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*Publications mathématiques de l'I.H.É.S.*, tome 33 (1967), p. 139-155

[http://www.numdam.org/item?id=PMIHES\\_1967\\_\\_33\\_\\_139\\_0](http://www.numdam.org/item?id=PMIHES_1967__33__139_0)

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# ALGEBRAIC EQUATIONS FOR NONSMOOTHABLE 8-MANIFOLDS

by NICOLAAS H. KUIPER <sup>(1)</sup>

## SUMMARY

The singularities of Brieskorn and Hirzebruch are used in order to obtain examples of algebraic varieties of complex dimension four in  $P^5(\mathbf{C})$ , which are homeomorphic to closed combinatorial 8-manifolds, but not homeomorphic to any differentiable manifold. Analogous nonorientable real algebraic varieties of dimension 8 in  $P^{10}(\mathbf{R})$  are also given. The main theorem states that every closed combinatorial 8-manifold is homeomorphic to a Nash-component with at most one singularity of some real algebraic variety. This generalizes the theorem of Nash for differentiable manifolds.

### § 1. Introduction. The theorem of Wall.

From the smoothing theory of Thom [1], Munkres [2] and others and the knowledge of the groups of differential structures on spheres due to Kervaire, Milnor [3], Smale and Cerf [4] follows a.o. that closed combinatorial  $n$ -manifolds for  $n \leq 7$  are *smoothable*. That is, they admit a combinatorially compatible differential structure. This structure is unique up to equivalence for  $n \leq 6$ . By a *manifold* we mean a connected closed combinatorial manifold. We will consider manifolds of dimension eight. In § 1, 2, and 3 all manifolds will be oriented. Let  $X$  be an oriented 8-manifold and  $X^k$  the  $k$ -skeleton of some triangulation of  $X$ . If the number of vertices is  $N$ , then let  $X^0$  be the set of end-points of  $N$  orthonormal unitvectors in euclidean vector  $N$ -space  $E^N$ . The simplices of  $X^k$  are then fixed and  $X$  lies embedded in  $E^N$ . For any  $W \subset X \subset E^N$  and  $\delta > 0$  we define the neighbourhood  $U(W, \delta) = \{x \in X \mid \text{distance}(x, W) < \delta\}$  of  $W$  in  $X$ .

For small  $\delta$ , say  $\delta < N^{-1}$ ,  $U(X^6, \delta)$  can be given a differential structure  $\mathcal{D}$  and this is unique up to equivalence. Next we construct a differential structure on  $U(X^7, \delta^2)$  which equals the first structure  $\mathcal{D}$  on  $U(X^6, \delta^2)$ . For that we have to define for every 7-simplex  $\Delta_7$  of  $X^7$  some differential structure on  $U(\Delta_7, \delta^2)$  which agrees with  $\mathcal{D}$  on  $U(X^6, \delta^2) \cap U(\Delta_7, \delta^2) = U(\partial\Delta_7, \delta^2)$ . This is possible in essentially 28 different ways, because the difference between two such smoothings corresponds with a smoothing

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of  $S^7 \times (0, 1)$  or of  $S^7$ , hence with an element of the group  $\Gamma_7 \simeq \mathbf{Z}_{28}$  of differential structures on  $S^7$  modulo those that can be extended to the 8-ball.

For each oriented 8-simplex  $\Delta_8$  of  $X^8$  we have now smoothed some neighbourhood of the boundary  $U(\partial\Delta_8, \delta^2) \cap \Delta_8$ . If this smoothing, restricted to  $U(X^7, \delta^2) \cap \Delta_8$ , can be extended over  $\Delta_8$ , then we assign to  $\Delta_8$  the element  $0 \in \Gamma_7$ . More generally, following M. Hirsch [5], we observe that any smooth oriented 7-manifold in  $U(\partial\Delta_8, \delta^2) \cap \Delta_8$ , which is combinatorially isotopic to  $\partial\Delta_8$ , will have the same structure of an exotic 7-sphere, and so it determines an element  $\gamma(\Delta_8) \in \Gamma_7$ . Indeed any two such manifolds are  $h$ -cobordant with some third that bounds a combinatorial 8-disc containing both, and hence all three are diffeomorphic.

In the top dimensional case the sheaf of coefficients (local coefficients of Munkres) is constant and it can be identified with  $\Gamma_{n-1} = \Gamma_7$ . This is the case for  $X$  orientable as well as  $X$  nonorientable.

The function  $\gamma$  on oriented simplices in  $X^8$ :

$$\gamma : \Delta_8^{(i)} \rightarrow \gamma(\Delta_8^{(i)}) \in \Gamma_7$$

is a cochain, which is a cocycle as there are no 9-simplices in  $X$ . So the value of  $\gamma$  on the fundamental cycle of the oriented  $X$  is

$$\gamma([X]) = \sum_i \gamma(\Delta_8^{(i)}) \in \Gamma_7.$$

If we change our choice of differential structure at one of the 7-simplices by  $\xi \in \Gamma_7$ , then the cochain value in the two adjacent 8-simplices alters by  $\xi$  and  $-\xi$  respectively, and we obtain a cohomologous cochain with unaltered value  $\gamma([X]) \in \Gamma_7$ . This element represents a cohomology-class  $\bar{\gamma}(X) \in H^8(X, \Gamma_7) = \Gamma_7$  which is *an invariant of the combinatorial 8-manifold*.

As  $X$  is connected, there is one choice of differential structures in the 7-simplices such that  $\gamma(\Delta_8^{(i)}) = 0$  for all except at most one of the 8-simplices. We transport all obstruction to smoothing to *one* 8-simplex. Then on that 8-simplex the value of the cochain is  $\bar{\gamma}(X)$ . We see that the nonsmoothability of an 8-manifold can be concentrated in an arbitrarily small neighbourhood  $N(p)$  of any point  $p$ . Any subdivision of the given triangulation, for which  $N(p)$  is interior to an 8-simplex therefore gives the same value for  $\bar{\gamma}(X)$ , which is then an invariant not only of the triangulation but of the combinatorial structure of  $X$ . From the above procedure follows :

*Lemma 1. — The 8-manifold  $X$  has a compatible smoothing if and only if the combinatorial invariant  $\bar{\gamma}(X) \in \Gamma_7 \simeq \mathbf{Z}_{28}$  vanishes.*

If  $X$  and  $Y$  are 8-manifolds, then the *connected sum*  $X \# Y$  is the oriented 8-manifold obtained by deleting from  $X$  and  $Y$  each one 8-simplex and identifying the boundaries, say linear on each 7-simplex of this boundary, so that a connected manifold is obtained and the injections of the remaining parts of  $X$  and  $Y$  are imbedded in  $X \# Y$  with preservation of orientation. The negative of  $X$  is the same non-oriented manifold with the other orientation.

**Lemma 2.** — For any 8-manifold  $X$ ,  $X \# (-X)$  is smoothable.

*Proof.* — Let  $\Delta$  and  $\Delta'$  be two 8-simplices of a triangulation of  $X$ ,  $p$  an interior point of  $\Delta'$  with neighbourhood  $U(p, \delta) \subset \Delta'$ . Take a smoothing of  $X - U(p, \delta)$ , in which  $\partial\Delta$  is a smooth usual 7-sphere. Take the smooth connected sum of  $X$  and  $-X$  along  $\partial\Delta \subset X$  and  $-\partial\Delta \subset -X$ . The combinatorial manifold  $X \# (-X)$  has then a natural smoothing, except in  $U(p, \delta) \subset X - (\Delta)$  and in the corresponding neighbourhood in  $(-X) - (-\Delta)$ .

The cochain on the triangulation of  $X \# (-X)$  has values  $\gamma([X])$  and  $-\gamma([X])$  on the two exceptional 8-simplices and zero elsewhere. Hence  $\bar{\gamma}(X \# (-X)) = 0$  and lemma 2 follows from lemma 1.

The 8-manifolds  $X$  and  $Y$  are called *equi-smoothable* or *equal modulo smooth manifolds*,  $X \sim Y$ , if  $X \# (-Y)$  is smoothable.

**Lemma 3.** — *Equi-smoothability is an equivalence relation.*

*Proof.* — Applying the above procedure of concentrating the essential contribution of the cochain  $\gamma$  into one 8-simplex, to the 8-manifolds  $X$  and  $Y$ , it follows immediately from lemma 1 that

$$X \sim Y \Leftrightarrow \bar{\gamma}(X) = \bar{\gamma}(Y).$$

*The Theorem of C. T. C. Wall [6].* — The equi-smoothability classes of oriented 8-manifolds (also called the combinatorial modulo smoothable 8-manifold classes) form a group isomorphic with  $\Gamma_7 \simeq (\mathbf{Z}_{28}, +)$  under connected sum  $\#$ .

*Proof.* — Again by the choice of special cochains for  $X$  and  $Y$  one sees:

$$\bar{\gamma}(X \# Y) = \bar{\gamma}(X) + \bar{\gamma}(Y).$$

Then  $\bar{\gamma}$  defines a homomorphism of the associative semi-group of oriented 8-manifolds with connected sum, onto  $\Gamma_7 \simeq \mathbf{Z}_{28}$ . By the proof of lemma 3 the equivalence classes are the 28 fibres of this map.

## § 2. Topological invariance of $\bar{\gamma}$ .

D. Sullivan [20] proved that any two combinatorial structures on a simply connected closed topological manifold of dimension  $\geq 6$  without 2-torsion in  $H^3(-, \mathbf{Z})$ , are combinatorially equivalent (Hauptvermutung). Hence  $\bar{\gamma}$  is a topological invariant for such manifolds.

C. T. C. Wall kindly brought to my attention that the topological invariance of the rational Pontrjagin classes, obtained by Novikov [12], implies that  $\bar{\gamma}$  is a topological invariant for an even larger class of 8-manifolds. This can be seen as follows.

Borel and Hirzebruch proved in [7], p. 494, that for a smooth closed oriented manifold  $X$

$$\hat{A}(X, d/2) = e^{d/2} \sum_{j=0}^{\infty} \hat{A}_j(p_1, \dots, p_j)[X]$$

is an integer. Here  $d \in H^2(X, \mathbf{Z})$  is any element which reduces in  $H^2(X, \mathbf{Z}_2)$  to the

second Whitney-class  $w_2(X)$ . So we have to assume the existence of  $d$ . For complex manifolds  $d$  exists and can be taken to be the first Chern class  $c_1$ .

One finds, with

$$\hat{A}_1 = \frac{1}{24} p_1, \quad \hat{A}_2 = \frac{2^{-7}}{45} (-4p_2 + 7p_1^2),$$

and with the formula for the signature:

$$\sigma(X) = \frac{1}{45} (7p_2 - p_1^2) [X],$$

that

$$\hat{A}\left(X, \frac{d}{2}\right) = \left(\frac{p_1^2 - 4\sigma}{896} - \frac{d^2 p_1}{192} + \frac{d^4}{384}\right) [X].$$

We now prove the formula

$$(*) \quad \bar{\gamma}(X) \equiv -28 \hat{A}\left(X, \frac{d}{2}\right) \pmod{28}.$$

*Proof.* — Let  $W$  be Milnor's example of a parallelisable 8-manifold with as boundary the exotic 7-sphere  $\partial W$  that represents the generator of  $\Gamma_7$ .  $M$  is the closed combinatorial manifold obtained by closing  $W$  with an 8-ball. Then  $\sigma(M) = 8$ .  $Y = X \# M \# \dots \# M$  is the connected sum of  $X$  and  $m$  copies of  $M$ .

One obtains, because  $X$  and  $Y$  have  $p_1^2$  and  $d$  in common,

$$\hat{A}\left(Y, \frac{d}{2}\right) = \hat{A}\left(X, \frac{d}{2}\right) - \frac{4m}{896} \sigma(M) = \hat{A}\left(X, \frac{d}{2}\right) - \frac{m}{28}.$$

By § 1,  $Y$  has a smoothing compatible with the given combinatorial structure for exactly one value of  $m \pmod{28}$ . This value is given by

$$0 = \bar{\gamma}(Y) = \bar{\gamma}(X) + m \pmod{28}.$$

For that value of  $m$  we also have

$$0 = \hat{A}\left(Y, \frac{d}{2}\right) = \hat{A}\left(X, \frac{d}{2}\right) - \frac{m}{28} = 0 \pmod{1},$$

and the formula follows.

Consequently the right hand side of  $(*)$  is mod 28 independent of the choice of  $d$ , as long as  $d$  reduces to  $w_2(X)$ . Then it depends only on the rational Pontrjagin class  $p_1$ , on the signature  $\sigma$  and on  $w_2(X)$ , which are all topological invariants.

Finally  $\bar{\gamma}(X)$ , the left hand side of  $(*)$ , is therefore also a topological invariant. We summarize:

*Theorem 1.* — If the oriented closed 8-manifold  $X$  is simply connected and has no 2-torsion in  $H^3(X, \mathbf{Z})$ , or if  $w_2(X)$  is the reduction of a  $\mathbf{Z}$ -cohomology class  $d$ , and  $\bar{\gamma}(X) \neq 0$ , then  $X$  has no smoothing.  $\bar{\gamma}$  is a topological invariant for such spaces  $X$ .

### § 3. Complex algebraic varieties as examples.

Brieskorn [8], Milnor [9] and Hirzebruch [10], using Pham [11], have studied isolated singularities of complex algebraic varieties, for which some neighbourhood of the singular point has the natural topological and combinatorial structure of a cone over a smooth possibly exotic 7-sphere which bounds it. In particular this is the case for the singularity at  $0 \in \mathbf{C}^5$  of the affine variety ([10])

$$(1) \quad \left. \begin{aligned} f_1(z_1, \dots, z_5) &= z_1^{n-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0 \\ n &= 6k > 0 \end{aligned} \right\}.$$

The intersection of (1) with  $|z| = \sqrt{\sum_{i=1}^5 |z_i|^2} \leq 9c$ , is for small  $c > 0$  homeomorphic with an 8-ball, and its boundary, obtained as intersection of (1) and

$$(2) \quad |z| = c$$

is the exotic sphere with value

$$(3) \quad k \cdot 1 \in \mathbf{Z}_{28} \simeq \Gamma_7.$$

For  $k=1$  the generator with value  $1 \in \mathbf{Z}_{28} \simeq \Gamma_7$  is found.

If we embed  $\mathbf{C}^5$  as the complement of a hyperplane, the so called "hyperplane at infinity" in  $\mathbf{P}^5(\mathbf{C})$ , then (1) can be considered as the affine equation of an algebraic variety in  $\mathbf{P}^5(\mathbf{C})$ .

In homogeneous coordinates  $z_1, \dots, z_5, w$ , it has the equation

$$z_1^{n-1} + z_2^3 w^{n-4} + (z_3^2 + z_4^2 + z_5^2) w^{n-3} = 0.$$

This algebraic variety has, apart from the old singularity, many more singularities namely at infinity ( $w=0$ ). In order to avoid new extra singularities we modify our function  $f_1$  and choose the new function  $f$  as follows:

$$(4) \quad \begin{aligned} f &= z_1^{n-1} + z_2^3 + \sum_{i=3}^5 z_i^2 + \sum_{j=1}^5 \lambda^{j-1} z_j^n \\ \lambda &\in \mathbf{R} \subset \mathbf{C}, n = 6k \end{aligned}$$

This function is locally near  $0 \in \mathbf{C}^5$  equivalent to  $f_1$  by a holomorphic change of coordinates of the kind

$$z'_j = \Phi_j(z_j) \quad (j=1, \dots, 5)$$

with

$$\begin{aligned} (\Phi_1(u))^{n-1} &= u^{n-1} + u^n \\ (\Phi_2(u))^3 &= u^3 + \lambda u^n \\ (\Phi_i(u))^2 &= u^2 + \lambda^{i-1} u^n \quad (i=3, 4, 5). \end{aligned}$$

Therefore the affine variety  $f=0$  has near  $0 \in \mathbf{C}^5$  a singularity with the same local properties as mentioned for  $f_1=0$  (take  $c$  small). We now search for the singularities on the variety  $f=0$ . They obey the equations:

$$(5) \quad \partial_1 f = \partial_2 f = \partial_3 f = \partial_4 f = \partial_5 f = 0$$

and

$$(6) \quad f=0.$$

Solution of  $z_1, \dots, z_5$  from (5) and substitution in (6) yields for

$$(z_1, \dots, z_5) \neq (0, \dots, 0) = 0 \in \mathbf{C}^5$$

and for different choices of the solutions, rational algebraic equations, which can be combined into one rational algebraic equation. It expresses a necessary condition on  $\lambda$ , for having at least one more singular point on (6). So for only a finite number of values of  $\lambda$  there are other singularities. In particular for  $\lambda = e$ , an arbitrary transcendental number, the only singularity on the affine variety is  $0 \in \mathbf{C}^5$ .

We imbed  $\mathbf{C}^5$  as the complement of the hyperplane  $w=0$  in the complex projective 5-space  $\mathbf{P}^5(\mathbf{C})$  and close the image of the affine variety. Then we obtain the algebraic variety  $V_k \subset \mathbf{P}^5(\mathbf{C})$  with equation in homogeneous coordinates  $(z_1, \dots, z_5, w)$ :

$$(7) \quad V_k : z_1^{n-1}w + z_2^3w^{n-3} + \sum_{i=3}^5 z_i^2w^{n-2} + \sum_{j=1}^5 e^{j-1}z_j^n = 0$$

$V_k$  has clearly no singularities at infinity ( $w=0$ ). It has exactly one singular point  $p=(0, 0, 0, 0, 0, 1)$ , and  $V_k - \{p\}$  is a smooth 8-manifold. Then by the result of Brieskorn  $V_k$  is a topological manifold at  $p$ , as well as all over. However, it also has its natural triangulation as an algebraic real 8-dimensional variety, where near  $p$  the triangulation is obtained by triangulating the cone on the (possibly) exotic 7-sphere described above. We compute the invariant  $\bar{\gamma}(V_k)$  as follows. Take a triangulation such that  $p$  is interior point of some 8-simplex. Take in  $V_k - \{p\}$  the differential structure from the (there!) differential manifold  $V_k$ . Then the cochain  $\gamma$  so obtained has value zero on all simplices outside  $p$ . At  $p$  the value is therefore  $\bar{\gamma}(V_k) = k \cdot 1 \in \mathbf{Z}_{28}$ .

*Theorem 2.* — Every class of combinatorial modulo smoothable 8-manifolds can be represented by a complex algebraic hypersurface  $V_k \subset \mathbf{P}^5(\mathbf{C})$ ,  $k = 1, 2, \dots, 28$ . Among these, only  $V_{28}$  is homeomorphic to a smooth manifold. In particular the algebraic variety  $V_1 \in \mathbf{P}^5(\mathbf{C})$  with affine equation

$$(8) \quad z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 + \sum_{j=1}^5 e^{j-1}z_j^6 = 0$$

is a topological 8-manifold without any smoothing.

*Proof.* — By theorem 1 it is sufficient to prove the existence of  $d \in H^2(V_k, \mathbf{Z})$  reducing to  $w_2(V_k) \in H^2(V_k, \mathbf{Z}_2)$ . For a sufficiently close approximation of equation (8) we can obtain a complex manifold without singularity  $Y$  with the following properties.

There exists a manifold with boundary  $V_k^0$  obtained from  $V_k$  by deleting a small disc containing the singularity.  $i: V_k^0 \rightarrow V_k$  is the inclusion. There is an embedding  $j: V_k^0 \rightarrow Y$ , because outside some neighbourhood of the singularity,  $V_k$  and  $Y$  are near to each other with first derivatives included and hence diffeomorphic. Now the first Chern class  $c_1(Y) \in H^2(Y, \mathbf{Z})$  reduces to  $w_1(Y) \in H^2(Y, \mathbf{Z}_2)$ . Hence  $d = (i^*)^{-1}j^*c_1(Y) \in H^2(V_k, \mathbf{Z})$  reduces to  $w_1(V_k) = (i^*)^{-1}j^*w_1(Y) \in H^2(V_k, \mathbf{Z}_2)$ .

#### § 4. Nonorientable 8-manifolds

We first recall the nonorientable version of Wall's theorem.

*Theorem.* — *The connected nonorientable closed combinatorial 8-manifolds modulo smooth manifolds form a group of two elements.*

*Proof.* — Let  $X$  be a nonorientable connected 8-manifold with  $k$ -skeleton  $X^k$ . We smooth some neighbourhood of  $X^7$  as before, such that the non-smoothability of  $X$  is concentrated in one 8-simplex  $\Delta$ . On this oriented 8-simplex let it be given by  $x \in \Gamma_7$ . As  $X$  is nonorientable, there exists (assuming the triangulation of  $X$  fine enough) a sequence of 8-simplices  $\Delta^{(i)}$ ,  $i = 1, \dots, N+1$  with  $\Delta^{(1)} = \Delta^{(N+1)} = \Delta$ ,  $\Delta_7^{(i)} = \Delta^{(i)} \cap \Delta^{(i+1)}$  is a common face, such that the union  $\bigcup_{i=1}^N \Delta^{(i)}$  is a nonorientable neighbourhood of a closed curve in  $X$ . Any element  $y \in \Gamma_7$  can be represented by a change of smoothing in the oriented face  $\Delta_7^{(1)}$  of  $\Delta$ , which can be neutralized with respect to smoothability of  $\Delta^{(2)}$  by a suitable change of smoothing in  $\Delta_7^{(2)}$ . Etc. After coming back to  $\Delta^{(N+1)} = \Delta$  the non-smoothability is again completely concentrated in the oriented 8-simplex  $\Delta$ , but represented with value  $x - y + (-y) = x - 2y \in \Gamma_7$ .

In the nonorientable case the 8-simplices of  $X^{(8)}$  have no preferred orientation. Then reducing the constant local coefficient sheaf  $\Gamma_7 \simeq \mathbf{Z}_{28}$ , modulo 2, there remains from the theory in § 1, a  $\mathbf{Z}_2$ -cocycle  $\gamma(X, \mathbf{Z}_2)$  in  $H^8(X, \mathbf{Z}_2)$  which is an invariant of the nonorientable manifold  $X$ . In order to be able to smooth  $X$ , it is necessary that  $\gamma(X, \mathbf{Z}_2)$  vanishes. But above we have seen that it is also sufficient: Take  $y$  such that  $2y = x \in \Gamma_7$ . From the construction as in § 1 it is seen that  $\gamma(X \# Y, \mathbf{Z}_2) = \gamma(X, \mathbf{Z}_2) + \gamma(Y, \mathbf{Z}_2) \in \mathbf{Z}_2$ , for  $X$  and  $Y$  orientable or not. Then the theorem follows. Formally the obstruction to smoothing lies in  $H^8(X, \Gamma_7) = H_0(X, \text{orientation} \otimes \Gamma_7) = H_0(X, \mathbf{Z}_2) = \mathbf{Z}_2$ .

*Theorem 2.* — *The real algebraic 8-variety  $W_1 \in \mathbf{P}^{10}(\mathbf{R})$  in real projective 10-space with affine equations in  $x_1, y_1, \dots, x_5, y_5$ :*

$$(9) \quad z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 + \sum_{j=1}^5 e^{j-1} z_j^6 = 0$$

$$z_j = x_j + iy_j$$

*is a closed nonorientable combinatorial 8-manifold without compatible smoothing. It represents the nonsmoothable class.*

*N.B.* — In this nonorientable case we cannot decide that the manifold is not homeomorphic to any smooth manifold (with a different combinatorial structure).

*Proof.* —  $W_1$  is an algebraic real variety with exactly one singularity of type  $\pm 1 \in \Gamma_7$ . At this singularity  $W_1$  is a combinatorial manifold and not smooth. There remains to prove that  $W_1$  is nonorientable.

Take a suitable diffeomorphism of  $\mathbf{C}^5 = \mathbf{R}^{10}$  onto the open ball  $|z| < 1$ , which



commutes with rotations, leaves each real half ray from 0 invariant, and is the identity near  $0 \in \mathbf{C}^5$ . Let  $\mathring{W}$  be the image of  $W \cap \mathbf{R}^{10}$ .  $\mathring{W}$  can be closed by the 7-manifold  $\partial\mathring{W}$ :

$$(\sum_j e^{j-1} z_j^6 = 0) \cap (\sum_j z_j \bar{z}_j = 1).$$

The diametrical map  $\delta : (z_1, \dots, z_5) \rightarrow (-z_1, \dots, -z_5)$  leaves  $\mathring{W}$  invariant and preserves orientation in  $\mathring{W}$  as well as in  $\mathbf{R}^{10}$ . Now  $W$  is essentially obtained from  $\mathring{W} \cup \partial\mathring{W}$  by identifying diametrical points in  $\partial\mathring{W}$ . [This is analogous to obtaining  $P^{10}(\mathbf{R})$  from  $\sum_j z_j \bar{z}_j \leq 1$  by identifying diametrical points on  $\sum_j z_j \bar{z}_j = 1$ .] Hence  $W$  is nonorientable.

*Remark.* — If a manifold with one singularity is “exotic” at that singularity, then it still may globally admit some smoothing. For example this is the case with the variety  $W_2 \subset P^{10}(\mathbf{R})$  with real affine equations

$$\left. \begin{aligned} z_1^{11} + z_2^3 + z_3^2 + z_4^2 + z_5^2 + \sum_{j=1}^5 e^{j-1} z_j^{12} &= 0 \\ z_j &= x_j + iy_j \end{aligned} \right\}.$$

It has the same exotic singularity at 0 as  $V_2$ . The same holds for any nonorientable 8-manifold with one singularity, in case that singularity is like that of  $V_k$  for some even  $k \neq 0 \pmod{28}$ .

*Exercise.* — If the oriented 8-manifold  $X$  admits an orientation reversing combinatorial involution without fixed point, then it has a smoothing.

## § 5. Formulation of the main theorem. A lemma on polynomial approximation.

A closed connected,  $C^\infty$ -manifold  $X$ ,  $C^\infty$ -embedded in  $\mathbf{R}^n$ , is called a *Nash manifold*, if there exists a polynomial map  $g : \mathbf{R}^n \rightarrow \mathbf{R}^q$  for some  $q$ , and  $X \subset g^{-1}(0) \subset \mathbf{R}^n$  with  $\dim X = \dim g^{-1}(0)$ . A  $C^\infty$ -map  $f : X \rightarrow Y$  between Nash manifolds  $X \subset \mathbf{R}^m$  and  $Y \subset \mathbf{R}^n$  is called a *morphism*, if its graph  $\{(x, f(x)) : x \in X\} \subset \mathbf{R}^m \times \mathbf{R}^n = \mathbf{R}^{m+n}$  is a Nash manifold. We now recall the classical

*Theorem of Nash* [13, 14]. — Every closed  $C^\infty$ -manifold  $X$  admits the structure of a Nash manifold and this structure is unique up to isomorphism.

As every closed combinatorial manifold of dimension  $k \leq 7$  has a compatible  $C^\infty$ -manifold structure (unique for  $k \leq 6$ ), it also has a Nash-manifold structure (unique for  $k \leq 6$ ). On the 7-sphere  $S^7$  there are 28 Nash-manifold structures as there are 28 differential structures.

If  $g : \mathbf{R}^n \rightarrow \mathbf{R}^q$  is a polynomial map and  $X$  is a real analytic closed subset of  $g^{-1}(0)$  of the same dimension as  $g^{-1}(0)$ , then  $X$  is called a Nash space and also a Nash component of  $g^{-1}(0)$ . A Nash space  $X$  which is a topological manifold, and except at one point  $x_0$  a  $C^\infty$ -manifold, will be called a *Nash manifold with one singularity at  $x_0$* . Examples are described in theorems 2 and 3 above. (In order to meet the definition strictly we have to embed  $P^5(\mathbf{C})$  and  $P^{10}(\mathbf{R})$  as real algebraic varieties in  $\mathbf{R}^N$  for some  $N$ .)

In the remaining part of this paper we prove an analogue of Nash's theorem:

*Main theorem 4.* — Every closed combinatorial 8-manifold  $X$  has the structure of a Nash manifold with one singularity, embedded in  $\mathbf{R}^{16}$ . It is a Nash component of the algebraic set  $g^{-1}(0)$  for some polynomial map  $g: \mathbf{R}^{16} \rightarrow \mathbf{R}^q$ .

We first prove an important lemma which we need later. For any  $C^\infty$ -function  $f: W \rightarrow \mathbf{R}^q$ , defined on a neighbourhood  $W$  of  $0$  in  $\mathbf{R}^n$ , and for any natural number  $s$ , we denote by  $f_s$  the polynomial function of degree  $s$ , which at  $0 \in \mathbf{R}^n$  has all derivatives of orders  $\leq s$  in common with  $f$ .  $f_s$  is therefore the Taylor series of  $f$  at  $0$ , up to and included terms of degree  $s$ .

*Lemma 4.* — Let  $W$ , with closure  $\bar{W}$ , and  $W'$  be bounded open sets in  $\mathbf{R}^n$  and  $0 \in W \subset \bar{W} \subset W'$ ;  $s \geq 0$ ;  $\varepsilon > 0$ ;  $|x| = \sqrt{\sum_{i=1}^n (x_i)^2}$  for  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ .

For any  $C^\infty$ -function  $f: W' \rightarrow \mathbf{R}^q$ , there exists a polynomial function  $\psi: W \rightarrow \mathbf{R}^q$  with the same Taylor- $s$ -part at  $0$ :

$$\psi_s = f_s$$

and  $\varepsilon$ -near to  $f$  on  $W$  in the  $C^s$ -metric:

$$|(\partial_\alpha \psi)(x) - (\partial_\alpha f)(x)| < \varepsilon \quad \text{for} \quad |\alpha| \leq s, x \in W.$$

Here, if  $\alpha$  is the multiindex  $\alpha = i_1, \dots, i_r$ , then

$$\partial_\alpha = \partial_{i_1} \dots \partial_{i_r} \quad \text{and} \quad |\alpha| = r \geq 0.$$

*Proof.* — Because we can  $C^\infty$ -extend the restriction of  $f$  to  $\bar{W}$  over  $\mathbf{R}^n$ , we may just as well assume that  $W$  and  $W'$  are bounded open balls with centre in  $0 \in \mathbf{R}^n$ . It is well known that given  $f$  and  $\delta > 0$ , there exists a polynomial function  $\Phi$ , for which

$$(10) \quad |\partial_\alpha (f - \Phi)(x)| < \delta \quad \text{for} \quad |\alpha| \leq s, x \in W.$$

We refer to Graves [16] and only recall that  $\Phi$  can be obtained for example for a sufficiently large integer  $m$ , and

$$c_m^{-1} = \int_{|u| \leq m} (1 - u^2/m^2)^{m^4} du, \quad u^2 = \langle u, u \rangle,$$

as the convolution (an averaging process):

$$\Phi(x) = \int_{\mathbf{R}^n} f(u) \cdot c_m \cdot [1 - (x-u)^2/m^2]^{m^4} du$$

with  $f(x) = 0$  by definition for  $x \notin W'$ .

$\Phi$  is then a polynomial function of highest degree  $\leq 2m^4$ .

From (10) we obtain in particular at  $0 \in \mathbf{R}^n$ :

$$|\partial_\alpha (f - \Phi)(0)| = |\partial_\alpha (f_s - \Phi_s)(0)| < \delta \quad \text{for} \quad |\alpha| \leq s.$$

By integrating along half rays starting at  $0 \in \mathbf{R}^n$  we see that a constant  $C > 0$  exists, such that

$$|\partial_\alpha (f_s - \Phi_s)(x)| < C\delta \quad \text{for} \quad |\alpha| \leq s, x \in \bar{W}.$$

$C$  depends only on  $\bar{W}$ , and not on  $f$  or  $\Phi$ .

Let  $\delta$  be so small that  $(1 + C)\delta < \varepsilon$ . The required function is then

$$(11) \quad \psi = \Phi + (f_s - \Phi_s).$$

It has the properties:

$$(\partial_\alpha \psi)(0) = (\partial_\alpha f_s)(0) = (\partial_\alpha f)(0) \quad \text{for} \quad |\alpha| \leq s,$$

and

$$|(\partial_\alpha \psi)(x) - (\partial_\alpha f)(x)| \leq |(\partial_\alpha (\Phi - f))(x)| + |\partial_\alpha (f_s - \Phi_s)(x)| \leq \delta + C\delta \leq \varepsilon \quad \text{for} \quad |\alpha| \leq s, x \in W.$$

## § 6. Construction of an embedding of the closed combinatorial 8-manifold $X$ in $\mathbf{R}^{16}$ as a $C^\infty$ -manifold with one specific singularity at $0 \in \mathbf{R}^{16}$ .

This construction follows completely the proof of Whitney's embedding theorem for  $C^\infty$ -manifolds. We smooth (see § 1) the complement  $X - U_0$  of an open 8-ball  $U_0$  in  $X$ . The boundary  $\partial U_0$  is an exotic 7-sphere representing  $\bar{\gamma}(X) \in \Gamma_7 = \mathbf{Z}_{28}$ . Let  $k = \bar{\gamma}(X) \bmod 28$  and  $0 < k < 28$ . (In the case  $k=0$ ,  $X$  is smoothable and we are done by Nash's theorem.) The closed 8-ball  $\bar{U}_0$  is embedded by a map  $i_0$  onto the standard model with  $n = 6k$ :

$$(12) \quad i_0(\bar{U}_0) = \{z \mid f_1(z) = z_1^{n-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0 \text{ and } |z| \leq 9c\} \subset \mathbf{C}^5 = \mathbf{R}^{10}.$$

For later use we define

$$(13) \quad U_0(t) = i_0(\{z \mid |z| < t\}), \quad 0 < t \leq 9c;$$

$\partial U_0$  has by virtue of  $i_0$  an induced differential structure, which represents  $\bar{\gamma}(X)$  by the choice of  $n = 6k$ .

We can assume the two smoothings of  $\partial U_0$  to be equal, and  $\bar{U}_0$  and  $X - U_0$  can be glued along their common boundary to obtain a  $C^\infty$ -manifold with one singularity (12) at  $x_0$ . From now on we assume this structure in the symbol  $X$ . Next we construct an embedding of  $X$  in some euclidean space.

It is easy to see that a  $C^\infty$ -map

$$\kappa'_0 : \mathbf{C}^5 \rightarrow \mathbf{C}^5 \times \mathbf{R} = \mathbf{R}^{11}$$

$$\text{exists with } \left\{ \begin{array}{ll} \kappa'_0(z) = (z, 0) & \text{for } |z| \leq 8c, \\ \kappa'_0(z) = (0, 1) & \text{for } |z| \geq 9c, \\ \kappa'_0 \text{ is a } C^\infty\text{-embedding} & \text{for } |z| < 9c. \end{array} \right.$$

The composition  $\kappa'_0 \circ i_0$ , extended by the constant map  $\kappa_0(x) = (0, 1)$  for  $x \notin U_0(9c)$ , determines a map

$$\kappa_0 : X \rightarrow \mathbf{R}^{11},$$

which is  $C^\infty$  on  $X - \{x_0\}$ ,  $C^\infty$ -embedding on  $U_0(9c) - \{x_0\}$ , and "standard" (see (12)) on  $U_0(8c)$ .

For any point  $x \in X - U_0(9c)$  there is an 8-ball neighbourhood  $U_x \subset X - U_0(8c)$  and a  $C^\infty$ -map

$$\kappa_x : U_x \rightarrow \mathbf{R}^9$$

onto the 8-sphere

$$S^8 = \{(u_1, \dots, u_9) \in \mathbf{R}^9 : \sum_{j=1}^9 u_j^2 = 2u_9\},$$

such that the restriction  $\kappa_x|_{U_x}$  is a  $C^\infty$ -diffeomorphism onto  $S^8 - \{o\}$ , and  $\kappa_x(y) = o \in \mathbf{R}^9$  for  $y \notin U_x$ .

A finite number of the neighbourhoods  $U_0$  and  $U_x$ , say  $U_0, U_1, \dots, U_L$  cover  $X$ . Then we obtain a map

$$\kappa : X \rightarrow \mathbf{R}^{11+9L}$$

defined by  $\kappa(x) = (\kappa_0(x), \kappa_1(x), \dots, \kappa_L(x))$ .  $\kappa$  is an embedding of  $X$  onto a  $C^\infty$ -manifold with one standard singularity  $\kappa(U_0(8c)) \subset \mathbf{R}^{10} \times o \subset \mathbf{R}^{11+9L}$  near  $\kappa(x_0) = o$ .

Finally we decrease the dimension of the target space in the usual manner as follows. The set of chords and tangents of  $\kappa(X - \{x_0\})$  is the  $C^\infty$ -image of a 17-manifold. It is nowhere dense for  $11+9L > 17$  by Sard's theorem. We then can project  $\kappa(X)$  from some point into that linear subspace of  $\mathbf{R}^{11+9L}$  on which the last coordinate vanishes, and we obtain an analogous embedding. This process can be repeated until we get an embedding in  $\mathbf{R}^{17}$ . One more projection yields an immersion with isolated transversal self-intersections in  $\mathbf{R}^{16}$ . The self intersections can be removed by Whitney's method [15], to obtain the required embedding. Observe that during this process the embedding of  $U_0(8c)$  remains unchanged.

From now on we identify  $X$  with  $\kappa(X) \subset \mathbf{R}^{16}$ , the embedded manifold with standard part  $U_0(8c) = \kappa(U_0(8c)) \subset \mathbf{R}^{10} \times o$ . So we have a diagram of inclusions:

$$(14) \quad \begin{array}{ccc} U_0(8c) & \longrightarrow & \mathbf{R}^{10} \times o \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathbf{R}^{16} \end{array}$$

## § 7. $C^\infty$ -equations for $X \subset \mathbf{R}^{16}$ (1).

In this paragraph we define a diagram of  $C^\infty$ -maps

$$(15) \quad \begin{array}{ccc} W & \xrightarrow{\bar{\alpha}} & A_8 \subset G_8 \times \mathbf{R}^{16} \xrightarrow{p_2} \mathbf{R}^{16} \\ & & \downarrow p \\ X^8 - U_0(c) & \xrightarrow{\alpha} & G_8 \end{array}$$

W an open neighbourhood of  $X \subset \mathbf{R}^{16}$ , such that

$$(16) \quad X^8 = (p_2 \bar{\alpha})^{-1}(o) \cap W.$$

(1) The constructions in § 7 and § 8 are analogous to those of Thom [14] concerning smooth manifolds.

The map  $p_2 \bar{\alpha}$  therefore determines a set of 16 equations for  $X^8$ .  $\bar{\alpha}$  has a *singularity* at 0, and it is *transversal* to  $G_8 \subset A_8$  at all other points of  $W$ .

We first define  $\bar{\alpha}$  on certain parts  $W(t)$  of  $W$ .

Consider the normal bundle of  $X - U_0(c)$  in  $\mathbf{R}^{16}$ . The normal exponential map  $\text{nexp} : (x, v) \mapsto x + v \in \mathbf{R}^{16}$  is a  $C^\infty$ -map of its total space into  $\mathbf{R}^{16}$ . Here  $v$  is a normal vector at  $x \in X - U_0(c)$ .

There is a constant  $\varepsilon_1$  such that the restriction of  $\text{nexp}$  to the space of normal vectors of length smaller than  $\varepsilon_1$ , is a diffeomorphism, with image the tubular neighbourhood (=total  $\varepsilon_1$ -ball bundle space):

$$W(t) = \text{nexp} \{ (x, v) \mid x \in X - U_0(t), |v| < \varepsilon_1 \} \quad c \leq t \leq 9c.$$

The projection of the  $\varepsilon_1$ -ball bundle is called

$$\mu : W(t) \rightarrow X^8 - U_0(t).$$

Let  $G_8$  be the Grassmann manifold of all 8-dimensional vector subspaces of  $\mathbf{R}^{16}$ , and  $A_8$  the total space of the corresponding open  $\varepsilon_1$ -ball bundle  $p$ :

$$\begin{array}{c} A_8 = \{ (g, v) \in G_8 \times \mathbf{R}^{16} \mid v \in g, |v| < \varepsilon_1 \} \subset G_8 \times \mathbf{R}^{16} \\ \downarrow p \\ G_8 = G_8 \times 0. \end{array}$$

The tangent vector spaces at different points of  $\mathbf{R}^{16}$  are all identified with  $\mathbf{R}^{16}$ .  $\mu$  is induced from  $p$  by the natural bundle map

$$\begin{array}{ccc} W(t) & \xrightarrow{\bar{\beta}} & A_8 \\ \mu \downarrow & & \downarrow p \\ X^8 - U_0(t) & \xrightarrow{\beta} & G_8 \end{array}$$

where  $\beta(x)$  is the normal vector space at  $x$  in  $\mathbf{R}^{16}$  with respect to  $X^8$ , and

$$\bar{\beta}(\text{nexp}(x, v)) = (\beta(x), v).$$

$G_8$  is identified with the 0-section of  $p$ .

We will modify the map  $\beta$  and obtain a map  $\alpha$  which is constant near  $\partial U_0(5c)$ .

The space  $\overline{U_0(8c)} - U_0(5c)$  is diffeomorphic with  $\Sigma^7 \times I$ , the product space of the exotic 7-sphere  $\Sigma^7$  and a segment. Because  $U_0(8c)$  is contained in  $\mathbf{R}^{10} \times 0 \subset \mathbf{R}^{16}$ , the normal bundle of this product-space part of  $X$  is the direct sum of a trivialized 6-plane bundle and a orientable trivial 2-plane bundle. Then by fibre-bundle theory [ $\pi_7(G_{2,8}) = 0$  for  $G_{2,8}$  the Grassmann space of 2-planes in  $\mathbf{R}^{10}$ ] there is a  $C^\infty$ -map  $\alpha$ , whose restriction to  $\overline{U_0(8c)} - U_0(5c)$  is a homotopy:

$$X^8 - U_0(5c) \xrightarrow{\alpha} G_8,$$

such that: 
$$\alpha(x) = \begin{cases} \beta(x) & \text{for } x \in X - U_0(7c) \\ g_0 & \text{for } x \in U_0(6c) - U_0(5c). \end{cases}$$

$g_0$  is the 8-plane  $o \times \mathbf{R}^2 \times \mathbf{R}^6 \subset \mathbf{R}^{16}$ ;

$\alpha(x)$  is an 8-plane containing the 6-plane,  $o \times o \times \mathbf{R}^6 \subset \mathbf{R}^{16}$  for all  $x \in U_0(8c) - U_0(5c)$ .

The bundle induced from  $p$  by  $\alpha$  is equivalent to that induced from  $p$  by  $\beta$ . Hence we may identify both induced bundles and we have an orthogonal  $\varepsilon_1$ -ball bundle map, which is a  $C^\infty$ -map of pairs,

$$(17) \quad \begin{array}{ccc} W(5c) & \xrightarrow{\bar{\alpha}} & A_8 \\ \downarrow u & & \downarrow p \\ X^8 - U_0(5c) & \xrightarrow{\alpha} & G_8 \end{array}$$

and

$$(17') \quad \alpha(U_0(6c) - U_0(5c)) = g_0, \quad \bar{\alpha}(w) = \bar{\beta}(w) \quad \text{for } w \in W(7c).$$

$A_8$  contains the fibre  $p^{-1}(g_0)$  which is an  $\varepsilon$ -ball in  $\mathbf{R}^8$ . We will in the sequel extend the fibre-bundle map over the base space  $U_0(5c) - U_0(3c)$  by a map  $(\bar{\alpha}, \alpha)$  for which  $\alpha$  takes the constant value  $g_0$ . We will further extend  $\bar{\alpha}$  over some neighbourhood of  $U_0(3c)$  in  $\mathbf{R}^{16}$ , by a map with all values in the  $\varepsilon$ -ball  $p^{-1}(g_0)\mathbf{R}^8$ .

In order to define  $\bar{\alpha}$  near the singularity, we start from the map

$$f_1 : \mathbf{R}^{10} \rightarrow \mathbf{R}^2,$$

which was defined in terms of complex variables by (1) in § 3. Observe that for  $t \leq 8c$ :

$$f_1^{-1}(o) \cap \{z \mid |z| < t\} = U_0(t) \subset \mathbf{R}^{10} = \mathbf{R}^{10} \times o \subset \mathbf{R}^{16}.$$

Near to the singularity, that is for some small enough neighbourhood of  $U_0(c)$  in  $\mathbf{R}^{16}$ , we define

$$(18) \quad \bar{\alpha} = f_1 \times \text{id} : \mathbf{R}^{10} \times \mathbf{R}^6 \rightarrow \mathbf{R}^2 \times \mathbf{R}^6.$$

Here  $\text{id}$  is the identity map of  $\mathbf{R}^6$ .

For small  $\varepsilon > 0$  and  $B(\varepsilon) = \{y \in \mathbf{R}^2 \mid |y| < \varepsilon\}$ ,  $f_1$  determines a framing and a trivial fibre bundle with fibres diffeomorphic to  $B(\varepsilon)$ , group the group of diffeomorphisms of  $B(\varepsilon)$ , base space  $\overline{U_0(4c)} - U_0(c)$ , and fibre over  $x$ :

$$F_x = f_1^{-1}(B(\varepsilon)) \cap \{n \exp(x, v) \mid v \text{ normal vector at } x\}.$$

$F_x$  is contained in a unique linear two-dimensional variety  $L_x \subset \mathbf{R}^{10}$ . Let the framing map  $\pi_x : B(\varepsilon) \rightarrow L_x$  be defined as the inverse of

$$(f_1|_{F_x}) : F_x \rightarrow B(\varepsilon).$$

We want to modify  $\pi_x$  (hence  $f_1$ ) to obtain isometries for  $x \in U_0(4c) - U_0(3c)$ . Because the oriented differentiable embeddings of  $B(\varepsilon)$  in  $\mathbf{R}^2$  with fixed origin, retract

by deformation into the orthogonal group  $SO(2)$ , there is for each  $x$  a homotopy of  $C^\infty$ -embeddings

$$\pi_{x,t} : B(\varepsilon) \rightarrow L_x, \quad \pi_{x,t}(0) = x,$$

starting with  $\pi_{x,0} = \pi_x$  and ending with an isometry  $\pi_{x,1}$ . We can choose it such that the mapping  $\pi_{x,t}$  depends  $C^\infty$  on  $x$  and  $t$ , and  $\pi_{x,t}$  is constant with respect to  $t$  for  $0 < t < \frac{1}{3}$  and for  $\frac{2}{3} < t \leq 1$ .

Now we are ready to replace  $f_1$  by a new map  $\alpha_0$ . For  $x \in \overline{U_0(4c)} - U_0(c)$  let  $t$  implicitly be given by

$$x \in \partial U_0(c + 3tc), \quad 0 \leq t \leq 1.$$

Let  $y \in \pi_{x,t}(B(\varepsilon)) \subset L_x \subset \mathbf{R}^{10}$ .

$$\text{Now put} \quad \alpha_0(y) = (\pi_{x,t})^{-1}(y) \in \mathbf{R}^2.$$

We continue the definition of  $\bar{\alpha}$ . In some neighbourhood of  $\overline{U_0(4c)} - U_0(0)$  in its tubular normal bundle space in  $\mathbf{R}^{16}$ , we put

$$(19) \quad \bar{\alpha} = \alpha_0 \times \text{id}.$$

Here again  $\text{id}$  is the identity map of  $\mathbf{R}^6$ . Observe that (19) agrees with (18).

Over the part  $\overline{U_0(4c)} - U_0(3c)$  and over the part  $\overline{U_0(6c)} - U_0(5c)$  the mapping  $\bar{\alpha}$  into  $\mathbf{R}^8 = \mathbf{R}^2 \times \mathbf{R}^6 = p^{-1}(g_0)$  determines orthogonal trivializations of the normal tangent bundle, each splitting of the same trivial trivialization in the vector spaces *parallel* to  $0 \times \mathbf{R}^6 \subset \mathbf{R}^{16}$ . Recall for this that  $\overline{U_0(8c)} \subset \mathbf{R}^{10} \times 0 \subset \mathbf{R}^{16}$ . These trivializations therefore reduce to trivializations of 2-plane bundles essentially over seven-spheres. They are homotopic.

The trivializations of the tubular neighbourhoods over  $\overline{U_0(4c)} - U_0(3c)$  and over  $\overline{U_0(6c)} - U_0(5c)$  as orthogonal  $\varepsilon_1$ -ball bundles (for  $\varepsilon_1$  small enough), correspond one-to-one to the orthogonal trivializations of the normal tangent bundles. Therefore  $\bar{\alpha}$  can be extended over the normal tubular bundle over  $\overline{U_0(5c)} - U_0(4c)$  in  $\mathbf{R}^{16}$  by a map into  $\mathbf{R}^8$ , which is also isometric on each fibre.

Taking the map  $\bar{\alpha}$  of differentiability class  $C^\infty$  we have obtained, with (17), (18) and (19), for some neighbourhood  $W$  of  $X$  in  $\mathbf{R}^{16}$ , the map

$$(15) \quad w \xrightarrow{\bar{\alpha}} A_8.$$

The restriction to  $W(3c)$  is an orthogonal bundle map:

$$(15 a) \quad \begin{array}{ccc} W(3c) & \xrightarrow{\bar{\alpha}} & A_8 \\ \downarrow & & \downarrow \\ X - U_0(3c) & \xrightarrow{\alpha} & G_8 \end{array}$$

The restriction to  $W - W(6c)$  is into  $p^{-1}(g_0) \subset \mathbf{R}^8$ :

$$(15\ b) \quad W - W(6c) \xrightarrow{\bar{\alpha}} p^{-1}(g_0) \subset \mathbf{R}^8.$$

The restriction to  $W - W(c)$  is:

$$(15\ c) \quad f_1 \times \text{id}.$$

Now the diagram (15) and the properties mentioned after (15) follow immediately.

## § 8. Algebraic equations for $X$ .

We consider again diagram (15). Let  $G_8$  be embedded as an algebraic submanifold in some euclidean space  $\mathbf{R}^M$ . The normal exponential map defines a tubular neighbourhood  $Y$  with radius  $\varepsilon$  (sufficiently small) of  $G_8$ , and with an *algebraic* orthogonal projection (a retraction)

$$(20) \quad \rho : Y \rightarrow G_8 \subset Y^M$$

$\rho(y)$  is the point in  $G_8$  that is nearest to  $y \in Y$ . We now extend diagram (15) by natural inclusions

$$(21) \quad \begin{array}{c} W \xrightarrow{\bar{\alpha}} A_8 \subset G_8 \times \mathbf{R}^{16} \subset Y \times \mathbf{R}^{16} \subset \mathbf{R}^{M+16} \\ \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow p_1 \quad \quad \quad \downarrow p_2 \\ G_8 \xrightarrow{c} Y \quad \quad \quad \mathbf{R}^{16} \end{array}$$

The retraction  $\rho$  in (20) can be covered by a *retraction*  $\bar{\rho}$ , which is also algebraic:

$$(22) \quad \begin{array}{ccc} Y \times \mathbf{R}^{16} & \xrightarrow{\bar{\rho}} & A_8 \\ \downarrow p_1 & & \downarrow p \\ Y & \xrightarrow{\rho} & G_8 \end{array}$$

It is defined by the condition that  $\bar{\rho}(y, z)$  is the orthogonal projection of the point  $(\rho(y), z) \in Y \times \mathbf{R}^{16}$  which lies in the euclidean 16-space  $p_1^{-1}(\rho(y))$ , into the euclidean sub-8-space  $p^{-1}(\rho(y)) \subset A_8$ .

We now call  $W : W'$ , and let  $W \subset \bar{W} \subset W'$  be a smaller analogous neighbourhood of  $X \subset \mathbf{R}^{16}$ . Then we apply lemma 4 to the map

$$W' \xrightarrow{i\bar{\alpha}} Y \times \mathbf{R}^{16} \subset \mathbf{R}^M \times \mathbf{R}^{16}.$$

We obtain a polynomial map  $\psi$ , arbitrary  $C^S$ -near to  $i\bar{\alpha}$  on  $W$  and with the



same  $s$ -jet at  $o \in W \in \mathbf{R}^{16}$ . The image of this  $s$ -jet therefore lies in  $A_8$ ! We now have the noncommutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\psi} & Y \times \mathbf{R}^{16} \\ & \searrow \bar{\alpha} \quad \nearrow i & \\ & A_8 & \\ & \nwarrow \bar{\rho} & \nearrow p_1 \\ & & \mathbf{R}^{16} \end{array}$$

Let  $g$  be the algebraic map  $g = p_2 i \bar{\rho} \psi : W \rightarrow \mathbf{R}^{16}$ .

Then  $X_1 = (\bar{\sigma}\psi)^{-1}(G_8) = g^{-1}(o) \subset W$  is the required Nash-manifold with one singularity. It is a real analytic manifold with one singularity, locally defined by algebraic equations. (It is an open problem also for smoothable  $n$ -manifolds, whether  $X_1$  can be a topological component, or even the whole, of the set of zeros of a set of polynomials. See [13]. If  $X_1$  has a trivial tangent bundle then it can be a topological component.)

The maps  $\bar{\alpha}$  and  $\bar{\rho}\psi : W \rightarrow A_8$  have the same  $s$ -jet at  $o$ . From Malgrange's preparation theorem [17], as applied to ideals of  $C^\infty$ -functions by Tougeron [18] and Mather [19], it follows that there exists for  $s$  large enough a  $C^\infty$ -diffeomorphism  $\zeta : U \rightarrow \zeta(U)$ , defined on some neighborhood  $U$  of  $o$  in  $\mathbf{R}^{16}$ , as  $C^2$ -near as we please to the identity map, such that  $\zeta[(p_2 \bar{\alpha})^{-1}(o) \cap U] = g^{-1}(o) \cap \zeta(U)$ . The singularities of  $X$  and  $X_1$  at  $o$  are therefore of the same exotic kind. The restriction  $\bar{\alpha}|(W - \{o\})$  is transversal to  $G_8 \subset A_8$ . Hence for any choice of neighborhood  $U'$  of  $o$  in  $\mathbf{R}^{16}$  also the restriction  $\bar{\rho}\psi|(W - U')$  is transversal to  $G_8$  in case  $\psi$  is  $C^1$ -near enough to  $i\bar{\alpha}$ .

The map which assigns to any point of  $X - U_0(c')$  ( $c'$  small) the unique nearest point of  $X_1$ , defines a diffeomorphism,  $C^2$ -near to the identity map restricted to  $X - U_0(c')$ . This diffeomorphism can be extended over  $X - \{o\}$  such that it equals  $\zeta$  near  $o$ .

Consequently  $X$  and  $X_1$  are combinatorially equivalent, and  $g$  is the polynomial map required in theorem 4.

We conclude with the formulation of two problems:

*Problem.* — Which combinatorial 8-manifolds admit a complex manifold structure with one Hirzebruch singularity?

*Problem.* — Which combinatorial 8-manifolds can be embedded as Nash manifold with one Hirzebruch singularity in a low dimensional euclidean space?

#### REFERENCES

- [1] R. THOM, Des variétés triangulées aux variétés différentiables, *Proc. Int. Congr. Math. Edinburgh*, 1958, 248-255.
- [2] J. MUNKRES, Obstructions to imposing differentiable structures, *Ill. J. Math.*, 8 (1964), 361-376.
- [3] M. A. Kervaire and J. W. MILNOR, Groups of homotopy spheres I, *Ann. Math.*, 77 (1963), 504-537.
- [4] J. CERF, La nullité de  $\pi_0$  (diff.  $S^3$ ), *Sém. H. Cartan*, 15 (1962-63).
- [5] M. HIRSCH, Obstruction theories for smoothing manifolds and maps, *Bull. A.M.S.*, 69 (1963), 352-356.
- [6] C. T. C. WALL, Cobordism of combinatorial  $n$  manifolds for  $n \leq 8$ , *Proc. Cambridge Phil. Soc.*, 60 (1964), 807-811.

- [7] A. BOREL-F. HIRZEBRUCH, Characteristic classes and homogeneous spaces, *A. J. Math.*, 81 (1959), 315-382 (Part. II) and 82 (1960), 491-504 (Part. III).
- [8] E. BRIESKORN, Beispiele zur Differentialtopologie von Singularitäten, *Inv. Math.*, 2 (1966), 1-14.
- [9] F. HIRZEBRUCH, Singularities and exotic spheres, *Séminaire Bourbaki*, 314 (November 1966).
- [10] J. W. MILNOR, *On isolated singularities of hypersurfaces*, Mimeographed, Princeton, 1966.
- [11] F. PHAM, Formules de Picard-Lefschetz généralisées et ramification des intégrales, *Bull. Soc. Math. de France*, 93 (1965), 333-367.
- [12] S. P. NOVIKOV, Topological invariance of rational Pontrjagin classes, *Doklady*, 163 (1965), 298-300.
- [13] J. NASH, Real algebraic manifolds, *Ann. Math.*, 56 (1952), 405-421.
- [14] R. THOM, Approximation algébrique des applications différentiables, *Colloque de topologie de Strasbourg*, décembre 1954, 5 pages.
- [15] H. WHITNEY, The self-intersections of a smooth  $n$ -manifold in  $2n$ -space, *Ann. Math.*, 45 (1944), 220-246.
- [16] L. M. GRAVES, Some general approximation theorems, *Ann. Math.*, 42 (1941), 281-292.
- [17] B. MALGRANGE, *Séminaire Cartan*, 1962-63, No. 11, 12, 13, 22. See also: *Ideals of differentiable functions*, Tata Institute, Bombay and Oxford University Press, and *Théorie locale des fonctions différentiables*, *Int. Math. Congress Moscow*, 1966.
- [18] J. C. TOUGERON, Thèse à Rennes. Compare *C. R. Acad. Sc. Paris*, 262 (A et B) (1966), 563-565.
- [19] J. MATHER, Thesis, *Work to appear in Ann. of Math.*, Princeton.
- [20] O. SULLIVAN, On the Hauptvermutung for manifolds, *Bull. A.M.S.*, 73 (1967), 598-600.

*Manuscrit reçu le 30 juin 1967.*