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GRADED BRAUER GROUPS AND K-THEORY WITH LOCAL COEFFICIENTS

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The Bott periodicity theorem shows that the real K-theory, KO, and the complex K-theory, KU, are generalised cohomology theories graded by $\mathbf{Z}_8$ and $\mathbf{Z}_2$ respectively. Our aim is to define a "K-theory with local coefficients" $K^\alpha(X)$ (K denotes either KO or KU) which shall generalize the usual groups $K^n(X)$, $n \in \mathbf{Z}_8$ or $n \in \mathbf{Z}_2$.

The ordinary cohomology with local coefficients $H^n(X, \alpha)$ is defined for $(n, \alpha) \in \mathbf{Z} \times H^1(X, \mathbf{Z}_2)$. At least when X is a connected finite CW-complex, $KO^\alpha(X)$ is defined for $\alpha \in \mathbf{Z}_8 \times H^1(X, \mathbf{Z}_2) \times H^2(X, \mathbf{Z}_2)$ and $KU^\alpha(X)$ is defined for

$$\alpha \in \mathbf{Z}_2 \times H^1(X, \mathbf{Z}_2) \times \text{Tors}(H^3(X, \mathbf{Z})),$$

and these may be called K-theory with local coefficients. The sets which index K-theory appear here naturally as "graded Brauer groups" associated with the space X; these groups were essentially studied by Serre [7] and Wall [17]. These graded Brauer groups, together with another less important set (§ 2), seem to index all reasonable "K-theories with local coefficients".

One motivation for this work is that it gives a complete satisfactory "Thom isomorphism" theorem in K-theory. More precisely, if V is a real vector bundle on X, the KO-theory of its Thom space is isomorphic to $KO^\alpha(X)$, $\alpha^{-1} = (d(V), w_1(V), w_2(V))$, where $d(V) \equiv \dim(V) \bmod 8$ and where the $w_i(V)$ are the first two Stiefel-Whitney classes of V. However not all α are of this form.

GBrO(X), the real graded Brauer group of X, is in fact the direct sum of $\mathbf{Z}_8$ with an extension of $H^1(X, \mathbf{Z}_2)$ by $H^2(X, \mathbf{Z}_2)$ if X is a connected finite CW-complex ⁽²⁾. (The extension splits set-theoretically only.) If $\alpha, \alpha' \in \text{GBrO}(X)$, there is a product $KO^\alpha(X) \otimes KO^{\alpha'}(X) \rightarrow KO^{\alpha\alpha'}(X)$; this is constructed by means of Fredholm operators in Hilbert space. So $\bigoplus_\alpha KO^\alpha(X)$ is a graded ring; GBrU(X), the complex graded Brauer group of X, has analogous properties.

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⁽²⁾ This result has been found independently by R. R. PATTERSON.

More generally, if $\text{GBr}(\mathbf{X})$ denotes either $\text{GBrO}(\mathbf{X})$ or $\text{GBrU}(\mathbf{X})$ (as appropriate), the tensor product of bundles of algebras induces a composition:

$$\hat{\otimes} : \text{GBr}(\mathbf{X}) \times \text{GBr}(\mathbf{X}') \rightarrow \text{GBr}(\mathbf{X} \times \mathbf{X}').$$

Then one can define a “cup product”:

$$\mathbf{K}^\alpha(\mathbf{X}) \times \mathbf{K}^{\alpha'}(\mathbf{X}') \rightarrow \mathbf{K}^{\alpha \hat{\otimes} \alpha'}(\mathbf{X} \times \mathbf{X}').$$

The same techniques enable us to define “Adams operations”:

$$\psi^n : \mathbf{KU}^\alpha(\mathbf{X}) \rightarrow \mathbf{KU}^{\alpha n}(\mathbf{X}) \otimes \Omega_n$$

for odd n , with $\Omega_n = \mathbf{Z}[\omega]/(\Phi_n(\omega))$, where Φ_n is the n -th cyclotomic polynomial. For $n \equiv 1 \pmod{4}$, there are also operations:

$$\psi^n : \mathbf{KO}^\alpha(\mathbf{X}) \rightarrow \mathbf{KO}^{\alpha n}(\mathbf{X}) \otimes \Omega_n.$$

If $\alpha = 0$, these operations are essentially the ordinary ψ^n operations.

1. Graded algebras

k will denote either of the fields $\mathbf{R}$ or $\mathbf{C}$. By a *graded k -algebra* we shall mean a $\mathbf{Z}_2$ -graded associative k -algebra $A = A_0 \oplus A_1$ with unit and of finite dimension as a vector space.

The *graded radical* $\mathfrak{r}(A)$ of A is defined to be the intersection of its maximal left graded ideals. It is a graded ideal. The graded radical of $A/\mathfrak{r}(A)$ is 0 and so $A/\mathfrak{r}(A)$ is said to be *semisimple*. The same argument as that used for ungraded algebras in Chapter 4 of [2] shows that:

Lemma 1. — *A semisimple graded k -algebra is a product of simple graded k -algebras.*

The simple graded k -algebras A may be classified as follows. If $A_1 = 0$ and $k = \mathbf{R}$, A must be isomorphic to $\mathbf{M}_n(\mathbf{R})$, $\mathbf{M}_n(\mathbf{H})$ or $\mathbf{M}_n(\mathbf{C})$ for some integer n . If $A_1 = 0$ and $k = \mathbf{C}$, A must be isomorphic to $\mathbf{M}_n(\mathbf{C})$ for some integer n .

Otherwise, according to Lemma 3 of [17], either A is simple (as an ungraded algebra) or A_0 is simple and there exists an element $u \in Z(A) \cap A_1$ such that $A_1 = A_0 \cdot u$ and $u^2 = 1$ ($Z(A)$ denotes the centre of A). In either event, if $k = \mathbf{R}$, either $Z(A) \cap A_0 = \mathbf{R}$ and A is *central* in the sense of [17], or $Z(A) \cap A_0 = \mathbf{C}$ and A is a simple central graded $\mathbf{C}$ -algebra. If $k = \mathbf{C}$, A is necessarily central.

In [17] Wall has classified the central simple k -algebras; a list of their isomorphism classes is given below. Note that if u is an element of an algebra A such that $u^2 = \pm 1$, we write $Z(u) = \{a \in A \mid a \cdot u = u \cdot a\}$ and $Z^*(u) = \{a \in A \mid a \cdot u = -u \cdot a\}$; $\mathbf{H}$ denotes the quaternion division $\mathbf{R}$ -algebra and $1, i, j, k$ is its usual basis; $1 \mapsto 1$ and $i \mapsto i$ will specify an embedding $\mathbf{C} \rightarrow \mathbf{H}$; I_n will denote the $n \times n$ unit matrix.

Simple central graded $\mathbf{R}$ -algebras are classified by their *type* (an element of $\mathbf{Z}_8$)

and their *size* (either a positive integer n , or an unordered pair of positive integers (p, q)). The eight types are as follows:

- [1; n] $A = \mathbf{M}_n(\mathbf{C}); A_0 = \mathbf{M}_n(\mathbf{R}); A_1 = i \cdot \mathbf{M}_n(\mathbf{R}).$
- [2; n] $A = \mathbf{M}_n(\mathbf{H}); A_0 = \mathbf{M}_n(\mathbf{C}) = Z(u); A_1 = Z^*(u); u = i \cdot I_n.$
- [3; n] $A = \mathbf{M}_n(\mathbf{H}) \oplus \mathbf{M}_n(\mathbf{H}); A_0 = \mathbf{M}_n(\mathbf{H}); A_1 = u \cdot \mathbf{M}_n(\mathbf{H}).$
(Here $u \in Z(A) \cap A_1$ is such that $u^2 = 1$.)
- [4; p, q] $A = \mathbf{M}_{p+q}(\mathbf{H}); A_0 = Z(u); A_1 = Z^*(u).$
(Here u is the diagonal matrix whose first p diagonal entries are 1 and whose last q diagonal entries are -1). Let $[4; n] = [4; n, n]$. Let $[4; n, 0] = \mathbf{M}_n(\mathbf{H}).$
- [5; n] $A = \mathbf{M}_{2n}(\mathbf{C}); A_0 = \mathbf{M}_n(\mathbf{H}); A_1 = u \cdot \mathbf{M}_n(\mathbf{H}).$
(The embedding $A_0 \rightarrow A$ is specified thus: for each $M \in \mathbf{M}_n(\mathbf{R}) \subset \mathbf{M}_n(\mathbf{H})$:

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}; \quad i \cdot M \mapsto \begin{pmatrix} iM & 0 \\ 0 & -iM \end{pmatrix};$$

$$j \cdot M \mapsto \begin{pmatrix} 0 & M \\ -M & 0 \end{pmatrix}; \quad k \cdot M \mapsto \begin{pmatrix} 0 & iM \\ iM & 0 \end{pmatrix}$$

$$u = i \cdot I_{2n}.)$$
- [6; n] $A = \mathbf{M}_{2n}(\mathbf{R}); A_0 = Z(u); A_1 = Z^*(u).$
(Here u is the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ where $I = I_n$.)
- [7; n] $A = \mathbf{M}_n(\mathbf{R}) \oplus \mathbf{M}_n(\mathbf{R}); A_0 = \mathbf{M}_n(\mathbf{R}); A_1 = u \cdot \mathbf{M}_n(\mathbf{R}).$
(Here $u \in Z(A) \cap A_1$ is such that $u^2 = 1$.)
- [8; p, q] $A = \mathbf{M}_{p+q}(\mathbf{R}); A_0 = Z(u); A_1 = Z^*(u).$
(Here u is the diagonal matrix whose first p diagonal entries are 1 and whose last q diagonal entries are -1). Let $[8; n] = [8; n, n]$. Let $[8; n, 0] = \mathbf{M}_n(\mathbf{R}).$

Simple central graded $\mathbf{C}$ -algebras are classified by their *type* (an element of $\mathbf{Z}_2$) and their *size* (either a positive integer n , or an unordered pair of positive integers (p, q)). The two types are as follows:

- (1; n) $A = \mathbf{M}_n(\mathbf{C}) \oplus \mathbf{M}_n(\mathbf{C}); A_0 = \mathbf{M}_n(\mathbf{C}); A_1 = u \cdot \mathbf{M}_n(\mathbf{C}).$
(Here $u \in Z(A) \cap A_1$ is such that $u^2 = 1$.)
- (2; p, q) $A = \mathbf{M}_{p+q}(\mathbf{C}); A_0 = Z(u); A_1 = Z^*(u).$
(Here u is the diagonal matrix whose first p diagonal entries are 1 and whose last q diagonal entries are -1). Let $(2; n) = (2; n, n)$. Let $(2; n, 0) = \mathbf{M}_n(\mathbf{C}).$

If A and B are two graded k -algebras, their *graded tensor product* $C = A \hat{\otimes} B$ has underlying graded vector space $A \otimes_k B$ and has its product subject to the rule:

$$(a \hat{\otimes} b_j) \cdot (a_i \hat{\otimes} b) = (-1)^{ij} (a \cdot a_i) \hat{\otimes} (b_j \cdot b) \quad \text{for } a_i \in A_i \text{ and } b_j \in B_j.$$

For example, $[t; n] \hat{\otimes} [t'; n'] = [t + t'; a(t, t') \cdot n \cdot n']$, where $a(t, t')$ is 1, 2, 4 or 8. The following lemma is Theorem 2 of [17]:

Lemma 2. — *The graded tensor product of two simple central graded k -algebras is simple and central.*

It may be verified that the type of such a tensor product is the sum of the types of the factors. Both the lemma and this additivity property are valid also for simple central algebras A with $A_1 = 0$ provided that $\mathbf{M}_n(\mathbf{H})$ is assigned type $4 \in \mathbf{Z}_8$, $\mathbf{M}_n(\mathbf{R})$ is assigned type $0 \in \mathbf{Z}_8$ and $\mathbf{M}_n(\mathbf{C})$ is assigned type $0 \in \mathbf{Z}_2$.

Let $V = \mathbf{R}^{p+q}$ equipped with the quadratic form:

$$Q(x_1, \dots, x_{p+q}) = -x_1^2 - \dots - x_p^2 + x_{p+1}^2 + \dots + x_{p+q}^2.$$

The *Clifford algebra* $C(Q) = C^{p,q}$ is defined to be the quotient of the tensor algebra $T(V)$ by the ideal generated by all elements of the form $x \otimes x - Q(x)$. It is naturally $\mathbf{Z}_2$ -graded. It has dimension 2^{p+q} ; $C^{p,q} \hat{\otimes} C^{r,s} = C^{p+r, q+s}$. See [9] for more details. As $C^{1,0} = \mathbf{C}$ and $C^{0,1} = \mathbf{R} \oplus \mathbf{R}$, $C^{p,q}$ is a simple central graded $\mathbf{R}$ -algebra of type $p-q$. Complex Clifford algebras can also be constructed.

2. Bundles of simple central $\mathbf{R}$ -algebras

Let X be a paracompact connected space. Let A be a graded $\mathbf{R}$ -algebra. Let $\text{Aut}(A)$ be the Lie group of $\mathbf{R}$ -automorphisms of A . A *bundle of A 's* on X is a fibre bundle with base X , fibre A and group $\text{Aut}(A)$. The isomorphism classes of these bundles form the set $H^1(X, \underline{\text{Aut}}(A))$, where the underline denotes “sheaf of continuous functions”. If A is one of the algebras mentioned explicitly in the last section, it is possible to find a compact closed subgroup $\text{Aut}_0(A)$ of $\text{Aut}(A)$ such that the induced map $H^1(X, \underline{\text{Aut}}_0(A)) \rightarrow H^1(X, \underline{\text{Aut}}(A))$ is bijective for all X . (This follows from the theory of p. 51 of [8] and some explicit checking; the details will not be needed.)

Consider first bundles of $\mathbf{M}_n(\mathbf{R})$'s and $\mathbf{M}_n(\mathbf{H})$'s over X (for all n). The automorphism groups are $\mathbf{PGL}_n(\mathbf{R})$ and $\mathbf{GL}_n(\mathbf{H})/\mathbf{R}^*$ respectively (by the Skolem-Noether theorem; see p. 66 of [2]). The exact sequence:

$$1 \rightarrow \mathbf{R}^* \rightarrow \mathbf{GL}_n(\mathbf{R}) \rightarrow \mathbf{PGL}_n(\mathbf{R}) \rightarrow 1$$

defines a coboundary map $w_2: H^1(X, \mathbf{PGL}_n(\mathbf{R})) \rightarrow H^2(X, \mathbf{R}^*) = H^2(X, \mathbf{Z}_2)$; w_2 is similarly defined in the quaternionic case. If $\mathcal{A}$ and $\mathcal{B}$ are two such bundles, so is $\mathcal{A} \otimes \mathcal{B}$, and an argument similar to the proof of Lemma 4 below shows that

$w_2(\mathcal{A} \otimes \mathcal{B}) = w_2(\mathcal{A}) + w_2(\mathcal{B})$. A bundle $\mathcal{A}$ of $\mathbf{M}_n(\mathbf{R})$'s is said to be *negligible* if it is of the form $\text{END}(\mathbf{E})$ for some real vector bundle $\mathbf{E}$ or equivalently if $w_2(\mathcal{A}) = 0$.

The isomorphism classes of such bundles form a commutative monoid under $\otimes$; the classes of negligible bundles form a submonoid. The *orthogonal Brauer group* of $\mathbf{X}$, $\text{BrO}(\mathbf{X})$, is defined to be the quotient. There is an injective homomorphism, natural in the obvious sense, $\Phi : \text{BrO}(\mathbf{X}) \rightarrow \mathbf{Z}_2 \oplus \text{H}^2(\mathbf{X}, \mathbf{Z}_2)$. (This shows that $\text{BrO}(\mathbf{X})$ is in fact a group. Φ is defined as the product of w_2 with the map which assigns $o \in \mathbf{Z}_2$ to bundles of $\mathbf{M}_n(\mathbf{R})$'s and $1 \in \mathbf{Z}_2$ to bundles of $\mathbf{M}_n(\mathbf{H})$'s.)

This remains valid if $\mathbf{X}$ is paracompact and has finitely many components provided that $\mathbf{Z}_2$ is replaced by $\text{H}^0(\mathbf{X}, \mathbf{Z}_2)$ where appropriate; analogous situations later will not be commented on. The following theorem will be proved in § 4:

Theorem 3. — *Let $\mathbf{X}$ be a finite CW-complex. Then:*

$$\Phi : \text{BrO}(\mathbf{X}) \rightarrow \text{H}^0(\mathbf{X}, \mathbf{Z}_2) \oplus \text{H}^2(\mathbf{X}, \mathbf{Z}_2)$$

is an isomorphism.

The automorphism group of $\mathbf{A} = [8; n] = \mathbf{M}_{2n}(\mathbf{R})$ is that subgroup of

$$\text{Aut}(\mathbf{M}_{2n}(\mathbf{R})) = \mathbf{GL}_{2n}(\mathbf{R}) / \mathbf{R}^*$$

which leaves invariant the subspaces $\mathbf{A}_0$ and $\mathbf{A}_1$. Call it $\mathbf{E}_n$. It may be verified that $\mathbf{E}_n = \mathbf{E}'_n / \mathbf{R}^*$, where $\mathbf{E}'_n = \mathbf{F}_n \cup \mathbf{F}'_n \subset \mathbf{GL}_{2n}(\mathbf{R})$, and where $\mathbf{F}_n = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbf{GL}_n(\mathbf{R}) \right\}$, and $\mathbf{F}'_n = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mid a, b \in \mathbf{GL}_n(\mathbf{R}) \right\}$. Finally, $\mathbf{G}_n$ is defined by the exactness and commutativity of the diagram:

$$\begin{array}{ccccccc} & & \mathbf{I} & & \mathbf{I} & & \\ & & \downarrow & & \downarrow & & \\ \mathbf{I} & \longrightarrow & \mathbf{R}^* & \longrightarrow & \mathbf{F}_n & \longrightarrow & \mathbf{G}_n \longrightarrow \mathbf{I} \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{I} & \longrightarrow & \mathbf{R}^* & \longrightarrow & \mathbf{E}'_n & \longrightarrow & \mathbf{E}_n \longrightarrow \mathbf{I} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{Z}_2 & = & \mathbf{Z}_2 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathbf{I} & & \mathbf{I} & & \end{array}$$

Hence, if $\mathcal{A}$ is a bundle of $[8; n]$'s on $\mathbf{X}$, this diagram defines its *characteristic classes* $w_1(\mathcal{A}) \in \text{H}^1(\mathbf{X}, \mathbf{Z}_2)$ and $w_2(\mathcal{A}) \in \text{H}^2(\mathbf{X}, \mathbf{R}^*) = \text{H}^2(\mathbf{X}, \mathbf{Z}_2)$.

A bundle $\mathcal{A}$ of $[8; n]$'s on X is said to be *negligible* if

$$\mathcal{A}_0 = \text{HOM}(V, V) \oplus \text{HOM}(W, W) \quad \text{and} \quad \mathcal{A}_1 = \text{HOM}(V, W) \oplus \text{HOM}(W, V)$$

for some real vector bundles V and W . The exact cohomology sequences obtained from the diagram show that $\mathcal{A}$ is negligible if and only if both $w_1(\mathcal{A}) = 0$ and $w_2(\mathcal{A}) = 0$.

Let $\text{HO}(X)$ be the set $H^1(X, \mathbf{Z}_2) \times H^2(X, \mathbf{Z}_2)$. The rule:

$$(a, b) \cdot (a', b') = (a + a', b + b' + a \cdot a')$$

gives it the structure of a 4-torsion abelian group; it is an extension of H^1 by H^2 . Then, for $\mathcal{A}$ as above, set $w(\mathcal{A}) = (w_1(\mathcal{A}), w_2(\mathcal{A}))$.

Lemma 4. — *If $\mathcal{A}$ and $\mathcal{B}$ are bundles of $[8; n]$'s and $[8; n']$'s respectively on X , then $w(\mathcal{A} \hat{\otimes} \mathcal{B}) = w(\mathcal{A}) \cdot w(\mathcal{B})$.*

Proof. — The maps $E'_n \rightarrow \mathbf{Z}_2$ and $E'_{n'} \rightarrow \mathbf{Z}_2$ will both be denoted by $a \mapsto \bar{a}$. Choose an open cover $\mathcal{U} = \{U_i\}$ of X such that the restrictions of $\mathcal{A}$ and $\mathcal{B}$ to each U_i are product bundles. Set $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. It is convenient to use the same symbol for a function as for its restriction to a smaller domain of definition. Hence there exist functions $\alpha_{ij} : U_{ij} \rightarrow E_n$ such that: 1) $\alpha_{ii} = 1$; 2) $\alpha_{ij} \cdot \alpha_{ji} = 1$; and 3) $\alpha_{ij} \cdot \alpha_{jk} \cdot \alpha_{ki} = 1$ on U_{ijk} . These functions determine $\mathcal{A}$ up to isomorphism in the usual way. Now choose functions $a_{ij} : U_{ij} \rightarrow E'_n$ such that: 4) $a_{ii} = 1$; 5) $a_{ij} \cdot a_{ji} = 1$; and 6) $a_{ij} \mapsto \alpha_{ij}$ under the morphism $E'_n \rightarrow E_n$. Then $a_{ij} \cdot a_{jk} \cdot a_{ki} = A_{ijk} : U_{ijk} \rightarrow \mathbf{R}^*$ and the set $\{A_{ijk}\}$ forms a Čech 2-cocycle. Note that $w_1(\mathcal{A})$ is specified by the Čech 1-cocycle $\{\bar{a}_{ij}\}$ and that $w_2(\mathcal{A})$ is specified by the Čech 2-cocycle $\{A_{ijk}\}$. Let b_{ij} and B_{ijk} be the corresponding objects for $\mathcal{B}$. If $\mathcal{C} = \mathcal{A} \hat{\otimes} \mathcal{B}$ has corresponding objects c_{ij} and C_{ijk} , $c_{ij} = (a_{ij} \hat{\otimes} 1) \cdot (1 \hat{\otimes} b_{ij})$. Hence $\bar{c}_{ij} = \bar{a}_{ij} + \bar{b}_{ij}$, which shows that $w_1(\mathcal{C}) = w_1(\mathcal{A}) + w_1(\mathcal{B})$. Further:

$$\begin{aligned} C_{ijk} &= (a_{ij} \hat{\otimes} 1) \cdot (1 \hat{\otimes} b_{ij}) \cdot (a_{jk} \hat{\otimes} 1) \cdot (1 \hat{\otimes} b_{jk}) \cdot (a_{ki} \hat{\otimes} 1) \cdot (1 \hat{\otimes} b_{ki}) \\ &= (A_{ijk} \hat{\otimes} 1) \cdot (1 \hat{\otimes} B_{ijk}) \cdot (\bar{b}_{ij} \cdot \bar{a}_{jk} + \bar{b}_{jk} \cdot \bar{a}_{ki} + \bar{b}_{ij} \cdot \bar{a}_{ki}), \end{aligned}$$

where the field $\mathbf{Z}_2$ is considered as a subgroup of $\mathbf{R}^*$. The result that:

$$w_2(\mathcal{C}) = w_2(\mathcal{A}) + w_2(\mathcal{B}) + w_1(\mathcal{A}) \cdot w_1(\mathcal{B})$$

follows from the fact that the third bracketed term represents the cup product.

If $\mathcal{A}$ is a bundle of $[t; n]$'s on X , and $\mathcal{B}$ is a product bundle of $[8-t; n']$'s on X , the above lemma shows that $w(\mathcal{A} \hat{\otimes} \mathcal{B})$ is independent of n' . This is defined to be $w(\mathcal{A}) = (w_1(\mathcal{A}), w_2(\mathcal{A}))$. Likewise, if $\mathcal{A}$ is a bundle of $[4; p, q]$'s or $[8; p, q]$'s on X , $w(\mathcal{A})$ is defined; in this case $w_1(\mathcal{A}) = 0$ if $p \neq q$. Likewise, if $\mathcal{A}$ is a bundle of $\mathbf{M}_n(\mathbf{H})$'s or $\mathbf{M}_n(\mathbf{R})$'s on X , $w(\mathcal{A})$ is defined to be $(0, w_2(\mathcal{A}))$. Now Lemma 4 implies:

Lemma 5. — *If $\mathcal{A}$ and $\mathcal{B}$ are bundles of simple central graded $\mathbf{R}$ -algebras on the para-compact space X , $w(\mathcal{A} \hat{\otimes} \mathcal{B}) = w(\mathcal{A}) \cdot w(\mathcal{B})$.*

The set of isomorphism classes of such graded bundles is a commutative monoid under $\hat{\otimes}$; the set of classes of negligible bundles is a submonoid; the *orthogonal graded Brauer group* of X , $\text{GBrO}(X)$, is defined to be the quotient. (Note that the same group is obtained by considering only bundles of $[t; n]$'s for all t, n .) If X is connected, let $\tau : \text{GBrO}(X) \rightarrow \mathbf{Z}_8$ be the homomorphism "type"; define Ψ to be $\tau \times w$. Note that Ψ is injective. Let $i : \text{BrO}(X) \rightarrow \text{GBrO}(X)$ be the obvious injection. Then the following diagram, in which the right hand vertical arrow is the product of the standard injections, commutes:

$$\begin{array}{ccc} \text{BrO}(X) & \xrightarrow{\Phi} & \mathbf{Z}_2 \oplus H^2(X, \mathbf{Z}_2) \\ \downarrow i & & \downarrow \\ \text{GBrO}(X) & \xrightarrow{\Psi} & \mathbf{Z}_8 \oplus \text{HO}(X) \end{array}$$

The definition of Ψ and the commutativity of the diagram immediately extend to the case when X is paracompact and has finitely many components.

Theorem 6. — *Let X be a finite CW-complex. Then:*

$$\Psi : \text{GBrO}(X) \rightarrow H^0(X, \mathbf{Z}_8) \oplus \text{HO}(X)$$

is an isomorphism.

Proof. — It is already known that Ψ is injective, and it suffices to assume that X is connected. For $a \in H^1(X, \mathbf{Z}_2)$ and $b \in H^2(X, \mathbf{Z}_2)$ it is required to construct a bundle $\mathcal{A}$ of $[8; n]$'s on X such that $w(\mathcal{A}) = (a, b)$. By Theorem 3 there exists a bundle $\mathcal{B}$ of $\mathbf{M}_n(\mathbf{R})$'s such that $v(\mathcal{B}) = b$; the diagram shows that $w(\mathcal{B} \hat{\otimes} [8; 1]) = (0, b)$, where $[8; 1]$ is the product bundle of $[8; 1]$'s. Let V be a real line bundle on X such that $w_1(V) = a$ and let W be the Whitney sum of V with $X \times \mathbf{R}^7$. Then Lemma 7 below shows that $w(C(W)) = (a, 0)$. Now Lemma 4 shows that $w(C(W) \otimes \mathcal{B}) = (a, b)$, proving the theorem.

If V is a real vector bundle on the compact space X , its *Clifford bundle* $C(V)$ ([9], § 1.1) is a bundle of central simple graded $\mathbf{R}$ -algebras. This construction defines a homomorphism $c : \text{KO}(X) \rightarrow \text{GBrO}(X)$; c need not be surjective. In fact, if c is surjective, the classical fibration

$$B \text{Spin}(n) \rightarrow BSO(n) \rightarrow K(\mathbf{Z}_2, 2)$$

has a cross-section for every n . This is clearly impossible (use Steenrod squares for instance).

Lemma 7. — *Let V be a real vector bundle on the paracompact space X , provided by a negative definite quadratic form. Then $w_i(V) = w_i(C(V))$ for $i = 1, 2$.*

Proof. — Clearly either $w_1(C(V)) = 0$ for all X and V or $w_1(C(V)) = w_1(V)$. To show that the first alternative is false, consider the case when $X = \mathbf{P}_1(\mathbf{R})$ and V is the

Hopf line bundle. Then $C(V) = I \oplus V$ and $\mathcal{A} = C(V) \hat{\otimes} [7; 1]$ is such that $\mathcal{A}_0 = I \oplus V$ and $w_2(\mathcal{A}) = 0$. (An easy direct check shows that $w_2(C(L)) = w_2(C(L) \hat{\otimes} [7; 1]) = 0$ for any line bundle L on any X .) As V is not stably trivial, $\mathcal{A}_0$ cannot be isomorphic to $\text{HOM}(W, W) \oplus \text{HOM}(W', W')$ for any vector bundles W, W' . Hence $w_1(C(V)) \neq 0$. Similarly, there exist universal constants $a, b \in \mathbf{Z}_2$ such that $w_2(C(V)) = a \cdot w_1(V)^2 + b \cdot w_2(V)$. Consider the case when $X = K(\mathbf{Z}_2 \times \mathbf{Z}_2, 1)$ (or its 3-skeleton); let L_1 and L_2 be line bundles on X such that $\pi_1 = w_1(L_1)$ and $\pi_2 = w_1(L_2)$ generate $H^1(X, \mathbf{Z}_2)$. Then, if $V = L_1 \oplus L_2$,

$$\begin{aligned} a \cdot (\pi_1 + \pi_2)^2 + b \cdot \pi_1 \cdot \pi_2 &= w_2(C(V)) = w_2(C(L_1) \otimes C(L_2)) \\ &= w_2(C(L_1)) + w_2(C(L_2)) + w_1(C(L_1)) \cdot w_1(C(L_2)) = \pi_1 \cdot \pi_2. \end{aligned}$$

Hence $a = 0$ and $b = 1$, as required.

Remark. — We will not deal with bundles of non-central simple graded $\mathbf{R}$ -algebras, i.e. with bundles of $(1; n)$'s or $(2; p, q)$'s considered as $\mathbf{R}$ -algebras. It is possible to define equivalence classes of these and obtain a “graded Brauer set” on which the graded orthogonal Brauer group acts. This set has no group structure.

3. Bundles of simple central $\mathbf{C}$ -algebras

Again, X will denote a paracompact connected space. If A is a graded $\mathbf{C}$ -algebra, a bundle of A 's on X is defined analogously to the real case.

Consider first bundles of $\mathbf{M}_n(\mathbf{C})$'s on X (for all n); $\mathbf{M}_n(\mathbf{C})$ has $\mathbf{C}$ -automorphism group $\mathbf{PGL}_n(\mathbf{C})$. The lower exact row of the commutative diagram:

$$\begin{array}{ccccccc} I & \longrightarrow & \mathbf{Z}_n & \longrightarrow & \mathbf{SL}_n(\mathbf{C}) & \longrightarrow & \mathbf{PGL}_n(\mathbf{C}) \longrightarrow I \\ & & \downarrow & & \downarrow & & \parallel \\ I & \longrightarrow & \mathbf{C}^* & \longrightarrow & \mathbf{GL}_n(\mathbf{C}) & \longrightarrow & \mathbf{PGL}_n(\mathbf{C}) \longrightarrow I \end{array}$$

defines a coboundary map $v : H^1(X, \mathbf{PGL}_n(\mathbf{C})) \rightarrow H^2(X, \mathbf{C}^*) = H^3(X, \mathbf{Z})$. The upper row shows that the image of v is n -torsion. Once again it may be proved that $v(\mathcal{A} \otimes \mathcal{B}) = v(\mathcal{A}) + v(\mathcal{B})$. A bundle $\mathcal{A}$ of $\mathbf{M}_n(\mathbf{C})$'s is said to be *negligible* if it is isomorphic to $\text{END}(E)$ for some complex vector bundle E or equivalently if $v(\mathcal{A}) = 0$.

The isomorphism classes of such bundles form a commutative monoid under $\otimes$; the classes of negligible bundles form a submonoid; the *unitary Brauer group* of X , $\text{BrU}(X)$, is defined to be the quotient and is indeed a group. The following theorem will be proved in § 4:

Theorem 8 (Serre). — *Let X be a finite CW-complex. Then $v : \text{BrU}(X) \rightarrow H^3(X, \mathbf{Z})$ is injective. Its image is the torsion subgroup.*

The automorphism group of $A=(2; n)$ is that subgroup of

$$\text{Aut}(\mathbf{M}_{2n}(\mathbf{C})) = \mathbf{GL}_{2n}(\mathbf{C})/\mathbf{C}^*$$

which leaves invariant the subspaces A_0 and A_1 . Call it E_n (no confusion will arise with E_n of § 2) and construct E'_n , F_n and G_n as before. Then a diagram like the first diagram of § 2 but with $\mathbf{C}^*$ replacing $\mathbf{R}^*$ may be used to define maps $u_1 : H^1(X, \mathbf{PGL}_n(\mathbf{C})) \rightarrow H^1(X, \mathbf{Z}_2)$ and $u_2 : H^1(X, \mathbf{PGL}_n(\mathbf{C})) \rightarrow H^2(X, \mathbf{C}^*) = H^3(X, \mathbf{Z})$. Hence if $\mathcal{A}$ is a bundle of $(2; n)$'s on X , its characteristic classes $u_1(\mathcal{A}) \in H^1(X, \mathbf{Z}_2)$ and $u_2(\mathcal{A}) \in H^3(X, \mathbf{Z})$ are defined. As $u_2(\mathcal{A}) = v(\mathcal{A}^*)$, where $\mathcal{A}^*$ is the bundle of $\mathbf{M}_{2n}(\mathbf{C})$'s underlying $\mathcal{A}$, $2n \cdot u_2(\mathcal{A}) = 0$.

A bundle $\mathcal{A}$ of $(2; n)$'s on X is said to be *negligible* if

$$\mathcal{A}_0 = \text{HOM}(V, V) \oplus \text{HOM}(W, W) \quad \text{and} \quad \mathcal{A}_1 = \text{HOM}(V, W) \oplus \text{HOM}(W, V)$$

for some complex vector bundles V and W . Once again, $\mathcal{A}$ is negligible if and only if both $u_1(\mathcal{A}) = 0$ and $u_2(\mathcal{A}) = 0$. Let $HU(X)$ be the set $H^1(X, \mathbf{Z}_2) \times \text{Tors}(H^3(X, \mathbf{Z}))$. If $\beta : H^2(X, \mathbf{Z}_2) \rightarrow H^3(X, \mathbf{Z})$ is the Bockstein, the rule $(a, b) \cdot (a', b') = (a + a', b + b' + \beta(a, a'))$ gives it the structure of a torsion abelian group; it is an extension of H^1 by $\text{Tors}(H^3)$. Then, for $\mathcal{A}$ as above, set $u(\mathcal{A}) = (u_1(\mathcal{A}), u_2(\mathcal{A}))$. The following lemma is proved in the same way as Lemma 4:

Lemma 9. — *If $\mathcal{A}$ and $\mathcal{B}$ are bundles of $(2; n)$'s and $(2; n')$'s respectively on X , then $u(\mathcal{A} \otimes \mathcal{B}) = u(\mathcal{A}) \cdot u(\mathcal{B})$.*

As before, this lemma enables us to define $u(\mathcal{A})$ when $\mathcal{A}$ is a bundle of $(1; n)$'s, $(2; p, q)$'s or $\mathbf{M}_n(\mathbf{C})$'s. Now Lemma 9 implies:

Lemma 10. — *If $\mathcal{A}$ and $\mathcal{B}$ are bundles of simple central graded $\mathbf{C}$ -algebras on the compact space X , $u(\mathcal{A} \hat{\otimes} \mathcal{B}) = u(\mathcal{A}) \cdot u(\mathcal{B})$.*

As before, the unitary graded Brauer group of X , $\text{GBrU}(X)$ may be defined. If X is connected, a homomorphism $\Psi : \text{GBrU}(X) \rightarrow \mathbf{Z}_2 \oplus HU(X)$ and an injection $i : \text{BrU}(X) \rightarrow \text{GBrU}(X)$ may be defined. Once again, the following diagram commutes:

$$\begin{array}{ccc} \text{BrU}(X) & \xrightarrow{v} & \text{Tors}(H^3(X, \mathbf{Z})) \\ \downarrow i & & \downarrow \\ \text{GBrU}(X) & \xrightarrow{\Psi} & \mathbf{Z}_2 \oplus HU(X) \end{array}$$

Theorem 11. — *Let X be a finite CW-complex. Then:*

$$\Psi : \text{GBrU}(X) \rightarrow H^0(X, \mathbf{Z}_2) \oplus HU(X)$$

is an isomorphism.

Proof. — The proof is analogous to that of Theorem 6 and so will be omitted.

4. The existence theorems

This section is devoted to the proofs of Theorems 3 and 8. Throughout it X will denote a finite connected CW-complex. Note that the injectivity of Φ and v has already been proved.

Lemma 12. — *If V is a k -vector bundle on X , there exists a vector bundle W on X such that $V \otimes W$ is trivial.*

Proof. — Let X have dimension d and V have rank r . Then $([V] - r) \in K(X)$ is nilpotent (see p. 127 of [3]). Let N be an integer such that $([V] - r)^N = 0$. If

$$x = r^{N-1} - r^{N-2} \cdot ([V] - r) + \dots + (-1)^{N-1} \cdot ([V] - r)^{N-1},$$

$M \cdot x \cdot [V] = M \cdot r^N \in K(X)$, where the integer M is chosen such that $M \cdot x$ has rank $\rho = M \cdot r^{N-1} > d$. So there exists a vector bundle W' such that $[W'] - n = M \cdot x$ for some positive integer n . Now the injection $\mathbf{O}(\rho) \rightarrow \mathbf{O}(\rho + n)$ induces isomorphisms $\pi_i(\mathbf{O}(\rho)) \rightarrow \pi_i(\mathbf{O}(\rho + n))$ for $i \leq d$, and hence induces a bijection $[X, B_{\mathbf{O}(\rho)}] \rightarrow [X, B_{\mathbf{O}(\rho+n)}]$. Hence there exists a vector bundle W such that $[W] = M \cdot x$. As $[V \otimes W] = M \cdot r^N > d$, $V \otimes W$ is trivial. (If $k = \mathbf{C}$, $\mathbf{O}$ must be replaced by $\mathbf{U}$ in this proof.)

Now, for definiteness, assume that $k = \mathbf{R}$. Let $K'PO(X)$ be the quotient of the commutative monoid of isomorphism classes of $\mathbf{R}$ -vector bundles on X with composition induced by the tensor product by the submonoid consisting of the classes of trivial bundles. By Lemma 12 it is a commutative group. Further, it is divisible. For if m is an integer and with the notation of the above proof, the formal binomial expansion of $r^{N-1} \cdot m^{N-1} \cdot (N-1)! \cdot (1 + ([V] - r)/r)^{1/m}$ may be used as above to construct a vector bundle V_m such that $V_m^{\otimes m}$ is a multiple of V . Further, it is torsion-free, and hence is a $\mathbf{Q}$ -vector space. For if V is a vector bundle such that $V^{\otimes m}$ is trivial of rank r^m , Lemma 12 shows that there is a vector bundle W such that $T = W \otimes (V^{\otimes (m-1)} \oplus r \cdot V^{\otimes (m-2)} \oplus \dots \oplus r^{m-1})$ is trivial and $[V \otimes T] = [r \cdot T] \in KO(X)$. So again, for a suitable positive integer M , $M \cdot V \otimes T \approx M \cdot r \cdot T$, and so V has image 0 in $K'PO(X)$.

Let $KPO(X)$ be the quotient of the commutative monoid of isomorphism classes of bundles of $\mathbf{M}_n(\mathbf{R})$'s on X (for all n) with composition induced by the tensor product by the submonoid consisting of the classes of trivial bundles. As the tensor product of such a bundle with its opposed bundle is negligible, Lemma 12 shows that $KPO(X)$ is a group. The endomorphism bundle construction induces a homomorphism $i: K'PO(X) \rightarrow KPO(X)$. The characteristic class w_2 defines a homomorphism $w_2: KPO(X) \rightarrow H^2(X, \mathbf{Z}_2)$. The sequence:

$$0 \rightarrow K'PO(X) \xrightarrow{i} KPO(X) \xrightarrow{w_2} H^2(X, \mathbf{Z}_2)$$

is clearly exact. Now the group $\mathbf{PO}_n = \mathbf{O}(n)/\mathbf{Z}_2$ acts on $\mathbf{R}^n \otimes \mathbf{R}^n$ in the obvious way; hence a bundle $\mathcal{A}$ of $\mathbf{M}_n(\mathbf{R})$'s on X induces a vector bundle of rank n^2 ; this construction induces a homomorphism $2j: KPO(X) \rightarrow K'PO(X)$. As $j \cdot i = 1$ the sequence splits.

If $X = \mathbf{S}^p$, the p -sphere, w_2 is surjective. This statement is trivial unless $p = 2$. If $\mathcal{A} = C(V) \hat{\otimes} [6; 1]$ (considered as a bundle of $\mathbf{M}_4(\mathbf{R})$'s), where V is the Hopf complex line bundle (considered as a real 2-plane bundle) on $X = \mathbf{S}^2 = \mathbf{P}^1(\mathbf{C})$, Lemma 7 shows that $w_2(\mathcal{A})$ is the non-zero element of $H^2(X, \mathbf{Z}_2)$. Now $K'PO(\)$, $KPO(\)$ and $H^2(\ , \mathbf{Z}_2)$ are all half exact homotopy functors; a theorem of Brown (see p. 7.1 of [5]) shows that $j \oplus w_2 : KPO(X) \rightarrow K'PO(X) \oplus H^2(X, \mathbf{Z}_2)$ is an isomorphism for all X .

It suffices to prove Theorem 3 in the case when X is connected. The surjectivity of w_2 implies the surjectivity of Φ , which therefore is an isomorphism.

Remark. — Brown's theorem and the "Pontryagin character" may now be used to show that $K'PO(X) \approx \prod_{i \geq 0} H^{4i}(X, \mathbf{Q})$. This yields the periodicity of the homotopy of the direct limit $\mathbf{PO}$ of the groups $\mathbf{PO}_n$.

As Theorem 8 is proved in [7] we will omit the modifications needed to make the above argument work in the case $k = \mathbf{C}$.

5. K-Theory with local coefficients

Let $\mathcal{A}$ be a bundle of A 's on the compact space X , where A is a graded k -algebra (§ 1). Denote by $\mathcal{E}^{\mathcal{A}}(X)$ the category of graded k -vector bundles which are projective $\mathcal{A}$ -modules in the obvious sense, with morphisms of degree 0. $\bar{\mathcal{E}}^{\mathcal{A}}(X)$ is the category whose objects are those of $\mathcal{E}^{\mathcal{A}}(X)$ but whose morphisms are not necessarily of degree 0. Both $\mathcal{E}^{\mathcal{A}}(X)$ and $\bar{\mathcal{E}}^{\mathcal{A}}(X)$ are "prebanach categories" (see [11]) and the forgetful functor $\varphi : \mathcal{E}^{\mathcal{A}}(X) \rightarrow \bar{\mathcal{E}}^{\mathcal{A}}(X)$ is a Banach functor. The Grothendieck group $K^{\mathcal{A}}(X)$ is the K -group $K(\varphi)$ of the Banach functor φ . For example, if $\mathcal{A} = k$ (the product bundle), $K^{\mathcal{A}}(X)$ is isomorphic to the well-known group $K(X)$. More generally, if $\mathcal{A} = C^{p,q}$, $K^{\mathcal{A}}(X)$ is the group $K^{p,q}(X) = K^{p-q}(X)$ introduced in [11].

If $\alpha \in \text{GBr}(X)$ (which means $\text{GBrO}(X)$ if $k = \mathbf{R}$, $\text{GBrU}(X)$ if $k = \mathbf{C}$), and if $\mathcal{A}$ is a bundle of central simple graded k -algebras of class α , it will be shown later that $K^{\mathcal{A}}(X)$ depends only on α . It is defined to be $K^{\alpha}(X)$.

If $\mathfrak{r}_x$ denotes the graded radical of $\mathcal{A}_x$ where $x \in X$, $\mathcal{R} = \bigcup \mathfrak{r}_x$ is a sub-bundle of graded ideals of $\mathcal{A}$ and $\mathcal{A}/\mathcal{R}$ is a bundle of $A/\mathfrak{r}(A)$'s, i.e. of semisimple graded k -algebras. Define a functor $\theta : \mathcal{E}^{\mathcal{A}}(X) \rightarrow \mathcal{E}^{\mathcal{A}/\mathcal{R}}(X)$ by the formula $\theta(E) = (\mathcal{A}/\mathcal{R}) \otimes_{\mathcal{A}} E$ for $E \in \mathcal{E}^{\mathcal{A}}(X)$.

Proposition 13. — The functor θ induces an isomorphism $\theta_* : K^{\mathcal{A}}(X) \rightarrow K^{\mathcal{A}/\mathcal{R}}(X)$.

Proof. — Following § 2.1 of [9] we give another description of $K^{\mathcal{B}}(X)$ for every $\mathcal{B}$. Consider triples $(E, \varepsilon_1, \varepsilon_2)$ where E is a $\mathcal{B}$ -module and where ε_1 and ε_2 are gradings of E . This means that ε_1 and ε_2 are two involutions of E (regarded as an ordinary bundle) such that $\varepsilon_i b = b \varepsilon_i$ for $b \in \mathcal{B}_0$ and $\varepsilon_i b = -b \varepsilon_i$ for $b \in \mathcal{B}_1$. Moreover (E, ε_i) is assumed to be a graded projective module over $\mathcal{B}$. A triple $(E, \varepsilon_1, \varepsilon_2)$ is called *elementary* if ε_1 is homotopic to ε_2 among the gradings of E . The group $K^{\mathcal{B}}(X)$ is then the quotient of the monoid constructed with such triples by the equivalence relation generated by

the addition of elementary triples (see § 2.1 of [9] for a proof of an analogous statement). The *cosuspension* $\Sigma\mathcal{B}$ of $\mathcal{B}$ (not the suspension of [13]) is defined to be $C^{0,1}\hat{\otimes}\mathcal{B}$. A graded projective module over $\mathcal{B}$ may be thought of as an ordinary projective module over $\Sigma\mathcal{B}$. Hence every element of $K^{\mathcal{B}}(X)$ can be written $(E, \varepsilon_1, \varepsilon_2)$ where $E = (\Sigma\mathcal{B})^n$ for a certain n and where ε_1 is the grading induced by the canonical generator of $C^{0,1}$. In this context, θ_* is simply defined by the formula $\theta_*((\Sigma\mathcal{A})^n, \varepsilon_1, \varepsilon_2) = ((\Sigma(\mathcal{A}/\mathcal{R}))^n, \bar{\varepsilon}_1, \bar{\varepsilon}_2)$, where $\bar{\varepsilon}_i$ is the image of ε_i . Let $\eta(t)$ be a homotopy between $\eta(0) = \bar{\varepsilon}_1$ and $\eta(1) = \bar{\varepsilon}_2$. The argument of Lemma 1.3 of [14] shows the existence of a continuous family $\zeta(t)$ of gradings of $(\Sigma\mathcal{A})^n$ such that $\bar{\zeta}(t) = \eta(t)$ and $\zeta(0) = \varepsilon_1$. If we put $\lambda(t) = 1 + t\zeta(1)\varepsilon_2$, $\lambda(t)$ is invertible by Nakayama's lemma. Then the homotopy $\xi(t)$ given by $\xi(t) = \zeta(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $\xi(t) = (\lambda(2t-1))^{-1}\zeta(1)\lambda(2t-1)$ for $\frac{1}{2} \leq t \leq 1$ connects ε_1 and ε_2 in

the set of gradings of E . Hence θ_* is injective. Now, for every bundle $\mathcal{B}$ of graded algebras, let $F(n, \mathcal{B})$ be the Banach space of endomorphisms $\varepsilon : (\Sigma\mathcal{B})^n \rightarrow (\Sigma\mathcal{B})^n$ such that $\varepsilon b = (-1)^i b \varepsilon$ for every $b \in \mathcal{B}_i \subset \Sigma\mathcal{B}$. Let $\text{Grad}(n, \mathcal{B}) \subset F(n, \mathcal{B})$ be the subset of endomorphisms ε such that $\varepsilon^2 = 1$. There are obvious maps $F(n, \mathcal{A}) \rightarrow F(n, \mathcal{A}/\mathcal{R})$ and $\text{Grad}(n, \mathcal{A}) \rightarrow \text{Grad}(n, \mathcal{A}/\mathcal{R})$; ε_1 is a canonical base point for $\text{Grad}(n, \mathcal{A})$ whilst $\bar{\varepsilon}_1$ is one for $\text{Grad}(n, \mathcal{A}/\mathcal{R})$. If $\eta \in F(n, \mathcal{A}/\mathcal{R})$ and if $\gamma = \eta\bar{\varepsilon}_1$, $\gamma b = b\gamma$ for every $b \in \mathcal{A}/\mathcal{R}$. Hence $F(n, \mathcal{A}) \rightarrow F(n, \mathcal{A}/\mathcal{R})$ is surjective. Let $((\Sigma(\mathcal{A}/\mathcal{R}))^n, \bar{\varepsilon}_1, \eta)$ be a triple which specifies an element of $K^{\mathcal{A}/\mathcal{R}}(X)$. Now there exists an $\varepsilon \in F(n, \mathcal{A})$ such that $\bar{\varepsilon} = \eta$ and $\varepsilon^2 = 1 + k$, where $k \in \mathbf{M}_n(\mathcal{R})$ is nilpotent. So the square root $\sqrt{1+k} = 1 + \frac{1}{2}k - \dots$ exists. Put $\varepsilon_2 = \varepsilon\sqrt{1+k}$; then $\theta_*((\Sigma\mathcal{A})^n, \varepsilon_1, \varepsilon_2) = ((\Sigma(\mathcal{A}/\mathcal{R}))^n, \bar{\varepsilon}_1, \eta)$. Hence θ_* is surjective and the proof is complete.

Remark. — In order to calculate $K^{\mathcal{A}}(X)$ it is now sufficient to assume that $\mathcal{A}$ is semisimple. As $K^{\mathcal{A} \times \mathcal{B}}(X)$ is isomorphic to $K^{\mathcal{A}}(X) \oplus K^{\mathcal{B}}(X)$, it is further sufficient to assume that all the simple factors of the fibre of $\mathcal{A}$ are isomorphic. It is false, unfortunately, that all bundles of semisimple graded k -algebras are products of bundles of simple graded k -algebras.

The definition of $K^{\mathcal{A}}(X)$ may be generalized in many directions in the usual way. First of all, we introduced *relative groups* $K^{\mathcal{A}}(X, Y)$ when Y is a closed subspace of X (consider triples $(E, \varepsilon_1, \varepsilon_2)$ such that $\varepsilon_1|_Y = \varepsilon_2|_Y$ as in § 2.1 of [9]); for Y empty we recover the definition of $K^{\mathcal{A}}(X)$. If $\mathcal{C}$ is a Banach category, $\mathcal{C}$ -vector bundles may be considered instead of ordinary vector bundles (cf. [9]). Denote by $K^{\mathcal{A}}(X; \mathcal{C})$ ($K^{\mathcal{A}}(X, Y; \mathcal{C})$ in the relative case) the group so obtained. An interesting example (see below) is the category $\mathcal{C} = \check{\mathcal{H}}$ of [10]. Finally, if G is a compact Lie group acting continuously on X and $\mathcal{A}$, we may consider “ G - $\mathcal{A}$ -vector bundles” (i.e. there is the relation $g.(a.e) = (g.a).(g.e)$ for $g \in G$, $a \in \mathcal{A}_x$, $e \in E_x$, $x \in X$). In this way a group $K_G^{\mathcal{A}}(X)$ is obtained. A slight generalisation may be obtained by considering augmented groups as in [15].

$K^{\mathcal{A}}(X)$ is a homotopy invariant in the following sense. Let $\mathcal{A} \times I$ be a bundle of algebras over $X \times I$, where I is the unit interval. The inclusions $i_0: X \rightarrow X \times \{0\} \subset X \times I$ and $i_1: X \rightarrow X \times \{1\} \subset X \times I$ are homotopic. If a is an element of $K^{\mathcal{A} \times I}(X \times I)$, $i_0^*(a) = i_1^*(a)$.

6. The Thom isomorphism

By “abus de langage” let us denote by $\mathcal{A}, E, \dots$ the inverse images $f^*\mathcal{A}, f^*E, \dots$ for any map $f: Y \rightarrow X$. As in [9], there is a homomorphism:

$$t: K^{\Sigma\mathcal{A}}(X; \mathcal{C}) \rightarrow K^{\mathcal{A}}(X \times \mathbf{D}^1, X \times \mathbf{S}^0; \mathcal{C})$$

defined by the formula $t(E, \varepsilon_1, \varepsilon_2) = (E', \varepsilon_1(\theta), \varepsilon_2(\theta))$. Here E' is E regarded as an $\mathcal{A}$ -module and $\varepsilon_i(\theta)$ is the grading of E' defined over the point $\theta \in \mathbf{D}^1 = [0, \pi]$ by $\varepsilon_i(\theta) = \varepsilon \cos \theta + \varepsilon_i \sin \theta$ where $\varepsilon \in C^{0,1} \subset C^{0,1} \hat{\otimes} \mathcal{A}$ is the canonical generator of $C^{0,1}$.

Theorem 14. — For every Banach category $\mathcal{C}$, t is an isomorphism.

Proof. — The proof is analogous to that of Theorem (2.2.2) of [9].

Remark. — If ΣX denotes the pair $(X \times \mathbf{D}^1, X \times \mathbf{S}^0)$, the theorem takes the striking form $K^{\Sigma\mathcal{A}}(X) \approx K^{\mathcal{A}}(\Sigma X)$. No analogue of this theorem in algebraic K-theory is known. Theorem 14 is, of course, still true for all the generalizations mentioned in § 5. Also, if V is a vector bundle on X with a positive quadratic form Q , the methods of [9] define a homomorphism $t: K^{\mathcal{A} \hat{\otimes} C(V)}(X; \mathcal{C}) \rightarrow K^{\mathcal{A}}(B(V), S(V); \mathcal{C})$.

Theorem 15. — The generalized homomorphism t is an isomorphism.

Proof. — This may be proved as in [9] by using Mayer-Vietoris arguments (cf. [18]).

To show that $K^{\mathcal{A}}(X)$, where $\mathcal{A}$ is a bundle of simple central graded k -algebras on X , depends only on the class of $\mathcal{A}$ in $G\text{Br}(X)$, it is convenient to interpret $K^{\mathcal{A}}(X)$ as the *graded Grothendieck group* of the category $\mathcal{E}^{\mathcal{A}}(X)$ (cf. § 2.1 of [9]).

Theorem 16. — Let $E = E_0 \oplus E_1$ be a $\mathbf{Z}_2$ -graded k -vector bundle on X with graded endomorphism bundle $\text{END}(E)$. If $\mathcal{A}$ is a bundle of graded k -algebras on X , the additive functor $\varphi: \mathcal{E}^{\mathcal{A}}(X) \rightarrow \mathcal{E}^{\mathcal{A} \hat{\otimes} \text{END}(E)}(X)$ defined by $\varphi(F) = F \hat{\otimes} E$ is an equivalence of graded Banach categories. In particular, φ induces an isomorphism $\varphi_*: K^{\mathcal{A}}(X) \rightarrow K^{\mathcal{A} \hat{\otimes} \text{END}(E)}(X)$.

Proof. — As the question is local there is no loss of generality in assuming that $E_0 = X \times k^p$ and $E_1 = X \times k^q$. So $\mathcal{A} \hat{\otimes} \text{END}(E) = \mathbf{M}_{p+q}(\mathcal{A})$ with a certain grading. If μ' is a homomorphism from $\varphi(F)$ to $\varphi(F')$, linear algebra shows that μ' is of the form $\varphi(\mu)$. Hence φ is fully faithful. It remains to prove that φ is essentially surjective; it suffices to prove that $\mathcal{A} \otimes \text{END}(E)$ is isomorphic to some $\varphi(F)$. For $F = \mathcal{A} \otimes E$, this is satisfied.

Definition 17. — For $\alpha \in G\text{Br}(X)$, $K^\alpha(X)$ is defined (up to canonical isomorphism) as the group $K^{\mathcal{A}}(X)$ for any $\mathcal{A}$ with class α . (This is justified by Theorem 16.)

Let $\varphi : \mathcal{C} \rightarrow \mathcal{C}'$ be a quasi-surjective Serre functor (in the sense of [9]) between two Banach categories. As in [9] and [14] we can define a connecting homomorphism $\partial : K^{\mathcal{A}}(X \times \mathbf{D}^1, X \times \mathbf{S}^0; \mathcal{C}') \rightarrow K^{\mathcal{A}}(X; \varphi)$. This yields an exact sequence:

$$K^{\mathcal{A}}(X \times \mathbf{D}^1, X \times \mathbf{S}^0; \mathcal{C}) \rightarrow K^{\mathcal{A}}(X \times \mathbf{D}^1, X \times \mathbf{S}^0; \mathcal{C}') \rightarrow \\ \rightarrow K^{\mathcal{A}}(X; \varphi) \rightarrow K^{\mathcal{A}}(X; \mathcal{C}) \rightarrow K^{\mathcal{A}}(X; \mathcal{C}').$$

The homomorphism t of Theorem 14 may be used to obtain the theorem:

Theorem 18. — *There exists an exact sequence:*

$$K^{\Sigma \mathcal{A}}(X; \mathcal{C}) \rightarrow K^{\Sigma \mathcal{A}}(X; \mathcal{C}') \rightarrow K^{\mathcal{A}}(X; \varphi) \rightarrow K^{\mathcal{A}}(X; \mathcal{C}) \rightarrow K^{\mathcal{A}}(X; \mathcal{C}').$$

The following theorem is proved in the same way:

Theorem 19. — *If $\mathcal{A}$ is a bundle of graded k -algebras on the compact space X and Y is a closed subspace of X , there exists an exact sequence:*

$$K^{\Sigma \mathcal{A}}(X; \mathcal{C}) \rightarrow K^{\Sigma \mathcal{A}}(Y; \mathcal{C}) \rightarrow K^{\mathcal{A}}(X, Y; \mathcal{C}) \rightarrow K^{\mathcal{A}}(X; \mathcal{C}) \rightarrow K^{\mathcal{A}}(Y; \mathcal{C}).$$

7. The multiplicative structure

As in [10], let $\mathcal{H}$ be the Banach category of k -Hilbert spaces and let $\check{\mathcal{H}}$ be the Banach category with the same objects as $\mathcal{H}$ but with $\check{\mathcal{H}}(H_1, H_2) = \mathcal{H}(H_1, H_2) / \mathcal{K}(H_1, H_2)$ where $\mathcal{K}(H_1, H_2)$ is the space of all completely continuous maps $H_1 \rightarrow H_2$; $\varphi : \mathcal{H} \rightarrow \check{\mathcal{H}}$ will denote the canonical functor and $\mathcal{E}$ is the Banach category of finite dimensional k -vector spaces; X again denotes a compact space.

Let $\kappa : K^{\mathcal{A}}(X; \mathcal{E}) \rightarrow K^{\mathcal{A}}(X; \varphi)$ be defined (as in [10]) by $\kappa(E, \varepsilon_1, \varepsilon_2) = (E, \varepsilon_1, \varepsilon_2)$. Then the following is proved in the same way as Proposition 5 of [10]:

Proposition 20. — κ is an isomorphism. Also

$$\kappa^{-1} \cdot \partial : K^{\Sigma \mathcal{A}}(X; \check{\mathcal{H}}) \rightarrow K^{\mathcal{A}}(X; \mathcal{E}) \approx K^{\mathcal{A}}(X)$$

is an isomorphism.

More generally, if we have an exact sequence of prebanach categories (cf. [13]) $0 \rightarrow \mathcal{C}' \rightarrow \mathcal{C} \rightarrow \mathcal{C}'' \rightarrow 0$, there is an exact sequence:

$$K^{\Sigma \mathcal{A}}(X; \mathcal{C}') \rightarrow K^{\Sigma \mathcal{A}}(X; \mathcal{C}) \rightarrow K^{\Sigma \mathcal{A}}(X; \mathcal{C}'') \rightarrow K^{\mathcal{A}}(X; \mathcal{C}') \rightarrow K^{\mathcal{A}}(X; \mathcal{C}) \rightarrow \dots$$

In particular, if $\mathcal{C}$ is *flabby* (“flasque”, see Definition 3 of [13]),

$$K^{\Sigma \mathcal{A}}(X; \mathcal{C}'') \approx K^{\mathcal{A}}(X; \mathcal{C}').$$

Hence there is an isomorphism $K^{\mathcal{A}}(X; \mathcal{C}) \approx K^{\Sigma \mathcal{A}}(X; S\mathcal{C})$ for every Banach category $\mathcal{C}$.

As in [10] we can define a group $\bar{K}^{\mathcal{A}}(X)$ by considering self-adjoint Fredholm

operators. More precisely, consider couples (E, D) , where E is a Hilbert bundle on X which is a graded $\mathcal{A}$ -module (i.e. a $\Sigma\mathcal{A}$ -module) and where $D : E \rightarrow E$ is a *quasi-graduation* of E . This means that D is a continuous family of self-adjoint Fredholm operators commuting with elements of $(\Sigma\mathcal{A})_0$ and anti-commuting with elements of $(\Sigma\mathcal{A})_1$. Then $\bar{K}^{\mathcal{A}}(X)$ is the Grothendieck group associated to the commutative monoid of homotopy classes of such couples. Using spectral theory as in [10], we can define an isomorphism $u : \bar{K}^{\mathcal{A}}(X) \rightarrow K^{\Sigma\mathcal{A}}(X; \check{\mathcal{H}})$. Eventually the following analogue of Theorem 6 of [10] is obtained:

Theorem 21. — $j = \kappa^{-1} \cdot \partial \cdot u : \bar{K}^{\mathcal{A}}(X) \rightarrow K^{\mathcal{A}}(X)$ is an isomorphism.

The methods of [12] give an explicit inverse to this isomorphism. This fact is not needed here.

The groups $\bar{K}^{\mathcal{A}}(X)$ enable a *cup product* to be defined:

$$K^{\mathcal{A}}(X) \otimes K^{\mathcal{A}'}(X') \rightarrow K^{\mathcal{A} \hat{\otimes} \mathcal{A}'}(X \times X').$$

The formula $(E, D) \cup (E', D') = (E \hat{\otimes} E', D \hat{\otimes} 1 + 1 \hat{\otimes} D')$ is used, where $\hat{\otimes}$ means the *graded completed* tensor product. The multiplication is associative and distributive with respect to the addition. It is commutative in the following sense: define $T : \mathcal{A} \hat{\otimes} \mathcal{A}' \rightarrow \mathcal{A}' \hat{\otimes} \mathcal{A}$, covering the canonical isomorphism $X \times X' \rightarrow X' \times X$, by $T(x \otimes x') = (-1)^{\delta(x)\delta(x')} \cdot x' \otimes x$, where δ denotes the degree. Define $T' : E \hat{\otimes} E' \rightarrow E' \hat{\otimes} E$ similarly. Then the pair (T', T) define an isomorphism.

In particular, if $X = X'$ and $\mathcal{A}$ and $\mathcal{A}'$ are bundles of central simple graded k -algebras, this product composed with the restriction to the diagonal defines a product $K^{\alpha}(X) \otimes K^{\alpha'}(X) \rightarrow K^{\alpha\alpha'}(X)$, where α, α' are respectively the images of $\mathcal{A}, \mathcal{A}'$ in $\text{GBr}(X)$.

Remark. — In the case when $\mathcal{A}$ and $\mathcal{A}'$ are Clifford algebras of vector bundles, the cup product is usually defined by using the Thom isomorphism (Theorem 15). The methods of [10] and [14] show that all reasonable compatibilities between the two definitions hold.

8. Adams operations

Let $\mathcal{A}$ be a bundle of graded k -algebras on the compact space X . An action of the symmetric group $\mathfrak{S}_n$ on $\mathcal{A}^{\hat{\otimes} n}$ is defined by the formula (in which the elements a_i are supposed to be homogeneous):

$$\sigma \cdot (a_1 \hat{\otimes} a_2 \hat{\otimes} \dots \hat{\otimes} a_n) = (-1)^N \cdot a_{\sigma(1)} \hat{\otimes} a_{\sigma(2)} \hat{\otimes} \dots \hat{\otimes} a_{\sigma(n)},$$

where $N = N(\sigma; a_1, a_2, \dots, a_n)$ is the number of inversions induced by σ of two a_i 's of odd degree. Note that:

$$N(\tau; a_1, a_2, \dots, a_n) + N(\sigma; a_{\tau(1)}, a_{\tau(2)}, \dots, a_{\tau(n)}) \equiv N(\sigma\tau; a_1, a_2, \dots, a_n) \pmod{2};$$

this states that an $\mathfrak{S}_n$ -action is indeed defined by this rule and may be verified by examining the case when $\mathcal{A} = \mathbb{C}^{1,0}$. In this case the action of $\mathfrak{S}_n$ on $\mathcal{A}^{\hat{\otimes} n} = \mathbb{C}^{n,0}$ is simply that induced by the action of $\mathfrak{S}_n$ on $\mathbb{R}^n$.

Definition 22. — The symmetric tensor power $\varphi_n : K^{\mathcal{A}}(X) \rightarrow K_{\mathfrak{S}_n}^{\mathcal{A}^{\hat{\otimes} n}}(X)$ is defined by the formula:

$$\varphi_n(E, D) = (E^{\hat{\otimes} n}, D \hat{\otimes}_I \hat{\otimes} \dots \hat{\otimes}_I I + \dots + I \hat{\otimes} \dots \hat{\otimes}_I I \hat{\otimes} D).$$

Remarks. — The action of $\mathcal{A}^{\hat{\otimes} n}$ on $E^{\hat{\otimes} n}$ is given by the formula (in which the elements are homogeneous) $(a_1 \hat{\otimes} \dots \hat{\otimes} a_n) \cdot (e_1 \hat{\otimes} \dots \hat{\otimes} e_n) = (-1)^M \cdot a_1 e_1 \otimes \dots \otimes a_n e_n$ where $M = M(a_1, \dots, a_n; e_1, \dots, e_n)$ is the number of inversions of odd degree elements in the permutation $(a_1, \dots, a_n, e_1, \dots, e_n) \mapsto (a_1, e_1, a_2, e_2, \dots, a_n, e_n)$. (The example of $\mathbb{C}^{1,0}$ considered as a module over itself shows that this is well-defined, and that, if $\sigma \in \mathfrak{S}_n$, $\lambda \in \mathcal{A}^{\hat{\otimes} n}$, $e \in E^{\hat{\otimes} n}$, $\sigma \cdot (\lambda \cdot e) = (\sigma \cdot \lambda) \cdot (\sigma \cdot e)$. Hence φ_n is well-defined.) Note that if $\mathcal{A}$ is a Clifford bundle $C(V)$, φ_n is the algebraic translation of the map $K(V) \rightarrow K_{\mathfrak{S}_n}(V^n)$ of [4] (cf. [15]).

It is now convenient to make the following definition. Let $f : B \rightarrow C$ be an isomorphism of algebras. Let P be a B -module and Q be a C -module. A group (iso)morphism $\bar{f} : P \rightarrow Q$ will be called an (iso)morphism of modules if $\bar{f}(\lambda \cdot x) = f(\lambda) \cdot \bar{f}(x)$ for $\lambda \in B$ and $x \in P$. The example that we are interested in is when $B = (\mathcal{A} \hat{\otimes} \mathcal{A}')^{\hat{\otimes} n}$ (n copies); $C = (\mathcal{A} \hat{\otimes} \dots \hat{\otimes} \mathcal{A}) \hat{\otimes} (\mathcal{A}' \hat{\otimes} \dots \hat{\otimes} \mathcal{A}')$ (n copies in each bracket); $P = (E \hat{\otimes} E')^{\hat{\otimes} n}$ and $Q = E^{\hat{\otimes} n} \hat{\otimes} E'^{\hat{\otimes} n}$. Here E is an $\mathcal{A}$ -module, E' an $\mathcal{A}'$ -module and $\mathcal{A}$ and $\mathcal{A}'$ are bundles of graded k -algebras on X and X' respectively. $f : B \rightarrow C$ is defined by the formula $f(a_1 \hat{\otimes} a'_1 \hat{\otimes} \dots \hat{\otimes} a_n \hat{\otimes} a'_n) = (-1)^M (a_1 \hat{\otimes} \dots \hat{\otimes} a_n) \hat{\otimes} (a'_1 \hat{\otimes} \dots \hat{\otimes} a'_n)$ with M as above, and the isomorphism $\bar{f}$ is defined by an analogous formula. Hence there is induced a canonical isomorphism $T : K_{\mathfrak{S}_n}^{(\mathcal{A} \hat{\otimes} \mathcal{A}')^{\hat{\otimes} n}}(X \times X') \rightarrow K_{\mathfrak{S}_n}^{\mathcal{A}^{\hat{\otimes} n} \hat{\otimes} \mathcal{A}'^{\hat{\otimes} n}}(X \times X')$. Then the operations φ_n are seen to have the multiplicative property expressed by the commutativity of:

$$\begin{array}{ccc} K^{\mathcal{A}}(X) \times K^{\mathcal{A}'}(X') & \longrightarrow & K^{\mathcal{A} \hat{\otimes} \mathcal{A}'}(X \times X') \\ \downarrow \varphi_n^{\mathcal{A}} \times \varphi_n^{\mathcal{A}'} & & \downarrow \varphi_n \\ & & K_{\mathfrak{S}_n}^{(\mathcal{A} \hat{\otimes} \mathcal{A}')^{\hat{\otimes} n}}(X \times X') \\ & & \downarrow T \\ K_{\mathfrak{S}_n}^{\mathcal{A}^{\hat{\otimes} n}}(X) \times K_{\mathfrak{S}_n}^{\mathcal{A}'^{\hat{\otimes} n}}(X') & \longrightarrow & K_{\mathfrak{S}_n}^{\mathcal{A}^{\hat{\otimes} n} \hat{\otimes} \mathcal{A}'^{\hat{\otimes} n}}(X \times X') \end{array}$$

The groups $K_{\mathfrak{S}_n}^{\mathcal{A}^{\hat{\otimes} n}}(X)$ seem to be very hard to compute in general. For example, if X is a point and $\mathcal{A} = \mathbb{C}^{1,0}$, the explicit determination of this group is related to the

explicit determination of the representation ring of a certain group $\mathfrak{F}'_n$, which double covers $\mathfrak{S}_n$. Part of this task has been done by Schur [16].

In order to define more acceptable invariants, the following basic idea is useful. Let G be a subgroup of $\mathfrak{S}_n$ such that there exists a homomorphism $\mu : G \rightarrow U(\mathcal{A}^{\hat{\otimes} n})$ (U denotes the group of invertible elements of degree 0) such that $\sigma \cdot a = \mu(\sigma) \cdot a \cdot \mu(\sigma^{-1})$ for $\sigma \in G$ and $a \in \mathcal{A}^{\hat{\otimes} n}$. Then we can "twist" the action of G on every G - $\mathcal{A}^{\hat{\otimes} n}$ -module P in the following way: set $\sigma * m = \mu(\sigma^{-1}) \sigma \cdot m$. Note that for all $m \in P$, $\sigma \in G$, $\tau \in G$ and $a \in \mathcal{A}^{\hat{\otimes} n}$, $\sigma * (\tau * m) = (\sigma \tau) * m$ and $\sigma * (am) = a(\sigma * m)$.

Lemma 23. — *If $\mathcal{E}^{\mathcal{A}^{\hat{\otimes} n}}$ is the category of graded G - $\mathcal{A}^{\hat{\otimes} n}$ -modules with the trivial action of G on $\mathcal{A}^{\hat{\otimes} n}$ and if $K_{(G)}^{\mathcal{A}^{\hat{\otimes} n}}$ is the associated graded Grothendieck group (G is as above), the graded categories $\mathcal{E}_{(G)}^{\mathcal{A}^{\hat{\otimes} n}}$ and $\mathcal{E}_G^{\mathcal{A}^{\hat{\otimes} n}}$ are isomorphic. In particular, $K_{(G)}^{\mathcal{A}^{\hat{\otimes} n}}(X) \approx K_G^{\mathcal{A}^{\hat{\otimes} n}}$.*

Proof. — Define $\varphi : \mathcal{E}_G^{\mathcal{A}^{\hat{\otimes} n}}(X) \rightarrow \mathcal{E}_{(G)}^{\mathcal{A}^{\hat{\otimes} n}}$ by $\varphi(P) = P'$ and $\varphi(f) = f'$, where P' is P as an $\mathcal{A}^{\hat{\otimes} n}$ -module with the action $*$ of G and where f' coincides with f on the underlying $\mathcal{A}^{\hat{\otimes} n}$ -modules. Since $\mu(\sigma)$ is of degree 0, the new action of G is of degree 0 and the functor is well-defined. Likewise, an inverse φ' to φ may be constructed.

Two cases are of special interest. Firstly, that when $G = \mathbf{Z}_n$, the cyclic group generated by the permutation $g = (1 \ 2 \ \dots \ n)$; n will always be taken to be *odd*; if further $n = 4p + 1$, the second case is when $G = \mathfrak{D}_n$, the dihedral group generated by g and the permutation $t = (1 \ 4p)(2 \ 4p-1) \dots (2p \ 2p+1)$. $\mathbf{Z}_n$ is normal in $\mathfrak{D}_n$; the quotient group is $\mathbf{Z}_2$ and it acts on $\mathbf{Z}_n$ by inversion. $\mathfrak{D}_n$ is generated by g and t with the relations $g^n = t^2 = 1$ and $tg = g^{-1}t$.

Lemma 24. — *Let $\mathcal{A}$ be a bundle of central simple graded k -algebras on X . If n is odd, there exists a homomorphism $\mu : \mathbf{Z}_n \rightarrow U(\mathcal{A}^{\hat{\otimes} n})$ such that $\sigma \cdot a = \mu(\sigma) a \mu(\sigma^{-1})$ for $\sigma \in \mathbf{Z}_n$ and $a \in \mathcal{A}^{\hat{\otimes} n}$. If $n = 4p + 1$, there also exists a homomorphism $\mu' : \mathfrak{D}_n \rightarrow U(\mathcal{A}^{\hat{\otimes} n})$ extending μ such that $\sigma \cdot a = \mu'(\sigma) a \mu'(\sigma^{-1})$ for $\sigma \in \mathfrak{D}_n$ and $a \in \mathcal{A}^{\hat{\otimes} n}$.*

Proof. — The Schur multipliers $H^2(\mathbf{Z}_n, \mathbf{R}^*)$, $H^2(\mathbf{Z}_n, \mathbf{C}^*)$ and $H^2(\mathfrak{D}_n, \mathbf{C}^*)$ are all zero; [16] and a little calculation show that $H^2(\mathfrak{S}_n, \mathbf{R}^*) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and that the restriction to $H^2(\mathfrak{D}_n, \mathbf{R}^*) = \mathbf{Z}_2$ maps the element corresponding to $\mathfrak{F}'_n$ to 0. The Skolem-Noether theorem and some explicit checking show that if A is any of the algebras listed in § 1, all automorphisms σ of A are induced by conjugation by a unit $x \in A_0 \cup A_1$, determined up to a central homogeneous unit. As $g^n = 1$, its action must be induced by a unit $x \in A_0$. As t is the product of an even number of conjugate elements of $\mathfrak{S}_n$, its action must also be induced by a unit $x \in A_0$. The result now follows from the theory of projective representations.

Denote by $v : K_G^{\mathcal{A}^{\hat{\otimes} n}}(X) \rightarrow K_{(G)}^{\mathcal{A}^{\hat{\otimes} n}}(X)$ the isomorphism of Lemma 23 for $G = \mathbf{Z}_n$

or $\mathfrak{D}_n$ and n as above. Then it is easily seen that the following diagram commutes:

$$\begin{array}{ccc}
 K_G^{\mathcal{A} \hat{\otimes} n}(X) \times K_G^{\mathcal{A}' \hat{\otimes} n}(X') & \longrightarrow & K_G^{\mathcal{A} \hat{\otimes} n \hat{\otimes} \mathcal{A}' \hat{\otimes} n}(X \times X') \approx K_G^{(\mathcal{A} \hat{\otimes} \mathcal{A}') \hat{\otimes} n}(X \times X') \\
 \downarrow v \times v & & \downarrow v \\
 K_{(G)}^{\mathcal{A} \hat{\otimes} n}(X) \times K_{(G)}^{\mathcal{A}' \hat{\otimes} n}(X') & \longrightarrow & K_{(G)}^{\mathcal{A} \hat{\otimes} n \hat{\otimes} \mathcal{A}' \hat{\otimes} n}(X \times X') \approx K_{(G)}^{(\mathcal{A} \hat{\otimes} \mathcal{A}') \hat{\otimes} n}(X \times X')
 \end{array}$$

We now treat in detail the case when $k = \mathbf{C}$ and $G = \mathbf{Z}_n$. The case when $k = \mathbf{R}$ and $G = \mathfrak{D}_n$ will be dealt with later.

The following lemma (which may be generalised to the case when $\mathbf{Z}_n$ is replaced by any other finite group) is proved in the same way as Proposition (2.1) of [4], i.e. by use of the canonical projection operators in the group algebra.

Lemma 25. — *Let F be a complex Hilbert bundle on X with an action of the cyclic group $\mathbf{Z}_n$. Then the bundle F splits into the sum $F_0 \oplus F_1 \oplus \dots \oplus F_{n-1}$. The generator g of $\mathbf{Z}_n$ acts on F_r by the multiplication by ω^r , where $\omega = \exp(2\pi i/n)$.*

If $D : F \rightarrow F'$ is a morphism of $\mathbf{Z}_n$ -bundles, we shall write $D_r : F_r \rightarrow F'_r$ for the restriction of D to F_r and F'_r . Let Ω_n be the subring of $\mathbf{C}$ generated over $\mathbf{Z}$ by the above ω . In other words, $\Omega_n = \mathbf{Z}[\omega]/(\Phi_n(\omega))$, where Φ_n is the n -th cyclotomic polynomial.

Definition 26. — *If (F, D) is an element of $KU_{(\mathbf{Z}_n)}^{\mathcal{A} \hat{\otimes} n}(X)$, $\text{Tr}(F, D)$, its trace, is the element $\sum_{r=0}^{n-1} (F_r, D_r) \otimes \omega^r$ of $KU^{\mathcal{A} \hat{\otimes} n}(X) \otimes \Omega_n$.*

Proposition 27. — *The “trace function” Tr is multiplicative. In other words, if $(F, D) \in KU_{(\mathbf{Z}_n)}^{\mathcal{A} \hat{\otimes} n}(X)$ and $(F', D') \in KU_{(\mathbf{Z}_n)}^{\mathcal{A}' \hat{\otimes} n}(X')$, the elements $\text{Tr}(F, D) \cup \text{Tr}(F', D')$ and $\text{Tr}(F \hat{\otimes} F', D \hat{\otimes} I + I \hat{\otimes} D')$ are equal when we identify $KU^{\mathcal{A} \hat{\otimes} n \otimes \mathcal{A}' \hat{\otimes} n}(X \times X')$ and $KU^{(\mathcal{A} \hat{\otimes} \mathcal{A}') \hat{\otimes} n}(X \times X')$ by the canonical isomorphism.*

Proof. — The proposition is a direct consequence of the definitions and of the fact that $\omega^n \omega^m = \omega^{n+m}$.

Lemma 28. — *Let $d > 1$ be a divisor of n . Let the couples $(G_0, D_0), \dots, (G_{d-1}, D_{d-1})$ be as in § 7 and suppose that the generator g of $\mathbf{Z}_n$ acts on $(F, D) = (G_0 \oplus \dots \oplus G_{d-1}, D_0 \oplus \dots \oplus D_{d-1})$ by the matrix $\gamma = (\gamma_{ij})$, where the only non-zero γ_{ij} are $\gamma_{10} = \gamma_0, \gamma_{21} = \gamma_1, \dots, \gamma_{d-1, d-2} = \gamma_{d-2}$ and $\gamma_{0, d-1} = \gamma_{d-1}$. Then $\text{Tr}(F, D) = 0$.*

Proof. — As G_0 and G_i can be identified by the isomorphism $\gamma_{i-1} \dots \gamma_0$, there is no loss of generality in assuming that $G_0 = G_i, D_0 = D_1 = \dots = D_{d-1}$ and that each γ_i is the identity. Then $\text{Ker}(g - \omega^{n/d}) \approx G_0$. As $\sum_r \omega^{rn/d} = 0$, $\text{Tr}(F, D) = 0$.

Definition 29. — The “Adams operation”:

$$\psi^n: KU^{\mathcal{A}}(X) \rightarrow KU^{\hat{\otimes}^n \mathcal{A}}(X) \otimes \Omega_n$$

is the composite:

$$KU^{\mathcal{A}}(X) \xrightarrow{\varphi_n^{\mathcal{A}}} KU_{\mathfrak{S}_n}^{\hat{\otimes}^n \mathcal{A}}(X) \rightarrow KU_{\mathbf{Z}_n}^{\hat{\otimes}^n \mathcal{A}}(X) \xrightarrow{v} KU_{(\mathbf{Z}_n)}^{\hat{\otimes}^n \mathcal{A}}(X) \xrightarrow{\text{Tr}} KU^{\hat{\otimes}^n \mathcal{A}}(X) \otimes \Omega_n.$$

Theorem 30. — The Adams operation ψ^n is additive, i.e. $\psi^n(x+y) = \psi^n(x) + \psi^n(y)$. Moreover, it is multiplicative in the sense that there is a commutative diagram:

$$\begin{array}{ccc} KU^{\mathcal{A}}(X) \otimes KU^{\mathcal{A}'}(X') & \xrightarrow{\quad U \quad} & KU^{\hat{\otimes} \mathcal{A} \otimes \mathcal{A}'}(X \times X') \\ \downarrow \psi^n \otimes \psi^n & & \downarrow \psi^n \\ KU^{\hat{\otimes}^n \mathcal{A}}(X) \otimes KU^{\hat{\otimes}^n \mathcal{A}'}(X') & \xrightarrow{\quad U \quad} KU^{\hat{\otimes}^n \mathcal{A} \otimes \mathcal{A}'}(X \times X') \otimes \Omega_n \xleftarrow{\quad T \quad} KU^{(\mathcal{A} \otimes \mathcal{A}') \hat{\otimes}^n}(X \times X') \otimes \Omega_n \end{array}$$

Proof. — The second assertion is a direct consequence of the diagrams following Definition 22 and Lemma 24, and of Proposition 27. Write $x = (E_0, D_0)$ and $y = (E_1, D_1)$. Then $(E_0 \oplus E_1)^{\hat{\otimes} n} = \bigoplus (E_{i_1} \hat{\otimes} E_{i_2} \hat{\otimes} \dots \hat{\otimes} E_{i_n})$, where the direct sum is taken over all i_r . This expression is the sum of $E_0^{\hat{\otimes} n}$, $E_1^{\hat{\otimes} n}$ and of bundles of the form $G = G_0 \oplus \dots \oplus G_{d-1}$ ($d > 1$), where d divides n . Here the action of $\mathbf{Z}_n$ on G is as in Lemma 28. So $\text{Tr}(F, D) = 0$. Hence $\psi^n(x+y) = \psi^n(x) + \psi^n(y)$, as required.

Proposition 31. — Let $E = E_0 \oplus E_1$ be a $\mathbf{Z}_2$ -graded $\mathbf{C}$ -vector bundle on X with graded endomorphism bundle $\mathcal{A} = \text{END}(E)$. Let λ be the isomorphism φ_* of Theorem 16. Then the following diagram is commutative:

$$\begin{array}{ccccc} KU(X) & \xrightarrow{\psi^n} & KU(X) & \xrightarrow{i} & KU(X) \otimes \Omega_n \\ \downarrow \lambda & & & & \downarrow \lambda \otimes 1 \\ KU^{\mathcal{A}}(X) & \xrightarrow{\psi^n} & & & KU^{\hat{\otimes}^n \mathcal{A}}(X) \otimes \Omega_n \end{array}$$

where the upper ψ^n is the ordinary Adams operation $([1], [4])$ and i is the inclusion.

Proof. — If $\mathcal{A} = \mathbf{C}$, the argument of [4] shows that our ψ^n is the usual ψ^n operation. For other $\mathcal{A}$, λ is defined by the cup product with E or $E^{\hat{\otimes} n}$ as appropriate. The action of $\mathfrak{S}_n$ on $\mathcal{A}^{\hat{\otimes} n}$ is induced by the natural action of $\mathfrak{S}_n$ on $E^{\hat{\otimes} n}$ and is thus given by inner automorphisms. This implies that the action of $\mathbf{Z}_n$ on $(v\varphi_n^{\mathcal{A}})(E, D)$ is trivial. The result follows.

Remark. — If $\mathcal{B}$ is another bundle of simple central graded $\mathbf{C}$ -algebras, exactly the same proof shows the commutativity of the diagram:

$$\begin{array}{ccc} \mathrm{KU}^{\mathcal{B}}(\mathrm{X}) & \xrightarrow{\psi^n} & \mathrm{KU}^{\widehat{\mathcal{B}}^n}(\mathrm{X}) \otimes \Omega_n \\ \downarrow \lambda & & \downarrow \lambda \\ \mathrm{KU}^{\mathcal{B} \widehat{\otimes} \mathcal{A}}(\mathrm{X}) & \xrightarrow{\psi^n} & \mathrm{KU}^{(\mathcal{B} \widehat{\otimes} \mathcal{A})^n}(\mathrm{X}) \otimes \Omega_n \approx \mathrm{KU}^{\widehat{\mathcal{B}}^n \widehat{\otimes} \mathcal{A}^n}(\mathrm{X}) \otimes \Omega_n \end{array}$$

This shows that ψ^n is essentially an operation from $\mathrm{KU}^\alpha(\mathrm{X})$ to $\mathrm{KU}^{\alpha^n}(\mathrm{X}) \otimes \Omega_n$ for $\alpha \in \mathrm{GBrU}(\mathrm{X})$.

Now consider the case when $k = \mathbf{R}$ and $n = 4p + 1$. Following End [6], we will define operations ψ^n in this case. In view of the preceding discussion, we need define only:

$$\mathrm{Tr} : \mathrm{KO}_{(\mathfrak{D}_n)}^{\widehat{\mathcal{A}}^n}(\mathrm{X}) \rightarrow \mathrm{KO}^{\widehat{\mathcal{A}}^n}(\mathrm{X}) \otimes \Omega_n.$$

If F is a (real) Hilbert bundle on X with an action of $\mathfrak{D}_n$, set $F' = F \otimes \mathbf{C}$ and $F'_s = \mathrm{Ker}(g - \omega^s)$. Let c denote the complex conjugation on F' . Then its restriction $c_s : F'_s \rightarrow F'_{-s}$ is $\mathbf{C}$ -anti-linear whilst the restriction of t , $t_{-s} : F'_{-s} \rightarrow F'_s$ is $\mathbf{C}$ -linear. So F'_s is naturally isomorphic to the complexification of the *real* Hilbert bundle $F_s = \mathrm{Ker}(tc - 1) \cap \mathrm{Ker}(g - \omega^s)$. If $D : F \rightarrow G$ is a morphism of real Hilbert $\mathfrak{D}_n$ -bundles, write $D_s : F_s \rightarrow G_s$ for the restriction. If (F, D) are as in § 8, define:

$$\mathrm{Tr}(F, D) = \sum_s (F_s, D_s) \otimes \omega^s.$$

As in Definition 29, the ψ^n operation is defined to be the composite:

$$\mathrm{KO}^{\mathcal{A}}(\mathrm{X}) \xrightarrow{\varphi_n^{\mathcal{A}}} \mathrm{KO}_{\mathfrak{D}_n}^{\widehat{\mathcal{A}}^n}(\mathrm{X}) \longrightarrow \mathrm{KO}_{\mathfrak{D}_n}^{\widehat{\mathcal{A}}^n}(\mathrm{X}) \xrightarrow{v} \mathrm{KO}_{(\mathfrak{D}_n)}^{\widehat{\mathcal{A}}^n}(\mathrm{X}) \xrightarrow{\mathrm{Tr}} \mathrm{KO}^{\widehat{\mathcal{A}}^n}(\mathrm{X}) \otimes \Omega_n.$$

It is easy to extend all the propositions proved above to the real case. Moreover, the following diagram, in which c is the complexification, commutes:

$$\begin{array}{ccc} \mathrm{KO}^{\mathcal{A}}(\mathrm{X}) & \xrightarrow{\psi^n} & \mathrm{KO}^{\widehat{\mathcal{A}}^n}(\mathrm{X}) \otimes \Omega_n \\ \downarrow c & & \downarrow c \\ \mathrm{KU}^{\mathcal{A} \otimes \mathbf{C}}(\mathrm{X}) & \xrightarrow{\psi^n} & \mathrm{KU}^{(\mathcal{A} \otimes \mathbf{C})^n}(\mathrm{X}) \otimes \Omega_n \approx \mathrm{KU}^{\widehat{\mathcal{A}}^n \otimes \mathbf{C}}(\mathrm{X}) \otimes \Omega_n \end{array}$$

The following diagram, in which r is the realification, also commutes (cf. [6]):

$$\begin{array}{ccc}
 \mathrm{KU}^{\mathcal{A} \otimes \mathbb{C}}(\mathrm{X}) & \xrightarrow{\psi^n} & \mathrm{KU}^{(\mathcal{A} \otimes \mathbb{C})^{\hat{\otimes} n}}(\mathrm{X}) \otimes \Omega_n \approx \mathrm{KU}^{\hat{\otimes} n}(\mathrm{X}) \otimes \Omega_n \\
 \downarrow r & & \downarrow r \\
 \mathrm{KO}^{\mathcal{A}}(\mathrm{X}) & \xrightarrow{\psi^n} & \mathrm{KO}^{\hat{\otimes} n}(\mathrm{X}) \otimes \Omega_n
 \end{array}$$

Remark. — Simple examples show that Ω_n is necessary to define ψ^n in both cases.

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