\otimes_k d + ub\beta_i\otimes_k[d, \delta_i]) \\
&\qquad + \textstyle\sum_i (vb(\psi(d')\beta_i)\otimes_k\delta_i - v\beta_i(\psi(\delta_i)b)\otimes_k d' + vb\beta_i\otimes_k[d', \delta_i]) \\
&= u[b\times d, m]^\# + v[b\times d', m]^\#.
\end{aligned}$$

Similarly for the B variable. Thus for each m there exists a k-linear map

$$[\ , m] : B\otimes_k L \to B\otimes_k L$$

such that $[b\otimes_k d, m] = [b\times d, m]^\#$, and in particular $[b\otimes_k d, \beta\otimes_k\delta] = [b\times d, \beta\times\delta]'$.

Now let $\gamma' = \mathrm{Id}_B\otimes_k\psi : B\otimes_k L \to L(B/k)$. The map γ' is clearly B-linear, and $B\otimes_k L$ is generated as a B-module by the elements $1\otimes_k d$, where $d\in L$.

If $d, d', d''\in L$, then

$$\begin{aligned}
&[1\otimes d, [1\otimes d', 1\otimes d'']] + [1\otimes d', [1\otimes d'', 1\otimes d]] + [1\otimes d'', [1\otimes d, 1\otimes d']] \\
&\qquad = 1\otimes[d, [d', d'']] + 1\otimes[d', [d'', d]] + 1\otimes[d'', [d, d']] = 0.
\end{aligned}$$

Clearly $[1\otimes d, 1\otimes d] = 0$ and $[1\otimes d, 1\otimes d'] = -[1\otimes d', 1\otimes d]$. If $b, b'\in B$, then

$$\begin{aligned}
[b\otimes d, b'\otimes d'] &= b(\psi(d)b')\otimes_k d' - b'(\psi(d')b)\otimes_k d + bb'\otimes_k[d, d'] \\
&= b(\psi(d)b')(1\otimes_k d') - b'(\psi(d')b)(1\otimes_k d) + bb'\otimes_k[1\otimes d, 1\otimes d'].
\end{aligned}$$

Thus Lemma (2.5) shows that $B\otimes_k L$ has the structure of a B-k-Lie algebra.

To complete the proof, we shall show that if $\eta : B\otimes_k L \to B\otimes_A L$ is the quotient map which carries $b\otimes_k d$ to $b\otimes_A d$, then the kernel of η is a k-Lie algebra ideal in $B\otimes_k L$ which is mapped to zero by γ'.

Note first that $\ker(\eta)$ is generated as an A-submodule of $B\otimes_k L$ by the elements of the form $ab\otimes_k d - b\otimes_k ad$, where $a\in A$, $b\in B$, $d\in L$. If $\beta\otimes_k\delta\in B\otimes_k L$, then

$$\begin{aligned}\eta[\beta\otimes_k\delta, ab\otimes_k d - b\otimes_k ad] &= \eta([\beta\otimes_k\delta, ab\otimes_k d] - [\beta\otimes_k\delta, b\otimes_k ad])\\ &= \eta(\beta(\psi(\delta)(ab))\otimes_k d - ab(\psi(d)\beta)\otimes_k\delta + ab\beta\otimes_k[\delta, d]\\ &\quad -\beta(\psi(\delta)b)\otimes_k ad\\ &\quad + b(\psi(ad)\beta)\otimes_k\delta - \beta b\otimes_k[\delta, ad])\\ &= \beta(\psi(\delta)a)b\otimes_A d + \beta a(\psi(\delta)b)\otimes_A d\\ &\quad - ab(\psi(d)\beta)\otimes_A\delta + ab\beta\otimes_A[\delta, d]\\ &\quad - a\beta(\psi(\delta)b)\otimes_A d\\ &\quad + ba(\psi(d)\beta)\otimes_A\delta - \beta b(\psi(\delta)a)\otimes_A d - \beta ab\otimes_A[\delta, d] = 0.\end{aligned}$$

Note that γ' clearly carries $ab\otimes_k d - b\otimes_k ad$ to zero in $L(B/k)$. This completes the proof.

Lemma (**2.7**). — *Suppose that* A *and* k *are rings and suppose* A *is a* k*-algebra. Assume that* S *is a multiplicatively closed set in* A *containing* 1. *There is an* $A[S^{-1}]$*-*k*-Lie algebra map* $\theta : A[S^{-1}]\otimes_A L(A/k) \to L(A[S^{-1}]/k)$. *Further if* A *is noetherian and* $D^1(A/k)$ *is finitely generated the map* θ *is an isomorphism.*

Proof. — Suppose $d\in L(A/k)$ and suppose $\varphi : A \to A[S^{-1}]$ is the canonical map. We set $\theta'(d)(\varphi(a).\varphi(s)^{-1}) = \frac{-1}{\varphi(s)^2}\varphi(a)\varphi(ds) + \frac{1}{\varphi(s)}\varphi(da)$. One checks easily that $\theta'(d)$ is a k-derivation on $A[S^{-1}]$. The map $\theta'(d)$ is clearly A-linear. Further, if $a\in A$, then $\theta'(d)(\varphi(a)) = \varphi(da)$. It is easily seen that $\theta'[d, d'] = [\theta'(d), \theta'(d')]$. We can extend θ' to an $A[S^{-1}]$-module map from $A[S^{-1}]\otimes_A L(A/k)$ to $L(A[S^{-1}]/k)$ by setting

$$\theta(\varphi(s)^{-1}\otimes_A d) = \varphi(s)^{-1}.\theta'(d).$$

We wish first to show that $A[S^{-1}]$ is now an $A[S^{-1}]\otimes_A L(A/k)$-algebra with structure map θ. Thus suppose $a, a'\in A$ and $s, s'\in S$. If $d, d'\in L(A/k)$,

$$\begin{aligned}&\theta\left[\frac{\varphi(a)}{\varphi(s)}\otimes d, \frac{\varphi(a')}{\varphi(s')}\otimes d'\right]\\ &= \theta\left(\frac{\varphi(a)}{\varphi(s)}\left(\theta'(d)\frac{\varphi(a')}{\varphi(s')}\right)\otimes_A d - \frac{\varphi(a')}{\varphi(s')}\left(\theta'(d')\frac{\varphi(a)}{\varphi(s)}\right)\otimes d' - \frac{\varphi(a)}{\varphi(s)}\frac{\varphi(a')}{\varphi(s')}\otimes_A[d, d']\right)\\ &= \left(\frac{\varphi(a)}{\varphi(s)}\left(\theta'(d)\frac{\varphi(a')}{\varphi(s')}\right)\theta'(d) - \frac{\varphi(a')}{\varphi(s')}\left(\theta'(d')\frac{\varphi(a)}{\varphi(s)}\right)\theta'(d') + \frac{\varphi(a)}{\varphi(s)}\frac{\varphi(a')}{\varphi(s')}\theta'[d, d']\right)\\ &= \left[\theta\left(\frac{\varphi(a)}{\varphi(s)}\otimes_A d\right), \theta\left(\frac{\varphi(a')}{\varphi(s')}\otimes_A d'\right)\right]\end{aligned}$$

where one derives the last equality by an application of the formula of Lemma (2.1). This completes the proof of the first assertion.

To prove the second assertion we need only note that when $D^1(A/k)$ is finitely presented, then

$$\mathrm{Hom}_{A[S^{-1}]}(A[S^{-1}]\otimes_A D^1(A/k), A[S^{-1}]) \cong A[S^{-1}]\otimes_A \mathrm{Hom}(D^1(A/k), A)$$

where the isomorphism is given by θ. (See [9]).

We would also like to remark here that if L is an A-k-Lie algebra and if B and B′ are L-algebras, then $B\otimes_A B'$ is a $(B\otimes_A L)$-algebra such that

$$(\beta\otimes_A\delta)(b\otimes_A b') = \beta\delta(b)\otimes_A b' + \beta b\otimes_A \delta(b').$$

Thus as in (iii) of Definition (2.2)

$$\begin{aligned}[][\beta\otimes_A\delta, \beta'\otimes_A\delta'](b\otimes_A b') &= (\beta\delta(\beta')\otimes\delta' - \beta'\delta'(\beta)\otimes\delta + \beta\beta'\otimes[\delta,\delta'])(b\otimes_A b')\\ &= \beta\delta(\beta')\delta'(b)\otimes_A b' + \beta\delta(\beta')b\otimes_A\delta'(b')\\ &\qquad -\beta'\delta'(\beta)\delta(b)\otimes_A b' - \beta'\delta'(\beta)b\otimes_A\delta(b')\\ &\qquad +\beta\beta'[\delta,\delta']b\otimes_A b' + \beta\beta' b\otimes_A[\delta,\delta']b'\\ &= \beta\delta(\beta')(1\otimes_A\delta')(b\otimes_A b') - \beta'\delta'(\beta)(1\otimes_A\delta)(b\otimes_A b')\\ &\qquad + \beta\beta'(1\otimes_A[\delta,\delta'])(b\otimes_A b').\end{aligned}$$

If $b\in B$ and $\beta\otimes_A\delta\in B\otimes_A L$, then

$$(\beta\otimes_A\delta)(b\otimes_A 1) = \beta\delta(b)\otimes_A 1 = (\beta\otimes_A\delta(b))(1\otimes_B 1).$$

Definition **(2.8)**. — Suppose A and k are rings and suppose that A is a k-algebra. Assume that L is an A-k-Lie algebra and suppose that B is a nonscalar k-algebra which is a left A-module. We shall say that B is an *A-k-enveloping algebra* for L if:

1) If φ is the A-k-structure map for L, then there exists an A-linear map $\rho_B : L\to B$ such that $\rho_B(d)(ax) = (\varphi(d)a)x + a\rho_B(d)x$, $a\in A$, $x\in B$.

2) If $d, d'\in L$, then $\rho_B[d,d'] = \rho_B(d)\rho_B(d') - \rho_B(d')\rho_B(d)$. We shall denote this last bracket by $[[\ ,\ ]]$.

Remark. — Suppose $a, b\in A$ and $d, d'\in L$. Then

$$\begin{aligned}[][[a\rho_B(d), b\rho_B(d')]] &= a(\varphi(d)b)\rho_B(d') + ab\rho_B(d)\rho_B(d')\\ &\qquad - b(\rho_B(d')a)\rho_B(d) - ab\rho_B(d')\rho_B(d) = \rho_B([ad, bd']).\end{aligned}$$

As usual one gives also the

Definition **(2.9)**. — Suppose L is an A-k-Lie algebra. An enveloping algebra E for L is said to be a *universal enveloping algebra* if given any enveloping algebra B for L there exists a unique map $\psi : E\to B$ such that

(i) ψ is a k-algebra map;
(ii) ψ is a left A-module map;
(iii) $\psi\circ\rho_E = \rho_B$.

We now wish to show that if L is an A-k-Lie algebra, then L has a universal A-k-enveloping algebra. The proof is entirely routine.

Theorem (**2.10**). — *Suppose that* A *and* k *are rings and that* A *is a* k-*algebra. If* L *is an* A-k-*Lie algebra, then there exists a universal enveloping* A-k-*algebra* E(L) *for* L. *Further* $E(L) \cong T/I$ *where* T *is the tensor algebra* $\sum_{j=1}^{\infty} \otimes_k^j L$ *and* I *is the two sided ideal in* T *generated by the elements* $d\otimes d' - d'\otimes d - [d, d']$ *for* $d, d' \in L$ *and the elements* $d(a\theta) - (\varphi(d)a)\,\theta - ad\theta$ *for* $\theta \in T$.

Proof. — Suppose B is an A-k-enveloping algebra for L. Then $\rho_B : L \to B$ is an A-linear map from L to B. This map is then necessarily k-linear, and if $d_1, \ldots, d_r \in L$, then $\psi'(d_1 \times \ldots \times d_r) = d_1 \ldots d_r \in B$ is clearly r-multilinear over k. We can extend the map $\rho_B : L \to B$ uniquely to a k-module map $\psi'' : \sum_{j=1}^{\infty} \otimes^j L \to B$ such that

$$\psi''(d_1 \otimes \ldots \otimes d_r) = d_1 \ldots d_r.$$

Also $$\psi''(ad_1 \otimes \ldots \otimes d_r) = a(d_1 \ldots d_r) = a\psi''(d_1 \otimes \ldots \otimes d_r),$$

thus ψ'' is A-linear from T to B. The map ψ'' is clearly a k-algebra map. Since ψ'' vanishes on the generators for I, this shows that there exists a unique extension of ψ'' to a k-algebra map $\psi : E(L) \to B$ with the desired properties.

We shall denote $E(L(A/k))$ by $E(A/k)$.

Lemma (**2.11**). — *Suppose* k *is a ring and assume that* M *is a* k-*module generated over* k *by elements* m_α, $\alpha \in \mathfrak{A}$. *Assume that there exists a* k-*bilinear function* $\varphi : M \times M \to M$ *such that* $\varphi(m_\alpha, \varphi(m_\beta, m_\gamma)) = \varphi(\varphi(m_\alpha, m_\beta), m_\gamma))$. *Then* M *has the structure of a nonscalar* k-*algebra with multiplication* $x.y = \varphi(x, y)$.

Proof:

$$\begin{aligned}
(\sum_\alpha x_\alpha m_\alpha).(\sum_\beta y_\beta m_\beta.\sum_\gamma z_\gamma m_\gamma) &= \sum_\alpha x_\alpha m_\alpha(\sum_{\beta,\gamma} y_\beta z_\gamma(m_\beta m_\gamma)) \\
&= \sum_{\alpha,\beta,\gamma} x_\alpha y_\beta z_\gamma(m_\alpha(m_\beta m_\gamma)) \\
&= \sum_{\alpha,\beta,\gamma} x_\alpha y_\beta z_\gamma(m_\alpha m_\beta)m_\gamma \\
&= \sum_\gamma z_\gamma(\sum_{\alpha,\beta} x_\alpha y_\beta m_\alpha m_\beta)m_\gamma \\
&= \sum_\gamma z_\gamma((\sum_\alpha x_\alpha m_\alpha)(\sum_\beta y_\beta m_\beta))m_\gamma \\
&= ((\sum_\alpha x_\alpha m_\alpha)(\sum_\beta y_\beta m_\beta))(\sum_\gamma z_\gamma m_\gamma).
\end{aligned}$$

We shall find the following notation convenient in the remainder of this paper. Suppose n and m are integers. We shall denote by $F(n, m)$ the set of strictly increasing functions from the set $\{1, \ldots, n\}$ to the set $\{1, \ldots, m\}$. If $\alpha \in F(n, m)$, then we shall denote by $c\alpha$ the function in $F(m-n, m)$ which has as image in $\{1, \ldots, m\}$ the complement of the range of α in $\{1, \ldots, m\}$. If α is onto, then $c\alpha$ is defined to be the empty function.

Lemma (**2.12**). — *Suppose that* A *and* k *are rings such that* A *is a* k*-algebra. Assume that* L *is an* A-k*-Lie algebra. If* B *is an enveloping algebra for* L *and* $d_1, \ldots, d_r \in$ L, *then for each* $x \in$ B *and each* $a \in$ A

$$\rho_B(d_r) \ldots \rho_B(d_1)(a.x) = \sum_{u=0}^{r} \Big(\sum_{\alpha \in F(u,r)} \big((\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{r-u} \rho_B(d_{c\alpha(w)})\big)x\Big).$$

In this expression $\prod_{v=1}^{u} \theta(d_{\alpha(v)})$ denotes the composition of the k-endomorphisms of A given by the $\theta(d_{\alpha(j)})$, where θ is the A structure map of L, and the product is written from right to left. That is, $\prod_{j=1}^{r} d_j = d_r \ldots d_1$. Further we set $\prod d_{\text{empty}} = \text{identity}$. Finally the same formula holds for $a, x \in$ A and $d_1, \ldots, d_r \in$ L(A/k).

Proof. — We shall prove the assertion by induction on r. For $r = 1$

$$\begin{aligned}
\rho_B(d_1)(ax) &= \sum_{u=0}^{1} \sum_{\alpha \in F(u,1)} \big((\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big)\big(\prod_{w=1}^{1-u} \rho_B(d_{c\alpha(w)})\big) . x \\
&= a . \rho_B\big(\prod_{w=1}^{1} d_{c\alpha(w)}\big)x + \sum_{\alpha \in F(1,1)} \big((\prod_{v=1}^{1} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{0} \rho_B(d_{c\alpha(w)})\big)x \\
&= a . \rho_B(d_1)x + (\theta(d_1)a) . x.
\end{aligned}$$

Proceeding by induction

$$\begin{aligned}
(\rho_B(d_r)(\rho_B(d_{r-1}) \ldots \rho_B(d_1)))(ax) &= \rho_B(d_r)\Big(\sum_{u=0}^{r-1} \sum_{\alpha \in F(u,r-1)} \big((\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{r-u-1} \rho_B(d_{c\alpha(w)})\big)x\Big) \\
&= \sum_{u=0}^{r-1} \sum_{\alpha \in F(u,r-1)} \big(\theta(d_r)(\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{r-u-1} \rho_B(d_{c\alpha(w)})\big)x \\
&\quad + \sum_{u=0}^{r-1} \sum_{\alpha \in F(u,r-1)} \big((\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big) . \rho_B(d_r)\big(\prod_{w=1}^{r-u-1} \rho_B(d_{c\alpha(w)})\big)x \\
&= \sum_{u=1}^{r} \sum_{\alpha \in F(u,r),\ \alpha(u)=r} \big((\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{r-u} \rho_B(d_{c\alpha(w)})\big)x \\
&\quad + \sum_{u=0}^{r-1} \sum_{\alpha \in F(u,r),\ \alpha(u) \neq r} \big((\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{r-u} \rho_B(d_{c\alpha(w)})\big)x \\
&= \sum_{\alpha \in F(0,r),\ \alpha(0) \neq r} \big((\prod_{v=1}^{0} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{r} \rho_B(d_{c(\alpha)w})\big)x \\
&\quad + \sum_{u=1}^{r-1} \sum_{\alpha \in F(u,r)} \big((\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{r-u} \rho_B(d_{c\alpha(w)})\big)x \\
&\quad + \sum_{u=r} \sum_{\alpha \in F(r,r),\ \alpha(r)=r} \big(\prod_{v=1}^{u} \theta(d_{\alpha(v)})a\big) . \big(\prod_{w=1}^{r-u} \rho_B(d_{c\alpha(w)})\big)x \\
&= \sum_{u=0}^{r} \sum_{\alpha \in F(u,r)} \big((\prod_{v=1}^{u} \theta(d_{\alpha(v)}))a\big) . \big(\prod_{w=1}^{r-u} \rho_B(d_{c\alpha(w)})\big)x.
\end{aligned}$$

This completes the proof.

Theorem (**2.13**). — *Suppose* L *is an* A-k-*Lie algebra and suppose that* B *is an* L-*algebra with structure map* $\theta : L \to L(B/k)$. *Then* $B\otimes_A E(L)$ *has the structure of an enveloping algebra for* $B\otimes_A L$. *Further* $B\otimes_A E(L) \cong E(B\otimes_A L)$.

Proof. — We consider the B-module $B\otimes_A \sum_{j=1}^{\infty} \otimes_k^j L = B\otimes_A \mathsf{T}(L)$. In order to put a k-algebra structure on $B\otimes_A \mathsf{T}(L)$ it will suffice to define a k-bilinear map "." from $(B\otimes_A \mathsf{T})\times(B\otimes_A \mathsf{T})$ to $B\otimes_A \mathsf{T}$ such that for k-generators of the form $x = b\otimes_A d_1 \dots d_r$, $y = \beta\otimes_A \delta_1 \dots \delta_s$ and $z = \zeta\otimes_A \theta_1 \dots \theta_t$ we have the relation $x(yz) = (xy)z$.

We define a k-bilinear map from $(B\otimes_A \otimes^r L)\times(B\otimes_A \otimes^s L)$ to $\sum_{t=1}^{r+s} B\otimes_A \otimes^t L$ by setting

$$(b\otimes_A d_r \dots d_1).(\beta\otimes_A \delta_s \dots \delta_1) = \sum_{j=0}^{r} \sum_{\alpha\in F(j,r)} b\Big(\prod_{v=1}^{j} \theta(d_{\alpha(v)})\Big)\beta\otimes_A\Big(\prod_{w=1}^{r-j} d_{c\alpha(w)}\Big)\delta_s \dots \delta_1.$$

This product may be extended to a k-bilinear map from $(B\otimes_A \mathsf{T})\times(B\otimes_A \mathsf{T})$ to $B\otimes_A \mathsf{T}$.

We turn to the question of associativity. Thus suppose that $b\otimes\delta_s \dots \delta_1$ and $c\otimes\theta_t \dots \theta_1$ are elements of $B\otimes_A \mathsf{T}$. Suppose $d_1, \dots, d_r \in L$. Then

$$\begin{aligned}
(1\otimes d_1)&((b\otimes_A \delta_s \dots \delta_1)(c\otimes_A \theta_t \dots \theta_1))\\
&= (1\otimes d_1)\Big(\sum_{u=0}^{s} \sum_{\alpha\in F(u,s)} b\Big(\prod_{v=1}^{u} \theta(\delta_{\alpha(v)})\Big)c\otimes_A\Big(\prod_{w=1}^{s-u} \delta_{c\alpha(w)}\Big).\theta_t \dots \theta_1\Big)\\
&= \sum_{u=0}^{s} \sum_{\alpha\in F(u,s)} \theta(d_1)\Big(b\Big(\prod_{v=1}^{u} \theta(\delta_{\alpha(v)})\Big)c\Big)\otimes_A\Big(\prod_{w=1}^{s-u} \delta_{c\alpha(w)}\Big)\theta_t \dots \theta_1\\
&\quad + \sum_{u=1}^{s} \sum_{\alpha\in F(u,s)} b\Big(\prod_{v=1}^{u} \theta(\delta_{\alpha(v)})\Big)c\otimes_A d_1\Big(\prod_{w=1}^{s-u} \delta_{c\alpha(w)}\Big)\theta_t \dots \theta_1\\
&= \sum_{u=0}^{s} \sum_{\alpha\in F(u,s)} (\theta(d_1)b)\Big(\prod_{v=1}^{u} \theta(\delta_{\alpha(v)})\Big)c\otimes_A\Big(\prod_{w=1}^{s-u} \delta_{c\alpha(w)}\Big)\theta_t \dots \theta_1\\
&\quad + \sum_{u=0}^{s} \sum_{\alpha\in F(u,s)} b\Big(\theta(d_1)\prod_{v=1}^{u} \theta(\delta_{\alpha(v)})\Big)c\otimes_A\Big(\prod_{w=1}^{s-u} \delta_{c\alpha(w)}\Big)\theta_t \dots \theta_1\\
&\quad + \sum_{u=0}^{s} \sum_{\alpha\in F(u,s)} b\Big(\prod_{v=1}^{u} \theta(\delta_{\alpha(v)})\Big)c\otimes_A d_1\Big(\prod_{w=1}^{s-1} \delta_{c\alpha(w)}\Big)\theta_t \dots \theta_1.
\end{aligned}$$

On the other hand

$$\begin{aligned}
((1\otimes d_1)&(b\otimes_A \delta_s \dots \delta_1))(c\otimes\theta_t \dots \theta_1)\\
&= (d_1 b\otimes_A \delta_s \dots \delta_1 + b\otimes d_1\delta_s \dots \delta_1)(c\otimes_A \theta_t \dots \theta_1)\\
&= (d_1 b\otimes_A \delta_s \dots \delta_1)(c\otimes_A \theta_t \dots \theta_1) + (b\otimes\delta_{s+1} \dots \delta_1)(c\otimes_A \theta_t \dots \theta_1)
\end{aligned}$$

where $\delta_{s+1}=d_1$. Thus

$$((1\otimes d_1)(b\otimes_A\delta_s\dots\delta_1))(c\otimes\theta_t\dots\theta_1)=\sum_{j=0}^{s}\sum_{\alpha\in F(j,s)}(d_1b)(\prod_{u=1}^{j}\theta(\delta_{\alpha(u)}))c\otimes_A(\prod_{w=1}^{s-j}\delta_{c\alpha(w)})\theta_t\dots\theta_1$$

$$+\sum_{j=0}^{s+1}\sum_{\alpha\in F(j,s+1)}b(\prod_{w=1}^{j}\theta(\delta_{\alpha(w)}))c\otimes(\prod_{v=1}^{s+1-j}\delta_{c\alpha(v)})\theta_t\dots\theta_1$$

$$=\sum_{j=0}^{s}\sum_{\alpha\in F(j,s)}(\theta(d_1)b)(\prod_{u=1}^{j}\theta(\delta_{\alpha(u)}))c\otimes_A(\prod_{w=1}^{s-j}\delta_{c\alpha(w)})\theta_t\dots\theta_1$$

$$+\sum_{j=0}^{s+1}\sum_{\alpha\in F(j,s+1),\ \alpha(j)=s+1}b(\theta(d_1)\prod_{w=1}^{j-1}\theta(\delta_{\alpha(u)}))c\otimes_A(\prod_{w=1}^{s-(j-1)}\delta_{c\alpha(w)})\theta_t\dots\theta_1$$

$$+\sum_{j=0}^{s}\sum_{\alpha\in F(j,s+1),\ \alpha(j)\leq s}b(\prod_{u=1}^{j}\theta(\delta_{\alpha(u)}))c\otimes_A d_1(\prod_{w=1}^{s-j}\delta_{c\alpha(w)})\theta_t\dots\theta_1$$

$$=\sum_{j=0}^{s}\sum_{\alpha\in F(j,s)}(\theta(d_1)b)(\prod_{v=1}^{j}\theta(\delta_{\alpha(v)}))c\otimes_A(\prod_{w=1}^{s-j}\theta(\delta_{c\alpha(w)}))\theta_t\dots\theta_1$$

$$+\sum_{j=1}^{s}\sum_{\alpha\in F(j,s)}b(\theta(d_1)\prod_{u=1}^{j}\theta(\delta_{\alpha(u)}))c\otimes_A(\prod_{w=1}^{s-j}\delta_{c\alpha(w)})\theta_t\dots\theta_1$$

$$+\sum_{j=0}^{s}\sum_{\alpha\in F(j,s)}b(\prod_{u=1}^{j}\theta(\delta_{\alpha(u)}))c\otimes_A d_1(\prod_{w=1}^{s-j}\delta_{c\alpha(w)})\theta_t\dots\theta_1.$$

More generally if $\psi=\theta_1+\dots+\theta_u$ and $\varphi=\varphi_1+\dots+\varphi_v$, with each φ_i and θ_j of the form $\beta\otimes\theta_1\dots\theta_w$

$$(1\otimes d_1)(\varphi\cdot\psi)=(1\otimes d_1)(\sum_{i,j}\varphi_i\theta_j)$$
$$=(\sum_{i,j}(1\otimes d_1)\varphi_i)\theta_j=((1\otimes d_1)\varphi)\psi.$$

Now if $d_r,\dots,d_1\in L$, then if one uses induction

$$(1\otimes d_r\dots d_1)(\psi\varphi)=((1\otimes d_r)\dots(1\otimes d_1))(\psi\varphi)$$
$$=((1\otimes d_r)(1\otimes d_{r-1})\dots(1\otimes d_1))(\psi\varphi)$$
$$=(1\otimes d_r)(((1\otimes d_{r-1})\dots(1\otimes d_1))(\psi\varphi))$$
$$=(1\otimes d_r)((((1\otimes d_{r-1})\dots(1\otimes d_1))\psi)\varphi)$$
$$=((1\otimes d_r)(((1\otimes d_{r-1})\dots(1\otimes d_1))\psi))\varphi$$
$$=(((1\otimes d_r)\dots(1\otimes d_1))\psi)\varphi.$$

Thus we have an associative k-algebra structure on $B\otimes_A T(L)$. Note that we have shown that $(b\otimes d_r\dots d_k)\cdot(\beta\otimes\delta_1\dots\delta_s)=\sum_j b'_j\otimes\theta_j\delta_1\dots\delta_s$ where $b'_j\in B$ and $\theta_j\in T(L)$.

Next suppose $R=b\otimes d_r\dots d_1$, $S=\beta\otimes\delta_s\dots\delta_1$ and assume $d, d'\in L$. Then

$$\begin{aligned}
R(d.d'-d'.d-[d,d'])S &= (b\otimes d_r\dots d_1).(dd'S-d'dS-[d,d']S)\\
&= (b\otimes d_r\dots d_1)(dd'(\beta\otimes\delta_s\dots\delta_1)-d'd(\beta\otimes\delta_s\dots\delta_1)-[d,d']\beta\otimes\delta_s\dots\delta_1)\\
&= (b\otimes d_r\dots d_1).(d(\theta(d')\beta\otimes\delta_s\dots\delta_1+\beta\otimes d'\delta_s\dots\delta_1))\\
&\qquad -d'(\theta(d)\beta\otimes\delta_s\dots\delta_1+\beta\otimes d\delta_s\dots\delta_1)\\
&\qquad -\theta(d)\theta(d')\beta\otimes\delta_s\dots\delta_1+\theta(d')\theta(d)\beta\otimes\delta_s\dots\delta_1\\
&\qquad -\beta\otimes[d,d']\delta_s\dots\delta_1)\\
&= (b\otimes d_r\dots d_1).(\theta(d)\theta(d')\beta\otimes\delta_s\dots\delta_1+\theta(d')\beta\otimes d\delta_s\dots\delta_1\\
&\qquad +\theta(d)\beta\otimes d'\delta_s\dots\delta_1+\beta\otimes dd'\delta_s\dots\delta_1\\
&\qquad -\theta(d')\theta(d)\beta\otimes\delta_s\dots\delta_1-\theta(d)\beta\otimes d'\delta_s\dots\delta_1\\
&\qquad -\theta(d')\beta\otimes d\delta_s\dots\delta_1-\beta\otimes d'd\delta_s\dots\delta_1\\
&\qquad -\theta(d)\theta(d')\beta\otimes\delta_s\dots\delta_1+\theta(d')\theta(d)\beta\otimes\delta_s\dots\delta_1\\
&\qquad -\beta\otimes[d,d']\delta_s\dots\delta_1)\\
&= (b\otimes d_r\dots d_1).(\beta\otimes_A(dd'-d'd-[d,d'])\delta_s\dots\delta_1)\\
&= \sum_u \beta'_u\otimes_A\theta_u(dd'-d'd-[d,d'])\delta_s\dots\delta_1.
\end{aligned}$$

Further if $\psi\in\mathbf{T}(L)$ and $a\in A$, then

$$\begin{aligned}
\rho &= R(d(a\psi)-(\varphi(d)a)\psi-ad\psi)S\\
&= R(d(a\psi)-(\varphi(d)a)\psi-ad\psi)(\beta\otimes\delta_s\dots\delta_1)\\
&= R(d(a\psi)(\beta\otimes\delta_s\dots\delta_1)-((\varphi(d)a)\psi)(\beta\otimes\delta_s\dots\delta_1)-(ad\psi)(\beta\otimes\delta_s\dots\delta_1)).
\end{aligned}$$

Suppose that $\psi=d_r\otimes_k\dots\otimes_k d_1$. Thus

$$a\psi=ad_r\otimes_k d_{r-1}\otimes\dots\otimes d_1 \quad\text{and}\quad d(a\psi)=d\otimes ad_r\otimes\dots\otimes d_1.$$

Set $\zeta=d_{r-1}\otimes\dots\otimes d_1$; thus

$$\begin{aligned}
\rho = R((d\otimes_k ad_r)\zeta(\beta\otimes_A\delta_s\otimes_k\dots\otimes_k\delta_1)\\
-(\varphi(d)a)(d_r\zeta)(\beta\otimes_A\delta_s\otimes_k\dots\otimes_k\delta_1))-ad\otimes(d_r\zeta)(\beta\otimes_A\delta_s\otimes\dots\otimes_k\delta_1)).
\end{aligned}$$

The expressions $\zeta.(\beta\otimes_A\delta_s\otimes_k\dots\otimes_k\delta_1)$ are linear combinations of expressions with the same form as $\beta\otimes\delta_s\dots\delta_1$. Thus ρ is a linear combination of expressions of the form

$$\begin{aligned}
\rho' &= R((d\otimes_k ad_r)(\beta\otimes_A\delta_s\dots\delta_1)-(\varphi(d)a)d_r(\beta\otimes\delta_s\dots\delta_1)-(ad\otimes d_r)(\beta\otimes\delta_s\dots\delta_1))\\
&= R(\theta(d)\theta(ad_r)\beta\otimes\delta_s\dots\delta_1+\theta(ad_r)\beta\otimes d\delta_s\dots\delta_1+\theta(d)\beta\otimes(ad_r)\delta_s\dots\delta_1+\beta\otimes d(ad_r)\delta_s\dots\delta_1\\
&\qquad -(\varphi(d)a)\theta(d_r)\beta\otimes\delta_s\dots\delta_1-(\varphi(d)a)\beta\otimes d_r\delta_s\dots\delta_1\\
&\qquad -\theta(ad)\theta(d_r)\beta\otimes\delta_s\dots\delta_1-\theta(d_r)\beta\otimes(ad)\delta_s\dots\delta_1\\
&\qquad -\theta(ad)\beta\otimes d_r\delta_s\dots\delta_1-\beta\otimes(ad)d_r\delta_s\dots\delta_1)\\
&= R(a\theta(d)\theta(d_r)\beta\otimes\delta_s\dots\delta_1+(\varphi(d)a)\theta(d_r)\beta\otimes\delta_s\dots\delta_1\\
&\qquad +a\theta(d)\beta\otimes d_r\delta_s\dots\delta_1+\beta\otimes d(ad_r)\delta_s\dots\delta_1\\
&\qquad -(\varphi(d)a)\theta(d_r)\beta\otimes\delta_s\dots\delta_1-(\varphi(d)a)\beta\otimes d_r\delta_s\dots\delta_1\\
&\qquad -a(\theta(d)\theta(d_r)\beta)\otimes\delta_s\dots\delta_1-a\theta(d)\beta\otimes d_r\delta_s\dots\delta_1-a\beta\otimes dd_r\delta_s\dots\delta_1)\\
&= R(\beta\otimes d(ad_r)\delta_s\dots\delta_1-(\varphi(d)a)\beta\otimes d_r\delta_s\dots\delta_1-a\beta\otimes dd_r\delta_s\dots\delta_1)\\
&= R(\beta\otimes(d(ad_r)\delta_s\dots\delta_1-(\varphi(d)a)d_r\delta_s\dots\delta_1-(ad)d_r\delta_s\dots\delta_1)).
\end{aligned}$$

Thus in each case an element of the form $R\gamma S$ is in $B\otimes\mathfrak{J}$ if $\gamma\in\mathfrak{J}$, where $\mathfrak{J}$ is the ideal in $\mathsf{T}(L)$ generated by the relation $dd'-d'd-[d, d']$ and $d(a\rho)-(\varphi(d)a)\rho-ad\rho$. It follows therefore that $B\otimes\mathfrak{J}$ is an ideal in $B\otimes_A \mathsf{T}(L)$ with our multiplication. Hence $B\otimes_A E(L)$ inherits the k-algebra structure of $B\otimes_A \mathsf{T}(L)$.

To complete the theorem we must show that $B\otimes_A E(L)$ is a universal enveloping algebra of $B\otimes_A L$.

First $B\otimes_A E(L)$ is an enveloping algebra for $B\otimes_A L$. The B-k-structure map for $B\otimes_A L$ is given by $\varphi(b\otimes_A d)(\beta)=b(\theta(d)\beta)$. If $\rho : B\otimes \mathsf{T}(L)\rightarrow B\otimes_A E(L)=E$ is the quotient map, then for $\beta\otimes\delta_s\ldots\delta_1$ in $B\otimes_A E(L)$, $b\in B$, and $c\otimes d\in B\otimes_A L$, it follows that

$$\begin{aligned}\rho_{B\otimes_A E(L)}(c\otimes d)(b.\beta\otimes\delta_s\ldots\delta_1) &= c((\theta(d)b)\beta\otimes\delta_s\ldots\delta_1+b\theta(d)\beta\otimes\delta_s\ldots\delta_1+b\beta\otimes d\delta_s\ldots\delta_1)\\ &=(\varphi(c\otimes d)b)\beta\otimes\delta_s\ldots\delta_1+b\rho_{B\otimes_A E(L)}(c\otimes d)(\beta\otimes\delta_s\ldots\delta_1).\end{aligned}$$

Further

$$\begin{aligned}\rho[b\otimes d, b'\otimes d'] &= \rho(b(\theta(d)b')\otimes d'-b'(\theta(d')b)\otimes d+bb'\otimes[d, d'])\\ &=\rho(b(\theta(d)b')\otimes d')-\rho(b'(\theta(d')b)\otimes d)+\rho(bb'\otimes[d, d'])\\ &=\rho(b(\theta(d)b')\otimes d')-\rho(b'(\theta(d')b)\otimes d)+\rho(bb'\otimes(dd'-d'd))\end{aligned}$$

since $dd'-d'd=[d, d']$ in $E(L)$. However a simple computation shows this last expression is $\rho(b\otimes d)\rho(b'\otimes d')-\rho(b'\otimes d')\,\rho(b\otimes d)$.

Finally, suppose that U is an enveloping algebra for $B\otimes_A L$. Then if we use the map $\lambda : d\mapsto \rho_U(1\otimes d)$, $d\in L$, U is an enveloping algebra for L. Thus there exists a unique A-k-algebra map $h : E(L)\rightarrow U$ such that $h\circ\rho_{E(L)}=\rho_U$. We may extend h uniquely to a B-module map h' from $B\otimes_A E(L)$ to U by setting $h'(b\otimes_A\delta)=b.h(\delta)$. If $d\in L$ and $b\in B$, then

$$h'(\rho_{B\otimes_A E(L)}(b\otimes_A d))=h'(b\otimes_A \rho_E(d))=bh(\rho_E(d))=b\rho_U(d)=\rho_U(b\otimes_A d).$$

In order to complete the proof that $B\otimes_A E(L)$ is the universal enveloping algebra for $B\otimes_A L$, it will suffice to show that h' is a B-k-algebra map. For this we need only show that h' is a k-algebra map. Since h' is k-linear it will suffice to show that

$$h'((b\otimes d_r\ldots d_1)(\beta\otimes\delta_s\ldots\delta_1))=h'(b\otimes d_r\ldots d_1).h'(\beta\otimes\delta_s\ldots\delta_1).$$

But

$$\begin{aligned}&h'((b\otimes\rho_E(d_r)\ldots\rho_E(d_1))(\beta\otimes\rho_E(\delta_s)\ldots\rho_E(\delta_1)))\\ &\quad=h'(\sum_{u=0}^{r} b\sum_{\alpha\in F(u,r)}(\prod_{v=1}^{u}\theta(d_{\alpha(v)}))\beta\otimes(\prod_{w=1}^{r-u}\rho_E(d_{c\alpha(w)}))\rho_E(\delta_s)\ldots\rho_E(\delta_1))\\ &\quad=\sum_{u=0}^{r} b\sum_{\alpha\in F(u,r)}(\prod_{v=1}^{u}\theta(d_{\alpha(v)}))\beta h(\prod_{w=1}^{r-u}\rho_E(d_{c\alpha(w)}))\rho_E(\delta_s)\ldots\rho_E(\delta_1))\\ &\quad=\sum_{u=0}^{r} b\sum_{\alpha\in F(u,r)}(\prod_{v=1}^{u}\theta(d_{\alpha(v)}))\beta.(\prod_{w=1}^{r-u}\rho_U(1\otimes d_{c\alpha(w)}))\rho_U(1\otimes\delta_s)\ldots\rho_U(1\otimes\delta_1)\\ &\quad=b(\rho_U(1\otimes d_r)\ldots\rho_U(1\otimes d_1))(\beta\rho_U(1\otimes\delta_s)\ldots\rho_U(1\otimes\delta_1))\end{aligned}$$

by Lemma (2.12). Thus $h'((b\otimes d_r\ldots d_1)(\beta\otimes\delta_s\ldots\delta_1))=h'(b\otimes d_r\ldots d_1)h'(\beta\otimes\delta_s\ldots\delta_1)$. This completes the proof.

Corollary (**2.14**). — *If* A *is a k-algebra which is noetherian, if* S *is a multiplicative closed set and if* $D^1(A/k)$ *is finitely generated, then* $E(L(A[S^{-1}]/k)) = A[S^{-1}] \otimes_A E(L(A/k))$.

Theorem (**2.15**). — *Suppose that* A *and k are rings such that* A *is a k-algebra. Assume that* L *is an* A-*k-Lie algebra. Assume that there exists a commutative k-Lie algebra* $C \subset L$ *such that* C *is a subalgebra of* L, *such that* C *is a finitely generated free k-module, and such that* L *is freely generated as an* A-*module by* C. *Then* $E(L/k) \cong S_A[L]^+$ *where* " + " *denotes the augmentation ideal of* $S_A[L]$.

Proof. — Suppose that $d_1, \ldots, d_n$ is a k-basis for C. Because $C \subset L$, and there exists a Lie algebra map $\theta : L \to L(A/k)$, there is also a k-Lie algebra map from C to $L(A/k)$. Thus A is a C-algebra, and hence $A \otimes_k C$ is an A-k-Lie algebra. The map $\gamma : a \otimes_k c \mapsto a.c \in L$ is an A-k-Lie algebra map which carries $A \otimes_k C$ onto L. Because $A \otimes_k C$ and L are finitely generated free and of the same rank, $A \otimes_k C = L$. Thus $E(L) = E(A \otimes_k C) = A \otimes_k E(C)$. However, E(C) is well known to be $S_k[C]^+$ (see [11], p. 163).

Corollary (**2.15**). — *Suppose that* A *is a regular local ring with a maximal ideal* $\mathfrak{M}$, *and suppose* A *contains a field k such that* $A/\mathfrak{M}$ *is separable algebraic over k. Assume that* $D^1(A/k)$ *is finitely generated. If* $\dim_{A/\mathfrak{M}} \mathfrak{M}/\mathfrak{M}^2 = t$, *then* $E(L(A/k)) = S_A[\mathrm{Der}^1(A/k)]^+$ *as* A-*modules, where* " + " *denotes the augmentation ideal of* $S_A[M]$.

Proof. — Our assumptions imply that $D^1(A/k)$ is a free module over A with a basis consisting of the elements $dx_1, \ldots, dx_s$ where the x_i are a minimal system of generators for the maximal ideal $\mathfrak{M}$ of A. Thus $L(A/k)$ is a free module with basis $\partial/\partial x_1, \ldots, \partial/\partial x_s$ where $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$.

Lemma (**2.16**). — *Suppose that k and* A *are rings and suppose that* B *is an* A-*algebra. Assume that* F *is a free* A-*module and that* $i : F \to B$ *is an* A-*linear map from* F *to* B. *If* f_α, $\alpha \in \mathfrak{A}$, *is a basis for* F, *and if* $b_\alpha \in B$, $\alpha \in \mathfrak{A}$, *is any collection of elements in* B, *then for a k-derivation* $\delta : A \to B$ *there exists a unique k-derivation* $\partial : \mathbf{S}_A[F] \to B$ *such that* $\partial(a) = \delta(a)$ *if* $a \in A$, *and* $\partial(f_\alpha) = b_\alpha$, $\alpha \in \mathfrak{A}$.

Proof. — There exists an exact sequence $0 \to N \to D^1(\mathbf{S}_A[F]/k) \to D^1(\mathbf{S}_A[F]/A) \to 0$ where N is generated by the image of $D^1(A/k)$ in $D^1(\mathbf{S}_A[F]/k)$ as an $\mathbf{S}_A[F]$-module. The $\mathbf{S}_A[F]$-module $D^1(\mathbf{S}_A[F]/A)$ is freely generated by the elements df_α, $\alpha \in \mathfrak{A}$. Further we have the direct sum decomposition

$$D^1(\mathbf{S}_A[F]/k) = (\mathbf{S}_A[F] \otimes_A D^1(A/k)) \oplus D^1(\mathbf{S}_A[F]/A)$$

Note that the A-linear map $i : F \to B$ extends uniquely to an A-algebra map (which we shall again denote by i) from $\mathbf{S}_A[F]$ to B.

Denote by β the A-derivation from $\mathbf{S}_A[F]$ to B which carries f_α to b_α and denote by θ the derivation from $\mathbf{S}_A[F]$ to B which carries f_α to zero. The derivations β and θ we may identify with the $\mathbf{S}_A[F]$-homomorphisms $\widehat{\beta}$ and $\widehat{\theta}$ from $D^1(\mathbf{S}_A[F]/k)$ to B, where $\widehat{\theta}$ is zero on $D^1(\mathbf{S}_A[F]/A)$ and $\widehat{\beta}$ is zero on $\mathbf{S}_A[F] \otimes_A D^1(A/k)$. We set $\widehat{\partial} = \widehat{\theta} \oplus \widehat{\beta}$ and $\partial = \widehat{\partial} \circ d$.

Lemma **(2.17)**. — *Suppose that* A *and* k *are rings, suppose that* A *is a* k*-algebra and suppose that* B *is an* A*-algebra. Assume that* M *is an* A*-module and that there exists a* k*-derivation* $d: A \to B$, *an* A*-linear map* $i: M \to B$ *and a* k*-linear map* $\delta: M \to B$ *such that for each* $m \in M$ *and* $a \in A$, $\delta(am) = a.\delta(m) + (da)i(m)$. *There exists a unique* k*-derivation* $\partial: \mathbf{S}_A[M] \to B$ *such that* $\partial(m) = \delta(m)$ *for* $m \in M$ *and such that* $\partial(a) = da$ *if* $a \in A$.

Proof. — Assume that F is a free module over A, and assume that there exists an exact sequence of A-modules $0 \to K \to F \xrightarrow{h} M \to 0$. Then there exists an exact sequence $0 \to \{K\} \to \mathbf{S}_A[F] \xrightarrow{H} \mathbf{S}_A[M] \to 0$ where H is an A-algebra map and H has as kernel the ideal $\{K\}$ in $\mathbf{S}_A[F]$ generated by the elements of K. The map $i \circ h: F \to B$ extends uniquely to an A-algebra map $j: \mathbf{S}_A[F] \to B$. The previous lemma shows that there exists a unique k-derivation $\Delta: \mathbf{S}_A[F] \to B$ such that $\Delta(a) = da$ if $a \in A$ and such that

$$\Delta(f) = i(H(f))$$

if $f \in F$. To complete the proof we need only show that Δ vanishes on the kernel of H. But if $k \in K$ and $P \in \mathbf{S}_A[F]$, then

$$\begin{aligned}\Delta(Pk) &= j(P)\Delta k + j(k)\Delta(P)\\ &= j(P)i(H(k)) = 0\end{aligned}$$

because j on F is $i \circ h$.

Suppose that L is an A-k-Lie algebra, and suppose that B is an L-algebra. Thus $\theta: L \to L(B/k)$ is such that for $a \in A$ and $d \in L$, $\theta(d)(a.1_B) = (da)1_B$. We then have a map $\rho_E \circ \theta$ which carries L into $E(B/k) = E(L(B/k))$. Since θ is A-linear and ρ_E is B-linear, $\rho_E \circ \theta$ is A-linear. Suppose that $a \in A$ and $x \in E(B/k)$. Then for $d \in L$,

$$\rho_E(\theta(d))(ax) = (\theta(d)(a.1_B))x + (a.1_B)\rho_E(\theta(d))x = (da.1_B)x + (a.1_B)\rho_E(\theta(d))x.$$

Further if $d, d' \in L$, then

$$(\rho_E \circ \theta)[d, d'] = \rho_E[\theta(d), \theta(d')] = \rho_E(\theta(d))\rho_E(\theta(d')) - \rho_E(\theta(d'))\rho_E(\theta(d)).$$

Therefore $E(B/k)$ is an enveloping algebra for L. It follows that there is a uniquely determined A-linear k-algebra map from E(L) to $E(B/k)$ which makes B an E(L)-module. We shall, in what follows, denote that map by $\varepsilon(\theta)$. In particular, then, if $d_1, \ldots, d_r \in L$, and $b \in B$

$$\varepsilon(\theta)(\rho_E(d_1) \ldots \rho_E(d_r)b) = \theta(d_1) \ldots \theta(d_r)(b).$$

We shall write $\alpha.x$ for $\varepsilon(\theta)(\alpha).x$, $x \in B$, when there is no fear of confusion.

Theorem **(2.18)**. — *Suppose* A, B *and* k *are rings and assume* A *and* B *are* k*-algebras. Suppose* L *is an* A-k*-Lie algebra and suppose* $J(L, A \otimes_k B) = \mathbf{S}_{A \otimes_k B}[B \otimes_k E(L) \otimes_k B]/\mathfrak{J}$ *where:*

(i) $B \otimes_k E(L) \otimes_k B$ *has the* $(A \otimes_k B)$*-structure determined by*

$$(a \otimes_k b).(b' \otimes \ell \otimes b'') = bb' \otimes_k a\ell \otimes_k b''$$

(ii) $\mathfrak{J}$ *is the ideal in* $\mathbf{S}_{A\otimes_k B}[B\otimes_k E(L)\otimes_k B]$ *generated by the elements*

a) $1\otimes e\otimes 1$ *if* $e\in E(L)$.

b) $1\otimes\rho_E(d_r)\ldots\rho_E(d_1)\otimes_k b_1 b_2$

$$-\sum_{j=1}^{r-1}\sum_{\alpha\in F(j,r)}(1\otimes_k\prod_{v=1}^{j}\rho_E(d_{\alpha(v)})\otimes_k b_1).(1\otimes_k\prod_{u=1}^{r-j}\rho_E(d_{c\alpha(u)})\otimes_k b_2)$$
$$-b_1\otimes_k\prod_{j=1}^{r}\rho_E(d_j)\otimes b_2-b_2\otimes_k\prod_{j=1}^{r}\rho_E(d_j)\otimes_k b_1.$$

The $(A\otimes_k B)$*-algebra has the structure of an* L*-algebra by an* A*-linear map*

$$\gamma:\ L\to L(J(L,A\otimes_k B)/k)$$

which is such that $\gamma(d)(b)$ *is the image of* $1\otimes d\otimes b$ *in* J.

Finally, the $(A\otimes_k B)$*-algebra* $J(L,A\otimes_k B)$ *is determined uniquely (to within* $(A\otimes_k B)$*-algebra isomorphism) by the following universal properties:*

(i) $J(L,A\otimes_k B)$ *is an* $(A\otimes_k B)$*-algebra, and there exists a map*

$$\gamma:\ L\to L(J(L,A\otimes_k B)/k)$$

which makes $E(J(L,A\otimes_k B)/k)$ *an enveloping algebra for* L *by* $\varepsilon(\gamma)$.

(ii) *If* T *is any* $(A\otimes_k B)$*-algebra which is an* L*-algebra by a structure map* $\theta: L\to L(T/k)$, *and with* $(A\otimes_k B)$*-algebra structure given by* $\lambda: A\otimes_k B\to T$, *then there exists a unique* $(A\otimes_k B)$*-algebra map* $j_{\theta,\lambda}: J(L,A\otimes_k B)\to T$ *such that for each* $\alpha\in E(L)$ *and each* $a\in A$, $b\in B$,

$$j_{\theta,\lambda}((\varepsilon(\gamma)(\alpha))(a\otimes_k b))=(\varepsilon(\theta)(\alpha))\lambda(a\otimes_k b).$$

Proof. — Denote by η the quotient map from $\mathbf{S}_{A\otimes_k B}[B\otimes_k E(L)\otimes_k B]$ to $J(L,A\otimes_k B)$. Suppose $d\in L$. If $x\in E(L)$ and $a\in A$, then $\rho_E(d)(ax)=(da)x+a\rho_E(d)x$, as we have seen in Lemma (2.12). We can define a correspondence from $B\otimes_k E(L)\otimes_k B$ to $J(L,A\otimes_k B)$ by setting

$$d^*(b'\otimes_k x\otimes_k b'')=\eta((1\otimes_k\rho_E(d)\otimes b')(1\otimes x\otimes b'')+b'\otimes_k\rho_E(d)x\otimes_k b'').$$

If b_1 and b_2 are elements of B, then

$$\begin{aligned}d^*((b_1+b_2)\otimes x\otimes b'')&=\eta((1\otimes_k\rho_E(d)\otimes_k b_1)(1\otimes x\otimes b'')+(1\otimes\rho_E(d)\otimes b_2)(1\otimes x\otimes b'')\\&\qquad+b_1\otimes_k\rho_E(d)\otimes_k b''+b_2\otimes_k\rho_E(d)\otimes_k b'')\\&=d^*(b_1\otimes x\otimes b'')+d^*(b_2\otimes x\otimes b'').\end{aligned}$$

Similarly, d^* is k-linear in E(L) and in the right B factor. Thus the correspondence determines a k-linear map from $B\otimes_k E(L)\otimes_k B$ to $J(L,A\otimes_k B)$ which we again denote by d^*.

Next suppose $a\in A$ and $b\in B$. Set $\delta(a\times b)=d(a)\otimes_k b+\eta(1\otimes\rho_E(ad)\otimes_k b)$. The map δ is k-bilinear and therefore δ extends to a k-linear map from $A\otimes_k B$ to $J(L,A\otimes_k B)$. Denote this extension again by δ.

The map δ is a k-derivation from $A\otimes_k B$ to $J(L, A\otimes_k B)$. To see this, suppose that a, $a'\in A$ and b, $b'\in B$. Then

$$\begin{aligned}\delta((a\otimes_k b)(a'\otimes_k b')) &= \delta(aa'\otimes_k bb') = ada'\otimes_k bb' + a'da\otimes_k bb' + (aa'\otimes_k 1)\eta(1\otimes\rho_E(d)\otimes bb'))\\ &= (a\otimes_k b)(da'\otimes_k b') + (a\otimes_k b).\eta(1\otimes\rho_E(a'd)\otimes b')\\ &\qquad + (a'\otimes_k b')(da\otimes_k b) + (a'\otimes_k b').\eta(1\otimes\rho_E(ad)\otimes_k b)\\ &= (a\otimes_k b)\delta(a'\otimes_k b') + (a'\otimes_k b')\delta(a\otimes_k b).\end{aligned}$$

Next assume $a\in A$, $b, b_1, b_2\in B$ and $x\in E(L)$. Then

$$\begin{aligned}d^*((a\otimes_k b)(b_1\otimes x\otimes b_2)) &= d^*(bb_1\otimes_k ax\otimes_k b_2)\\ &= \eta((b_1\otimes\rho_E(d)\otimes b)(1\otimes ax\otimes b_2)) + \eta((b\otimes\rho_E(d)\otimes b_1)(1\otimes ax\otimes b_2))\\ &\qquad + \eta(bb_1\otimes((da)x + a\rho_E(d)x)\otimes_k b_2)\\ &= \eta(1\otimes\rho_E(ad)\otimes b)\eta(b_1\otimes x\otimes b_2) + (da\otimes b)\eta(b_1\otimes x\otimes b_2)\\ &\qquad + (a\otimes_k b)(\eta(1\otimes\rho_E(d)\otimes b_1).\eta(1\otimes x\otimes b_2) + \eta(b_1\otimes\rho_E(d)x\otimes b_2))\\ &= \delta(a\otimes_k b)(b_1\otimes x\otimes b_2) + (a\otimes_k b)(d^*(b_1\otimes x\otimes b_2)).\end{aligned}$$

From this it follows easily that if $\alpha\in A\otimes_k B$ and $x\in B\otimes_k E(L)\otimes_k B$, then

$$d^*(\alpha x) = (\delta\alpha)x + \alpha d^*(x).$$

We can now apply the result of Lemma (2.16) which shows that there exists a unique k-derivation $d' : \mathbf{S}_{A\otimes_k B}[B\otimes_k E(L)\otimes_k B] \to J(L, A\otimes_k B)$ which coincides with d^* on $B\otimes_k E(L)\otimes_k B$ and which coincides with δ on $A\otimes_k B$.

Next we claim that d' factors through $J(L, A\otimes_k B)$. To show this, it will suffice for us to show that d' vanishes on each of the generators of $\mathfrak{J}$. Since the assertion is obvious for elements of type *a)*, we suppose $d_1, \ldots, d_r\in L$, and that $b_1, b_2\in B$. Then

$$\begin{aligned}d'(1\otimes\rho_E(d_r)\ldots\rho_E(d_1)\otimes b_1b_2) &- d\Big(\sum_{j=0,\ \alpha\in F(j,r)}^{r}(1\otimes\prod_{u=1}^{j}\rho_E(d_{\alpha(u)})\otimes b_1)(1\otimes\prod_{v=1}^{r-j}\rho_E(d_{c\alpha(v)})\otimes b_2)\Big)\\ &= \eta(1\otimes\rho_E(d)\rho_E(d_r)\ldots\rho_E(d_1)\otimes b_1b_2))\\ &\quad - \sum_{j=0}^{r}\sum_{\alpha\in F(j,r)}(1\otimes\rho_E(d)\prod_{u=1}^{j}\rho_E(d_{\alpha(u)})\otimes b_1)(1\otimes\prod_{v=1}^{r-j}\rho_E(d_{c\alpha(v)})\otimes b_2)\\ &\quad - \sum_{j=0}^{r}\sum_{\alpha\in F(j,r)}(1\otimes\prod_{u=1}^{j}\rho_E(d_\alpha)\otimes b_1)(1\otimes\rho_E(d)\prod_{v=1}^{r-j}\rho_E(d_{c\alpha(v)})\otimes b_2).\end{aligned}$$

Now denote d by d_{r+1}. Thus the above expression becomes

$$\begin{aligned}&\eta(1\otimes\rho_E(d_{r+1})\ldots\rho_E(d_1)\otimes b_1b_2)\\ &\qquad - \eta\Big(\sum_{j=1}^{r+1}\sum_{\alpha\in F(j,r+1),\ \alpha(j)=r+1}(1\otimes\prod_{u=1}^{j}\rho_E(d_{\alpha(u)})\otimes b_1).(1\otimes\prod_{v=1}^{r+1-j}\rho_E(d_{c\alpha(v)})\otimes b_2)\\ &\qquad - \sum_{j=0}^{r}\sum_{\alpha\in F(j,r+1),\ \alpha(j)\leq r}(1\otimes\prod_{u=1}^{j}\rho_E(d_{\alpha(u)})\otimes b_1).(1\otimes\prod_{v=1}^{r+1-j}\rho_E(d_{c\alpha(v)})\otimes b_2))\\ &= \eta(1\otimes\rho_E(d_{r+1})\ldots\rho_E(d_1)\otimes b_1b_2)\\ &\qquad - \eta\Big(\sum_{j=0}^{r+1}\sum_{\alpha\in F(j,r+1)}(1\otimes\prod_{u=1}^{j}\rho_E(d_{\alpha(u)})\otimes b_1).(1\otimes\prod_{v=1}^{r+1-j}\rho_E(d_{c\alpha(v)})\otimes b_2))\\ &= 0.\end{aligned}$$

We have now shown that there exists a function

$$\gamma : d \mapsto d'$$

from L to $L(J(L, A\otimes_k B)/k)$. The map γ is A-linear. To see this, suppose $a \in A$ and $d \in L$. Then we must show that $\gamma(ad) = a\gamma(d)$, and to show this it will suffice to show that $\gamma(ad)(a'\otimes b') = a\gamma(d)(a'\otimes b')$ and that $\gamma(ad)\eta(b'\otimes\theta\otimes b'') = a\gamma(d)\eta(b'\otimes\theta\otimes b'')$. But

$$\begin{aligned}\gamma(ad)(a'\otimes b') &= a(da')\otimes b' + \eta(1\otimes\rho_E(a'ad)\otimes b')\\ &= (a\otimes 1)(da'\otimes b' + \eta(1\otimes\rho_E(a'd)\otimes b')).\end{aligned}$$

Also

$$\begin{aligned}\gamma(ad)\eta(b'\otimes\theta\otimes b'') &= \eta((1\otimes\rho_E(ad)\otimes b')(1\otimes\theta\otimes b'')) + \eta(b'\otimes\rho_E(ad)\theta\otimes b'')\\ &= (a\otimes 1)\eta((1\otimes\rho_E(d)\otimes b')(1\otimes\theta\otimes b'')) + \eta(b'\otimes\rho_E(d)\otimes b'')).\end{aligned}$$

Next we claim that γ is a k-Lie algebra map. Thus suppose that $d, d' \in L$ and suppose that $a \in A$, $b \in B$. Then

$$\begin{aligned}[\gamma(d), \gamma(d')](a\otimes_k b) &= \gamma(d)\gamma(d')(a\otimes_k b) - \gamma(d')\gamma(d)(a\otimes_k b)\\ &= \gamma(d)(d'a\otimes b + \eta(1\otimes a\rho_E(d')\otimes b)) - \gamma(d')(da\otimes b + \eta(1\otimes a\rho_E(d)\otimes b))\\ &= dd'a\otimes b + \eta(1\otimes(d'a)\rho_E(d)\otimes b) + \eta(1\otimes\rho_E(d)(a\rho_E(d'))\otimes b)\\ &\qquad - d'da\otimes b - \eta(1\otimes(da)\rho_E(d')\otimes b) - \eta(1\otimes\rho_E(d')(a\rho_E(d))\otimes b)\\ &= dd'a\otimes b + \eta(1\otimes(d'a)\rho_E(d)\otimes b) + \eta(1\otimes(da)\rho_E(d')\otimes b)\\ &\qquad + \eta(1\otimes a\rho_E(d)\rho_E(d')\otimes b) - d'da\otimes b - \eta(1\otimes(da)\rho_E(d')\otimes b)\\ &\qquad - \eta(1\otimes(d'a)\rho_E(d)\otimes b) - \eta(1\otimes a\rho_E(d')\rho_E(d)\otimes b)\\ &= dd'a\otimes b + \eta(1\otimes a\rho_E(d)\rho_E(d')\otimes b)\\ &\qquad - d'da\otimes b - \eta(1\otimes a\rho_E(d')\rho_E(d)\otimes b)\\ &= dd'a\otimes b - d'da\otimes b + \eta(1\otimes a\rho_E[d, d']\otimes b)\\ &= [d, d']a\otimes b + \eta(1\otimes a\rho_E[d, d']\otimes b)\\ &= (\gamma[d, d'])(a\otimes b).\end{aligned}$$

Further, if $b, b' \in B$, and $x \in E(L)$, then

$$\begin{aligned}[\gamma(d), \gamma(d')]\eta(b\otimes x\otimes b') &= (\gamma(d)\gamma(d') - \gamma(d')\gamma(d))\eta(b\otimes x\otimes b')\\ &= \eta(1\otimes\rho_E(d)\rho_E(d')\otimes b)\eta(1\otimes x\otimes b')\\ &\qquad + \eta(1\otimes\rho_E(d')\otimes b)\eta(1\otimes\rho_E(d)x\otimes b')\\ &\qquad + \eta(1\otimes\rho_E(d)\otimes b)\eta(1\otimes\rho_E(d')x\otimes b') + \eta(b\otimes\rho_E(d)\rho_E(d')x\otimes b')\\ &\qquad - \eta(1\otimes\rho_E(d')\rho_E(d)\otimes b)\eta(1\otimes x\otimes b')\\ &\qquad - \eta(1\otimes\rho_E(d)\otimes b)\eta(1\otimes\rho_E(d')x\otimes b')\\ &\qquad - \eta(1\otimes\rho_E(d')\otimes b)\eta(1\otimes\rho_E(d)x\otimes b') - \eta(b\otimes\rho_E(d')\rho_E(d)x\otimes b')\\ &= \eta(1\otimes\rho_E[d, d']\otimes b)\eta(1\otimes x\otimes b') + \eta(b\otimes\rho_E[d, d']x\otimes b')\\ &= \eta[d, d'](b\otimes x\otimes b').\end{aligned}$$

Because $\gamma(d)(a\otimes 1) = da\otimes 1$, this completes the proof that $J(L, A\otimes_k B)$ has the structure of an L-algebra. $J(L, A\otimes_k B)$ now inherits the structure of an E(L)-algebra by $\varepsilon(\gamma)$, where $(\varepsilon(\gamma)\alpha)(b) = \eta(1\otimes\alpha\otimes b)$, $\alpha \in E(L)$ and $b \in B$.

Now suppose that T is an $(A\otimes_k B)$-algebra and suppose $\theta : L \to L(T/k)$ gives T the structure of an L-algebra. Suppose that φ is the A-k-structure for L and suppose $\lambda : A\otimes_k B \to T$ is the map giving T an $(A\otimes_k B)$-structure. Suppose $a\in A$ and assume $x\in E(T/k)$. If $d\in L$, then $\theta(d)(ax)=\theta(d)(\lambda(a\otimes 1)x)=(\theta(d)\lambda(a\otimes 1))x+\lambda(a\otimes 1)(\theta(d)x)$. We know that $E(T/k)$ is an A-k-enveloping algebra for L; thus there exists a map $\varepsilon(\theta) : E(L) \to E(T/k)$, such that $\varepsilon(\theta)\circ\rho_E=\theta$ on L. If we are to have that

$$j_{\theta,\lambda}((\varepsilon(\gamma)(\alpha))(a\otimes_k b))=(\varepsilon(\theta)(\alpha))\lambda(a\otimes_k b),$$

then for $b\in B$ we must have $j_{\theta,\lambda}(1\otimes\alpha\otimes b)=(\varepsilon(\theta)(\alpha))(1\otimes_k b)$. Thus we must have $j_{\theta,\lambda}\eta(b'\otimes\alpha\otimes b'')=\lambda(1\otimes b')((\varepsilon(\theta)\alpha)b'')$. This uniquely determines an $(A\otimes_k B)$-linear map μ from $B\otimes_k E(L)\otimes_k B$ to T where $\mu(b\otimes\alpha\otimes b')=\lambda(1\otimes b)((\varepsilon(\theta)\alpha)b')$. There is then a unique extension of μ to an $(A\otimes_k B)$-algebra map from $S_{A\otimes_k B}[B\otimes_k E(L)\otimes_k B]$ to T. Denote this extension also by μ. Our construction of μ shows that if $d\in L$, then

$$\begin{aligned}\mu\eta(da\otimes b+1\otimes ad\otimes b)&=\lambda(da\otimes b)+a\theta(d)b\\&=\theta(d)\lambda(a\otimes_k b).\end{aligned}$$

Assume that $d_1, \ldots, d_r\in L$ and suppose that $b_1, b_2\in B$. Lemma (2.12) shows that in T

$$\theta(d_r)\ldots\theta(d_1)(b_1b_2)=\sum_{u=0}^{r}\sum_{\alpha\in F(u,r)}(\prod_{v=1}^{u}\theta(d_{\alpha(v)}))b_1\cdot(\prod_{w=1}^{r-u}\theta(d_{c\alpha(w)}))b_2.$$

Clearly $\mu(1\otimes x\otimes 1)=0$. Thus μ factors uniquely through $J(L, A\otimes_k B)$. Denote this map by j. If $a\in A$, $b\in B$, and $d_r, \ldots, d_1\in L$, then

$$\begin{aligned}j((\varepsilon(\gamma)\rho_E(d_r)\ldots\rho_E(d_1))(a\otimes_k b))&=j(\eta(\sum_{u=0}^{r}\sum_{\alpha\in F(u,r)}((\prod_{v=1}^{u}d_{\alpha(v)})a)(1\otimes\prod_{w=1}^{r-u}\rho_E(d_{c\alpha(w)})\otimes b)))\\&=\sum_{u=0}^{r}\sum_{\alpha\in F(u,r)}((\prod_{v=1}^{u}d_{\alpha(v)})a)(\varepsilon(\theta)\prod_{w=1}^{r-u}\rho_E(d_{c\alpha(w)}))b\\&=\theta(d_r)\ldots\theta(d_1)\lambda(a\otimes_k b)=\varepsilon(\theta)(\rho_E(d_r)\ldots\rho_E(d_1))\lambda(a\otimes_k b).\end{aligned}$$

Since both $\varepsilon(\theta)$ and j are A-linear, this shows that for each element $\alpha\in E(L)$,

$$j((\varepsilon(\gamma)\alpha)(a\otimes_k b))=(\varepsilon(\theta)\alpha)\lambda(a\otimes_k b).$$

This completes the proof.

In what follows, if A, B, k, L and T are as in the above theorem, then we shall call the map $j_{\theta,\lambda} : J(L, A\otimes_k B)\to B$ the jet map of θ. In case $L=L(A/k)$ we shall denote $J(L, A\otimes_k B)$ by $J(A, B/k)$. In this case, if $h : B\to A$ is a k-algebra map, then A has the structure of an $(A\otimes_k B)$-algebra by the map $H(a\otimes_k b)=ah(b)$. A is therefore an $L(A/k)$-algebra which is also an $(A\otimes_k B)$-algebra. We shall denote $j_{\mathrm{Id},H}$ by j_k and we shall call $j_k h$ the *jet section map* of h.

Lemma **(2.19)**. — *Assume* A *and* k *are rings, and suppose that* L *and* L′ *are* A-k-*Lie algebras. Suppose that* B *is an* A-*algebra such that it is both an* L- *and an* L′-*algebra with structure maps* λ *and* λ' *respectively. If* $\theta : L \to L'$ *is an* A-k-*Lie algebra map such that for each* $d \in L$ *one has* $\lambda'(\theta(d)) = \lambda(d)$, *then the map* $\mathrm{Id} \otimes_A \theta : B \otimes_A L \to B \otimes_A L'$ *is a* B-k-*map.*

Proof. — The map $\mathrm{Id} \otimes_A \theta$ is by definition B-linear, hence we need only show that $\mathrm{Id} \otimes_A \theta$ is a k-Lie algebra map such that for $\delta \in B \otimes_A L$ and $b \in B$ it follows that $((\mathrm{Id} \otimes_A \theta)(\delta))(b) = \delta(b)$. If $\delta = \sum_i b_i \otimes d_i$, $d_i \in L$, then

$$\begin{aligned}((\mathrm{Id} \otimes_A \theta)(\sum_i b_i \otimes d_i))(b) &= (\sum_i b_i \otimes_A \theta(d_i))(b) \\ &= \sum_i b_i \lambda'(\theta(d_i))(b) = \sum_i b_i \lambda(d_i)(b) \\ &= (\sum_i b_i \otimes d_i)(b).\end{aligned}$$

Now assume that $b, b' \in B$ and that $d, d' \in L$. Then

$$\begin{aligned}(\mathrm{Id} \otimes \theta)[b \otimes d, b' \otimes d'] &= (\mathrm{Id} \otimes \theta)(b(\lambda(d)b') \otimes d' - b'(\lambda(d')b) \otimes d + bb' \otimes [d, d']) \\ &= b(\lambda(d)b') \otimes \theta(d') - b'(\lambda(d')b) \otimes \theta(d) + bb' \otimes [\theta(d), \theta(d')] \\ &= b(\lambda'(\theta(d))b') \otimes \theta(d') - b'(\lambda'(\theta(d'))b) \otimes \theta(d) + bb' \otimes [\theta(d), \theta(d')] \\ &= [b \otimes \theta(d), b' \otimes \theta(d')].\end{aligned}$$

To prove the next assertion we shall find it convenient at this time to introduce some notation. If n is a positive integer, then $\mathbf{2}^n$ will denote the collection of subsets of the set $\{1, \ldots, n\}$. If $\alpha \in \mathbf{2}^n$, then $|\alpha|$ will denote the cardinality of α, and if $|\alpha| = j$, and if $d_1, \ldots, d_r \in L$ (an A-k-Lie algebra) we shall denote by d_α the product in E(L), $d_{i(j)} \ldots d_{i(1)}$, where $i(1) \leqslant i(2) \leqslant \ldots \leqslant i(j)$ and $\alpha = (i(1), \ldots, i(j))$. If $|\alpha| = 0$, then set $d_\alpha = 1$. We shall denote by $G(j, n)$ the collection of subsets α of j elements of $\mathbf{2}^n$ such that: (i) $\bigcup_{u=1}^{j} \alpha(u) = \{1, \ldots, n\}$, (ii) $\alpha(u) \cap \alpha(v)$ is empty if $u \neq v$ and (iii) $\alpha(u)$ is nonempty for each u; that is α is a partition of $\{1, \ldots, n\}$ into j nonempty disjoint sets.

Lemma **(2.20)**. — *Assume* L *is an* A-k-*Lie algebra, and assume that* B *is an* L-*algebra. Suppose that* $d_r, \ldots, d_1 \in L$ *and that* s *is a unit in* B. *Then*

$$(d_r \ldots d_1)(s^{-1}) = \sum_{d=1}^{r} (-1)^j \frac{j!}{s^{j+1}} \sum_{\alpha \in G(j, r)} \prod_{u=1}^{j} (d_{\alpha(u)} s).$$

Proof. — If $d \in L$ and t is a unit in B, then $d(t^{-1}) = -(1/t^2)dt$ since $0 = d(t.t^{-1})$. Assume now that the formula is correct for $r-1$. That is

$$d_{r-1} \ldots d_1(s^{-1}) = \sum_{j=1}^{r-1} (-1)^j \frac{j!}{s^{j+1}} \sum_{\alpha \in G(j, r-1)} \prod_{u=1}^{j} (d_{\alpha(u)} s).$$

Thus

$$d_r \dots d_1(s^{-1})$$

$$= \sum_{j=1}^{r-1} (-1)^j j! \left(d_r\left(\frac{1}{s^{j+1}}\right) \sum_\alpha \prod_{u=1}^{j} (d_{\alpha(u)}s) + \sum_\alpha \frac{1}{s^{j+1}} d_r\left(\prod_{u=1}^{j} (d_{\alpha(u)}s)\right)\right)$$

$$= \sum_{j=1}^{r-1} (-1)^{j+1}(j+1)!\, \frac{1}{s^{j+2}} d_r(s)\left(\sum_{\alpha\in G(j,r-1)} \prod_{u=1}^{j} (d_{\alpha(u)}s)\right)$$

$$+ \sum_{j=1}^{r-1} (-1)^j \frac{j!}{s^{j+1}} \sum_\alpha \sum_{v=1}^{j} \prod_{u\neq v} (d_{\alpha(u)}s)(d_r d_{\alpha(v)}s) \qquad (\text{where } \Pi_{\text{empty}}\, d = 1)$$

$$= \sum_{j=2}^{r-1} (-1)^j \frac{j!}{s^{j+1}} \left(\sum_{\alpha\in G(j,r-1)} \sum_{v=1}^{j} \prod_{u\neq v} (d_{\alpha(u)}s)(d_r d_{\alpha(v)}s) + \sum_{\alpha\in G(j-1,r-1)} d_r(s) \prod_{u=1}^{j-1} (d_{\alpha(u)}s)\right)$$

$$+ (-1)^r r! \frac{1}{s^{r+1}} d_r(s) \sum_{\alpha\in G(r-1,r-1)} \prod_{u=1}^{r-1} (d_{\alpha(u)}s)$$

$$+ (-1)\frac{1!}{s^2} \sum_{\alpha\in G(1,r-1)} \sum_{v=1}^{1} \left(\prod_{u\neq 1} d_{\alpha(u)}s\right)(d_r d_{\alpha(1)}s)$$

$$= \sum_{j=2}^{r-1} (-1)^j \frac{j!}{s^{j+1}} \left(\sum_{\beta\in G(j,r),\ \beta(j)=r} \prod_{u=1}^{j} (d_{\beta(u)}s) + \sum_{u=1}^{j} \sum_{\beta\in G(j,r),\ r\in\beta(v),\ |\beta(v)|>1} \prod_{u=1}^{j} (d_{\beta(u)}s)\right)$$

$$+ (-1)^r r! \frac{1}{s^{r+1}} \prod_{u=1}^{r} (d_u s) - \frac{1}{s^2}(d_r \dots d_1 s)$$

$$= \sum_{j=1}^{r} (-1)^j \frac{j!}{s^{j+1}} \sum_{\alpha\in G(j,r)} \prod_{u=1}^{j} (d_{\alpha(u)}s).$$

The object $J(A, B/k)$ is the construction which we needed for the Boardman-Thom theory. However, the object constructed in the next theorem is one which displays the adjoint structure of the infinite jet space.

Theorem **(2.21)**. — *Suppose* A, B *and* k *are rings and assume* A *is a* k*-algebra with* B *an* A*-algebra by a map* $\lambda : A \to B$. *Assume that* L *is an* A-k*-Lie algebra. There is one and up to* B*-algebra isomorphisms, only one* B*-algebra (scalar)* $J(L, B)$ *satisfying the following conditions:*

1) *There exists an* A*-linear map* $\gamma : L \to L(J(L, B)/k)$ *which is the structure map for* $E(J(L, B)/k)$ *as an enveloping algebra for* L.

2) *If* T *is a* B*-algebra by* $\lambda : B \to T$ *such that* T *is an* L*-algebra by a structure map* $\theta : L \to L(T/k)$, *then there exists a unique* B*-algebra map* $j_{\theta,\lambda} : J(L, B) \to T$ *such that for each* $x \in E(L)$ *and each* $b \in B$, $j_{\theta,\lambda}((\varepsilon(\gamma)x)b) = (\varepsilon(\theta)x)(\lambda b)$.

Proof. — There exists an exact sequence $0 \to K \to A \otimes_k B \xrightarrow{\sigma} B \to 0$ where

$$\sigma(a \otimes_k b) = \lambda(a)b.$$

We set $J(L, B)=J(L, A\otimes_k B)/R$ where R is the $J(L, A\otimes_k B)$-ideal generated by the elements of K and the elements of the form $(\varepsilon(\gamma)x)(y)$, $y\in K$. Denote by ρ the quotient map from $J(L, A\otimes_k B)$ to $J(L, B)$. $J(L, B)$ now has the structure of a B-algebra by the map $b\to\rho(1\otimes_k b)$.

Suppose $x\in L$ and assume that $y\in R$. Then $y=\sum_\alpha r_\alpha\kappa_\alpha$ where $r_\alpha\in J(L, A\otimes_k B)$, and $\kappa_\alpha=(\varepsilon(\gamma)\theta_\alpha)s_\alpha$ where $\theta_\alpha\in E(L)$ and $s_\alpha\in K$. Thus

$$\begin{aligned}\gamma(x)(y)&=\sum_\alpha(\gamma(x)r_\alpha)\kappa_\alpha+\sum_\alpha r_\alpha\gamma(x)((\varepsilon(\gamma)\theta_\alpha)s_\alpha)\\&=\sum_\alpha(\gamma(x)r_\alpha)\kappa_\alpha+\sum_\alpha r_\alpha((\varepsilon(\gamma)(x.\theta_\alpha))s_\alpha).\end{aligned}$$

Because of this, R is closed under the action of E(L). If $\kappa\in K$, then $\varepsilon(\gamma)\kappa\in R$. Thus $\gamma(x)$ extends uniquely to a derivation from $J(L, B)$ to $J(L, B)$. We denote this extension by $\bar\gamma(x)$. The map $\bar\gamma$ is clearly A-linear and makes $J(L, B)$ into an L-algebra. We denote $\bar\gamma$ by γ_B.

To complete the proof of the theorem, we need only show that $J(L, B)$ (and γ_B) satisfy the universal properties of the assertion. Thus suppose that $\varphi: B\to T$ gives T the structure of a B-algebra, and suppose that $\theta: L\to L(T/k)$ gives T the structure of an L-algebra. The B-algebra T also has the structure of an $(A\otimes_k B)$-algebra by a map $\nu: a\otimes_k b\mapsto\varphi(\lambda(a).b)$. Theorem (2.20) shows that there exists a unique $(A\otimes_k B)$-algebra map $j_{\theta,\nu}: J(L, A\otimes_k B)\to T$ such that $j_{\theta,\nu}((\varepsilon(\gamma)x)(a\otimes_k b))=(\varepsilon(\theta)x)\nu(a\otimes_k b)$. If $y\in R$, then if we use the notation of the previous paragraph

$$\begin{aligned}j_{\theta,\nu}(\kappa+\sum_\alpha r_\alpha(\varepsilon(\gamma)\theta_\alpha)s_\alpha)&=\nu(\kappa)+\sum_\alpha j_{\theta,\nu}(r_\alpha)j_{\theta,\nu}((\varepsilon(\gamma)\theta_\alpha)s_\alpha)\\&=\sum_\alpha j_{\theta,\nu}(r_\alpha)(\varepsilon(\theta)\theta_\alpha)(\nu s_\alpha)=0.\end{aligned}$$

Therefore $j_{\theta,\nu}$ factors uniquely through $J(L, B)$. This completes the proof.

Corollary (**2.22**). — *Suppose that* L *is an* A-*k*-*Lie algebra. Denote by* ***L-Alg*** *the category with objects* A-*algebras which are* L-*algebras, and morphisms which are* E(L)-*algebra maps. Let* F *denote the forgetful functor to the category of* A-*algebras. Then* $J(L, \cdot)$ *is a left adjoint for* F.

Proof. — We need only remark that we have shown that each object of the category of A-algebras has an associated free object with respect to F. (See [5]).

Theorem (**2.23**). — *Suppose* A, B *and* k *are rings and assume* $\lambda: A\to B$ *gives* B *the structure of an* A-*algebra. If* L *is an* A-*k*-*Lie algebra and if* T *is a multiplicatively closed set which contains* 1 *in* B, *then* $B[T^{-1}]\otimes_B J(L, B)\cong J(L, B[T^{-1}])$. *Further if* S *is a multiplicatively closed set in* A, *then* $A[S^{-1}]\otimes_A J(L, B)\cong J(A[S^{-1}]\otimes_A L, A[S^{-1}]\otimes_A B)$.

Proof. — We shall first show that $B[T^{-1}]\otimes_B J(L, B)\cong J(L, B[T^{-1}])$. Suppose that $\theta: L\to L(A/k)$ is the L-structure map for L as an A-k-Lie algebra. Suppose $d\in L$. Then $\gamma(d)\in L(J(L, B)/k)$. Assume that $\tau: B\to B[T^{-1}]$ is the canonical map, and suppose that $\tau(t)^{-1}\otimes_B x\in B[T^{-1}]\otimes_B J(L, B)$. We set

$$\bar\gamma(d)(\tau(b)\tau(t)^{-1}\otimes_B x)=\tau(t)^{-1}\otimes((\gamma(d)b)x+b(\gamma(d)x))-\frac{\tau(b)}{\tau(t)^2}\otimes(\gamma(d)t)x.$$

We claim that $\bar{\gamma}(d)$ defines a k-derivation on $B[T^{-1}]\otimes_B J(L, B)$. Note that the map

$$\tau(b)\tau(t)^{-1}\otimes_B x \mapsto \tau(b)\tau(t)^{-1}\otimes_B \gamma(d)x + \tau(t)^{-1}\otimes(\gamma(d)b)x - \frac{\tau(b)}{\tau(t)^2}\otimes(\gamma(d)t)x$$

is bilinear in B, thus $\bar{\gamma}(d)$ does determine a map from $B[T^{-1}]\otimes_B J(B, L)$ to $B[T^{-1}]\otimes_B J(B, L)$. Further

$$\bar{\gamma}(d)((\tau(t)^{-1}\otimes_B x)(\tau(t')^{-1}\otimes_B x')) = \bar{\gamma}(d)(\tau(tt')^{-1}\otimes_B xx')$$

$$= -\frac{1}{\tau(t)^2\tau(t')}\otimes_B(\gamma(d)t)xx' + \frac{1}{\tau(tt')}\otimes_B(\gamma(d)x)x' - \frac{1}{\tau(t')^2\tau(t)}(\gamma(d)t')xx' + \frac{1}{\tau(tt')}\otimes_B x(\gamma(d)x')$$

$$= \left(\frac{1}{\tau(t')}\otimes x'\right)\bar{\gamma}(d)\left(\frac{1}{\tau(t)}\otimes x\right) + \left(\frac{1}{\tau(t)}\otimes_B x\right)\bar{\gamma}(d)\left(\frac{1}{\tau(t')}\otimes_B x'\right).$$

Next $\bar{\gamma}$ is A-linear since

$$\bar{\gamma}(ad)\left(\frac{1}{\tau(t)}\otimes_B x\right) = -\frac{1}{\tau(t)^2}\otimes_B(a\gamma(d)t)x + \frac{1}{\tau(t)}\otimes_B\gamma(ad)x$$

$$= a\bar{\gamma}(d)\left(\frac{1}{\tau(t)}\otimes_B x\right).$$

Also if $d, d'\in L$, then

$$\bar{\gamma}[d, d'](\tau(t)^{-1}\otimes_B x) = -\frac{1}{\tau(t)^2}\otimes_B(dd't - d'dt)x + \frac{1}{\tau(t)}\otimes_B(\gamma(d)\gamma(d')x) - \frac{1}{\tau(t)}\otimes_B(\gamma(d')\gamma(d)x)$$

$$= \frac{1}{\tau(t)^4}\otimes_B(\gamma(d)t^2)(\gamma(d')t)x - \frac{1}{\tau(t)^4}\otimes_B(\gamma(d')t^2)(\gamma(d)t)x$$

$$-\frac{1}{\tau(t)^2}\otimes_B(\gamma(d')t)(\gamma(d)x) + \frac{1}{\tau(t)^2}\otimes_B(\gamma(d')t)(\gamma(d)x)$$

$$-\frac{1}{\tau(t)^2}\otimes_B(\gamma(d)t)(\gamma(d')x) + \frac{1}{\tau(t)^2}\otimes_B(\gamma(d)t)(\gamma(d')x)$$

$$-\frac{1}{\tau(t)^2}\otimes(\gamma(d)\gamma(d')t)x + \frac{1}{\tau(t)^2}\otimes_B(\gamma(d')\gamma(d)t)x$$

$$+\frac{1}{\tau(t)}\otimes_B(\gamma(d)\gamma(d')x) - \frac{1}{\tau(t)}\otimes_B(\gamma(d')\gamma(d)x)$$

$$= \bar{\gamma}(d)\left(-\frac{1}{\tau(t)^2}\otimes_B(\gamma(d')t)x + \frac{1}{\tau(t)}\otimes_B\gamma(d')x\right)$$

$$-\bar{\gamma}(d')\left(-\frac{1}{\tau(t)^2}\otimes_B(\gamma(d)t)x + \frac{1}{\tau(t)}\otimes_B\gamma(d)x\right)$$

$$= [\bar{\gamma}(d), \bar{\gamma}(d')]\left(\frac{1}{\tau(t)}\otimes_B x\right).$$

The map $\bar{\gamma}$ thus gives $B[T^{-1}]\otimes_B J(L, B)$ the structure of an L-algebra.

To complete the proof of the isomorphism $B[T^{-1}]\otimes_B J(L, B)\cong J(L, B[T^{-1}])$ we need only demonstrate the universality of $B[T^{-1}]\otimes_B J(L, B)$. Thus suppose that Λ is a $B[T^{-1}]$-algebra by a map $\lambda : B[T^{-1}]\to\Lambda$ and suppose Λ is an L-algebra with structure map $\varphi : L\to L(\Lambda/k)$. Λ is also a B-algebra by the map $\mu=\lambda\circ\tau$. Thus, there exists a unique map $j_{\varphi,\mu} : J(L, B)\to\Lambda$ such that for $\alpha\in E(L)$ and $b\in B$

$$j_{\varphi,\mu}((\varepsilon(\gamma)\alpha)b)=(\varepsilon(\varphi)\alpha)\mu(b).$$

The map $j_{\varphi,\mu}$ extends uniquely to a map $\bar{j} : B[T^{-1}]\otimes_B J(L, B)\to\Lambda$ if we demand that $\bar{j}(\tau(t)^{-1})=\lambda(\tau(t)^{-1})$. Suppose that $d\in L$, $b\in B$ and $t\in T$. Then

$$\begin{aligned}\bar{j}(\bar{\gamma}(d)(\tau(b)\tau(t)^{-1})) &=\bar{j}\left(\bar{\gamma}(d)\left(\frac{1}{\tau(t)}\otimes_B b\right)\right)\\ &=\bar{j}\left(-\frac{1}{\tau(t)^2}\otimes(\gamma(d)t)b+\frac{1}{\tau(t)}\otimes\gamma(d)b\right)\\ &=-\lambda\left(\frac{1}{\tau(t)^2}\right)j_{\varphi,\mu}((\gamma(d)t)b)+\lambda\left(\frac{1}{\tau(t)}\right)j_{\varphi,\mu}(\gamma(d)b)\\ &=-\lambda\left(\frac{1}{\tau(t)^2}\right)(\varphi(d)\mu(t))\mu(b)+\lambda\left(\frac{1}{\tau(t)}\right)\varphi(d)\mu(b)\\ &=\varphi(d)(\lambda(\tau(b)\tau(t)^{-1})).\end{aligned}$$

Now suppose that $d_1, \ldots, d_r\in L$. Then

$$\begin{aligned}\bar{j}((\varepsilon(\gamma)d_r\ldots d_1)(\tau(b)\tau(t)^{-1})) &=\bar{j}(\bar{\gamma}(d_r)\ldots\bar{\gamma}(d_1)(\tau(b)\tau(t)^{-1}))\\ &=\bar{j}(\sum_{j=1}^{r}\sum_{\alpha\in G(j,r)}(\prod_{u=1}^{j}\bar{\gamma}(d_{\alpha(u)}))\tau(b)(\prod_{v=1}^{r-j}\bar{\gamma}(d_{c\alpha(v)}))\tau(t)^{-1})\\ &=\sum_{j=1}^{r}\sum_{\alpha\in G(j,r)}\bar{j}((\prod_{u=1}^{j}\bar{\gamma}(d_{\alpha(u)}))\tau(b))\bar{j}((\prod_{v=1}^{r-j}\bar{\gamma}(d_{c\alpha(v)}))\tau(t)^{-1}).\end{aligned}$$

But $(\prod_{u=1}^{j}\bar{\gamma}(d_{\alpha(u)}))\tau(b)=(\tau\otimes\mathrm{id})(\varepsilon(\gamma)(\prod_{u=1}^{j}d_{\alpha(u)}))b$, hence

$$\begin{aligned}\bar{j}((\prod_{u=1}^{j}\bar{\gamma}(d_{\alpha(u)}))\tau(b)) &=\bar{j}((\tau\otimes\mathrm{id})(\varepsilon(\gamma)(\prod_{u=1}^{j}d_{\alpha(u)}))b)\\ &=j_{\varphi,\mu}(\varepsilon(\gamma)(\prod_{u=1}^{j}d_{\alpha(u)})b)\\ &=(\varepsilon(\varphi)(\prod_{u=1}^{j}\gamma(d_{\alpha(u)})))\mu(b).\end{aligned}$$

Also

$$\bar{j}((\prod_{v=1}^{r-j}\bar{\gamma}(d_{c\alpha(v)}))\tau(t)^{-1})=\bar{j}\left(\sum_{w=1}^{r-j}(-1)^w\frac{w!}{\tau(t)^{w+1}}\sum_{\beta\in G(w,r-j)}\prod_{\sigma=1}^{w}(\bar{\gamma}(d^*_{\beta(\sigma)})\tau(t))\right)$$

where $d_x^* = d_{c\alpha(x)}$. Thus

$$\bar{j}((\prod_{v=1}^{r-j} \bar{\gamma}(d_{c\alpha(v)}))\tau(t)^{-1}) = \sum_{w=1}^{r-j} (-1)^w \frac{w!}{\lambda(\tau(t))^{w+1}} \sum_{\beta \in G(w, r-j)} \bar{j}(\prod_{\sigma=1}^{w} (\bar{\gamma}(d^*_{\beta(\sigma)})\tau(t)))$$

$$= \sum_{w=1}^{r-j} (-1)^w \frac{w!}{\lambda(\tau(t))^{w+1}} \sum_{\beta \in G(w, r-j)} \prod_{\sigma=1}^{w} ((\varepsilon(\varphi)\gamma(d^*_{\beta(\sigma)}))\mu(t))$$

$$= (\varepsilon(\varphi)(\prod_{v=1}^{r-j} \gamma(d_{c\alpha(v)})))(\mu(t))^{-1}.$$

Thus

$$\bar{j}((\varepsilon(\gamma)(d_r \ldots d_1))(\tau(b)\tau(t)^{-1})) = \sum_{j=1}^{r} \sum_{\alpha \in F(j,r)} (\varepsilon(\varphi)(\prod_{u=1}^{j} \gamma(d_{\alpha(u)}))\mu(b))(\varepsilon(\varphi)\prod_{v=1}^{r-j} \gamma(d_{c\alpha(v)}))(\mu(t))^{-1}$$

$$= (\varepsilon(\varphi)(\gamma(d_r) \ldots \gamma(d_1)))\lambda(\tau(b)\tau(t)^{-1}).$$

Now assume that S is a multiplicatively closed set in A. Assume that $\sigma : A \to A[S^{-1}]$ is the canonical map. Lemma (2.7) shows that there exists an $A[S^{-1}]$-k-map

$$\varphi : A[S^{-1}] \otimes_A L(J(L, B/k)) \to L(A[S^{-1}] \otimes_A J(L, B)/k).$$

Note that

$$A[S^{-1}] \otimes_A J(L, B) \cong B[\lambda(S)^{-1}] \otimes_B J(L, B)$$
$$\cong J(L, B[\lambda(S)^{-1}])$$
$$\cong J(L, A[S^{-1}] \otimes_A B).$$

The map $\gamma : L \to L(J(L, B)/k)$ gives $J(L, B)$ the structure of an L-algebra. We can extend γ to a map $\bar{\gamma} : A[S^{-1}] \otimes_A L \to A[S^{-1}] \otimes_A L(J(L, B)/k)$ by setting

$$\bar{\gamma}(\sigma(s)^{-1} \otimes_A d) = \sigma(s)^{-1} \otimes_A \gamma(d).$$

Lemma (2.18) shows that this is an $A[S^{-1}]$-k-map. The composition $\varphi \circ \bar{\gamma} = \gamma^*$ gives $A[S^{-1}] \otimes_A J(L, B)$ the structure of an $(A[S^{-1}] \otimes_A L)$-algebra.

We can now complete the proof of the theorem if we show that $A[S^{-1}] \otimes_A J(L, B)$ together with the map γ^* satisfies the universal conditions for $J(A[S^{-1}] \otimes_A L, A[S^{-1}] \otimes_A B)$. Suppose that Λ is an $(A[S^{-1}] \otimes_A B)$-algebra which is an $(A[S^{-1}] \otimes_A L)$-algebra. Assume $\lambda : A[S^{-1}] \otimes_A B \to \Lambda$ and suppose $\rho : A[S^{-1}] \otimes_A L \to L(\Lambda/k)$. The map $\rho' : d \mapsto \rho(1 \otimes d)$ gives Λ the structure of an L-algebra. Thus there exists a uniquely determined map $\bar{j} = j_{\rho', \lambda} : J(L, A[S^{-1}] \otimes_A B) \to \Lambda$ such that for $s \in S$, $b \in B$, $\alpha \in E(L)$

$$j_{\rho', \lambda}((\varepsilon(\gamma')\alpha)(\tau(s)^{-1} \otimes_A b)) = (\varepsilon(\rho)\alpha)\lambda(\tau(s)^{-1} \otimes_A b),$$

where $\gamma' : L \to L(J(L, A[S^{-1}] \otimes_A B)/k)$ is constructed from γ by the procedures of the first half of this theorem. Thus one sees easily that γ' and γ^* coincide (under the isomorphism $A[S^{-1}] \otimes_A J(L, B) \cong J(L, A[S^{-1}] \otimes_A B)$). We must show that if

$$x \in E(A[S^{-1}] \otimes_A L) \quad \text{then} \quad \bar{j}((\varepsilon(\gamma^*)x)(\tau(s)^{-1} \otimes_A b)) = (\varepsilon(\rho)x)\lambda(\tau(s)^{-1} \otimes_A b).$$

Since $E(A[S^{-1}]\otimes_A L)\cong A[S^{-1}]\otimes_A E(L)$ we may suppose that $x=\tau(s')^{-1}\otimes_A d_r\dots d_1$ where $d_i\in L$. Thus

$$\begin{aligned}\bar{j}((\varepsilon(\gamma^*)(\tau(s')^{-1}\otimes d_r\dots d_1))(\tau(s)^{-1}\otimes_A b)) &= \lambda(\tau(s')^{-1}\otimes_A 1)\bar{j}((\varepsilon(\gamma^*)(1\otimes d_r\dots d_1))(\tau(s)^{-1}\otimes_A b))\\ &= \lambda(\tau(s')^{-1}\otimes_A 1)(\varepsilon(\rho')(d_r\dots d_1))\lambda(\tau(s)^{-1}\otimes_A b)\\ &= (\varepsilon(\rho)(\tau(s')^{-1}\otimes d_r\dots d_1))\lambda(\tau(s)^{-1}\otimes_A b).\end{aligned}$$

The result which follows connects the $J(L, B)$ to the Thom-Boardman spaces of [2].

Theorem **(2.24)**. — *Assume that* A, B *and* k *are rings such that* A *is a* k*-algebra and such that* B *is an* A*-algebra. Assume that* C *is a finitely generated* k*-Lie algebra such that* A *and* B *are* C*-algebras and such that* $L(A/k)$ *is generated as an* A*-module by the sub-*k*-Lie algebra* C. *Further suppose that* $D(B/A)$ *is a free* B*-module with a basis* $db_1, \dots, db_r$ *and with dual basis* $\delta_1, \dots, \delta_r$ $(\in \mathrm{Hom}_B(D(B/A), B))$. *Assume that* $L(B/k)\cong B.L(A/k)\oplus L(B/A)$ *where the sum is an* A-k*-Lie algebra direct sum. Assume that* $C(b_i)=0$. *Suppose that* D *is a* k*-Lie algebra and suppose that* A *is a* D*-algebra with a structure map* $\theta : D\to C$. *Then*

$$J(A\otimes_k D, B)\cong \mathbf{S}_B[E(B\otimes_k D)\otimes_B D(B/A)].$$

Further, the map $\gamma : A\otimes_k D\to L(J(A\otimes_k D, B))$ *carries* $c\in D$ *to the derivation which sends* $b\in B$ *to* $\theta(c)b+\sum_j \delta_j(b)(c\otimes db_j)$. *Further* $\gamma(c)$ *carries* $e\otimes\varphi\in E(D)\otimes D(B/A)$ *to* $c.e\otimes\varphi$.

Proof. — We shall first construct the map

$$\gamma : A\otimes_k D\to L(\mathbf{S}_B[E(B\otimes_k D)\otimes_B D(B/A)]/k).$$

Set $db_j=\theta_j$. Thus if $\delta\in A\otimes_k D$ then $\delta=\sum_j a_j c_j$ where $c_j\in D$. If $b\in B$, then we set $\gamma(\delta)(b)=\sum a_j\theta(c_j)b+\sum_{j,t} a_j\delta_j(b)(c_j\otimes_B\theta_t)$. If $b, b'\in B$, then

$$\begin{aligned}\gamma(\delta)(bb') &= \sum_j a_j\theta(c_j)(bb')+\sum_{j,t} a_j\delta_t(bb')(c_j\otimes_B\theta_t)\\ &= \sum_j ba_j\theta(c_j)(b')+\sum_j b'a_j\theta(c_j)(b)\\ &\quad+\sum_{j,t} a_j b'\delta_t(b)(c_j\otimes_B\theta_t)\\ &\quad+\sum_{j,t} a_j b\delta_t(b')(c_j\otimes_B\theta_t)\\ &= b\gamma(\delta)(b')+b'\gamma(\delta)b.\end{aligned}$$

Thus γ carries $A\otimes_k D$ into $\mathrm{Hom}_B(D(B/k), \mathbf{S}_B[E(B\otimes_k D)\otimes_B D(B/A)])$. Further if $\delta\in A\otimes_k D$, then we may define $\gamma(\delta) : E(B\otimes_k D)\otimes_B D(B/A)\to \mathbf{S}_B[E(B\otimes_k D)\otimes_B D(B/A)]$ by setting $\gamma(\delta)(\varphi\otimes\theta_i)=\delta\varphi\otimes\theta_i$ and then extend by B-linearity. Thus if $b\in B$ and $\varphi\otimes\theta_i\in E(B\otimes_k D)\otimes_B D(B/A)$, we have

$$\gamma(\delta)(b.\varphi\otimes\theta_i)=\delta(b\varphi)\otimes\theta_i=\theta(\delta)(b)\varphi\otimes\theta_i+b.\delta\varphi\otimes\theta_i.$$

Lemma (2.16) shows that there exists a derivation from $S_B[E(B\otimes_k D)\otimes_B D(B/A)]$ to itself determined by $\gamma(\delta)$. We shall also denote this derivation by $\gamma(\delta)$. The map $\gamma : A\otimes_k D \to L(S_B[E(B\otimes_k D)\otimes_B D(B/A)]/k)$ is clearly A-linear. If $d, d'\in D$, then

$$\begin{aligned}\gamma[d, d'](b) &= \theta[d, d']b + \sum_t \delta_t(b)[d, d']\otimes\theta_t \\ &= [\theta(d), \theta(d')]b + \sum_t \delta_t(b)(dd'-d'd)\otimes\theta_t.\end{aligned}$$

On the other hand

$$\begin{aligned}[\gamma(d), \gamma(d')](b) &= \gamma(d)\gamma(d')b - \gamma(d')\gamma(d)b \\ &= \gamma(d)(\theta(d')b + \sum_t \delta_t(b)d'\otimes\theta_t) - \gamma(d')(\theta(d)b + \sum_t \delta_t(b)d\otimes\theta_t) \\ &= \theta(d)\theta(d')b + \sum_t \delta_t(\theta(d')b)d\otimes\theta_t + \sum_t \theta(d)\delta_t(b)d'\otimes\theta_t + \sum_t \delta_t(b)dd'\otimes\theta_t \\ &\quad - \theta(d')\theta(d)b - \sum_t \delta_t(\theta(d)b)d'\otimes\theta_t - \sum_t \theta(d')(\delta_t(b))d\otimes\theta_t - \sum_t \delta_t(b)d'd\otimes\theta_t \\ &= [\theta(d), \theta(d')]b + \sum_t \delta_t(b)[d, d']\otimes\theta_t + \\ &\quad \sum_t (\delta_t(\theta_t(d')b)d + \theta(d)\delta_t(b)d' - \delta_t(\theta(d)b)d' - \theta(d')(\delta_t(b))d)\otimes\theta_t.\end{aligned}$$

But $\theta(d')\in C$, and by assumption δ_t commutes with C. Thus the last sum vanishes, and $\gamma[d, d']b = [\gamma(d), \gamma(d')]b$. If $a, a'\in A$, then

$$\begin{aligned}\gamma[ad, a'd'] &= \gamma(a(da')d' - a'(d'a)d + aa'[d, d']) \\ &= a(da')\gamma(d') - a'(d'a)\gamma(d) + aa'[\gamma(d), \gamma(d')] \\ &= [\gamma(ad), \gamma(a'd')]\end{aligned}$$

on elements of B. If $\alpha\in E(B\otimes_k D)\otimes_B D(B/A)$ and $\alpha = \beta\otimes\delta$, then for d and d' in D it follows that $\gamma[d, d'](\beta\otimes\delta) = dd'\beta\otimes\delta - d'd\beta\otimes\delta = [\gamma(d), \gamma(d')]\beta\otimes\delta$. Because of this one can easily see that γ is an A-linear k-Lie algebra map.

Suppose now that T is a B-algebra by $\lambda : B\to T$ and suppose $\varphi : A\otimes_k D \to L(T/k)$ gives T the structure of an $(A\otimes_k D)$-algebra. If $d\in D$ and $\theta(d)\in C$, then

$$\gamma(d)(b_j) = \delta_j(b_j)(d\otimes db_j).$$

Therefore, the jet section condition (see Theorem (2.21)) shows that if j is a jet section from $S_B[E(B\otimes_k D)\otimes_B D(B/A)]$ we must have

$$\begin{aligned}j(d_{\alpha(1)}\dots d_{\alpha(r)}\otimes db_j) &= j(d_{\alpha(1)}\dots d_{\alpha(r-1)}(\gamma(d_{\alpha(r)})(b_j))) \\ &= (\varepsilon(\theta)(d_{\alpha(1)}\dots d_{\alpha(r-1)}))(\theta(d_{\alpha(r)})(\lambda b_j)).\end{aligned}$$

Thus the map from $E(D)\otimes_k\langle db_1, \dots, db_r\rangle$ is uniquely determined. There is then a unique extension to a B-linear map from $B\otimes_k E(D)\otimes_B D(B/A)$ to T. This extends uniquely to a B-algebra map from $S_B[E(B\otimes_k D)\otimes_B D(B/A)]$ to T.

This completes the proof.

§ 3. ITERATED SINGULARITIES

Suppose that M is a finitely generated module over a ring A. We shall denote by $\mathfrak{f}_j(M)$ the j^{th} Fitting invariant of M (see [8] or [12]). If $0 \to K \to F \to M \to 0$ is an exact sequence of R-modules where F is free with a basis $f_1, \ldots, f_n$ and if K is generated by the elements $\sum_j a_{ij} f_j$, then the r-th Fitting invariant of M is the ideal in R generated by the $(n-r)\times(n-r)$ subdeterminants of the matrix (a_{ij}).

Definition (**3.1**). — Assume that A and k are scalar rings and suppose $\Delta : A \to M$ is a k-derivation from A to a finitely generated A-module M. Suppose that j is a non-negative integer. Then we set

(i) $\mathfrak{Z}_j(M) = \mathfrak{f}_{j-1}(M)$

(ii) $M(j) = \dfrac{A}{\mathfrak{Z}_j(M)} \otimes_A \dfrac{M}{(\Delta \mathfrak{Z}_j(M))}$

If $i(1), \ldots, i(r)$ is a sequence of integers, then we define a sequence of modules and ideals as follows:

(iii) $\mathfrak{Z}(i(1)) = \mathfrak{Z}_{i(1)}(M)$

(iv) $M(i(1)) = \dfrac{A}{\mathfrak{Z}_{i(1)}(M)} \otimes_A \dfrac{M}{(\Delta \mathfrak{Z}_{i(1)}(M))}$

(v) if $M(i(1), \ldots, i(r-1))$ and $\mathfrak{Z}(i(1), \ldots, i(r-1))$ have been defined, then

$$\mathfrak{Z}(i(1), \ldots, 1(r)) = (\mathfrak{Z}(i(1), \ldots, i(r-1)), \mathfrak{Z}_{i(r)}(M(i(1), \ldots, i(r-1)))$$

and

$$M(i(1), \ldots, i(r)) = \frac{A}{\mathfrak{Z}(i(1), \ldots, i(r))} \otimes_A \frac{M(i(1), \ldots, i(r-1))}{\Delta \mathfrak{Z}(i(1), \ldots, i(r-1))}$$

We now follow the Boardman-Thom construction and give the following definition (see [2]).

Definition (**3.2**). — If A, k, M and Δ are as in definition (3.1) and if $i(1), \ldots, i(r)$ is a finite sequence of non-negative integers, then we define a subschema of Spec(A) which we shall denote by $\Sigma(\Delta; i(1), \ldots, i(r))$ as follows:

(i) the support of $\Sigma(\Delta; i(1), \ldots, i(r))$, that is the underlying topological space, consists of all $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathfrak{p} \supset \mathfrak{Z}(i(1), \ldots, i(r))$ and such that

$$\mathfrak{p} \not\supset \mathfrak{Z}(i(1), \ldots, i(j-1), i(j)+1) \quad \text{for} \quad 1 \leq j \leq r.$$

(ii) The structure sheaf of $\Sigma(\Delta; i(1), \ldots, i(r))$ is the sheaf of rings induced on $\Sigma(\Delta; i(1), \ldots, i(r))$ by the ring (i.e. A-algebra) $A/\mathfrak{Z}(i(1), \ldots, i(r))$.

Lemma (**3.3**). — *Suppose* M *is a finitely generated* A*-module and assume* $\Delta : A \to M$ *is a k-derivation from* A *to* M. *The set* $\Sigma(i(1), \ldots, i(r))$ *is empty unless* $i(1) \geq i(2) \geq \ldots \geq i(r)$.

Proof. — First assume that i and j are two non-negative integers. We shall show that $\Sigma(i,j)$ is empty unless $i \geq j$. Thus assume that $i<j$. If $\mathfrak{p} \supset \mathfrak{Z}(i,j)$, then

$$\mathfrak{p} \supset \mathfrak{f}_{j-1}(\mathrm{M}(i)) = \mathfrak{f}_{j-1}\left(\frac{\mathrm{A}}{\mathfrak{Z}(i)} \otimes_{\mathrm{A}} \frac{\mathrm{M}}{(\Delta\mathfrak{Z}(i))}\right).$$

Suppose that $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{M} \to 0$ is an exact sequence of A-modules where F is free with a base $f_1, \ldots, f_n$. Then there is an exact sequence of A-modules

$$\text{(E)} \qquad 0 \to (\mathrm{K}^*, \mathfrak{f}_{i-1}(\mathrm{M}).\mathrm{F}) \to \mathrm{E} \to \frac{\mathrm{A}}{\mathfrak{Z}(i)} \otimes_{\mathrm{A}} \frac{\mathrm{M}}{(\Delta\mathfrak{Z}(i))} \to 0$$

where K^* contains K and a set of elements in F which map to generators of $(\Delta\mathfrak{Z}(i))$. It is then clear that $\mathfrak{f}_{j-1}\left(\frac{\mathrm{A}}{\mathfrak{Z}(i)} \otimes_{\mathrm{A}} \frac{\mathrm{M}}{(\Delta\mathfrak{Z}(i))}\right)$ contains $\mathfrak{Z}(j) = \mathfrak{Z}_j(\mathrm{M})$.

Therefore if $\mathfrak{p}$ is a prime which contains $\Sigma(i,j)$, then $\mathfrak{p}$ also contains $\mathfrak{Z}(j)$. However, if $j>i$, then $\mathfrak{Z}(j) = \mathfrak{f}_{j-1}(\mathrm{M}) \supset \mathfrak{f}_i(\mathrm{M}) = \mathfrak{Z}(i+1)$. Thus $\mathfrak{p} \notin \Sigma(i,j)$. If in a sequence $i(1), \ldots, i(r)$ there is a j such that $i(j) < i(j+1)$, then we can apply the previous argument to $\mathrm{M}(i(1), \ldots, i(j-1))$.

Lemma **(3.4)**. — *Assume that* A *is a local ring (not necessarily noetherian) and suppose that* M *is a finitely generated* A*-module such that* $\mathfrak{f}_{r+1}(\mathrm{M}) = (1)$. *Then there is an exact sequence* $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{M} \to 0$ *where* F *is a free module of rank* $r+1$.

Proof. — Suppose that F is a free module with rank $\mathrm{F} = \dim \frac{\mathrm{M}}{\mathfrak{m}\mathrm{M}} = t$, where $\mathfrak{m}$ is the maximal ideal of A. We can then construct an exact sequence $0 \to \mathrm{K} \to \mathrm{F} \to \mathrm{M} \to 0$ where a basis for F maps to a minimal system of generators for M. Thus

$$\frac{\mathrm{A}}{\mathfrak{m}} \otimes_{\mathrm{A}} \mathrm{K} \to \frac{\mathrm{A}}{\mathfrak{m}} \otimes_{\mathrm{A}} \mathrm{F} \to \frac{\mathrm{M}}{\mathfrak{m}\mathrm{M}} \to 0$$

is exact. Because $\frac{\mathrm{A}}{\mathfrak{m}} \otimes_{\mathrm{A}} \mathfrak{f}_j(\mathrm{M}) = \mathfrak{f}_j\left(\frac{\mathrm{A}}{\mathfrak{m}} \otimes_{\mathrm{A}} \mathrm{M}\right)$, we know that $\mathfrak{f}_{r+1}\left(\frac{\mathrm{A}}{\mathfrak{m}} \otimes_{\mathrm{A}} \mathrm{M}\right) = (1)$; therefore $t = \dim \frac{\mathrm{M}}{\mathfrak{m}\mathrm{M}} \leq r+1$.

Theorem **(3.5)**. — *Assume* A *is a noetherian catenary local ring which is a k-algebra for some ring k. Suppose that* M *is a finitely presented* A*-module given by an exact sequence* $\mathrm{F}^s \to \mathrm{F}^r \xrightarrow{\rho} \mathrm{M} \to 0$ *where* F^s *and* F^r *are respectively free modules of ranks s and r. Suppose* $\Delta : \mathrm{A} \to \mathrm{M}$ *is a k-derivation. If* $i<j$ *are two non-negative integers, the closure in* Spec(A) *of the locally closed set* $\Sigma(i,j)$ *has rank* $\leq \text{rank } \mathfrak{f}_{i-1}(\mathrm{M}) + j(si-i+j)$.

Proof. — The definition of $\Sigma(i,j)$ shows that the closure of $\Sigma(i,j)$ is the closed set of the ideal $\mathfrak{Z}(i,j) = \mathfrak{f}_{j-1}\left(\frac{\mathrm{A}}{\mathfrak{f}_{i-1}(\mathrm{M})} \otimes_{\mathrm{A}} \frac{\mathrm{M}}{(\Delta\mathfrak{f}_{i-1}(\mathrm{M}))}\right)$. It follows from the fact that the sequence (E) of Lemma (3.3) is exact that $\mathfrak{Z}(i,j)$ contains a power of the ideal $\mathfrak{f}_{i-1}(M)$. Therefore the rank of $\mathfrak{Z}(i,j)$ is the rank of $\mathfrak{f}_{i-1}(M)$ plus the rank of $\dfrac{\mathfrak{Z}(i,j)+\mathfrak{f}_{i-1}(M)}{\mathfrak{f}_{i-1}(M)}$.

Now suppose $\mathfrak{p}\in\Sigma(i,j)$. Then $\mathfrak{p}\not\supset\mathfrak{f}_i(M)$, and it will suffice to compute the rank of

$$A_\mathfrak{p}.\mathfrak{Z}(i,j)=\mathfrak{f}_{j-1}\left((A_\mathfrak{p}/(A_\mathfrak{p}.\mathfrak{f}_{i-1}(M)))\otimes_A\frac{M}{\Delta\mathfrak{f}_{i-1}(M)}\right).$$

It is well known that Δ has an extension to a derivation from $A_\mathfrak{p}$ to $A_\mathfrak{p}\otimes_A M$ (see [14]). We shall denote this extension also by Δ. Then $\Delta(as^{-1})=s^{-1}\Delta a-s^{-2}a\Delta s$. We claim that

$$(A_\mathfrak{p}\otimes_A M)/(\mathfrak{f}_{i-1}(M)M,\ \Delta\mathfrak{f}_{i-1}(M))=(A_\mathfrak{p}/\mathfrak{f}_{i-1}(A_\mathfrak{p}\otimes_A M))\otimes_{A_\mathfrak{p}}\frac{A_\mathfrak{p}\otimes_A M}{(\Delta\mathfrak{f}_{i-1}(A_\mathfrak{p}\otimes_A M))}.$$

To demonstrate this we need only show that the submodules $(\mathfrak{f}_{i-1}(M).M,\ \Delta\mathfrak{f}_{i-1}(M))$ and $(A_\mathfrak{p}.\mathfrak{f}_{i-1}(M)).M,\ \Delta\mathfrak{f}_{i-1}(A_\mathfrak{p}\otimes_A(M))$ of $A_\mathfrak{p}\otimes_A M$ are equal. Note that

$$\mathfrak{f}_{i-1}(A_\mathfrak{p}\otimes_A M)=A_\mathfrak{p}.\mathfrak{f}_{i-1}(M),$$

thus if $y\in\mathfrak{f}_{i-1}(A_\mathfrak{p}\otimes_A M)$ we may write $y=s^{-1}x$ where $x\in\mathfrak{f}_{i-1}(M)$ and $s\notin\mathfrak{p}$. Thus $\Delta y=s^{-1}\Delta x-s^{-2}x\Delta s\in(A_\mathfrak{p}\mathfrak{f}_{i-1}(M)M,\ A_\mathfrak{p}.\Delta\mathfrak{f}_{i-1}(M))$. We have shown, therefore, that $(\mathfrak{f}_{i-1}(M).M,\ \Delta\mathfrak{f}_{i-1}(M))\supseteq(A_\mathfrak{p}\mathfrak{f}_{i-1}(M).M,\ \Delta\mathfrak{f}_{i-1}(A_\mathfrak{p}\otimes_A M))$. The opposite inequality is obvious. The isomorphism shows that we may suppose that $\mathfrak{f}_i(M)=(1)$.

If $\mathfrak{f}_i(M)=(1)$, then lemma (3.4) shows that there exists an exact sequence $0\to K\to F\xrightarrow{\varepsilon}M\to 0$ where $\operatorname{rank} F=i$. Further $\rho: F^r\to M$ is surjective, thus there exists a map $\sigma: F^r\to F$ such that $\varepsilon\circ\sigma=\rho$. If $f_1,\ldots,f_r$ is a basis for F^r, then the $\rho(f_i)$ generate M and we may choose a minimal system of generators for M from among the $\rho(f_i)$. This minimal system of generators for M consists of a sequence of elements which are the images under ε of a basis for F. Thus we may suppose σ is surjective. From this it follows that K is generated by s elements. Suppose that $e_1,\ldots,e_i$ is a basis for F and suppose that K is generated by $k_u=\sum_{t=1}^{i}a(ut)e_t$. The ideal $\mathfrak{f}_{i-1}(M)$ is then generated by the elements $a(uv)$, with $1\leq u\leq s$ and $1\leq v\leq i$. The remarks of the first paragraph of this proof show that we may assume that $\mathfrak{f}_{i-1}(M)=(0)$, thus that M is free of rank $=i$ (see [12]) and that $\Delta\mathfrak{f}_{i-1}(M)$ is a submodule generated by at most $s.i$ elements. Thus $\dfrac{\mathfrak{Z}(i,j)+\mathfrak{f}_{i-1}(M)}{\mathfrak{f}_{i-1}(M)}$ is an ideal generated by the $i-j+1$ by $i-j+1$ subdeterminants of an $i\times si$ matrix of elements of A. It follows (see [3] or [7]) that the rank of $\dfrac{\mathfrak{Z}(i,j)+\mathfrak{f}_{i-1}(M)}{\mathfrak{f}_{i-1}(M)}$ is $\leq(i-(i-j+1)+1).(si-(i-j+1)+1)=j(si-i+j)$.

§ 4. GENERIC SINGULARITIES

Suppose now that A, k are rings with A a k-algebra and suppose L is an A-k-Lie algebra. Assume that B is an A-algebra and that T is a B-algebra which is also an L-algebra by a map $\theta : L \to L(T/k)$. We shall suppose that K is a B-submodule of $D(B/k)$. The map θ induces a T-homomorphism

$$\widetilde{\theta} : \operatorname{Hom}_T(L(T/k), T) \to \operatorname{Hom}_T(T \otimes_A L, T).$$

Further since $L(T/k) = \operatorname{Hom}_T(D(T/k), T)$, there exists a map θ^- induced by θ from $D(T/k)$ to $\operatorname{Hom}_T(T \otimes_A L, T)$. Because T is a B-algebra there is a map

$$h : D(B/k) \to D(T/k).$$

The map $\theta^- h$ carries $D(B/k)$ to $\operatorname{Hom}_T(T \otimes_A L, T)$. We set

$$D(T \otimes_A L/K) = \operatorname{Hom}_T(T \otimes_A L, T)/(\theta^- h(K)).$$

Denote by ρ_T the quotient map from $\operatorname{Hom}_T(T \otimes_A L, T)$ to $D(T \otimes_A L/K)$ and suppose that $d_T : T \to D(T/k)$ is the canonical derivation of T. The map

$$\rho_T \circ \theta^- \circ d_T : T \to D(T \otimes_A L/K)$$

is a k-derivation from T to $D(T \otimes_A L/K)$ which we shall denote by $\Delta_L(T/K)$.

Definition **(4.1)**. — $\Sigma(T, L/K; i(1), \ldots, i(r)) = \Sigma(i(1), \ldots, i(r))$ of the definition (3.2) with M and Δ the module $D(T \otimes_A L/K)$ and the derivation $\Delta_L(T/K)$ respectively.

Suppose that A and B are regular local rings which contain a characteristic zero field k and which have maximal ideals generated by systems of parameters $(x_1, \ldots, x_v)$ and $(y_1, \ldots, y_w)$ respectively. If $k = A/(x_1, \ldots, x_v) = B/(y_1, \ldots, y_w)$ and if $D^1(A/k)$ and $D^1(B/k)$ are finitely generated, then the Lie algebra of A is generated by the commuting partial derivatives $\partial/\partial x_i$ and Theorem (2.24) shows that the jet space using the Lie algebra of A is the schema of a polynomial ring in elements $z_{j,\sigma}$ where $1 \leqq j \leqq w$ and σ is a sequence of nonnegative integers $(\sigma(1), \ldots, \sigma(v))$. Furthermore

$$(\partial/\partial x_t) z_{j,\sigma} = z_{j,\sigma'}$$

where $\sigma' = (\sigma(1), \ldots, \sigma(t)+1, \sigma(t+1), \ldots, \sigma(v))$. In this case a catalogue of the Boardman-Thom singularities and the singularities defined above is given in [2] on page 32.

Lemma **(4.2)**. — *Suppose that* S, T, A, B *and* k *are rings so that* B *is an* A-*algebra, such that* S *and* T *are* B-*algebras and such that* A *is a* k-*algebra. Assume that* L *is an* A-k-*Lie algebra which is a finitely generated projective* A-*module. If* S *and* T *are* L-*algebras and if* K *is a* B-*submodule of* $D(B/k)$, *then for an* L-*algebra map* $\varphi : T \to S$ *which is also a* B-*algebra map there exists an isomorphism* $\hat{\varphi} : S \otimes_T D(T \otimes_A L/K) \to D(S \otimes_A L/K)$.

Proof. — We shall consider first the case in which L is a free A-module. Suppose that $\lambda : B \to T$ and $\mu : B \to S$ are the B-algebra structure maps for T and S, and assume $\theta : L \to L(T/k)$ and $\nu : L \to L(S/k)$ are the L-structure maps for S and T respectively. Because φ is an L-algebra map we have that for $t \in T$ and $f \in L$, $\varphi(\theta(f))(t) = \nu(f)(\varphi(t))$. Suppose $h_X : D(B/k) \to D(X/k)$ for $X = S, T$. Then

$$S \otimes_T D(T \otimes_A L/K) = S \otimes_T (\mathrm{Hom}_T(T \otimes_A L, T)) / (h_T \theta^- (K)).$$

Denote by σ the natural S map from $S \otimes_T \mathrm{Hom}_T(T \otimes_A L, T)$ to $\mathrm{Hom}_S(S \otimes_A L, S)$. This map is an isomorphism since L is finitely generated and free. Therefore

$$S \otimes_T D(T \otimes_A L/K) \cong \mathrm{Hom}_S(S \otimes_A L, S) / (\sigma(S \otimes_T h_T \theta^- (K))).$$

If $b \in B$, then for $f \in L$ and $1 \otimes_A f \in S \otimes_A L$

$$\begin{aligned} \sigma(1 \otimes_T \theta^- h_T d_B(b))(1 \otimes_A f) &= (\theta^- h_T d_T(\lambda b))(f) \\ &= (\theta(f))(\lambda b) = \nu(f)(\mu b) \\ &= (\nu^- h_S d_S(\mu b))(1 \otimes_S f). \end{aligned}$$

It follows that for each element $y \in D(B/k)$, $\sigma(1 \otimes_T \theta^- h_T(y)) = \nu^- h_S(y)$, and therefore $\sigma(S \otimes_T \theta^- h_T(K)) = \nu^- h_S(K)$. Note that we have also shown that

$$1 \otimes \Delta_L(T/K) = \Delta_L(S/K).$$

We turn now to the case in which L is finitely generated and projective. Suppose $\mathfrak{p}$ is a prime in the ring A, suppose $T \to S$ is an A-algebra map and assume that M is an A-module. Then

$$A_\mathfrak{p} \otimes_A (S \otimes_T M) \cong (A_\mathfrak{p} \otimes_A S) \otimes_{A_\mathfrak{p} \otimes_A T} (A_\mathfrak{p} \otimes_A M).$$

To see this note first that $A_\mathfrak{p} \otimes_A M \cong A_\mathfrak{p} \otimes_A T \otimes_T M$. We now consider the change of rings given by the A-map $T \to A_\mathfrak{p} \otimes_A T$. Then $A_\mathfrak{p} \otimes_A S$ is both a T-and an $A_\mathfrak{p} \otimes_A T$-module. Thus (see [4], p. 116) $(A_\mathfrak{p} \otimes_A S) \otimes_{A_\mathfrak{p} \otimes_A T} (A_\mathfrak{p} \otimes_A M) \cong (A_\mathfrak{p} \otimes_A S) \otimes_T M$. Therefore $A_\mathfrak{p} \otimes_A (S \otimes_T D(T \otimes_A L/K)) \cong (A_\mathfrak{p} \otimes_A S) \otimes_{A_\mathfrak{p} \otimes_A T} (A_\mathfrak{p} \otimes_A D(T \otimes_A L/K))$. If

$$A_\mathfrak{p} \otimes_A D(T \otimes_A L/K) \cong D(A_\mathfrak{p} \otimes_A T \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \otimes_A L)/K),$$

then $A_\mathfrak{p} \otimes_A L$ is free and we would have that

$$\begin{aligned} (A_\mathfrak{p} \otimes_A S) \otimes_T D(T \otimes_A L/K) &= (A_\mathfrak{p} \otimes_A S) \otimes_{A_\mathfrak{p} \otimes_A T} D((A_\mathfrak{p} \otimes_A T) \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \otimes_A L)/K) \\ &\cong D((A_\mathfrak{p} \otimes_A S) \otimes_{A_\mathfrak{p}} (A_\mathfrak{p} \otimes_A L)/K) \cong D(A_\mathfrak{p} \otimes_A S \otimes_A L/K). \end{aligned}$$

Further, if $A_\mathfrak{p} \otimes_A D(S \otimes_A L/K) \cong D(A_\mathfrak{p} \otimes_A S \otimes_A L/K)$, then the proof of the assertion would follow. Thus we need only show that for a projective Lie algebra L,

$$A_\mathfrak{p} \otimes_A D(T \otimes_A L/K) \cong D(A_\mathfrak{p} \otimes_A T \otimes_A L/K).$$

However

$$\begin{aligned} A_\mathfrak{p} \otimes_A D(T \otimes_A L/K) &\cong A_\mathfrak{p} \otimes_A ((\mathrm{Hom}_T(T \otimes_A L, T)) / (h_T \theta^- (K))) \\ &\cong (A_\mathfrak{p} \otimes_A \mathrm{Hom}_T(T \otimes_A L, T)) / (A_\mathfrak{p} \otimes_A h_T \theta^- (K)). \end{aligned}$$

But $\mathrm{Hom}_T(T\otimes_A L, T)\cong \mathrm{Hom}_A(L, T)$. Thus

$$A_p\otimes_A \mathrm{Hom}_T(T\otimes_A L, T)\cong A_p\otimes_A \mathrm{Hom}_A(L, T)\cong \mathrm{Hom}_{A_p}(A_p\otimes_A L, A_p\otimes_A T)$$

because L is finitely generated projective (and hence finitely related) and A_p is A-flat (see [9]). Thus $A_p\otimes_A \mathrm{Hom}_T(T\otimes_A L, T)\cong \mathrm{Hom}_{A_p\otimes_A T}(A_p\otimes_A T\otimes_A L, A_p\otimes_A T)$. We shall denote this map from $A_p\otimes_A \mathrm{Hom}_T(T\otimes_A L, T)$ to $\mathrm{Hom}_{A_p\otimes_A T}(A_p\otimes_A T\otimes_A L, A_p\otimes_A T)$ by ζ. We give $A_p\otimes_A T$ the structure of an $(A_p\otimes_A L)$-algebra by the map $1\otimes_A \theta$ and we can give $A_p\otimes_A T$ a B-algebra structure by the map $1\otimes_A h_T$. In order to complete the proof of the lemma it will suffice to show that the image by ζ of $A_p\otimes_A \theta^- h_T(K)$ is the image of $(1\otimes_A\theta)^-(1\otimes_A h_T)$. Thus suppose $b\in B$, assume $f\in L$ and suppose ε denotes $1\otimes_A 1$ in $A_p\otimes_A T$. Then

$$\zeta(1\otimes_A \theta^- h_T d_B(b))(\varepsilon\otimes_A f)=1\otimes_A(\theta^- h_T d_B(b))(f)=1\otimes_A\theta^- d_T(\lambda b)(f)$$
$$=1\otimes_A\theta(f)(\lambda b)=(1\otimes_A\theta(f))((1\otimes_A h_T)(b)).$$

This completes the proof.

The following is a more convenient restatement of the above using the universality of $J(L, B)$.

Theorem **(4.3)**. — *Suppose* A, B *and* k *are scalar rings such that* A *is a* k*-algebra and such that* B *is an* A*-algebra. Suppose that* L *is a finitely generated projective* A-k*-Lie algebra. Assume that* $J=J(L, B)$ *is the jet space of* B, *and suppose that* K *is a* B*-submodule of* $D(B/k)$. *If* $i(1)\geq\ldots\geq i(r)$ *is any sequence of non-negative integers, then the* $\Sigma(J, L/K; i(1), \ldots, i(r))$ *are subschemas of* Spec(J). *Further if* T *is a* B*-algebra which is also an* L*-algebra by structure maps* $\theta: L\to L(T/k)$ *and* $\lambda: B\to T$, *then the jet section* $j_{\theta,\lambda}: J(L, B)\to T$ *induces a morphism* $J(\theta,\lambda): \mathrm{Spec}(T)\to\mathrm{Spec}(J)$ *such that*

$$\Sigma(J, L/K; i(1), \ldots, i(r))\times_{J(L,B)} T=\Sigma(T, L/K; i(1), \ldots, i(r)).$$

(Because of this the subschemas $\Sigma(J, L/K; i(1), \ldots, i(r))$ will be called the *generic singularities* of type $i(1), \ldots, i(r)$ for L/K on B).

We shall continue by computing the codimension of the $\Sigma(J, L/K; i(1), \ldots, i(r))$. We shall use the concept of codimension introduced by Berthelot in [1]. Thus if $i: Y\to X$ is an immersion of a schema Y defined in an open set U of X by an Ideal $\mathscr{J}$ of $\mathcal{O}_X$, then the immersion is said to be regular if for each $x\in i(Y)$ there exists an exact sequence $E\xrightarrow{u}\mathscr{J}.\mathcal{O}_{X,x}\to 0$ where E is a free finitely generated $\mathcal{O}_{X,x}$-module such that the Koszul complex of E is acyclic. Then, if i is a regular immersion, the $\mathcal{O}_Y$-Module $\mathscr{J}/\mathscr{J}^2$ is locally free of finite type. The rank of $\mathscr{J}/\mathscr{J}^2$ is called the codimension of Y in X. The Module $\mathscr{J}/\mathscr{J}^2$ will be denoted by $\mathscr{N}_{X/Y}$.

Definition **(4.4)**. — Assume A and B are k-algebras, suppose L is an A-k-Lie algebra and assume B is an A-algebra. If K is a B-submodule of $D(B/k)$, we shall denote by $\Sigma(J(L, B)/K; i(1), \ldots, i(s))$ the subschema $\Sigma(J(L, B), L/K; i(1), \ldots, i(s))$ of $\operatorname{Spec}(J(B, B))$.

Theorem (4.5). — *Assume that* A *and* B *are k-algebras such that* $\lambda: A \to B$ *is a k-algebra map. Assume that k is a field. Suppose that* $D(B/A)$ *is a finitely generated projective* B*-module. Suppose that for each prime* $\mathfrak{p}$ *in* $\operatorname{Spec}(B)$ *the following conditions are satisfied*:

(i) *There is a finitely generated k-algebra* $C(\mathfrak{p})$ *such that* $L(A/k)$ *is generated by the subalgebra* $C(\mathfrak{p})$.

(ii) $D(B_\mathfrak{p}/k)$ *is a free* $B_\mathfrak{p}$*-module with a basis* $db_1(\mathfrak{p}), \ldots, db_r(\mathfrak{p})$ *with dual basis* $\delta_i(\mathfrak{p}) \in \operatorname{Hom}_{B_\mathfrak{p}}(D(B_\mathfrak{p}/A), B_\mathfrak{p})$.

(iii) $L(B_\mathfrak{p}/k) \cong B_\mathfrak{p} L(A/k) \oplus L(B_\mathfrak{p}/A)$.

(iv) $C(\mathfrak{p}).b_i = 0$.

Suppose that L *is an* A-*k*-*Lie algebra such that* $A_{\mathfrak{p}\cap A} \otimes_A L = A_{\mathfrak{p}\cap A} \otimes_A D(\mathfrak{p})$, *where* $D(\mathfrak{p})$ *is a free k-Lie algebra with a map* $\theta: D(\mathfrak{p}) \to C(\mathfrak{p})$. *Suppose that* K *is a* B*-submodule of* $D(B/k)$ *such that for each* $\mathfrak{p} \in \operatorname{Spec}(B/k)$, $B_\mathfrak{p} \otimes_B K$ *is freely generated by* $db_1(\mathfrak{p}), \ldots, db_t(\mathfrak{p})$, $t \leq r$. *Then the map* $i: \Sigma(J(L, B)/K; i(1), \ldots, i(s)) \to \operatorname{Spec}(J(L, B))$ *is a regular immersion. Further, if* B *is an integral domain, then the* $\Sigma(J(L, B)/K; i(1), \ldots, i(s))$ *are irreducible reduced subschemas of* $\operatorname{Spec}(J(L, B))$.

Proof. — Suppose that $\mathfrak{P}$ is a prime of $\operatorname{Spec}(J(L, B))$ which is in

$$\Sigma(J(L, B)/K; i(1), \ldots, i(s)).$$

Set $\mathfrak{p} = \mathfrak{P} \cap B$. Then if we set $S = B - \mathfrak{p}$ we have that

$$B_\mathfrak{p} \otimes_B J(L, B) = J(L, B)_S \subset J(L, B)_\mathfrak{P}.$$

Further $J(L, B)_\mathfrak{P} = (B_\mathfrak{p} \otimes_B J(L, B))_T$ for a multiplicatively closed set T in $B_\mathfrak{p} \otimes_B J(L, B)$. Thus we may suppose that B is a local ring. Set $\mathfrak{p}' = A \cap \mathfrak{p}$. Then

$$J(L, B) = J(L, B)_{A-\mathfrak{p}'} = A_{\mathfrak{p}'} \otimes_A J(L, B) = J(A_{\mathfrak{p}'} \otimes_A L, B).$$

Thus we may suppose that both A and B are local such that the map $\lambda: A \to B$ is a local map (for this terminology see [9]). It follows from the hypotheses that

$$J(L, B) \cong \mathbf{S}_B[E(L) \otimes_B D(B/A)],$$

and that the symmetric algebra is a polynomial ring over B which has as independent generators a k-basis for $E(D(\mathfrak{p})) \otimes_k F$, where F denotes the k-submodule of $D(B_\mathfrak{p}/A)$ generated by the elements $db_i(\mathfrak{p})$. Thus we may suppose that $\mathfrak{P}$ is a prime dominating the maximal ideal of B in the polynomial ring $\mathbf{S}_B[E(L) \otimes_B D(B/A)]$.

The Poincaré-Birkhoff-Witt theorem (see [11]) shows that we may choose as a k-basis for $E(D(\mathfrak{p}))$ the monomials $d_1^{\alpha(1)} \ldots d_u^{\alpha(u)}$ where $d_1, \ldots, d_u$ is a basis for $D(\mathfrak{p})$. We shall say that the monomial $d_1^{\alpha(1)} \ldots d_u^{\alpha(u)}$ has filtration degree $\alpha(1) + \ldots + \alpha(u)$. The algebra $J(L, B)$ is therefore a polynomial ring in indeterminates $Z(\alpha, t)$ where $\alpha = (\alpha(1), \ldots, \alpha(u))$ is a sequence of non-negative integers and $1 \leq t \leq r$. Further

$d_i(d_1^{\alpha(1)} \ldots d_u^{\alpha(u)}) = d_1^{\alpha(1)} \ldots d_i^{\alpha(i)+1} \ldots d_u^{\alpha(u)} + \theta$, where θ is a linear combination of monomials of degree at most $|\alpha| = \alpha(1) + \ldots + \alpha(u)$. Thus we have that

$$d_i Z(\alpha, v) = Z(\alpha(1), \ldots, \alpha(i)+1, \alpha(i+1), \ldots, \alpha(u), v) + \Theta$$

where Θ is a linear combination of $Z(\beta, \nu)$ such that $\sum_j \beta(j) \leq \sum_j \alpha(j)$. Now suppose P′ is a polynomial in $B[Z(\alpha, j)]$ such that $Z(\beta, v)$ does not occur in P′ and such that $|\beta|$ is the highest filtration degree which occurs in P. If $d_1^{\alpha(1)} \ldots d_u^{\alpha(u)}$ is a standard monomial of $E(D/\mathfrak{p})$, and if $\gamma(i) = \alpha(i) + \beta(i)$, then $Z(\gamma, v)$ does not occur in $d_1^{\alpha(1)} \ldots d_u^{\alpha(u)} P'$. To see this, note first that $|\gamma|$ is the highest filtration degree which can occur in $d_1^{\alpha(1)} \ldots d_u^{\alpha(u)} P'$. If $Z(\varphi, v)$ occurs in P, then $d_1^{\alpha(1)} \ldots d_u^{\alpha(u)} Z(\varphi, v) = Z(\alpha + \varphi, v) + \theta$ where $(\alpha + \varphi)(i) = \alpha(i) + \varphi(i)$ and Θ is a polynomial of $B[Z(\gamma, j)]$ in which the largest filtration degree occurring is $|\alpha + \varphi| - 1$.

In what follows, we shall say that a collection of elements $\zeta(j) \in J(L, B)_{\mathfrak{p}}$ are *admissible of type s* if they satisfy the following conditions:

(i) $\zeta(j) = \dfrac{P'}{Q}$ where $P', Q \in B[Z(\alpha, \nu)]$ and the highest filtration degree occurring in either P′ or Q is s.

(ii) $\zeta(j) = \sum_w A(j, w) Z(\alpha(w), \nu(w)) + \Theta(j)$ where $|\alpha(w)| = s$, $1 \leq j, w \leq z$ (some integer).

(iii) $\det(A(j, w))$ is a unit in $J(L, B)_{\mathfrak{P}}$.

(iv) $A(j, w) = \dfrac{U(j, w)}{V(j, w)}$, $\Theta(j) = \dfrac{R(j)}{S(j)}$ where $U(j, w)$, $V(j, w)$, $R(j)$ and $S(j)$ are polynomials in $J(L, B)$, and there is no occurrence of $Z(\alpha(w), \nu(w))$ in $U(j, z)$, $V(j, z)$, $R(j)$ or $S(j)$ for all w, j and z.

Set $\delta_i(j)$ = Kronecker delta. If $\zeta(j)$, $1 \leq j \leq z$ are admissible of type s, then the elements $d_i\zeta(j) = \zeta_i'(j)$ are admissible of type $s+1$. Thus

$$\begin{aligned} d_i\zeta(j) &= \sum_w A(j, w)(Z(\alpha(w) + \delta_i, \nu(w)) + \Phi(w)) + \sum_w d_i A(j, w) + d_i \Theta(j) \\ &= \sum_w A(j, w) Z(\alpha(w) + \delta_i, \nu(w)) + \Theta'(j). \end{aligned}$$

Further there is no occurrence of $Z(\alpha(w) + \delta_i, \nu(w))$ in $A(j, w)$ or $\Theta'(j)$, because there is no occurrence of $Z(\alpha(w), \nu(w))$ in $U(j, z)$, $V(j, z)$, $R(j)$ or $S(j)$.

We shall prove the assertion by using an induction on the integer s of $\Sigma(J(L, B)/K; i(1), \ldots, i(s))$. We claim that for each $s \geq 0$, there exists a ring $B(s)$, an ideal $\mathfrak{S}(s)$ and a collection of elements $\zeta(j)$, $1 \leq j \leq \sigma(s)$, which are admissible of type $s+1$ such that:

(i) $B(s) \subset J(L, B)_{\mathfrak{P}}$ and $\zeta(j) \in J(L, B)_{\mathfrak{P}}$.

(ii) $B(s)[\zeta(j), Z(\alpha, \nu) : \alpha(1) + \ldots + \alpha(u) > s+1] = J_s$ is a polynomial ring in the independent indeterminates $\zeta(j)$, $Z(\alpha, \nu)$.

(iii) $J(L, B)_{\mathfrak{P}}$ is a localization of J_s.

(iv) There exists a module $M(s)$ over J_s and an exact sequence

$$0 \to R(s) \to J_s \otimes_J \mathrm{Hom}_J(J \otimes_A L, J) \to M(s) \to 0$$

where $R(s)$ is given by a matrix $N(s)$ satisfying the following conditions:

a) $N(s)$ has τ rows with entries which are either 0, 1 or one of the $\zeta(j)$.

b) There are $i(s+1)$ columns of $N(s)$ which consist of entries which are distinct $\zeta(j)$'s.

c) The remaining $u-i(s+1)$ columns consist of 0's and 1's with only one non-zero entry in each column, and such that the columns are linearly independent.

(v) $J(L, B)_{\mathfrak{P}} \otimes_{J_s} \dfrac{J_s}{\mathfrak{S}(s)} \otimes_{J_s} M(s) = D(J \otimes_A L/K)(i(1), \ldots, i(s))$ and

$$(\mathfrak{Z}(i(1), \ldots, i(s-1), \mathfrak{S}(s).))J(L, B)_{\mathfrak{P}} = \mathfrak{Z}(i(1), \ldots, i(s+1)).$$

Furthermore, the ideal $\mathfrak{S}(s)$ is generated by algebraically independent indeterminates of J_s. Finally if $\zeta(j) \in \mathfrak{S}(s)$ and $\zeta(j)$ has type $\leq s-1$, then $d_v \zeta(j) \in \mathfrak{S}(s)$ for each j.

Suppose $h : D(B/k) \to D(J/k)$ is the map induced by the B-algebra structure on J. Because the Lie algebra L has an A-basis which consists of the elements $d_1, \ldots, d_u$, the J-module $\mathrm{Hom}_T(J \otimes_A L, J)$ has a basis consisting of the elements $d_1^*, \ldots, d_u^*$ dual to the d_i. The submodule K of $D(B/k)$ is generated by $db_1(\mathfrak{p}), \ldots, db_t(\mathfrak{p})$. Further

$$\theta^- h(db_j(\mathfrak{p}))(d_v) = \theta(d_v)b_j(\mathfrak{p}) + \sum_i \delta_i(\mathfrak{p})(db_j)Z(\delta_v, j)$$
$$= Z(\delta_v, j) + \theta(d_v)b_j(\mathfrak{p}).$$

Therefore $D(J \otimes_A L/K)$ has the form $\mathrm{Hom}_J(J \otimes_A L, J)/R$ where R is spanned by the elements $\sum_{v=1}^{u} (Z(\delta_v, j) + \theta(d_v)b_j(\mathfrak{p}))d_v^*$, $1 \leq j \leq t$. Set $\zeta(v, j) = Z(\delta_v, j) + \theta(d_v)b_j(\mathfrak{p})$. The $\zeta(v, j)$ are admissible indeterminates of type 1. Because $\mathfrak{P} \in \Sigma(J(L, B)/K; i(1), \ldots, i(s))$, $\mathfrak{f}_{i(1)}(D(J \otimes_A L/K)) \not\subset \mathfrak{P}$, thus one of the $(u-i(1)) \times (u-i(1))$ subdeterminants of the matrix of $\zeta(v, j)$ (the presentation of R) is a unit in $J(L, B)_{\mathfrak{P}}$. We may without loss of generality assume that $\xi = \det(\zeta(v, j))$ is a unit for $1 \leq v, j \leq u-i(1)$. Denote by $X(\alpha(0), \ldots, \alpha(u-i(1)))$ the determinant of the $\alpha(0)$ to $\alpha(u-i(1))$ columns of the $(u-i(1)) \times u$ matrix with v-th row vectors $(\zeta(v, 1), \ldots, \zeta(v, u))$. Set $G = \mathrm{Hom}_J(J \otimes_A L, J)$. The submodule of $J[\xi^{-1}] \otimes_J G$ generated by the elements $\kappa(j) = \sum_{v=1}^{u} \zeta(v, j)d_v^*$ is the same as the submodule generated by the elements

$$\kappa'(j) = \sum_{v=1}^{u} (X(1 \ldots (j-1)\ v(j+1) \ldots (t-i(1)))/\xi)d_v^* \qquad 1 \leq j \leq u-i(1)$$

(see [10] for example). Set $u' = u - i(1)$. Then

$$X(1 \ldots (j-1)v(j+1) \ldots u') = \pm \det(\zeta(i, l))$$

where $1 \leq i \leq u'$ and $l \in \{1, \ldots, (j-1), \nu, (j+1), \ldots, u'\}$, thus

$$(*) \qquad X(1 \ldots (j-1)v(j+1) \ldots u') = \pm \sum_{w=1}^{u'} \Delta(w, j)\zeta(w, v)$$

where $\Delta(w, j)$ is the cofactor of $\zeta(w, j)$ in ξ. Because ξ is a unit in $J(L, B)_{\mathfrak{P}}$, the matrix formed of the $\Delta(w, j)$ also has a determinant which is a unit in $J(L, B)_{\mathfrak{P}}$. Set

$$z(j, v) = X(1 \ldots (j-1)v(j+1) \ldots u')/\xi,$$

where $s+1 \leq v \leq u$. The elements $z(j, v)$ are rational functions with denominator ξ and numerator of the form $(*)$. Thus $z(j, v) = \sum_w A(j, v; \mu, w)(Z(\delta_\mu, w) + \theta(d_\mu)b_w(\mathfrak{p}))$, and $A(j, v; \mu, w)$ and ξ are polynomials in which $Z(\delta_\mu, w)$ does not occur when $u' \leq \mu \leq u$ and $1 \leq j \leq t$. Furthermore, because for a fixed v the matrix of the $\Delta(w, j)$ has a determinant which is a unit in $J(L, B)_{\mathfrak{P}}$, it follows that the $z(j, v)$ are admissible of type 1. The module $D(J \otimes_A L/K)[\xi^{-1}]$ over $J(L, B)[\xi^{-1}]$ has the form F/T where F is free on a basis $d_1^*, \ldots, d_u^*$ and T is spanned by the elements $\kappa'(j)$, $1 \leq j \leq u'$, and the elements $\kappa(j)$, $u'+1 \leq j \leq t$ (or the images of these elements in $J[\xi^{-1}] \otimes_J \mathrm{Hom}_J(J \otimes_A L, J)$). Note that

$$\kappa'(j) = d_j^* + \sum_{v=u'+1}^{u} z(j, v) d_v^*.$$

Thus $D(J \otimes_A L/K)[\xi^{-1}] \cong F'/T'$ where F' is free on $d_{u'+1}^*, \ldots, d_u^*$ and T' is generated by the elements $\sum_{v=u'+1}^{u} z(j, v) d_v^*$, $1 \leq j \leq u'$, and the elements

$$\sum_{v=u'+1}^{u} \left(\zeta(v, j) - \sum_{w=1}^{u'} \zeta(w, j) z(w, v)\right) d_v^* \qquad \text{for} \qquad u' \leq j \leq t.$$

Set $z(jv) = \zeta(v, j) - \sum_{w=1}^{u'} \zeta(w, j) z(wv)$. It is clear that the full collection of $z(jv)$, $1 \leq j \leq t$, $u'+1 \leq v \leq u$ is an admissible collection of elements of type 1. We let $B(1)'$ denote the sub-B-module of $J(L, B)$ generated by the $Z(\delta_u, w)$ which occur in the polynomial ξ. Set $B(1) = B(1)'[\xi^{-1}]$ and set

$$\begin{aligned} J_1 &= B(1)[Z(\alpha, \nu) : (\alpha, \nu) \neq (\delta_\mu, w), 1 \leq \mu \leq u'] \\ &= B(1)[z(j\nu), Z(\alpha, \nu) : |\alpha| \geq 2]. \end{aligned}$$

Set $M(0) = D(J \otimes_A L/K)[\xi^{-1}]$. Thus we have an exact sequence

$$0 \to R(0) \to F' \to M(0) \to 0$$

where $R(0) = T'$, and thus T' is presented by a $t \times i(1)$ matrix with entries $z(j\nu)$. Finally $\mathfrak{S}(0) = \mathfrak{f}_{i(1)-1}(M(0)) = (z(j\nu))$.

We now suppose that we have constructed $M(s)$, $B(s)$, $\mathfrak{S}(s)$ and thus J_s. By assumption

$$\frac{J(L, B)_{\mathfrak{P}}}{\mathfrak{Z}(i(1), \ldots, i(s))} \otimes_{J_s} M(s) = D(J \otimes_A L/K)(i(1), \ldots, i(s))_{\mathfrak{P}}.$$

Further

$$\mathfrak{Z}(i(1), \ldots, i(s+1)) = \left(\mathfrak{Z}(i(1), \ldots, i(s)), \mathfrak{f}_{i(s+1)-1} \cdot \left(\frac{J(L, B)_{\mathfrak{P}}}{\mathfrak{Z}(i(1), \ldots, i(s))} \otimes_{J_s} M(s)\right)\right).$$

We may suppose without loss of generality that the matrix which represents the kernel of the map from $J_s \otimes_J \text{Hom}(J \otimes_A L, J)$ to $M(s)$ is of the form $\begin{pmatrix} I & X \\ 0 & Y \end{pmatrix}$ where I is an $(u-i(s+1)) \times (u-i(s+1))$ identity matrix and the matrix $\begin{pmatrix} X \\ Y \end{pmatrix}$ is a $\tau \times i(s+1)$ matrix consisting of independent admissible elements of type $s+1$. Denote the algebraically independent entries of $N(s)$ (i.e. the entries in X and Y) by $z(\nu w)$

$$1 \leq \nu \leq \tau, \quad u-i(s+1)+1 \leq w \leq u.$$

Set $u' = u - i(s+1)$. The entries $z(\nu w)$ are admissible of type $s+1$ and thus the $d_i z(\nu w)$ are admissible of type $s+2$.

Because the matrix $\begin{pmatrix} X \\ Y \end{pmatrix}$ is $\tau \times i(s+1)$, it follows that $\mathfrak{Z}(i(1), \ldots, i(s+1))$ is generated by $\mathfrak{Z}(i(1), \ldots, i(s))$ and the elements $z(vw)$.

The module $D^* = D(J \otimes_A L/K)(i(1), \ldots, i(s+1))_{\mathfrak{P}}$ is then determined by an exact sequence

$$0 \to R'(s+1) \to \frac{J(L, B)_{\mathfrak{P}}}{\mathfrak{Z}(i(1), \ldots, i(s+1))} \otimes_J \text{Hom}_J(J \otimes_A L, J) \to D^* \to 0$$

where $R'(s+1)$ is presented (as an $J(L, B)_{\mathfrak{P}}/\mathfrak{Z}(i(1), \ldots, i(s+1))$-module) by the matrix of $R(s)$ and the image in $\dfrac{J(L, B)_{\mathfrak{P}}}{\mathfrak{Z}(i(1), \ldots, i(s+1))} \otimes_J \text{Hom}_J(J \otimes_A L, J)$ of the elements $\sum_{v=1}^{u} d_v z(jw) d_v^*$ for all j and w, and the images of the elements Δz for z running over a set of generators for $\mathfrak{Z}(i(1), \ldots, i(s+1))$. However, if a z of $\mathfrak{Z}(i(1), \ldots, i(s))$ is written in the form $\sum_j d_j(z) d_j^*$, then we know that $d_j z \in \mathfrak{Z}(i(1), \ldots, i(s+1))$. Thus a presentation for $R'(s+1)$ is given by the matrix of $R(s)$ to which we adjoin the rows $(d_1 z(jw), \ldots, d_\nu z(jw))$. Furthermore, the construction of

$$J_{\mathfrak{p}} \cdot \mathfrak{S}(s) = \mathfrak{Z}(i(1), \ldots, i(s+1))$$

shows that the elements $z(jw)$ are in the ideal $J_{\mathfrak{P}} \mathfrak{S}(s)$. Therefore

$$D^* = \frac{J(L, B)_{\mathfrak{P}}}{\mathfrak{Z}(i(1), \ldots, i(s+1))} \otimes_J \frac{M(s)}{(\sum_\nu d_\nu z(jw) d_\nu^*)}.$$

Because $\mathfrak{P} \in \Sigma(i(1), \ldots, i(s+2))$, it follows that $\mathfrak{f}_{i(s+2)}(D^*) \not\subset \mathfrak{P}$. However

$$\mathfrak{Z}(i(1), \ldots, i(s+1)) \subset \mathfrak{P},$$

thus the matrix with rows $(d_1 z(jw) \dots d_\nu z(jw))$ must contain a $(u-i(s+2)) \times (u-i(s+2))$ subdeterminant which is a unit in $J(B, L)_{\mathfrak{P}}$. We may now proceed precisely as we did in the case $s=0$, to construct the $B(s+1)$, $M(s+1)$, J_{s+1}, $\mathfrak{S}(s+1)$. In particular $B(s+1)$ will have the form $B(s)[z'(w)][\xi'^{-1}]$ for some admissible elements z' in $J(L, B)_{\mathfrak{P}}$. The ideal $\mathfrak{S}(s+1)$ will be generated by admissible elements in $B(s+1)[Z(\rho, j), z(\gamma)]$, where the $z(\gamma)$ are admissible elements chosen in the constructions of the rings J_w, $w \leq s$. Each of the ideals $\mathfrak{S}(\nu)$ has an acyclic Koszul complex over J_ν. Therefore, $J_{\nu+1}(\mathfrak{S}_\nu)$ has an acyclic Koszul complex. Furthermore, the ideal $\mathfrak{S}_{\nu+1}$ is generated by admissible elements which have type $\nu+2$, and $J_{\nu+1}(\mathfrak{S}_\nu)$ is generated by elements of type at most $\nu+1$. In particular $J_{\nu+1}/J_{\nu+1}(\mathfrak{S}_\nu)$ is a polynomial ring in a set of indeterminates among which the generators for $(J_{\nu+1}(S_\nu), \mathfrak{S}_{\nu+1})/J_{\nu+1}(\mathfrak{S}_\nu)$ occur. Thus the Koszul complex of $(J_{\nu+1}(\mathfrak{S}_\nu), \mathfrak{S}_{\nu+1})/J_{\nu+1}(\mathfrak{S}_\nu)$ is acyclic. However, the property of being a regular immersion is transitive (see [1]) thus the Koszul complex of $(J_{\nu+1}(\mathfrak{S}_\nu), \mathfrak{S}_{\nu+1})$ is acyclic, and hence the Koszul complex of $(J_{\nu+1}(\mathfrak{S}_1), \dots, J_{\nu+1}(\mathfrak{S}_\nu), \mathfrak{S}_{\nu+1})$ is acyclic. Since $J(L, B)_{\mathfrak{P}}$ is a localization of $J_{\nu+1}$, we can conclude that $\mathfrak{Z}(i(1), \dots, i(s))$ has an acyclic Koszul complex. This completes the proof.

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