PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

L. A. COBURN
R. G. DOUGLAS

C^* algebras of operators on a half-space

Publications mathématiques de l'I.H.É.S., tome 40 (1971), p. 59-68
<http://www.numdam.org/item?id=PMIHES_1971__40__59_0>

C*-ALGEBRAS OF OPERATORS ON A HALF-SPACE I

by L. A. COBURN (1) and R. G. DOUGLAS (2)

1. Introduction.

An equation of the form

$$(*) \qquad f(x)+\int_0^\infty k(x-t)f(t)\,dt=g(x)$$

is known as a Wiener-Hopf equation. Such equations occur not only in diverse areas of mathematics but are also important in applications. An immense literature is devoted to all aspects of their study (cf. [13]) and much is known concerning their solution for various function space domains and classes of kernels. A related equation, namely the discrete analogue of $(*)$

$$(**) \qquad \sum_{k=0}^{\infty} c_{n-k}a_k=b_n$$

gives rise to what are called Toeplitz operators.

If $\{c_k\}_{k=-\infty}^{\infty}$ are the Fourier coefficients of a continuous function on the unit circle, then the equation $(**)$ is pretty well understood. In [7] a study of this equation was made, based on the structure of the C*-algebra generated by the corresponding class of Toeplitz operators [3]. In [4] results of a similar study were announced for the class of Wiener-Hopf operators with " almost periodic symbol " on the reals based on the structure of the analogous C*-algebra and a real-valued topological index.

In this paper, we give the proofs of some of those results, using the more general setting of locally compact abelian groups. A semi-group is chosen in the dual group and the half-space taken to be the functions whose Fourier transforms are supported on this semigroup. The compression to the half-space of the multiplication operators with almost periodic symbol are the Wiener-Hopf operators and the C*-algebra which they generate is the principal object of study in this paper. Our main results are that this C*-algebra modulo its commutator ideal is isomorphic to the algebra of almost periodic functions on the group, that the commutant of this C*-algebra is the weak closure of its center, and that this center is generated by the " translations " which leave the semigroup invariant. Thus the question of the irreducibility of this C*-algebra depends entirely on the geometry of the semigroup.

---

(1) Research supported under N.S.F. GP-9654.
(2) Alfred P. Sloan, Fellow and research supported by a grant from N.S.F.

Under certain additional assumptions one can complete the analysis of this C^*-algebra by considering certain of its representations and introducing both an analytical and a topological index on the algebra. This will be done in the paper [5] which includes D. G. Schaeffer and I. M. Singer as co-authors.

2. Preliminaries.

Let G be a locally compact abelian group with dual group $\hat{G}$ and let Σ denote a fixed sub-semigroup of $\hat{G}$ which is, in addition, a Borel subset of $\hat{G}$. Let μ and $\hat{\mu}$ denote the normalized Haar measures on G and $\hat{G}$, respectively, and let $L^2(G)$ and $L^2(\hat{G})$ denote the usual Hilbert spaces of square-integrable complex functions. The Plancherel transform $\mathscr{F}$ is an isometry from $L^2(G)$ onto $L^2(\hat{G})$. We denote by $H^2(\Sigma)$ the subspace of $L^2(G)$ consisting of the functions f for which $\mathscr{F}f$ is in $L^2(\Sigma)$, that is, for which $\mathscr{F}f$ is supported on Σ. Let $P(=P_\Sigma)$ denote the (orthogonal) projection of $L^2(G)$ onto $H^2(\Sigma)$ and $\hat{P}(=\hat{P}_\Sigma)$ the projection of $L^2(\hat{G})$ onto $L^2(\Sigma)$.

If φ is a bounded measurable function on G, then there are (bounded, linear) operators L_φ and W_φ defined on $L^2(G)$ and $H^2(\Sigma)$ by

$$L_\varphi f = \varphi f \quad \text{and} \quad W_\varphi f = P(\varphi f) = PL_\varphi f,$$

respectively. We shall call W_φ a Wiener-Hopf operator. Note that $W_{\varphi+\psi} = W_\varphi + W_\psi$ and $W_\varphi^* = W_{\bar{\varphi}}$. If φ is a character γ in Σ, then the projection is unnecessary and $W_\gamma f = \gamma f$. In this paper we want to analyze the C^*-algebra $\mathscr{A} = \mathscr{A}(G, \Sigma)$ generated by $\{W_\gamma : \gamma \in \Sigma\}$.

In case G is the circle group $\mathbf{T}$, $\hat{G}$ is the integers $\mathbf{Z}$, and Σ is the semigroup $\mathbf{Z}^+$ of non-negative integers, the problem reduces to the study of the C^*-algebra generated by the simple unilateral shift. The latter algebra $\mathscr{A}(\mathbf{T}, \mathbf{Z}^+)$ was studied in [3], where it was shown that $\mathscr{A}(\mathbf{T}, \mathbf{Z}^+)$ contains the space of compact operators $\mathfrak{K}$ as a two-sided ideal and that the quotient algebra $\mathscr{A}(\mathbf{T}, \mathbf{Z}^+)/\mathfrak{K}$ is naturally $*$-isometrically isomorphic to the algebra $C(\mathbf{T})$ of continuous complex functions on $\mathbf{T}$ with the supremum norm. In fact, it is shown that every operator in $\mathscr{A}(\mathbf{T}, \mathbf{Z}^+)$ has a unique representation of the form

$$W_\varphi + K,$$

where K is compact and φ is in $C(\mathbf{T})$. Further, the natural isomorphism between $\mathscr{A}(\mathbf{T}, \mathbf{Z}^+)/\mathfrak{K}$ and $C(\mathbf{T})$ is given by

$$W_\varphi + \mathfrak{K} \leftrightarrow \varphi.$$

We show under a rather mild restriction on the semigroup Σ, that analogous results hold for $\mathscr{A}(G, \Sigma)$. In particular, *we assume throughout the paper that* $\hat{\mu}(\Sigma) > 0$ *and that* Σ *generates* $\hat{G}$.

The following consequence of this assumption will be used in the sequel. For E a subset of $\hat{G}$ and σ in $\hat{G}$ we define $\sigma E = \{\sigma\tau : \tau \in E\}$.

Lemma. — For each compact subset E *of* $\hat{G}$ *there exists* σ *in* Σ *such that* $\sigma E \subset \Sigma$.

Proof. — Since $\hat{\mu}(\Sigma) > 0$ it follows ([14], Cor. (20.17)) that $\Sigma \cdot \Sigma$ contains an open set U. Since Σ is a semigroup, $\Sigma \cdot \Sigma \subset \Sigma$ and hence Σ contains U. If γ is in U, then e (the identity in $\hat{G}$) is in $\gamma^{-1}U$. Let E be a compact subset of $\hat{G}$. Since $\{\kappa\gamma^{-1}U : \kappa \in E\}$ is an open cover of E, there are $\kappa_1, \ldots, \kappa_n$ in E such that $E \subset \bigcup_{i=1}^{n} \kappa_i \gamma^{-1} U$. Since Σ generates $\hat{G}$, there exist $\{\sigma_i\}_{i=1}^{n}$ and $\{\tau_i\}_{i=1}^{n}$ in Σ such that $\sigma_i \tau_i^{-1} = \kappa_i$ for $i = 1, 2, \ldots, n$. If we set $\tau = \tau_1 \tau_2 \ldots \tau_n \gamma$, then $\tau E \subset \bigcup_{i=1}^{n} \sigma_i \tau \tau_i^{-1} \gamma^{-1} U \subset \Sigma$ since each $\tau\tau_i^{-1}\gamma^{-1}$ is in Σ.

If we don't assume $\hat{\mu}(\Sigma) > 0$, then the space $H^2(\Sigma)$ consists of just the zero function. If $\hat{G}$ is connected, then the assumption that Σ generate $\hat{G}$ is not needed since Σ would already have to generate $\hat{G}$. If G is discrete, then the lemma implies any semigroup Σ satisfying $\hat{\mu}(\Sigma) > 0$ and generating $\hat{G}$ would have to be all of $\hat{G}$ so that this case is trivial.

Let $\mathfrak{C}(G, \Sigma)$ denote the closed two-sided ideal in $\mathscr{A}(G, \Sigma)$ generated by the commutators of elements in $\mathscr{A}(G, \Sigma)$. We shall prove that every element in $\mathscr{A}(G, \Sigma)$ has a unique representation of the form

$$W_\varphi + K,$$

where K is in $\mathfrak{C}$ and φ is an almost periodic function on G. Moreover, the mapping

$$W_\varphi + \mathfrak{C} \leftrightarrow \varphi$$

induces a $*$-isometrical isomorphism between $\mathscr{A}(G, \Sigma)/\mathfrak{C}(G, \Sigma)$ and the algebra AP(G) of almost periodic functions on G with the supremum norm. Since for compact G we have $AP(G) = C(G)$ these results generalize those of [3].

We adopt [12] as our reference for facts about topological groups and harmonic analysis and [6] for facts about C*-algebras. Although all groups under consideration are abelian, we shall adhere to multiplicative notation throughout.

3. The Spectral Inclusion Theorem.

For the case of Toeplitz or Wiener-Hopf operators it is well-known (cf. [2]) that for φ in $L^\infty(\mathbf{T})$ the spectrum $\sigma(L_\varphi)$ of the normal operator L_φ is contained in the approximate point spectrum of W_φ. We extend this result to out context. In [2] it is further shown that such a result implies many things about the relationship between the operators L_φ and W_φ. We shall state only those results which we shall need and refer the interested reader to that paper for the others.

Before proceeding we need to adopt the following conventions. If A is an operator on $L^2(G)$, then we let $\hat{A}$ denote the operator $\mathscr{F}A\mathscr{F}^{-1}$ on $L^2(\hat{G})$. Similarly, since $\mathscr{F}$ takes $H^2(\Sigma)$ onto $L^2(\Sigma)$, we can define for each B on $H^2(\Sigma)$ the corresponding operator $\widetilde{B} = \mathscr{F}B\mathscr{F}^{-1}$ on $L^2(\Sigma)$. We use $\widetilde{\mathscr{A}}(G, \Sigma)$ to denote the transformed algebra $\mathscr{A}(G, \Sigma)$.

Clearly, for γ in Σ we have $(\widetilde{W}_\gamma f)(\sigma) = f(\gamma^{-1}\sigma)$. For the bounded measurable function φ let $\mathrm{R}(\varphi)$ denote the essential range of φ, that is, the set of complex λ for which the function $1/(\varphi - \lambda)$ fails to be bounded a.e. Lastly, recall that the semigroup Σ induces a partial order $\leq$ on $\hat{\mathrm{G}}$ in which $\alpha \leq \beta$ if $\beta\alpha^{-1}$ is in Σ and that $\hat{\mathrm{G}}$ is a directed set with respect to this partial order.

We can now state and prove our first Theorem.

Theorem 1. — *If* φ *is in* $\mathrm{L}^\infty(\mathrm{G})$, *then*

$$\sigma(L_\varphi) = \mathrm{R}(\varphi) \subset \sigma(W_\varphi) \quad \textit{and} \quad \|W_\varphi\| = \|L_\varphi\| = \|\varphi\|_\infty.$$

Proof. — For the normal multiplication operator L_φ on $\mathrm{L}^2(\mathrm{G})$ the relations $\sigma(L_\varphi) = \mathrm{R}(\varphi)$ and $\|L_\varphi\| = \|\varphi\|_\infty$ are well-known. Further, since W_φ is the compression $PL_\varphi | \mathrm{H}^2(\Sigma)$ of L_φ to the subspace $\mathrm{H}^2(\Sigma)$, we have $\|W_\varphi\| \leq \|L_\varphi\|$. Moreover, since the norm of an operator majorizes its spectral radius with equality holding for normal operators (cf. [11]), the theorem will be completely proved when we establish that $\sigma(L_\varphi) \subset \sigma(W_\varphi)$. Lastly, since the spectrum of the normal operator L_φ consists entirely of approximate eigenvalues (cf. [11]), it suffices to show that if 0 is an approximate eigenvalue of L_φ, then 0 is also an approximate eigenvalue for W_φ.

To that end consider the net of operators $\{B_\alpha\}_{\alpha \in \Sigma}$ on $\mathrm{L}^2(\mathrm{G})$ defined for α in Σ by $B_\alpha = L_\alpha^* W_\varphi P L_\alpha$. From the definition of W_φ and the fact that L_α is unitary on $\mathrm{L}^2(\mathrm{G})$ we have

$$B_\alpha = (L_\alpha^* P L_\alpha) L_\varphi (L_\alpha^* P L_\alpha).$$

Using the Fourier transform we have

$$\mathscr{F}(L_\alpha^* P L_\alpha)\mathscr{F}^{-1} = \hat{L}_\alpha^* \hat{P} \hat{L}_\alpha,$$

where $(\hat{L}_\alpha f)(\sigma) = f(\sigma\alpha^{-1})$ for f in $\mathrm{L}^2(\hat{\mathrm{G}})$ and σ in $\hat{\mathrm{G}}$. If f is a function in $\mathrm{L}^2(\hat{\mathrm{G}})$ with compact support E, then by the Lemma in § 2 there exists α_0 in Σ so that $\alpha \geq \alpha_0$ implies $\alpha\mathrm{E} \subset \Sigma$ and hence $\hat{L}_\alpha^* \hat{P} \hat{L}_\alpha f = f$. Thus the net of contractions $\{\hat{L}_\alpha^* \hat{P} \hat{L}_\alpha\}_{\alpha \in \Sigma}$ converges in the strong operator topology to the identity I, since the functions with compact support are dense in $\mathrm{L}^2(\hat{\mathrm{G}})$. Therefore the net $\{B_\alpha\}_{\alpha \in \Sigma}$ converges strongly to L_φ and hence $\|L_\varphi\| \leq \varliminf_{\alpha \in \Sigma} \|B_\alpha\| \leq \|W_\varphi\|$.

Now suppose 0 is an approximate eigenvalue of L_φ, $\varepsilon > 0$ and f is a unit vector in $\mathrm{L}^2(\mathrm{G})$ such that $\|L_\varphi f\| < \frac{\varepsilon}{2}$. Choose α_0 and β_0 in Σ such that $\alpha \geq \alpha_0$ implies $\|W_\varphi P L_\alpha f\| = \|B_\alpha f\| < \frac{\varepsilon}{2}$ and such that $\beta \geq \beta_0$ implies $\|PL_\beta f\| = \|L_\beta^* P L_\beta f\| > \frac{1}{2}$. Then for $\gamma \geq \alpha_0\beta_0$ we have

$$\|W_\varphi(PL_\gamma f)\| < \frac{\varepsilon}{2} < \varepsilon \|PL_\gamma f\|.$$

Thus 0 is an approximate eigenvalue for W_φ and the proof if complete.

In recent work the second author has shown (cf. [9]) that the preceding result can be " explained " in the case of Toeplitz operators as follows: Let $\mathscr{T}$ be the C*-algebra generated by all of the operators $\{W_\varphi \mid \varphi \in L^\infty(\mathbf{T})\}$ and $\mathfrak{C}$ be the commutator ideal of $\mathscr{T}$. Then $\mathscr{T}/\mathfrak{C}$ is isometrically isomorphic to $L^\infty(\mathbf{T})$ with the map $W_\varphi + \mathfrak{C} \leftrightarrow \varphi$. Whether the corresponding statement is true in our context we do not know(1). The proof for the case of the circle group $\mathbf{T}$ depends on some relatively deep properties of the collection of inner functions [10]. No corresponding results are known in the more general group context.

4. Perturbations by Commutators.

Consider the mapping from AP(G) into $\mathscr{A}(G, \Sigma)/\mathfrak{C}(G, \Sigma)$ defined by

$$\varphi \mapsto W_\varphi + \mathfrak{C}(G, \Sigma).$$

From the preceding theorem it follows that this map is a contractive mapping between the two Banach algebras AP(G) and $\mathscr{A}(G, \Sigma)/\mathfrak{C}(G, \Sigma)$. In this section we show, in fact, that this map is an isometrical isomorphism. To do this we need the following

Lemma. — *If* φ *is in* $L^\infty(G)$ *and* K *is in* $\mathfrak{C}(G, \Sigma)$, *then* $\|W_\varphi + K\| \geq \|\varphi\|_\infty$.

Proof. — Observe first that for α in Σ, the semigroup $\alpha\Sigma$ satisfies our fundamental hypotheses. Thus, it follows from Theorem 1 that for $\varepsilon > 0$ and α in Σ, there exists f in $L^2(G)$ such that $\mathscr{F}f$ is in $L^2(\alpha\Sigma)$ and $\|P_{\alpha\Sigma}(\varphi f)\| > (\|W_\varphi\| - \varepsilon)\|f\|$.

Secondly, we need some information concerning the operators in $\mathfrak{C}(G, \Sigma)$. For $\alpha_1, \ldots, \alpha_N$ in Σ, $W_{\alpha_1} W^*_{\alpha_2} \ldots W_{\alpha_N}$ is in $\mathscr{A}(G, \Sigma)$, and the set $\mathscr{D}$ of finite linear combinations of operators of this form is dense in $\mathscr{A}(G, \Sigma)$. If we choose operators $A_1, A_2, \ldots, A_N$, $B_1, B_2, \ldots, B_N$, $C_1, C_2, \ldots, C_N$, $D_1, D_2, \ldots$ and D_N in $\mathscr{D}$, then the operator $K = \sum_{i=1}^{N} A_i(B_iC_i - C_iB_i)D_i$ is in $\mathfrak{C}(G, \Sigma)$ and the set $\mathscr{E}$ of operators of this form is dense in $\mathfrak{C}(G, \Sigma)$. Let K be such an operator with a particular presentation, and let C be the operator on $L^2(G)$ obtained by replacing each W_β or W^*_β in the given presentation for K, by the corresponding L_β or L^*_β. If $\alpha_1, \alpha_2, \ldots, \alpha_M$ is a list of the elements of Σ (counted multiply) which occur in the given presentation for K, then $\alpha = \alpha_1\alpha_2 \ldots \alpha_M$ is in Σ and for f in $L^2(G)$ such that $\mathscr{F}f$ is in $L^2(\alpha\Sigma)$, we have $Kf = Cf = 0$ since the L_β are commuting normal operators. Thus K has the property that $Kf = 0$ for every f in $L^2(G)$ such that $\mathscr{F}f$ is in $L^2(\alpha\Sigma)$. But this argument can be reversed to prove that given K in $\mathscr{E}$ there exists α in Σ with the property that $Kf = 0$ for every f in $L^2(G)$ such that $\mathscr{F}f$ is in $L^2(\alpha\Sigma)$.

We now prove the lemma. Let φ be in $L^\infty(G)$ and K be in $\mathfrak{C}(G, \Sigma)$. For $\varepsilon > 0$ choose K' in $\mathscr{E}$ such that $\|K - K'\| < \frac{\varepsilon}{3}$, choose α in Σ such that $\mathscr{F}f$ in $L^2(\alpha\Sigma)$ implies

(1) *Added in proof :* A proof that $\mathscr{T}/\mathfrak{C}$ is isometrically isomorphic to $L^\infty(G)$ can be given based on a recent result of J. Bunce in " The joint spectrum of commuting non-normal operators " (to appear).

$K'f=0$, and lastly choose such an f such that $||P_{\alpha\Sigma}(\varphi f)||>\left(||W_\varphi||-\frac{\varepsilon}{3}\right)||f||$. Then we have

$$\begin{aligned}||(W_\varphi+K)f|| &\geq ||(W_\varphi+K')f||-\frac{\varepsilon}{3}||f|| \\ &\geq ||W_\varphi f||-\frac{\varepsilon}{3}||f|| \\ &\geq ||P_{\alpha\Sigma}(\varphi f)||-\frac{\varepsilon}{3}||f||\geq(||W_\varphi||-\varepsilon)||f||\end{aligned}$$

so that $||W_\varphi+K||\geq||W_\varphi||$ and the lemma is proved.

We next use the preceding lemma to characterize the operators in $\mathscr{A}(\mathrm{G}, \Sigma)$.

Theorem 2. — The algebra $\mathscr{A}(\mathrm{G}, \Sigma)$ consists precisely of all operators of the form $W_\varphi+K$, where φ is an almost periodic function on G and K is in $\mathfrak{C}(\mathrm{G}, \Sigma)$. Moreover, this representation is unique.

Proof. — If φ is an almost periodic function on G and $\varepsilon>0$, then there exist [1] characters $\gamma_1, \gamma_2, \ldots, \gamma_{\mathrm{N}}$ in $\hat{\mathrm{G}}$ and complex scalars $\lambda_1, \lambda_2, \ldots, \lambda_{\mathrm{N}}$ such that

$$||\sum_{i=1}^{\mathrm{N}}\lambda_i\gamma_i-\varphi||_\infty<\varepsilon.$$

Since Σ generates $\hat{\mathrm{G}}$, it follows that there exists α_i and β_i in Σ such that $\gamma_i=\alpha_i^{-1}\beta_i$. Therefore, it follows from Theorem 1 that $||\sum_{i=1}^{\mathrm{N}}\lambda_i W_{\alpha_i^{-1}\beta_i}-W_\varphi||<\varepsilon$ and since $W^*_{\alpha_i}W_{\beta_i}=W_{\alpha_i^{-1}\beta_i}$ we have W_φ is in $\mathscr{A}(\mathrm{G}, \Sigma)$. Moreover, if ψ is also an almost periodic function on G, then a similar approximation of W_ψ and a simple computation shows that the difference $W_\varphi W_\psi-W_{\psi\varphi}$ is in $\mathfrak{C}(\mathrm{G}, \Sigma)$. Thus, the closure of the set of sums of the form $W_\varphi+K$ is seen to be a C*-algebra which must equal $\mathscr{A}(\mathrm{G}, \Sigma)$. Moreover, from the lemma we have $||W_\varphi+K||\geq||\varphi||$ which implies the representation is unique. Lastly, if $\{W_{\varphi_n}+K_n\}_{n=1}^\infty$ is a Cauchy sequence, then the preceding norm inequality shows that $\{W_{\varphi_n}\}$ is also a Cauchy sequence. Hence, $\lim_{n\to\infty}W_{\varphi_n}=W_\varphi$ for some almost periodic function φ and also $\lim_{n\to\infty}K_n=K$ for some K in $\mathfrak{C}(\mathrm{G}, \Sigma)$. This completes the proof.

Corollary. — The mapping $W_\varphi+\mathfrak{C}(\mathrm{G}, \Sigma)\leftrightarrow\varphi$ is a $*$-isometrical isomorphism between $\mathscr{A}(\mathrm{G}, \Sigma)/\mathfrak{C}(\mathrm{G}, \Sigma)$ and AP(G).

5. Commutant of $\mathscr{A}(\mathbf{G}, \Sigma)$.

In the classical case $\mathscr{A}(\mathbf{T}, \mathbf{Z}^+)$ is an irreducible algebra or equivalently the only operators that commute with $\mathscr{A}(\mathbf{T}, \mathbf{Z}^+)$ are the scalar operators. In the general case we show that the commutant $\mathscr{A}(\mathrm{G}, \Sigma)'$ of $\mathscr{A}(\mathrm{G}, \Sigma)$ is the von Neumann algebra generated by the operators W_γ for γ in $\hat{\mathrm{G}}$ that happen to be unitary. This latter result depends on a Theorem due to Calderon on spectral synthesis. As a corollary we obtain that the irreducibility of $\mathscr{A}(G, \Sigma)$ is equivalent to there existing no γ in $\hat{G}$ satisfying $\gamma\Sigma = \Sigma$ a.e.

We begin by considering the relationship between the isometric representation $\sigma \mapsto W_\sigma$ of Σ and the unitary representation $\sigma \mapsto L_\sigma$. Since $W_\sigma = L_\sigma | H^2(\Sigma)$ for σ in Σ, the latter unitary representation is an extension of the isometric representation. It is known (cf. [8]) that a unitary extension of an isometric representation always exists and that, moreover, if the extension is minimal, then it is uniquely determined. By minimality is meant that the smallest subspace of $L^2(G)$ containing $H^2(\Sigma)$ and reducing all of the L_σ is the space $L^2(G)$ itself.

Lemma. — The representation $\sigma \mapsto L_\sigma$ *of* Σ *is the minimal unitary extension of the isometric representation* $\sigma \mapsto W_\sigma$ *of* Σ.

Proof. — A subspace reduces L_σ if and only if it is invariant for L_σ and $L_{\sigma-1} = L_\sigma^*$. Since the semigroup generated by the collection $\Sigma \cup \Sigma^{-1}$ is G itself, the proof reduces to showing that the smallest closed translation invariant subspace of $L^2(\hat{G})$ containing $L^2(\Sigma)$ is $L^2(\hat{G})$ itself. Since Σ contains an open set U, then $L^2(\Sigma)$ contains the collection of all continuous functions which vanish outside of U. Using a partition of unity argument, the smallest translation invariant subspace containing $L^2(\Sigma)$ must contain all continuous functions with compact support. Since the latter set is dense in $L^2(\hat{G})$, the proof is complete.

We next recall a special case of a theorem in [8].

Lemma. — If A is an operator on $H^2(\Sigma)$ *that commutes with the collection* $\{W_\sigma : \sigma \in \Sigma\}$, *then there exists a unique operator B on* $L^2(G)$ *that commutes with the collection* $\{L_\sigma : \sigma \in \Sigma\}$ *and such that* $A = B | H^2(\Sigma)$.

Proof. — See Theorem 2 of [8].

We state our preliminary characterization of the commutant of $\mathscr{A}(G, \Sigma)$ after introducing some terminology. Let $H^\infty(\Sigma)$ denote the weak*-closed subalgebra of $L^\infty(G)$ consisting of those functions φ for which $L_\varphi(H^2(\Sigma)) \subset H^2(\Sigma)$. Moreover, let $R^\infty(\Sigma)$ denote the largest self-adjoint subalgebra of $H^\infty(\Sigma)$, that is, $R^\infty(\Sigma) = H^\infty(\Sigma) \cap \overline{H^\infty(\Sigma)}$. Lastly, let $Z(G, \Sigma)$ denote the commutant of $\mathscr{A}(G, \Sigma)$.

Theorem 3. — The algebra $Z(G, \Sigma)$ *is an abelian von Neumann algebra and equals* $\{W_\varphi : \varphi \in R^\infty(\Sigma)\}$. *Moreover,* $Z(G, \Sigma)$ *is the center of the von Neumann algebra generated by* $\mathscr{A}(G, \Sigma)$.

Proof. — We begin by observing that the commutant of the collection $\{L_\sigma : \sigma \in \Sigma\}$ is the operators $\{L_\varphi : \varphi \in L^\infty(G)\}$. To prove this we observe that as in the proof of the first lemma, if an operator B commutes with each L_σ for σ in Σ, then B commutes with each L_σ for σ in $\hat{G}$. Since the weak*-closed subspace of $L^\infty(G)$ generated by the characters is $L^\infty(G)$, and the weak*-topology on $L^\infty(G)$ coincides with the weak operator topology on $\{L_\varphi : \varphi \in L^\infty(G)\}$, we obtain the fact that B commutes with the algebra $\{L_\varphi : \varphi \in L^\infty(G)\}$. Moreover, the latter algebra is maximal abelian and hence B must belong to it.

Combining this fact with the preceding lemma we see that each operator A on $H^2(\Sigma)$ that commutes with $\{W_\sigma : \sigma\in\Sigma\}$ can be written uniquely $A=L_\varphi|H^2(\Sigma)$ for some φ in $L^\infty(G)$. Since $A(H^2(\Sigma))\subset H^2(\Sigma)$ it follows that φ is in $H^\infty(\Sigma)$. Therefore, if A commutes with $\mathscr{A}(G,\Sigma)$, then $A=L_\varphi|H^2(\Sigma)$ for some φ in $H^\infty(\Sigma)$. Since $\mathscr{A}(G,\Sigma)$ is self-adjoint, A^* also commutes with $\mathscr{A}(G,\Sigma)$ and $L_\varphi^*=L_{\bar\varphi}$ must leave $H^2(\Sigma)$ invariant. Hence φ is in $R^\infty(\Sigma)$ and the identification $Z(G,\Sigma)=\{W_\varphi : \varphi\in R^\infty(\Sigma)\}$ is complete. The remaining results follow immediately.

In order to obtain our final characterization of $Z(G,\Sigma)$ we need to study the spaces $H^\infty(\Sigma)$ and $R^\infty(\Sigma)$ a little more deeply. We begin with a lemma which puts our problem in the context of harmonic analysis.

Lemma. — The space $H^\infty(\Sigma)$ *is a weak*-closed translation invariant subalgebra of* $L^\infty(G)$.

Proof. — Since we already know that $H^\infty(\Sigma)$ is a weak*-closed subalgebra of $L^\infty(G)$, the only thing remaining to prove is that it is translation invariant and this will follow if we knew that $H^2(\Sigma)$ were translation invariant. This, however, is obvious since if f is in $H^2(\Sigma)$, x is in G, and f_x denotes the translate of f by x, then $\hat f_x(\gamma)=\gamma(x)\hat f(\gamma)$ for γ in $\hat G$. Since $\hat f$ is supported on Σ, it follows that $\hat f_x$ is also and hence f_x is in $H^2(\Sigma)$.

The algebra $H^\infty(\Sigma)$ seems to merit further study. Although it is defined in a manner analogous to the classical Hardy space it can, for example, contain non scalar real functions. Some trace, however, of analyticity should remain. One question one might ask is whether the maximal ideal space of $L^\infty(G)$ is the Šilov boundary for $H^\infty(\Sigma)$.

The spectrum of $H^\infty(\Sigma)$ is a closed subsemigroup Σ_0 of $\hat G$ which contains Σ. It is reasonable to expect that Σ_0 admits spectral synthesis and hence that the weak*-closed subspace spanned by Σ_0 is $H^\infty(\Sigma)$ itself. This follows from a result in [14], Theorem (7.5.6), when the identity is in the closure of the interior of Σ_0. While this is true in many examples, it is not true in general.

Lemma. — The spectrum $U(\Sigma)=\hat G\cap R^\infty(\Sigma)$ *of* $R^\infty(\Sigma)$ *is a closed subgroup of* $\hat G$.

Proof. — Since $R^\infty(\Sigma)$ is a subalgebra of $L^\infty(\Sigma)$ we see that $U(\Sigma)$ is a subsemigroup of $\hat G$. Moreover, $U(\Sigma)$ is a subgroup since $R^\infty(\Sigma)$ is self-adjoint. Lastly, $U(\Sigma)$ is a closed subgroup since the spectrum of a weak*-closed translation invariant subspace is always closed ([14], Th. (7.8.2)).

We next determine " geometrically " which γ are in $U(\Sigma)$.

Lemma. — The character γ *is in* $U(\Sigma)$ *if and only if* $\gamma\Sigma=\Sigma$ *a.e.*

Proof. — (In case the Haar measure on $\hat G$ is not σ-finite the relation $\gamma\Sigma=\Sigma$ a.e. is interpreted to mean $(\gamma\Sigma)\cap E=\Sigma\cap E$ a.e. for every measurable σ-finite subset E of $\hat G$.)

A character γ is in $U(\Sigma)$ if and only if both γ and $\bar\gamma$ are in $H^\infty(\Sigma)$, and the latter happens if and only if $H^2(\Sigma)$ is a reducing subspace for the operator L_γ. Therefore, γ is in $U(\Sigma)$ if and only if W_γ is unitary. If f is in $H^2(\Sigma)$, then $W_\gamma f=L_\gamma f$ and $\widehat{L_\gamma f}$ is the translate $(\hat f)_\gamma$ of $\hat f$ by γ. Since $L_\gamma f$ is in $H^2(\Sigma)$ it follows that the support of $(\hat f)_\gamma$ is contained in Σ. From this the result follows.