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K-THEORY AND STABLE ALGEBRA

by H. BASS

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INTRODUCTION

A vector space can be viewed, according to one's predilections, either as a module over a ring, or as a vector bundle over a space (with one point). If one seeks broad generalizations of the structure theorems of classical linear algebra, however, the satisfaction afforded by the topologists has, unhappily, no algebraic counterpart.

Our point of departure here is the observation that the topological version of linear algebra is a particular case of the algebraic one. Specifically, the study of continuous real vector bundles over a compact X, for example, is equivalent to the study of finitely generated *projective* C(X)-modules, C(X) being the ring of continuous real functions on X. This connection was first pointed out by Serre [28] in algebraic geometry, and recently in the above form by Swan [34]. Serre even translated a theorem from bundle theory into pure algebra, and he invented the techniques to prove it [29] (see § 8). His example made it clear how to translate large portions of homotopy theory into the same setting, and thus discover, if not prove, an abundance of natural theorems of topological origin.

What follows is the result of a first systematic attempt to exploit this idea. This investigation was inaugurated jointly by S. Schanuel and the author, and an announcement of the results of that earlier work was made in [9]. In particular, the topological background for the results of §§ 10-11 is pointed out there.

Generally speaking the method is as follows. The problem at hand is " locally trivial ", i.e. locally it can be solved by a simple parody of classical linear algebra. Furthermore, one proves an " approximation " lemma which asserts the existence of global data with prescribed behavior in a given local situation. Finally, in order to piece these together one uses a kind of " general position " argument. The ability to put things in general position imposes a dimensional restriction on the conclusions, and thus we determine a " stable range " for the problem.

The structures investigated here are those of projective modules (Chapter II) and of the general linear group (Chapter I). This can be thought of as analogous to the study of vector bundles on a space, and on its suspension, respectively. If we " stabilize ", we are led to consider analogues of the functors K^0 and K^1, respectively, of Atiyah and Hirzebruch [2], and a good deal of the formalism, in particular the exact sequence, of that analogy is developed in Chapter III. The natural extension of the functors K^i, $i>2$, to our algebraic context has so far evaded a definitive appearance.

In another direction, this point of view should be fruitful in studying other classical linear groups. For example, thinking of bundles with reduced structure group, one can consider non degenerate quadratic forms on projective modules, and the associated orthogonal groups. K^0 would then be the " Witt ring ", and Witt's theorem, for example, is analogous to our Theorem 9.3. The local part of this theory has recently been done by Klingenberg [25], but several serious problems continue to obstruct its globalization.

The most important applications of our machinery are to the linear groups over orders in semi-simple algebras finite dimensional over the rationals (§§ 19-21). When applied to $\mathbf{Z}\pi$, π a finite group, they give quantitative information on J. H. C. Whitehead's groups of simple homotopy types [36], results which elaborate on some earlier work of G. Higman [38]. They also shed some new light on the structure of $\mathbf{SL}(n, A)$, with A the ring of integers in a number field. Most striking is the fact that, for n suitably large, if H is a non central normal subgroup, then $\mathbf{SL}(n, A)/H$ is a *finite central* extension of **PSL**$(n, A/\mathfrak{q})$ for some ideal $\mathfrak{q}$. This information is the starting point for the proof, in [40], that every subgroup of finite index in **SL**$(n, \mathbf{Z})$, $n \geq 3$, contains a congruence subgroup (see § 21).

The author owes most of his mathematics to Serre's genius for asking the right question at the right time, and he records here his gratitude for having so profited. He is particularly thankful also to A. Heller, who has endured long audience on the present work, and who is responsible for numerous improvements in its exposition.

CHAPTER I

STABLE STRUCTURE OF THE LINEAR GROUPS

§ 1. Notation and lemmas.

The objectives of this chapter are Theorems 3.1 and 4.2 below, which purport to describe all the normal subgroups of the general linear group. We begin in this section by establishing notation and some general trivialities on matrices.

Let A be a ring. $\mathbf{GL}(n, A)$ is the group of invertible n by n matrices over A. Let e_{ij} denote the matrix with 1 in the $(i, j)^{th}$ coordinate, and zeroes elsewhere; we recall that $e_{ij}e_{kh} = \delta_{jk}e_{ih}$. Let $a, b \in A$ and $i \neq j$. Then $(1 + ae_{ij})(1 + be_{ij}) = 1 + (a+b)e_{ij}$; here $1 = 1_n$ denotes the n by n identity matrix. Thus, for i and j fixed, the matrices $1 + ae_{ij}$ form a subgroup of $\mathbf{GL}(n, A)$ isomorphic to the additive group of A. A matrix of the form $1 + ae_{ij}, i \neq j$, is called *elementary*, and we denote by $\mathbf{E}(n, A)$ the subgroup of $\mathbf{GL}(n, A)$ generated by all elementary matrices.

Now let $\mathfrak{q}$ (possibly $=$ A) be a two sided ideal in A. The $\mathfrak{q}$-congruence group is

$$\mathbf{GL}(n, A, \mathfrak{q}) = \ker(\mathbf{GL}(n, A) \to \mathbf{GL}(n, A/\mathfrak{q})).$$

Moreover we denote by $\mathbf{E}(n, A, \mathfrak{q})$ the *normal* subgroup of $\mathbf{E}(n, A)$ generated by all elementary matrices in $\mathbf{GL}(n, A, \mathfrak{q})$.

We shall identify $\mathbf{GL}(n, A)$ with a subgroup of $\mathbf{GL}(n+m, A)$ by identifying $\alpha \in \mathbf{GL}(n, A)$ with $\begin{vmatrix} \alpha & 0 \\ 0 & 1_m \end{vmatrix} \in \mathbf{GL}(n+m, A)$. This done, we set

$$\mathbf{GL}(A, \mathfrak{q}) = \bigcup_n \mathbf{GL}(n, A, \mathfrak{q}) \quad \text{and} \quad \mathbf{E}(A, \mathfrak{q}) = \bigcup_n \mathbf{E}(n, A, \mathfrak{q}).$$

When $\mathfrak{q} = A$ we write $\mathbf{GL}(A) = \mathbf{GL}(A, A)$ and $\mathbf{E}(A) = \mathbf{E}(A, A)$. $\mathbf{GL}(A)$ is called the *stable general linear group* over A.

Lemma **(1.1)** (" Homotopy Extension "). — *If* $A \to B$ *is a surjective ring homomorphism, then* $\mathbf{E}(n, A, \mathfrak{q}) \to \mathbf{E}(n, B, \mathfrak{q}B)$ *is surjective for all* n *and* $\mathfrak{q}$.

Proof. — If $1 + be_{ij}$ is B-elementary, and b is the image of $a \in A$, then $1 + ae_{ij}$ maps onto $1 + be_{ij}$. (There is an abuse of notation here which one should excuse, in that " 1 " and " e_{ij} " have two senses.) This shows that $\mathbf{E}(n, A) \to \mathbf{E}(n, B)$ is surjective. Now $\mathbf{E}(n, B, \mathfrak{q}B)$ is generated by elements $\tau^\sigma = \sigma^{-1}\tau\sigma$ with $\sigma \in \mathbf{E}(n, B)$ and τ $\mathfrak{q}B$-elementary; i.e. $\tau = 1 + qe_{ij}$, $q \in \mathfrak{q}B$. We can find $q' \in \mathfrak{q}$ with image q, and $\sigma' \in \mathbf{E}(n, A)$ with image σ (by the first part of the proof). Clearly $(1 + q'e_{ij})^{\sigma'}$ lifts τ^σ.

If H_1 and H_2 are subgroups of a group we denote by $[H_1, H_2]$ the subgroup generated by all $[h_1, h_2] = h_1^{-1}h_2^{-1}h_1h_2$, with $h_i \in H_i$.

Lemma **(1.2)**. — *If* i, j, *and* k *are distinct, then*

$$[1-ae_{ij},\ 1-be_{jk}]=1+abe_{ik}.$$

Proof. — $(1+ae_{ij})(1+be_{jk})(1-ae_{ij})(1-be_{jk})=$

$$(1+ae_{ij}+be_{jk}+abe_{ik})(1-ae_{ij}-be_{jk}+abe_{ik})=$$

$$(1+ae_{ij}+be_{jk}+abe_{ik})-(ae_{ij})-(be_{jk}+abe_{ik})+(abe_{ik})=1+abe_{ik}.$$

Corollary **(1.3)**. — *If* $\mathfrak{q}$ *and* $\mathfrak{q}'$ *are ideals and* $n\geq 3$, *then*

$$\mathbf{E}(n,\mathrm{A},\mathfrak{q}\mathfrak{q}')\subset[\mathbf{E}(n,\mathrm{A},\mathfrak{q}),\mathbf{E}(n,\mathrm{A},\mathfrak{q}')].$$

In particular, $\mathbf{E}(n,\mathrm{A},\mathfrak{q})=[\mathbf{E}(n,\mathrm{A}),\mathbf{E}(n,\mathrm{A},\mathfrak{q})]$.

Proof. — The right side is a normal subgroup of $\mathbf{E}(n,\mathrm{A})$ which, by (1.2), contains all $\mathfrak{q}\mathfrak{q}'$-elementary matrices.

Corollary **(1.4)**. — *Suppose* $n\geq 3$, *and let* H *be a subgroup of* $\mathbf{GL}(n,\mathrm{A})$ *normalized by* $\mathbf{E}(n,\mathrm{A})$. *If* T *is a family of elementary matrices in* H, *then* $\mathrm{H}\supset\mathbf{E}(n,\mathrm{A},\mathfrak{q})$, *where* $\mathfrak{q}$ *is the two-sided ideal generated by the coordinates of* $1-\tau$ *for all* $\tau\in\mathrm{T}$.

Proof. — If $\tau=1+ce_{hl}\in\mathrm{T}$ then, since $n\geq 3$, we can commutate with other elementary matrices and obtain all elementary matrices of the form $1+acbe_{ik}$. The $\mathbf{E}(n,\mathrm{A})$-invariant subgroup generated by these, as τ varies in T, is clearly $\mathbf{E}(n,\mathrm{A},\mathfrak{q})$.

Corollary **(1.5)**. — (i) $\mathbf{E}(n,\mathrm{A})=[\mathbf{E}(n,\mathrm{A}),\mathbf{E}(n,\mathrm{A})]$ *for* $n\geq 3$.

(ii) *If* A *is finitely generated as a* **Z**-*module, then* $\mathbf{E}(n,\mathrm{A})$ *is a finitely generated group for all* n.

(iii) *If* A *is finitely generated as a* **Z**-*algebra, then* $\mathbf{E}(n,\mathrm{A})$ *is a finitely generated group for all* $n\geq 3$.

Proof. — (i) is immediate from (1.3), setting $\mathfrak{q}=\mathrm{A}=\mathfrak{q}'$.

(ii) Is obvious, since $\mathbf{E}(n,\mathrm{A})$ is generated by a finite number of subgroups, each isomorphic to the additive group of A.

(iii) Let $a_0=1, a_1, \ldots, a_r$ generate A as a ring, and consider the elementary matrices $1+a_ie_{jk}$, $0\leq i\leq r$, and all $j\neq k$. These generate a group which, by (1.2), contains all $1+\mathrm{M}e_{jk}$, where M is a monomial in the a_i. Since A is additively generated by these M we catch all elementary matrices.

In special cases we get something for $n=2$.

Lemme **(1.6)**. — *Suppose* $1=u+v$ *with* u *and* v *units in* A. *Then*

$$\mathbf{E}(2,\mathrm{A},\mathfrak{q})\subset[\mathbf{GL}(2,\mathrm{A}),\mathbf{E}(2,\mathrm{A},\mathfrak{q})].$$

If, further, $u=w^2$ *with* $w\in$center A, *then* $\mathbf{E}(2,\mathrm{A},\mathfrak{q})=[\mathbf{E}(2,\mathrm{A}),\mathbf{E}(2,\mathrm{A},\mathfrak{q})]$.

Proof. — Given $q\in\mathfrak{q}$, set $b=v^{-1}q$, $\alpha=\begin{vmatrix}u & 0\\ 0 & 1\end{vmatrix}$, and $\beta=\begin{vmatrix}1 & b\\ 0 & 1\end{vmatrix}\in\mathbf{E}(2,\mathrm{A},\mathfrak{q})$. Then $[\alpha,\beta]=\begin{vmatrix}1 & (1-u)b\\ 0 & 1\end{vmatrix}=\begin{vmatrix}1 & q\\ 0 & 1\end{vmatrix}\in[\mathbf{GL}(2,\mathrm{A}),\mathbf{E}(2,\mathrm{A},\mathfrak{q})]$. Moreover, $w^{-1}\alpha=\begin{vmatrix}w & 0\\ 0 & w^{-1}\end{vmatrix}\in\mathbf{E}(2,\mathrm{A})$, by Lemma 1.7 below, and $[w^{-1}\alpha,\beta]=[\alpha,\beta]$.

Examples. — 1. If A contains a field with more than two elements, 1 is a sum of two units. Likewise if 2 is a unit in A.

2. If A is an integral domain containing a primitive n^{th} root, u, of 1, then $1+u+\ldots+u^i$ is a unit if $i+1$ is relatively prime to n. Since

$$1+u+\ldots+u^i=1+u(1+\ldots+u^{i-1})$$

we can write 1 as a sum of two units provided both i and $i+1$ are prime to n. Such an i always exists unless n is a power of two.

3. The commutator quotient of $\mathbf{GL}(2, \mathbf{Z})$ is a group of type $(2, 2)$, and that of $\mathbf{SL}(2, \mathbf{Z})$ is cyclic of order 12. More generally, the commutator quotient of $\mathbf{SL}(2, \mathbf{Z}/q\mathbf{Z})$ is cyclic of order d, where $d=\gcd(q, 12)$. If $\mathbf{Z}_p$ denotes the p-adic integers then the commutator quotient of $\mathbf{SL}(2, \mathbf{Z}_p)$ is cyclic of order 4 for $p=2$, 3 for $p=3$, and 1 for $p\neq 2$ or 3. In all of these examples $\mathbf{E}=\mathbf{SL}$.

4. For any ring A 1 is a sum of two units in $\mathbf{M}(n, \mathrm{A})$ for all $n>1$. For if $n>1$ let $\mathrm{B}=\mathrm{A}[\alpha]=\mathrm{A}[t]/(f(t))$, where $f(t)=t^n-t+1$. Then $1=\alpha(1-\alpha^{n-1})=\alpha(1-\alpha)g(\alpha)$ so α and $1-\alpha$ are units in B. Therefore they define A-automorphisms of $\mathrm{B}\cong\mathrm{A}^n$ whose sum is 1.

The next lemma plays a fundamental role in what follows.

Lemma **(1.7)** (" Whitehead Lemma "). — *Let* $a\in\mathbf{GL}(n, \mathrm{A})$ *and* $b\in\mathbf{GL}(n, \mathrm{A}, \mathfrak{q})$. *Then*

$$\begin{vmatrix} ab & 0 \\ 0 & 1 \end{vmatrix} \equiv \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \equiv \begin{vmatrix} ba & 0 \\ 0 & 1 \end{vmatrix} \qquad \bmod \mathbf{E}(2n, \mathrm{A}, \mathfrak{q})$$

and

$$\equiv \begin{vmatrix} 0 & a \\ -b & 0 \end{vmatrix} \qquad \bmod \mathbf{E}(2n, \mathrm{A})$$

(The congruences hold for either left or right cosets.)

Proof. — Write $b=1+q$; q is an $n\times n$ matrix with coordinates in $\mathfrak{q}$. Let

$$\alpha=\begin{vmatrix} ba & 0 \\ 0 & 1 \end{vmatrix}, \qquad \beta=\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}, \qquad \tau_1=\begin{vmatrix} 1 & (ba)^{-1}q \\ 0 & 1 \end{vmatrix}, \qquad \tau_2=\begin{vmatrix} 1 & -a^{-1}q \\ 0 & 1 \end{vmatrix},$$

$$\tau_3=\begin{vmatrix} 1 & 0 \\ -b^{-1}qa & 1 \end{vmatrix}, \qquad \text{and} \qquad \sigma=\begin{vmatrix} 1 & 0 \\ a & 1 \end{vmatrix}.$$

Then clearly $\tau=\tau_1\tau_2^{\sigma}\tau_3\in\mathbf{E}(2n, \mathrm{A}, \mathfrak{q})$. We begin by showing that $\alpha\tau=\beta$.

$$\alpha\tau_1=\begin{vmatrix} ba & q \\ 0 & 1 \end{vmatrix}, \qquad \alpha\tau_1\sigma^{-1}=\begin{vmatrix} ba-qa & q \\ -a & 1 \end{vmatrix}=\begin{vmatrix} a & q \\ -a & 1 \end{vmatrix},$$

$$\alpha\tau_1\sigma^{-1}\tau_2=\begin{vmatrix} a & -q+q \\ -a & 1+q \end{vmatrix}=\begin{vmatrix} a & 0 \\ -a & b \end{vmatrix},$$

$$\alpha\tau_1\sigma^{-1}\tau_2\sigma=\begin{vmatrix} a & 0 \\ ba-a & b \end{vmatrix}=\begin{vmatrix} a & 0 \\ qa & b \end{vmatrix}; \qquad \alpha\tau=\beta.$$

Taking $a=b^{-1}$ we have $\begin{vmatrix} b^{-1} & 0 \\ 0 & b \end{vmatrix} \in \mathbf{E}(2n, A, \mathfrak{q})$ and hence

$$\begin{vmatrix} ab & 0 \\ 0 & 1 \end{vmatrix} \equiv \begin{vmatrix} ab & 0 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} b^{-1} & 0 \\ 0 & b \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \qquad \text{mod } \mathbf{E}(2n, A, \mathfrak{q}).$$

Finally, modulo $\mathbf{E}(2n, A)$, we have

$$\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} \equiv \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & a \\ -b & 0 \end{vmatrix}.$$

For left cosets we need only observe that all subgroups involved are invariant under transposition.

Corollary **(1.8)**. — $[\mathbf{GL}(n, A), \mathbf{GL}(n, A, \mathfrak{q})] \subset \mathbf{E}(2n, A, \mathfrak{q})$.

Remark. — It would be useful if one could strengthen this corollary to say: $[\mathbf{GL}(n, A, \mathfrak{q}), \mathbf{GL}(n, A, \mathfrak{q}')] \subset \mathbf{E}(2n, A, \mathfrak{q}\mathfrak{q}')$, say even under the assumption that $\mathfrak{q}+\mathfrak{q}'=A$.

Combining (1.3) and (1.8) we have, for $n\geq 3$,

$$\mathbf{E}(n, A, \mathfrak{q}) = [\mathbf{E}(n, A), \mathbf{E}(n, A, \mathfrak{q})] \subset [\mathbf{GL}(n, A), \mathbf{GL}(n, A, \mathfrak{q})] \subset \mathbf{E}(2n, A, \mathfrak{q}).$$

Letting $n\to\infty$ we conclude:

Corollary **(1.9)**. — $\mathbf{E}(A, \mathfrak{q}) = [\mathbf{E}(A), \mathbf{E}(A, \mathfrak{q})] = [\mathbf{GL}(A), \mathbf{GL}(A, \mathfrak{q})]$. *In particular,* $\mathbf{E}(A) = [\mathbf{GL}(A), \mathbf{GL}(A)]$.

Remark. — The last conclusion is due to J. H. C. Whitehead [36, § 1]. Indeed, essentially, everything in this section is inspired by Whitehead's procedure in [36].

§ 2. The affine group.

Lemma **(2.1)**. — *For* $n\geq 2$ *the additive group generated by* $\mathbf{E}(n, A)$ *is the full matrix algebra,* $\mathbf{M}(n, A)$.

Proof. — It suffices to catch ae_{ij} for all $a\in A$, all i, j. If $i\neq j$, $ae_{ij}=(1+ae_{ij})-1$. For the diagonal elements we have, for example,

$$ae_{11}=(1+ae_{12})(1+e_{21})-1-ae_{12}-e_{21}.$$

Viewing A^n as a right A-module we can identify $\mathbf{GL}(n, A)$ with $\mathrm{Aut}_A(A^n)$.

Corollary **(2.2)**. — *For* $n\geq 2$ *the* $\mathbf{E}(n, A)$ *invariant subgroups of* A^n *are the* $\mathfrak{q}A^n$, *where* $\mathfrak{q}$ *ranges over all left ideals.* (Note that these are not sub-A-modules.)

Proof. — By the Lemma we can replace $\mathbf{E}(n, A)$ by $\mathbf{M}(n, A)$. Let

$$\alpha=(a_1, \ldots, a_n)\in A^n$$

and let $\mathfrak{q}=\Sigma A a_i$. It clearly suffices to show that $\mathbf{M}(n, A)\alpha=\mathfrak{q}A^n$. But the obvious use of coordinate projections, permutations, and left multiplications makes this evident.

Now $\mathbf{Aff}(n, A)$ is defined to be the semi-direct product $\mathbf{GL}(n, A)\times A^n$. If $\sigma\in\mathbf{GL}(n, A)$ and $\alpha\in A^n$ the multiplication is given by

$$(\sigma, \alpha)(\sigma', \alpha')=(\sigma\sigma', \sigma\alpha'+\alpha).$$

In particular

$$(\sigma, \alpha)^{-1} = (\sigma^{-1}, -\sigma^{-1}\alpha).$$

We identify $\mathbf{GL}(n, A)$ with $\mathbf{GL}(n, A) \times 0$, and A^n with $1 \times A^n$. The latter is an abelian normal subgroup. Since

$$(\sigma, 0)(1, \alpha)(\sigma, 0)^{-1} = (1, \sigma\alpha)$$

we see that for subgroups $H \subset \mathbf{GL}(n, A)$ and $S \subset A^n$, invariance of S under H is the same in both of its possible senses.

We shall often identify $(\sigma, \alpha) \in \mathbf{Aff}(n, A)$ with $\begin{vmatrix} \sigma & \alpha \\ 0 & 1 \end{vmatrix} \in \mathbf{GL}(n+1, A)$ (viewing α as a column vector). Note that this identifies $\mathbf{E}(n, A) \times A^n$ with a subgroup of $\mathbf{E}(n+1, A)$.

Proposition **(2.3)**. — *Let* H *be a subgroup of* $\mathbf{Aff}(n, A)$ *with projection* L *in* $\mathbf{GL}n(, A)$. *Then*

(i) $[H, A^n] = [L, A^n] = \Sigma_{\tau \in L}(1-\tau)A^n$.

(ii) *If* H *is normalized by* A^n *then* $[H, A^n] \subset H$ *so in this case* $H \cap A^n$ *trivial* $\Rightarrow$ H *trivial.*

(iii) *If* $n \geq 2$ *and* H *is normalized by* $\mathbf{E}(n, A)$, *then* $[H, A^n] = \mathfrak{q}A^n$ *for a unique left ideal* $\mathfrak{q}$.

Proof. — (i) If $(\tau, \beta) \in H$, so $\tau \in L$, and $\alpha = (1, \alpha) \in A^n$, then

$$[(\tau, \beta), (1, \alpha)] = (1, (1-\tau^{-1})\alpha).$$

(i) $\Rightarrow$ (ii) is trivial, and (iii) is a consequence of (2.2).

We shall use this proposition to show that, under suitable conditions, a subgroup $H \subset \mathbf{GL}(n, A)$ normalized by $\mathbf{E}(n, A)$ contains $\mathbf{E}(n, A, \mathfrak{q})$ for some $\mathfrak{q} \neq 0$. By (1.4) it will be sufficient to show that H contains a single non trivial elementary matrix. Of course we assume $n \geq 3$ in order to invoke (1.4).

Suppose first that there is a $\sigma \in H$ and a unit $u \in$ center A, such that $\sigma \neq u.1$, but $\sigma - u.1$ has a zero row or column. We first apply the following:

Remark. — Since **Z** is a Euclidean ring $\mathbf{E}(n, \mathbf{Z}) = \mathbf{SL}(n, \mathbf{Z})$ for all $n \geq 2$. Hence every " permutation " (i.e. a matrix with one non zero coordinate equal to ± 1 in each row and column) of determinant 1 lies in $\mathbf{E}(n, \mathbf{Z})$. By specialization, every such permutation lies in $\mathbf{E}(n, A)$ for any A.

Now back to σ and u above. We can conjugate with a permutation in $\mathbf{E}(n, A)$ and assume $\sigma - u.1$ has either the last row or column zero; say the last row. Then $\sigma = u\sigma'$ with $1 \neq \sigma' \in \mathbf{Aff}(n-1, A)$. Since commutating with σ neglects u we can use (2.3) to produce lots of elementary matrices (in the last column). If the last column of $\sigma - u.1$ vanishes we transpose the above argument.

Before proceeding further with our problem, let us compute the centralizer of $\{\tau = 1 + ae_{12} \mid a \in A\}$. Given σ, let α be the first column of σ and β the second row of σ^{-1}. Then $\sigma\tau\sigma^{-1} = 1 + \alpha a \beta$. Hence, if $\sigma\tau = \tau\sigma$ we have $\alpha a \beta = ae_{12}$. If $a = 1$ we conclude that $\alpha = \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and $\beta = (0, u^{-1}, 0, \ldots, 0)$. Then allowing general a we find $uau^{-1} = a$ so that $u \in \text{center } A$. Now changing e_{12} to e_{ij} we can already record the following:

Corollary **(2.4)**. — *If* $n \geq 2$, *the centralizer of* $\mathbf{E}(n, A)$ *consists of all matrices* $u.1$ *with u a unit in the center of* A.

Continuing our argument above, our objective is:

Lemma **(2.5)**. — *Let* $n \geq 3$ *and* H *a subgroup of* $\mathbf{GL}(n, A)$ *normalized by* $\mathbf{E}(n, A)$. *If* H *contains a non central element* σ *with some coordinate zero, then* $H \supset \mathbf{E}(n, A, \mathfrak{q})$ *for some* $\mathfrak{q} \neq 0$.

Proof. — After conjugating σ with a permutation in $\mathbf{E}(n, A)$ we can make a_1 or a_n zero, where $\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is the first column of σ. If σ commutes with all $\tau = 1 + ae_{12}$ then σ has the same first column as $u.1$ for some $u \in \text{center } A$, by the last paragraph, and this case was handled already above. Hence we may assume $a\tau$ can be chosen so that $\rho = \sigma\tau\sigma^{-1}\tau^{-1} \neq 1$. Then $\rho = \tau^{-1} + \alpha\gamma$, where $\gamma = a\beta\tau^{-1}$, and β is the second row of σ^{-1} (see computation above).

Case 1. $a_n = 0$. Then $\alpha\gamma$ has zero last row, so $\rho = \tau^{-1} + \alpha\gamma$ and τ^{-1} have the same last row. Hence we have $1 + \rho \in \mathbf{Aff}(n-1, A)$ and we can use (2.3) to get elementary matrices.

Case 2. $a_1 = 0$. Then the first rows of ρ and τ^{-1} agree, so ρ is not central and has an off diagonal zero (since τ^{-1} has first row $(1, -a, 0, \ldots, 0)$). Hence we can replace σ by ρ and obtain Case 1 again.

§ 3. Structure of GL(A).

Theorem **(3.1)** (Stable Structure Theorem.)

Let A *be any ring and* $\mathbf{GL}(A)$ *the stable general linear group over* A (see § 1 for definition).

a) *For all two-sided ideals* $\mathfrak{q}$,

$$\mathbf{E}(A, \mathfrak{q}) = [\mathbf{E}(A), \mathbf{E}(A, \mathfrak{q})] = [\mathbf{GL}(A), \mathbf{GL}(A, \mathfrak{q})].$$

b) *A subgroup* $H \subset \mathbf{GL}(A)$ *is normalized by* $\mathbf{E}(A) \Leftrightarrow$ *for some (necessarily unique)* $\mathfrak{q}$,

$$\mathbf{E}(A, \mathfrak{q}) \subset H \subset \mathbf{GL}(A, \mathfrak{q}).$$

H *is then automatically normal in* $\mathbf{GL}(A)$.

c) *If* $A \to B$ *is a surjective ring homomorphism and if* H *is normal in* $\mathbf{GL}(A)$, *the image of* H *is normal in* $\mathbf{GL}(B)$. (Note that $\mathbf{GL}(A) \to \mathbf{GL}(B)$ need not be surjective.)

Proof. — *a)* is just (1.9). *c)* is a consequence of *b)* since the image of H will be normalized by the image of $\mathbf{E}(A)$, which, by (1.1), is $\mathbf{E}(B)$. Moreover, the uniqueness of $\mathfrak{q}$ and normality of H in part *b)* both follows from $[\mathbf{GL}(A), H] = \mathbf{E}(A, \mathfrak{q}) \subset H$, which is a consequence of *a)*.

It remains to prove that if $H \subset \mathbf{GL}(A)$ is normalized by $\mathbf{E}(A)$, and if $\mathfrak{q}$ is the ideal generated by the coordinates of $1 - \tau$ for all $\tau \in H$, then $\mathbf{E}(A, \mathfrak{q}) \subset H$. Let $H_n = H \cap \mathbf{GL}(n, A)$ and let $\mathfrak{q}_n$ be the ideal generated by all coordinates of $1 - \tau$, $\tau \in H_n$. Viewing $H_n \subset \mathbf{Aff}(n+1, A)$ it follows from (2.3) and (1.4) that H_{n+1} contains enough elementary matrices to capture $\mathbf{E}(n+1, A, \mathfrak{q}_n)$. Hence

$$H \supset \cup_n \mathbf{E}(n+1, A, \mathfrak{q}_n) = \cup_n \mathbf{E}(A, \mathfrak{q}_n) = \mathbf{E}(A, \mathfrak{q}).$$

§ 4. Structure in the stable range.

One would like to recover the stable structure theorem for $\mathbf{GL}(n, A)$, $n < \infty$. Fortunately a sufficient condition for this can be formulated as a very simple axiom, one which we will verify in a rather general setting in Chapter II (Corollary 6.5 and Theorem 11.1).

Definition. — Let $\alpha = (a_1, \ldots, a_r)$ be an element of the right A-module A^r. We call α *unimodular* (in A^r) if $\Sigma_i A a_i = A$. This is clearly equivalent to the existence of a linear functional $f : A^r \to A$ such that $f(\alpha) = 1$. Let $n \geq 1$; we say *n defines a stable range for* $\mathbf{GL}(\mathbf{A})$ if, for all $r > n$, given $\alpha = (a_1, \ldots, a_r)$ unimodular in A^r, there exist $b_1, \ldots, b_{r-1}$ in A such that $(a_1 + b_1 a_r, \ldots, a_{r-1} + b_{r-1} a_r)$ is unimodular in A^{r-1}.

Examples. — If A is a semi-local ring, then $n = 1$ defines a stable range. If A is a Dedeking ring $n = 2$ works. If A is the coordinate ring of a d dimensional affine algebraic variety, e.g. if A is a polynomial ring in d variables over a field, then $n = d + 1$ defines a stable range for $\mathbf{GL}(A)$.

Lemma **(4.1)**. — *If n defines a stable range for* $\mathbf{GL}(A)$ *it does likewise for* $\mathbf{GL}(A/\mathfrak{q})$ *for all ideals* $\mathfrak{q}$.

Proof. — Suppose $r > n$. Writing $A' = A/\mathfrak{q}$, suppose $\alpha' = (a'_1, \ldots, a'_r)$ is unimodular in $(A')^r$; say $1 = \Sigma t'_i a'_i$. Lift t'_i and a'_i to t_i and a_i in A, so $1 = \Sigma t_i a_i + q$ with $q \in \mathfrak{q}$. Then $(a_1, \ldots, a_r, q) \in A^{r+1}$ is unimodular; so, by hypothesis, there exist $c_i \in A$ such that $(a_1 + c_1 q, \ldots, a_r + c_r q)$ is unimodular. Replacing a_i by $a_i + c_i q$, then, we can assume α is unimodular. Again by hypothesis there exist $b_i \in A$ rendering

$$(a_1 + b_1 a_r, \ldots, a_{r-1} + b_{r-1} a_r)$$

unimodular. The images, b'_i, of the b_i in A' now satisfy our requirements.

Theorem **(4.2)**. — *Suppose n defines a stable range for* $\mathbf{GL}(A)$.

For $r > n$ *and for all ideals* $\mathfrak{q}$:

a) *The orbits of* $\mathbf{E}(r, A, \mathfrak{q})$ *on the unimodular elements of* A^r *are the congruence classes modulo* $\mathfrak{q}$. *In particular* $\mathbf{E}(r, A)$ *is transitive.*

b) $\mathbf{GL}(r, A, \mathfrak{q}) = \mathbf{GL}(n, A, \mathfrak{q}) \mathbf{E}(r, A, \mathfrak{q})$.

c) $\mathbf{E}(r, A, \mathfrak{q})$ *is a normal subgroup of* $\mathbf{GL}(r, A)$.

For $r > \max(n, 2)$:

d) $\mathbf{E}(r, A, \mathfrak{q}) = [\mathbf{E}(r, A), \mathbf{E}(r, A, \mathfrak{q})] = [\mathbf{E}(r, A), \mathbf{GL}(r, A, \mathfrak{q})]$ *for all ideals* $\mathfrak{q}$.

e) *If* $H \subset \mathbf{GL}(r, A)$ *is normalized by* $\mathbf{E}(r, A)$, *then, for a unique ideal* $\mathfrak{q}$, $\mathbf{E}(r, A, \mathfrak{q}) \subset H$ *and the image of* H *in* $\mathbf{GL}(r, A/\mathfrak{q})$ *lies in the center.*

For $r \geq \max(2n, 3)$, *and for all ideals* $\mathfrak{q}$:

f) $\mathbf{E}(r, A, \mathfrak{q}) = [\mathbf{GL}(r, A), \mathbf{GL}(r, A, \mathfrak{q})]$.

Proof. — a) We first show that if $\alpha = (a_1, \ldots, a_r) = (1 + q_1, q_2, \ldots, q_r)$ is unimodular, with $q_i \in \mathfrak{q}$, then there is a $\tau \in \mathbf{E}(r, A, \mathfrak{q})$ such that $\tau\alpha = (1, 0, \ldots, 0)$. Writing 1 as a left linear combination of $a_1, \ldots, a_r$ and multiplying this equation on the left by $q_r = a_r$, the coefficient of a_r in the new equation is a right multiple, say q, of q_r, and hence in $\mathfrak{q}$. Thus a_r is in the left ideal generated by $a_1, \ldots, a_{r-1}, qa_r$, so $(a_1, \ldots, a_{r-1}, qa_r)$ is unimodular. Our hypotheses now say that we can find $a'_i = a_i + b_i q a_r$, $1 \leq i \leq r-1$, such that $(a'_1, \ldots, a'_{r-1})$ is unimodular. Set $\tau_1 = 1 + \Sigma_{i=1}^{r-1} b_i q e_{ir} \in \mathbf{E}(r, A, \mathfrak{q})$, and write $(a'_1, \ldots, a'_{r-1}) = (1 + q'_1, \ldots, q'_{r-1})$. Then $q'_i \in \mathfrak{q}$, $1 \leq i \leq r-1$, and

$$\tau_1\alpha = (1 + q'_1, \ldots, q'_{r-1}, q_r).$$

Writing 1 as a left linear combination of $a'_1, \ldots, a'_{r-1}$, and left multiplying this equation by $q'_1 - q_r$, we can write $q'_1 - q_r = \Sigma_{i=1}^{r-1} c_i a'_i$ with $c_i \in \mathfrak{q}$. Then if

$$\tau_2 = 1 + \Sigma_{i=1}^{r-1} c_i e_{ri} \in \mathbf{E}(r, A, \mathfrak{q}),$$

we have $\tau_2\tau_1\alpha = (1 + q'_1, \ldots, q'_{r-1}, q'_1)$. If $\sigma = 1 - e_{1r}$ then $\sigma\tau_2\tau_1\alpha = (1, q'_2, \ldots, q'_{r-1}, q'_1)$. Let $\tau_3 = 1 - (q'_2 e_{21} + \ldots + q'_{r-1} e_{(r-1)1} + q'_1 e_{r1})$; then $\tau_3\sigma\tau_2\tau_1\alpha = \beta = (1, 0, \ldots, 0)$, and $\tau_3 \in \mathbf{E}(r, A, \mathfrak{q})$. The presence of σ, which need not belong to $\mathbf{E}(r, A, \mathfrak{q})$, is harmless, since $\sigma^{-1}\beta = \beta$. Hence $\tau = \sigma^{-1}\tau_3\sigma\tau_2\tau_1 \in \mathbf{E}(r, A, \mathfrak{q})$ solves our problem.

Now for the general case. Setting $\mathfrak{q} = A$, the case above was already general, and we have thus shown that $\mathbf{E}(r, A)$ is transitive. Now let $\mathfrak{q}$ be arbitrary, and $\alpha \equiv \beta \bmod \mathfrak{q}$ unimodular elements of A^r. We can find $\sigma \in \mathbf{E}(r, A)$ so that $\sigma\beta = (1, 0, \ldots, 0)$. Since $\sigma\alpha \equiv \sigma\beta \bmod \mathfrak{q}$ the argument above provides a $\tau \in \mathbf{E}(r, A, \mathfrak{q})$ with $\tau\sigma\alpha = \sigma\beta$; hence $\sigma^{-1}\tau\sigma\alpha = \beta$.

b) Given $a \in \mathbf{GL}(r, A, \mathfrak{q})$, the last column of a is congruent, mod $\mathfrak{q}$, to $(0, \ldots, 0, 1)$. Hence, by part *a)* above, there is a $\tau_1 \in \mathbf{E}(r, A, \mathfrak{q})$ such that $\tau_1 a = \begin{vmatrix} a_1 & 0 \\ \alpha & 1 \end{vmatrix}$ with $a_1 \in \mathbf{GL}(r-1, A, \mathfrak{q})$, $\alpha \in \mathfrak{q}A^{r-1}$. Set $\tau_2 = \begin{vmatrix} 1 & 0 \\ -\alpha a_1^{-1} & 1 \end{vmatrix} \in \mathbf{E}(r, A, \mathfrak{q})$; then

$$\tau_2\tau_1 a = \begin{vmatrix} a_1 & 0 \\ 0 & 1 \end{vmatrix} \in \mathbf{GL}(r-1, A, \mathfrak{q}).$$

By induction we can continue reducing until reaching $\mathbf{GL}(n, A, \mathfrak{q})$.

c) $\mathbf{E}(r, A, \mathfrak{q})$ is generated by all $\tau^\sigma = \sigma^{-1}\tau\sigma$ with τ $\mathfrak{q}$-elementary and $\sigma \in \mathbf{E}(r, A)$. Given $\alpha \in \mathbf{GL}(r, A)$ we must show $\alpha^{-1}\tau^\sigma\alpha \in \mathbf{E}(r, A, \mathfrak{q})$. But $\alpha^{-1}\tau^\sigma\alpha = (\beta^{-1}\tau\beta)^\sigma$, where $\beta = \alpha^{\sigma^{-1}}$. Since (by definition) $\mathbf{E}(r, A, \mathfrak{q})$ is normalized by $\mathbf{E}(r, A)$, it suffices to show $\beta^{-1}\tau\beta \in \mathbf{E}(r, A, \mathfrak{q})$. By part *b)* we can write $\beta = \gamma\rho$, with $\gamma \in \mathbf{GL}(n, A)$ and $\rho \in \mathbf{E}(r, A)$. Again, then, it suffices to show $\gamma^{-1}\tau\gamma \in \mathbf{E}(r, A, \mathfrak{q})$. Write $\gamma = \begin{vmatrix} c & 0 \\ 0 & 1 \end{vmatrix}$, $c \in \mathbf{GL}(r-1, A)$, and $\tau = 1 + qe_{ij}$, $q \in \mathfrak{q}$.

Case 1. i or j is $=r$, say $i=r$. Then $\tau=\begin{vmatrix}1 & 0\\ t & 1\end{vmatrix}$, $t\in\mathfrak{q}A^{r-1}$ so $\gamma\tau\gamma^{-1}=\begin{vmatrix}1 & 0\\ tc^{-1} & 1\end{vmatrix}$ which is clearly in $\mathbf{E}(r, A, \mathfrak{q})$. Similarly for $j=r$.

Case 2. $i, j<r$. Let a be the i^{th} column of c and b the j^{th} row of c^{-1}. Then $ba=0$, being the $(i,j)^{th}$ coordinate of $c^{-1}c=1$ (recall $i\neq j$). One sees easily that $\gamma\tau\gamma^{-1}=\begin{vmatrix}1+aqb & 0\\ 0 & 1\end{vmatrix}$. Let $\tau_1=\begin{vmatrix}1 & 0\\ qb(1+aqb)^{-1} & 1\end{vmatrix}\in\mathbf{E}(r, A, \mathfrak{q})$, and

$$\sigma=\begin{vmatrix}1 & a\\ 0 & 1\end{vmatrix}\in\mathbf{E}(r, A).$$

We conclude by showing that $(\tau_1\gamma\tau\gamma^{-1})^\sigma\in\mathbf{E}(r, A, \mathfrak{q})$.

$$(\tau_1\gamma\tau\gamma^{-1})^\sigma=\begin{vmatrix}1 & -a\\ 0 & 1\end{vmatrix}\begin{vmatrix}1 & 0\\ qb(1+aqb)^{-1} & 1\end{vmatrix}\begin{vmatrix}1+aqb & 0\\ 0 & 1\end{vmatrix}\begin{vmatrix}1 & a\\ 0 & 1\end{vmatrix}$$
$$=\begin{vmatrix}1 & -a\\ 0 & 1\end{vmatrix}\begin{vmatrix}1+aqb & 0\\ qb & 1\end{vmatrix}\begin{vmatrix}1 & a\\ 0 & 1\end{vmatrix}$$
$$=\begin{vmatrix}1 & -a\\ qb & 1\end{vmatrix}\begin{vmatrix}1 & a\\ 0 & 1\end{vmatrix}=\begin{vmatrix}1 & a-a\\ qb & 1+qba\end{vmatrix}=\begin{vmatrix}1 & 0\\ qb & 1\end{vmatrix},$$

since $ba=0$.

d) Using $r\geq 3$ we have, from (1.3),

$$\mathbf{E}(r, A, \mathfrak{q})=[\mathbf{E}(r, A), \mathbf{E}(r, A, \mathfrak{q})]\subset[\mathbf{E}(r, A), \mathbf{GL}(r, A, \mathfrak{q})].$$

Let $\tau=1+ae_{ij}$ be one of the generators of $\mathbf{E}(r, A)$. Suppose, for some σ, that $[\tau^\sigma, \mathbf{GL}(r, A, \mathfrak{q})]\subset\mathbf{E}(r, A, \mathfrak{q})$. Then

$$[\tau, \mathbf{GL}(r, A, \mathfrak{q})]=[\tau^\sigma, \mathbf{GL}(r, A, \mathfrak{q})^\sigma]^{\sigma^{-1}}=[\tau^\sigma, \mathbf{GL}(r, A, \mathfrak{q})]^{\sigma^{-1}}\subset\mathbf{E}(r, A, \mathfrak{q})^{\sigma^{-1}}=\mathbf{E}(r, A, \mathfrak{q}).$$

Being free thus to choose σ, we can transform τ and assume $\tau=1+ae_{rj}$. Now by part *b)*, using $r>n$, $\mathbf{GL}(r, A, \mathfrak{q})=\mathbf{GL}(r-1, A, \mathfrak{q})\mathbf{E}(r, A, \mathfrak{q})$, so it suffices to check that $[\tau, \mathbf{GL}(r-1, A, \mathfrak{q})]\subset\mathbf{E}(r, A, \mathfrak{q})$. Let $\gamma=\begin{vmatrix}c & 0\\ 0 & 1\end{vmatrix}\in\mathbf{GL}(r-1, A, \mathfrak{q})$, and write $\tau=\begin{vmatrix}1 & 0\\ t & 1\end{vmatrix}$. Then

$$[\tau, \gamma]=\begin{vmatrix}1 & 0\\ -t & 1\end{vmatrix}\begin{vmatrix}c^{-1} & 0\\ 0 & 1\end{vmatrix}\begin{vmatrix}1 & 0\\ t & 1\end{vmatrix}\begin{vmatrix}c & 0\\ 0 & 1\end{vmatrix}=\begin{vmatrix}1 & 0\\ t(c-1) & 1\end{vmatrix}.$$

This is in $\mathbf{E}(r, A, \mathfrak{q})$ since $c\equiv 1 \bmod \mathfrak{q}$.

f) Since $\mathbf{GL}(r, A)=\mathbf{GL}(n, A)\mathbf{E}(r, A)$ and $\mathbf{GL}(r, A, \mathfrak{q})=\mathbf{GL}(n, A, \mathfrak{q})\mathbf{E}(r, A, \mathfrak{q})$ there remains only, by virtue of part *d)*, to show that $[\mathbf{GL}(n, A), \mathbf{GL}(n, A, \mathfrak{q})]\subset\mathbf{E}(r, A, \mathfrak{q})$. But this follows from (1.8), since $r\geq 2n$.

e) We begin by showing that if H contains a non central element, σ, then $H\supset\mathbf{E}(r, A, \mathfrak{q}')$ for some $\mathfrak{q}'\neq 0$. By (2.5) it suffices to produce, in H, a non central element with at least one coordinate zero. Let $\alpha=\begin{pmatrix}a_1\\ \vdots\\ a_r\end{pmatrix}$ be the first column of σ. Since $r>n$, our hypothesis on n permits us to add multiples of a_r to the other coordinates and render them unimodular. This can be accomplished by conjugating σ with a matrix of the form $\begin{vmatrix} 1_{r-1} & * \\ 0 & 1 \end{vmatrix} \in \mathbf{E}(r, A)$, so we can reduce to the case where $(a_1, \ldots, a_{r-1})$ is unimodular. We can then write $a_r = d_1 a_1 + \ldots + d_{r-1} a_{r-1}$, so that if $\lambda = \begin{vmatrix} 1_{r-1} & 0 \\ d & 1 \end{vmatrix}$, where $d = (d_1, \ldots, d_{r-1})$, we have $\lambda^{-1}\alpha = \begin{pmatrix} a_1 \\ \vdots \\ 0 \end{pmatrix}$.

Now in the paragraph preceding Corollary 2.4 we showed that if σ commutes with all $\tau = 1 + ae_{12}$, then α has only one non zero coordinate, so we can finish with (2.5). Hence we may assume there is such a τ for which $\rho = \sigma\tau\sigma^{-1}\tau^{-1} \neq 1$. Let β denote the second row of σ^{-1} and $\gamma = a\beta\tau^{-1}$. Then (just as in the proof of (2.5)) we have $\sigma\tau\sigma^{-1} = 1 + \alpha a\beta$, so $\rho = \tau^{-1} + \alpha\gamma$.

We now claim that ρ is not central. Otherwise $\rho = u.1$ with $u \in$ center A, by (2.4). With λ as above, the last row of $(\lambda^{-1}\alpha)\gamma$ is zero, so that $\lambda^{-1}\rho = \lambda^{-1}\tau^{-1} + (\lambda^{-1}\alpha)\gamma$ and $\lambda^{-1}\tau^{-1}$ have the same last row. Since $1 - \tau^{-1}$ is concentrated in the upper left 2×2 corner, and $1 - \lambda^{-1}$ in the last row, we see, using $r \geq 3$, that the $(r, r)^{th}$ coordinate of $\lambda^{-1}\tau^{-1}$ is 1. But the same coordinate of $u\lambda^{-1}$ is u. This shows that ρ central $\Rightarrow \rho = 1$, contrary to our choice of τ.

Now consider $\lambda^{-1}\rho\lambda = \lambda^{-1}\tau^{-1}\lambda + (\lambda^{-1}\alpha)(\gamma\lambda)$. This is a non central element of H whose last row agrees with that of $\lambda^{-1}\tau^{-1}\lambda = 1 - \varepsilon a\delta$, where ε is the first column of λ^{-1} and δ is the second row of $\lambda = \begin{vmatrix} 1_{r-1} & 0 \\ d & 1 \end{vmatrix}$. Hence δ has only one non zero coordinate (using $r > 2$), so the last row of $\lambda^{-1}\tau^{-1}\lambda$ has at most two non zero coordinates. We conclude then that $\lambda^{-1}\rho\lambda$ has some coordinate zero, and again we finish with (2.5).

For the proof of *e)* now, choose $\mathfrak{q}$ maximal so that $\mathbf{E}(r, A, \mathfrak{q}) \subset H$. Our problem is to show that the image, H′, of H in $\mathbf{GL}(r, A/\mathfrak{q})$ lies in the center of $\mathbf{GL}(r, A/\mathfrak{q})$. Since $\mathbf{E}(r, A)$ normalizes H, H′ is normalized by the image of $\mathbf{E}(r, A)$, which, by (1.1), is $\mathbf{E}(r, A/\mathfrak{q})$. Hence if H′ is not central Lemma 4.1 permits us to apply the first part of our argument to H′ and conclude that $\mathbf{E}(r, A/\mathfrak{q}, \mathfrak{q}'/\mathfrak{q}) \subset H'$ for some $\mathfrak{q}' \neq \mathfrak{q}$. Then the inverse image, $L = H\mathbf{GL}(r, A, \mathfrak{q})$, of H′ contains $\mathbf{E}(r, A, \mathfrak{q}')$. By part *d)* now

$$\mathbf{E}(r, A, \mathfrak{q}') = [\mathbf{E}(r, A), \mathbf{E}(r, A, \mathfrak{q}')] \subset [\mathbf{E}(r, A), L] \subset$$
$$\subset [\mathbf{E}(r, A), H][\mathbf{E}(r, A), \mathbf{GL}(r, A, \mathfrak{q})]^H = [\mathbf{E}(r, A), H]\mathbf{E}(r, A, \mathfrak{q}) \subset H,$$

contradicting the maximality of $\mathfrak{q}$.

§ 5. Dimension 0.

Corollary 6.5 tells us that $n = 1$ defines a stable range for $\mathbf{GL}(A)$ when A is semi-local, and it is in precisely this case that the restrictions $r \geq 3$ intervene effectively. The following refinements can be made:

Proposition **(5.1)**. — *Suppose* $n = 1$ *defines a stable range for* $\mathbf{GL}(A)$ (e. g. A can be any semi-local ring).

a) *If* A *is commutative, then, for all ideals* $\mathfrak{q}$ *and all* $r \geq 2$,

$$\mathbf{E}(r, A, \mathfrak{q}) = \mathbf{SL}(r, A, \mathfrak{q})(= \mathbf{SL}(r, A) \cap \mathbf{GL}(r, A, \mathfrak{q})).$$

In particular $\mathbf{E}(r, A) = \mathbf{SL}(r, A)$.

b) *Suppose* $1 = w^2 + v$ *with* w *and* v *units in the center of* A. *Then, for all* $r \geq 2$,

$$\mathbf{E}(r, A, \mathfrak{q}) = [\mathbf{E}(r, A), \mathbf{E}(r, A, \mathfrak{q})] = [\mathbf{GL}(r, A), \mathbf{GL}(r, A, \mathfrak{q})].$$

Proof. — *a)* Clearly $\mathbf{E}(r, A, \mathfrak{q}) \subset \mathbf{SL}(r, A)$. Now an element of $\mathbf{GL}(r, A, \mathfrak{q})$ is, by (4.2) *b)*, reducible modulo $\mathbf{E}(r, A, \mathfrak{q})$ to $\mathbf{GL}(1, A, \mathfrak{q})$, i.e. to a unit, and that unit is evidently the determinant of the original matrix. This proves *a)*.

b) For $r \geq 3$ *b)* is just parts *d)* and *f)* of Theorem 4.2. However, the proof there uses $r \geq 3$ only to invoke (1.3) and conclude that $\mathbf{E}(r, A, \mathfrak{q}) = [\mathbf{E}(r, A), \mathbf{E}(r, A, \mathfrak{q})]$. Our hypothesis permits us to use (1.6) instead for the same purpose when $r = 2$.

As a consequence of part *a)* above and (1.1) we have:

Corollary **(5.2)**. — *Let* $\mathfrak{q}$ *be an ideal in the commutative ring* A *for which* $A/\mathfrak{q}$ *is semi-local. Then, for all* $\mathfrak{q}' \supset \mathfrak{q}$, $\mathbf{E}(r, A, \mathfrak{q}') \rightarrow \mathbf{SL}(r, A/\mathfrak{q}, \mathfrak{q}'/\mathfrak{q})$ *is surjective for all* r. *In particular,* $\mathbf{SL}(r, A) \rightarrow \mathbf{SL}(r, A/\mathfrak{q})$ *is surjective.*

Remarks. — 1. For local rings some further refiniments of our results can be found in Klingenberg [24].

2. Let $\mathbf{GL}(r, A, \mathfrak{q})'$ be the inverse image in $\mathbf{GL}(r, A)$ of the center of $\mathbf{GL}(r, A/\mathfrak{q})$. One would like the following converse to part *e)* of Theorem 4.2: If

$$\mathbf{E}(r, A, \mathfrak{q}) \subset H \subset \mathbf{GL}(r, A, \mathfrak{q})'$$

then H is normal. This would follow from: $\mathbf{E}(r, A, \mathfrak{q}) = [\mathbf{GL}(r, A), \mathbf{GL}(r, A, \mathfrak{q})']$. For a commutative local ring this follows from (4.2) *f)*, since $\mathbf{GL}(r, A, \mathfrak{q})'$ is then generated by $\mathbf{GL}(r, A, \mathfrak{q})$ and the center of $\mathbf{GL}(r, A)$. Klingenberg's proof of this for non commutative A appears to contain a gap, due to the erroneous equation, " $g_J(a^{-1}b^{-1}a) = 1$ " on p. 77 of [24].

3. If A is the ring of integers in a number field then $n = 1$ defines a stable range for $\mathbf{GL}(A)$. Our results in this setting are, to some extent, well known. For example, the last part of Corollary 5.2 (familiar to function theorists for $A = \mathbf{Z}$) and condition (4.2) *b)* for $\mathfrak{q} = A$ were known long ago by Hurwitz. Moreover, Brenner [13] recognized (4.2) *b)* for $A = \mathbf{Z}$. However the commutator formulae appear to have escaped notice even for $A = \mathbf{Z}$. They turn out to be essential in the proof (see § 21) that every subgroup of finite index in $\mathbf{SL}(n, \mathbf{Z})$, $n \geq 3$, contains a congruence subgroup. The discovery of the generality of these formulae lies ultimately in J. H. C. Whitehead's work on simple homotopy types [36].

CHAPTER II

STABLE STRUCTURE OF PROJECTIVE MODULES

§ 6. Semi-local rings.

Let A be a ring. Throughout this chapter, " A-module " means right A-module, and " ideal ", unqualified, means two-sided ideal. Let P be an A-module, and $\alpha \in P$. We write $P^* = \mathrm{Hom}_A(P, A)$ and define

$$o(\alpha) = o_P(\alpha) = \{f(\alpha) \mid f \in P^*\}.$$

$o(\alpha)$ is a left ideal in A, and it is clear that $o(\alpha) = A$ if, and only if, the homomorphism $g : A \to P$, $g(a) = \alpha a$, has a left inverse. In this case we call α a *unimodular* element of P.

Lemma **(6.1)**. — *Let* $\sigma : Q \to P$ *be a homomorphism of* A-*modules with* Q *finitely generated and projective. Then* σ *has a left inverse* (i.e. is a monomorphism onto a direct summand) *if, and only if,* $\sigma^* : P^* \to Q^*$ *is an epimorphism.*

Proof. — σ has a left inverse $\Rightarrow \sigma^*$ has a right inverse $\Rightarrow \sigma^{**}$ has a left inverse $\Rightarrow \sigma$ has a left inverse, since $Q = Q^{**}$ is reflexive. Moreover, σ^* has a right inverse $\Leftrightarrow \sigma^*$ is an epimorphism, since Q^* is projective.

Denote by rad A the Jacobson radical of A.

Corollary **(6.2)**. — *Let* $\sigma, \tau : Q \to P$ *as in* (6.1), *and assume* $\mathrm{Im}(\sigma - \tau) \subset P.\mathrm{rad}\, A$. *Then* σ *has a left inverse* $\Leftrightarrow \tau$ *does.*

Proof. — Let $f \in \mathrm{Im}(\sigma - \tau)^* \subset Q^*$; then $f(Q) \subset \mathrm{rad}\, A$. Since Q if finitely generated and projective this implies that $f \in \mathrm{rad}\, A.Q^*$. Hence $\mathrm{Im}\, \sigma^* \subset \mathrm{Im}\, \tau^* + \mathrm{rad}\, A.Q^*$. Since Q^* is finitely generated, Nakayama's Lemma tells us that σ^* surjective $\Rightarrow \tau^*$ surjective. Using (6.1) now, this completes the proof.

Definition. — Call A *semi-local* if A/rad A is an Artin ring. It follows then that A/rad A is a finite product of full matrix algebras over division rings.

For the balance of this section, A *always denotes a semi-local ring*. The lemmas which follow contain the " zero dimensional " case of the general results to follow.

If S is a subset of an A-module, P, denote by (S) the submodule generated by S. We shall say

$$f\text{-rank}_A(S; P) \geq r$$

if (S) contains a direct summand of P isomorphic to A^r. We will suppress the subscript " A " when A is fixed by the context. In what follows A is a fixed semi-local ring, and P denotes an A-module.

Corollary **(6.3)**. — $f\text{-rank}(S; P) = f\text{-rank}((S) + P.\mathrm{rad}\, A; P)$.

Proof. — If f-rank$((S)+P.\mathrm{rad}\,A;\,P)\geq r$ there is a homomorphism $\sigma: A^r\to P$ having a left inverse, and with $\mathrm{Im}\,\sigma\subset(S)+P.\mathrm{rad}\,A$. Hence we can find $\tau: A^r\to P$ with $\mathrm{Im}\,\tau\subset(S)$ and $\mathrm{Im}(\sigma-\tau)\subset P.\mathrm{rad}\,A$. By (6.2) τ has then also a left inverse, so f-rank$(S;\,P)\geq r$. The reverse inequality is obvious.

The following simple result will play a fundamental role in what follows.

Lemma **(6.4)**. — *If* $b\in A$ *and* $\mathfrak{a}$ *is a left ideal such that* $Ab+\mathfrak{a}=A$, *then* $b+\mathfrak{a}$ *contains a unit.*

Proof. — Since units in A/rad A lift automatically to units in A we may assume A is semi-simple. Passing then to a simple factor we can reduce to the case $A=\mathrm{End}_D(V)$, where V is a finite dimensional right vector space over a division ring D. In this case $\mathfrak{a}$ can be described as the set of endomorphisms which annihilate some subspace, W, of V ($\mathfrak{a}=Ae$, $e^2=e$, and $W=\ker e$). The fact that $Ab+\mathfrak{a}=A$ guarantees that $\ker b\cap W=0$. Write $V=W\oplus W'=bW\oplus U$. Now we can clearly construct an automorphism u such that $u|W=b|W$ and $u(W')=U$. (Note that $W\cong bW\Rightarrow W'\cong U$.) Since $a=u-b$ annihilates W, $a\in\mathfrak{a}$ and we're done.

Corollary **(6.5)**. — $n=1$ *defines a stable range for* $\mathbf{GL}(A)$ (*in the sense of* § 4).

Proof. — By definition, we must show that if $Aa_1+\ldots+Aa_r=A$, $r>1$, then we can add multiples of a_r to $a_1,\ldots,a_{r-1}$ so that the resulting $r-1$ elements still generate the unit left ideal. Let $b=a_1$ and $\mathfrak{a}=Aa_2+\ldots+Aa_r$; then (6.4) provides us with a unit $u=a_1+b_2a_2+\ldots+b_ra_r$. Hence $u\in(A(a_1+b_ra_r)+Aa_2+\ldots+Aa_{r-1})$, as required.

The following is a technical little argument with two important corollaries: Let α and β be unimodular in $\beta A\oplus P$. Writing $\alpha=\beta b+\alpha_P$ we have $A=o(\alpha)=Ab+o(\alpha_P)$. By (6.4) $u=b+a$ is a unit for some $a\in o(\alpha_P)$. Let f be an endomorphism of $\beta A\oplus P$ such that $f(\beta)=0$, $f(P)\subset\beta A$, and $f(\alpha_P)=\beta a$. The existence of f follows from the definition of $o(\alpha_P)=o_P(\alpha_P)$. If $\varphi_1=1+f$ then $\varphi_1(\alpha)=\beta u+\alpha_P$. Now let g be the endomorphism killing P such that $g(\beta)=-\alpha_P u^{-1}$, and set $\varphi_2=1+g$. If $\varphi=\varphi_2\varphi_1$ then (i) $\varphi(\alpha)=\beta u$, (ii) φ leaves invariant all submodules containing $\beta A+\alpha A$, and (iii) since $f^2=0=g^2$, φ is an automorphism.

Corollary **(6.6)**. — $A\oplus P\cong A\oplus P'\Rightarrow P\cong P'$.

Proof. — Using the hypothesized isomorphism to identify the two modules, we can write $\beta A\oplus P=\alpha A\oplus P'$. With the φ constructed above we have, from (i), $\varphi(\alpha A)=\beta A$, so

$$P\cong(\beta A\oplus P)/\beta A=\varphi(\alpha A\oplus P')/\varphi(\alpha A)\cong(\alpha A\oplus P')/\alpha A\cong P'.$$

Corollary **(6.7)**. — *If* M *is a submodule of* P, *then*

$$f\text{-rank}(A^r\oplus M;\ A^r\oplus P)=r+f\text{-rank}(M;\ P).$$

Proof. — It suffices to treat the case $r=1$. Moreover, the left side is clearly at least as large as the right.

Let $\alpha_1,\ldots,\alpha_s\in\beta A\oplus M$ be a basis for a free direct summand of $\beta A\oplus P$. (β is assumed unimodular here of course.) Now construct φ as above with $\alpha=\alpha_1$. By (ii) above $\varphi(\alpha_i)\in\beta A\oplus M$ also, $1\leq i\leq s$, so we can replace α_i by $\varphi(\alpha_i)$ and, by (i), reduce to the case $\beta A = \alpha_1 A$. But then we can subtract multiples of α_1 from $\alpha_2, \ldots, \alpha_s$ and further render $\alpha_2, \ldots, \alpha_s \in M$. Thus, $f\text{-rank}(\beta A \oplus P) \geq s \Rightarrow f\text{-rank}(M;\, P) \geq s-1$, as required.

Corollary **(6.8)**. — *If* α *is an element, and* S *a subset, of* P, *then*

$$f\text{-rank}(S, \alpha;\, P) \leq 1 + f\text{-rank}(S;\, P).$$

Proof. — Suppose $f\text{-rank}(S, \alpha;\, P) \geq r$; i.e. there is a $\sigma : A^r \to P$ with a left inverse, and $\operatorname{Im} \sigma \subset (S) + \alpha A$. Let $f : A \oplus P \to P$ by $f(a, p) = \alpha a + p$. Then $\operatorname{Im}(f \mid (A \oplus (S))) = \alpha A + (S) \supset \operatorname{Im} \sigma$, so we can find $g : A^r \to A \oplus P$ such that $\sigma = fg$ and $\operatorname{Im} g \subset A \oplus (S)$. The left invertibility of σ implies that of g, and hence $f\text{-rank}(A \oplus (S);\, A \oplus P) \geq r$. The present Corollary now follows from the preceeding one.

Lemma **(6.9)**. — *Suppose* $\alpha_1, \ldots, \alpha_r \in P$ *and*

$$f\text{-rank}(\alpha_1, \ldots, \alpha_r;\, P) = t \leq s \leq r.$$

Then we can find $\beta_1, \ldots, \beta_s$ *of the form*

$$\beta_i = \alpha_i + \Sigma_{j>i} \alpha_j a_{ij}, \quad 1 \leq i \leq s,$$

such that $f\text{-rank}(\beta_1, \ldots, \beta_s;\, P) = t$.

Proof. — We induce on t, the case $t=0$ being trivial. If $t>0$ we can write $P = \beta A \oplus P'$ with β a unimodular element of P in $(\alpha_1, \ldots, \alpha_r)$. Writing $\alpha_i = \beta b_i + \alpha'_i$ then, with $\alpha'_i \in P'$, we must have $\Sigma_i b_i A = A$. By (6.4) (applied to the opposite ring of A, since we are now dealing with right ideals) we can find a unit $u = b_1 + \Sigma_{j>1} b_j a_{1j}$. Set $\beta_1 = \alpha_1 + \Sigma_{j>1} \alpha_j a_{1j} = \beta u + \beta'_1$. Then β_1 is unimodular in P (since $u \in o(\beta_1)$), so $P = \beta_1 A \oplus P_1$ for some P_1. Write $\alpha_i = \beta_1 c_i + \gamma_i$, $\gamma_i \in P_1$, $1 \leq i \leq r$. Note next that $(\alpha_1, \alpha_2, \ldots, \alpha_r) = (\beta_1, \alpha_2, \ldots, \alpha_r) = (\beta_1, \gamma_2, \ldots, \gamma_r)$. Hence, we conclude from (6.7) that $f\text{-rank}(\gamma_2, \ldots, \gamma_r;\, P_1) = t-1$. By induction, we can find

$$\delta_i = \gamma_i + \Sigma_{j>i} \gamma_j a_{ij}, \quad 2 \leq i \leq s,$$

such that $f\text{-rank}(\delta_2, \ldots, \delta_s;\, P_1) = t-1$. Writing

$$\beta_i = \alpha_i + \Sigma_{j>i} \alpha_j a_{ij} = \beta_1(c_i + \Sigma_{j>i} c_j a_{ij}) + \delta_i, \quad 2 \leq i \leq s,$$

we see that $(\beta_1, \beta_2, \ldots, \beta_s) = (\beta_1, \delta_2, \ldots, \delta_s)$, and this clearly solves our problem.

§ 7. Delocalization. The maximal spectrum.

As a basic reference for the material of this section, we refer to Bourbaki [12] or Grothendieck [22, Ch. 0].

Let A be a commutative ring and

$$X = \max(A),$$

the topological space whose points are the maximal ideals of A, and whose closed sets are the sets of all maximal ideals containing a given subset of A.

If M is an A-module and $x \in X$ (x is a maximal ideal in A) we let $M_x = A_x \otimes_A M$ denote the localization of M at x. Further, define

$$\operatorname{supp} M (= \operatorname{supp}_X M) = \{x \in X \mid M_x \neq 0\}.$$

The topology in X is better described, for our purposes, as follows: the closed sets in X are precisely those of the form supp M where M is some *finitely generated* A-module.

Let Λ be a finite A-algebra (i.e. finitely generated as an A-module). If P is a Λ-module, P_x is a module over the A_x-algebra, Λ_x, and our finiteness condition on Λ guarantees that Λ_x is semi-local, for all $x \in X$. Hence, if S is a subset of P, we can consider $f\text{-rank}_{\Lambda_x}(S; P_x)$, as defined in § 6. (Here we are confusing S with its image in P_x, but this should cause no difficulty.) We now define

$$f\text{-rank}_\Lambda(S; P) = \inf_{x \in X} f\text{-rank}_{\Lambda_x}(S; P_x),$$

and

$$f\text{-rank}_\Lambda(P) = f\text{-rank}_\Lambda(P; P).$$

When Λ is fixed, we shall suppress the subscript. Since the definition is local, (6.7) immediately yields:

Corollary **(7.1)**. — *If* M *is a submodule of the* Λ*-module* P,

$$f\text{-rank}(\Lambda^r \oplus M; \Lambda^r \oplus P) = r + f\text{-rank}(M; P).$$

More generally, we define the " singularities " of $S \subset P$ by

$$F_j(S; P) = \{x \in X \mid f\text{-rank}_{\Lambda_x}(S; P_x) < j\}.$$

Thus, $F_0(S; P) = \emptyset$ for any S, and $F_j(\emptyset; P) = X$ for all $j > 0$, for example. It is essential to our method that these sets be closed, and for this we need a " coherence " condition on P.

Lemma **(7.2)**. — *Suppose* P *is a direct summand of a direct sum of finitely presented* Λ*-modules. Then, for all subsets* S *of* P *and for all* j, $F_j(S; P)$ *is closed in* X.

(Recall that a finitely presented module is the cokernel of a homomorphism $\Lambda^r \to \Lambda^s$.)

Proof. — Let $x \in X - F_j(S; P)$. Then if $Q = \Lambda^j$ there is a homomorphism $\sigma : Q \to P$ such that $\operatorname{Im} \sigma \subset (S)$ and such that $\sigma_x : Q_x \to P_x$ has a left inverse. (One simply chooses elements in (S) whose images in P_x are a basis for a direct summand.) It suffices now to show that $\{y \in X \mid \sigma_y \text{ has a left inverse}\}$ is open, for the latter will then be a neighborhood of x in $X - F_j(S; P)$. Since $\operatorname{Im} \sigma$ is finitely generated, our hypothesis on P permits us to reduce this last question to the case where P itself is finitely presented. Under these conditions the natural homomorphism $(P^*)_y \to (P_y)^*$ is an isomorphism for all $y \in X$ (see [5, Lemma 3.3]); the second * here refers, of course, to the duality of Λ_y-modules. The same remark applies to the free module, Q, so that we can conclude from the commutative square,

$$\begin{array}{ccc} (P^*)_y & \xrightarrow{(\sigma^*)_y} & (Q^*)_y \\ \downarrow \cong & & \downarrow \cong \\ (P_y)^* & \xrightarrow{(\sigma_y)^*} & (Q_y)^*, \end{array}$$

an isomorphism $M_y \cong \operatorname{coker}(\sigma_y)^*$, where $M = \operatorname{coker}(\sigma^*)$. Now we have from (6.1) that $\{y \mid \sigma_y$ has a left inverse$\} = \{y \mid (\sigma_y)^*$ is surjective$\} = \{y \mid M_y = 0\} = X - \operatorname{supp} M$. Since M is finitely generated (being a quotient of Q^*), supp M is closed, and this completes the proof.

It is useful to note above that, since $M = 0 \Leftrightarrow M_y = 0$ for all y, σ has a left inverse $\Leftrightarrow \sigma_y$ does for all y. Applying this to $\sigma : A \to P$ with $\sigma(1) = \alpha$ we obtain:

Corollary **(7.3)**. — *With* P *as in* (7.2) *and* $\alpha \in P$, α *is unimodular in* P *if, and only if,* $F_1(\alpha; P) = \emptyset$.

Examples. — 1) Lemma 7.2 applies, notably, when P is either projective or finitely presented.

2) The inclusion $\mathbf{Z} \to \mathbf{Z}_{(p)}$, (rational p-adic integers) has a left inverse only at $(p) \in \max(\mathbf{Z})$.

3) If P is the $\mathbf{Z}$-submodule of $\mathbf{Q}$ generated by $\{p^{-1} \mid p \text{ prime}\}$, and if q is a fixed prime, then $F_1(q^{-1}; P) = \max(\mathbf{Z}) - \{(q)\}$ is not closed, whereas P is locally free (of rank one).

Finally, we recall some topological notions. In any topological space, X, a closed set F is called *irreducible* if $F \neq \emptyset$ and if $F = G \cup H$ with G and H closed $\Rightarrow F = G$ or H. We then call codim F$(= \operatorname{codim}_X F)$ the supremum of the lengths, n, of chains,

$$F_0 = F \subset F_1 \subset \ldots \subset F_n$$

of distinct closed irreducible sets in X. For arbitrary closed F we define codim F to be the infimum of the codimensions of the irreducible closed subsets of F, with the convention $\operatorname{codim} \emptyset = \infty$. The supremum of the codimensions of non empty closed subsets of X will be called, dim X.

We call X *noetherian* if the closed sets satisfy the descending chain condition. In this case every non empty closed set F is a *finite* union of irreducible closed sets. Such a representation of F, when made irredundant, is unique up to order, and we call the intervening irreducible closed sets the *irreducible components* of F.

For example, if A is a commutative semi-local ring, max(A) is a finite discrete space, and hence a noetherian space of dimension zero. If A is only noetherian, then max(A) is a noetherian space of dimension $\leq$Krull dimension $A = \dim \operatorname{spec}(A)$.

§ 8. Serre's theorem.

For the next four sections we fix the following data:

(8.1) $\begin{cases} \text{A is a commutative ring for which} \\ X = \max(A) \text{ is a noetherian space.} \\ \Lambda \text{ is a finite A-algebra.} \end{cases}$

The next theorem is a slight generalization of earlier versions. We include it for the sake of completeness, and also for the proof below, which is perhaps a little more manageable then its predecessors.

Theorem (**8.2**). — (Serre [29]; see also [5]). *Suppose* P *is a direct summand of a direct sum of finitely presented* Λ*-modules. Then, if* $f\text{-rank}_\Lambda(P) > d = \dim X$, $P \cong \Lambda \oplus P'$ *for some* P'.

We will derive the theorem from the two lemmas below, wherein P is always assumed to satisfy the coherence condition of the theorem. This assumption permits us to invoke (7.2) which ensures that the various " singular sets " intervening in the proofs are closed. All codimensions refer to codimension in X.

Lemma I. — *If* $f\text{-rank}\ P \geq r$, *there exist* $\alpha_1, \ldots, \alpha_r \in P$ *such that*

$$\operatorname{codim} F_j(\alpha_1, \ldots, \alpha_r; P) \geq (r+1) - j, \qquad \textit{all } j \geq 0.$$

Lemma II. — *If* $\alpha_1, \ldots, \alpha_r \in P$, $r > 1$, *and* $k \in \mathbf{Z}$, *are such that*

$$\operatorname{codim} F_j(\alpha_1, \ldots, \alpha_r; P) \geq k - j, \qquad 1 \leq j \leq r,$$

then there exist $\beta_i = \alpha_i + \alpha_r a_i$, *for suitable* $a_i \in A$, $1 \leq j \leq r-1$, *such that*

$$\operatorname{codim} F_j(\beta_1, \ldots, \beta_{r-1}; P) \geq k - j, \qquad 1 \leq j \leq r-1.$$

Proof that I and II ⇒ (8.2): Apply I to P, with $r = d+1$, to obtain $\alpha_1, \ldots, \alpha_r \in P$ such that $\operatorname{codim} F_j(\alpha_1, \ldots, \alpha_r; P) \geq (r+1) - j$ for all $j \geq 0$. Now, with $k = r+1$, apply II, $(r-1)$-times. The result is a single element, $\beta \in P$, such that $\operatorname{codim} F_1(\beta; P) \geq k - 1 = r = d+1$. Since $d = \dim X$ this implies $F_1(\beta; P) = \emptyset$, so, by (7.3), β is unimodular, and this completes the proof.

Proof of I. — We induce on r, the case $r = 0$ being trivial. Suppose $\alpha_1, \ldots, \alpha_r$ have been constructed as in the lemma, and we want α_{r+1} (assuming $f\text{-rank}\ P \geq r+1$). For $0 \leq j \leq r$, let $\{D_{j\nu}\}$ be the " largest " irreducible components of $F_{j+1}(\alpha_1, \ldots, \alpha_r; P)$, i.e. those of smallest codimension, $(r+1) - (j+1)$. Of course there may be none, but that's all the better. Since $\operatorname{codim} F_j(\alpha_1, \ldots, \alpha_r; P) \geq (r+1) - j$, $D_{j\nu} \not\subset F_j(\alpha_1, \ldots, \alpha_r; P)$ for all ν. It follows (since $D_{j\nu}$ is irreducible) that we can choose

$$x(j, \nu) \in D_{j\nu} - [F_j(\alpha_1, \ldots, \alpha_r; P) \cup \textstyle\bigcup_{\mu \neq \nu} D_{j\mu}].$$

Since $x(j,\nu) \in F_{j+1}, \notin F_j$ we have $f\text{-rank}_{\Lambda_{x(j,\nu)}}(\alpha_1, \ldots, \alpha_r; P_{x(j,\nu)}) = j \leq r$, for $0 \leq j \leq r$, and all ν. It follows now from our hypothesis on P and (6.7) that we can choose $\beta_{j\nu} \in P$ such that $f\text{-rank}_{\Lambda_{x(j,\nu)}}(\alpha_1, \ldots, \alpha_r, \beta_{j\nu}; P_{x(j,\nu)}) \geq j+1$. Since $x(j,\nu)$ are (clearly) all distinct, we can write $1 = \Sigma_{j,\nu} e_{j\nu}$ in A, with $e_{j\nu} \equiv \delta_{(j,\nu)(h,\mu)} \bmod x(h,\mu)$. Setting $\alpha_{r+1} = \Sigma_{j,\nu} \beta_{j\nu} e_{j\nu}$ we have $\alpha_{r+1} \equiv \beta_{j\nu} \bmod x(j,\nu)$ so, by (6.3), $f\text{-rank}_{\Lambda_{x(j,\nu)}}(\alpha_1, \ldots, \alpha_{r+1}; P_{x(j,\nu)}) \geq j+1$, $0 \leq j \leq r$. Hence $x(j,\nu) \notin F_{j+1}(\alpha_1, \ldots, \alpha_{r+1}; P)$ (by definition) so also $D_{j\nu} \not\subset F_{j+1}(\alpha_1, \ldots, \alpha_{r+1}; P)$. On the other hand, $F_{j+1}(\alpha_1, \ldots, \alpha_{r+1}; P) \subset F_{j+1}(\alpha_1, \ldots, \alpha_r; P)$, and since the former contains no component of extreme codimension of the latter, we conclude, as desired, $\operatorname{codim} F_{j+1}(\alpha_1, \ldots, \alpha_{r+1}; P) \geq (r+1) - (j+1) + 1 = (r+2) - (j+1)$, $0 \leq j \leq r$. For the remaining values of j, $F_0 = \emptyset$ has infinite codimension, and if $j > r$, $(r+2) - (j+1) \leq 0$, whereas all codimensions are ≥ 0.

Proof of II. — For $0 \leq j < r$ let $\{D_{j\nu}\}$ be the irreducible components of $F_{j+1}(\alpha_1, \ldots, \alpha_r; P)$ of lowest possible codimension, $k - (j+1)$. Then

$$D_{j\nu} \not\subset F_j(\alpha_1, \ldots, \alpha_r; P)$$

so we can find

$$x(j,\nu)\in D_{j\nu}-[F_j(\alpha_1,\ldots,\alpha_r;P)\cup U_{\mu\neq\nu}D_{j\mu}];$$

thus $f\text{-rank}_{\Lambda_{x(j,\nu)}}(\alpha_1,\ldots,\alpha_r;P_{x(j,\nu)})=j<r$, for $0\leq j<r$. By (6.9) we can find elements $a_{ij\nu}\in\Lambda_{x(j,\nu)}$ such that, if $\beta_{ij\nu}=\alpha_i+\alpha_r a_{ij\nu}\in P_{x(j,\nu)}$, we have

$$f\text{-rank}_{\Lambda_{x(j,\nu)}}(\beta_{1j\nu},\ldots,\beta_{(r-1)j\nu};P_{x(j,\nu)})=j.$$

If we modify our choice of $a_{ij\nu}$ modulo the radical of $\Lambda_{x(j,\nu)}$ we can even choose the $a_{ij\nu}\in\Lambda$, and hence $\beta_{ij\nu}\in P$. This is permissible by (6.3). Now choose $e_{j\nu}\equiv\delta_{(j,\nu)(h,\mu)}\bmod x(h,\mu)$, as in the proof of I, set $a_i=\Sigma_{j,\nu}a_{ij\nu}e_{j\nu}$, and put $\beta_i=\alpha_i+\alpha_r a_i$, $1\leq i\leq r-1$. Since $\beta_i\equiv\beta_{ij}\bmod x(j,\nu)$ we have, by (6.3) again, $f\text{-rank}_{\Lambda_{x(j,\nu)}}(\beta_1,\ldots,\beta_{r-1};P_{x(j,\nu)})=j$, $0\leq j\leq r$, all ν. The form of the β's makes it evident that $(\beta_1,\ldots,\beta_{r-1},\alpha_r)=(\alpha_1,\ldots,\alpha_{r-1},\alpha_r)$, so it follows from (6.8) that $F_j(\beta_1,\ldots,\beta_{r-1};P)\subset F_{j+1}(\alpha_1,\ldots,\alpha_r;P)$. Since we have arranged that the former contains no irreducible components of lowest possible codimension, $k-(j+1)$, of the latter, we conclude, as desired, that

$$\operatorname{codim} F_j(\beta_1,\ldots,\beta_{r-1};P)\geq k-(j+1)+1=k-j,\qquad 0\leq j<r.$$

Counterexamples. — 1) To see the necessity of the coherence condition in Serre's Theorem, we can let $P=Q\oplus Q$, where Q is the $\mathbf{Z}$-submodule of the rationals generated by $\{p^{-1}\mid p\text{ prime}\}$. Here $f\text{-rank}_{\mathbf{Z}}(P)=2$ and $\dim\max(\mathbf{Z})=1$, but P has no projective direct summands.

2) The following example of Serre shows that P can even be made finitely generated and locally free. Let X be a Cantor set on the real line and y a point of X which is a limit point from both the left and right in X. Set $D_0=\{x\in X\mid x\geq y\}$, $D_1=\{x\in X\mid x\leq y\}$. With $A=C(X)$, the ring of continuous real valued functions on X, it is well known that $\max(A)=X$ and is hence of dimension zero. Moreover A is locally a field so that A-modules are locally free. Let $\mathfrak{a}_i$ be the ideal of functions vanishing on D_i, $i=0,1$, and let $P=(A/\mathfrak{a}_0)\oplus(A/\mathfrak{a}_1)$. Since $\operatorname{supp} A/\mathfrak{a}_i=D_{1-i}$ we see that P is locally free of rank one except at $\{y\}=D_0\cap D_1$, where it has rank two. Hence, if P had a free direct summand, the complementary summand would have support $\{y\}$. If $(f_0,f_1)\in P$ were in this summand, $f_i\in A/\mathfrak{a}_i=C(D_{1-i})$, f_i would be a function on D_{1-i} vanishing everywhere except, perhaps, at y. Since y is a limit in D_{1-i}, f_i has to vanish also at y, and hence $f_i=0$. Thus the complement would have to be zero, and we have a contradiction.

§ 9. Cancellation.

A, X, and Λ are as in (8.1). " Cancellation " refers to Theorem 9.3 below.

Theorem **(9.1)**. — *Let Q and P be projective Λ-modules, $\mathfrak{a}$ a left ideal, and $\alpha=\alpha_Q+\alpha_P\in Q\oplus P$ an element such that* $o(\alpha)+\mathfrak{a}=\Lambda$. (See § 6 for definition of $o(\alpha)$.) *Suppose, moreover, that f-rank $P>d=\dim X$. Then there is a homomorphism $f:Q\to P$ such that* $o(f(\alpha_Q)+\alpha_P)+\mathfrak{a}=\Lambda$.

Proof. — We induce on d, and the case $d=0$ will be subsumed in the general induction step.

Our hypothesis makes Serre's theorem available, to the effect that $P=\beta'\Lambda\oplus P'$ for some unimodular $\beta'\in P$; say $\alpha_P=\beta' b+\alpha'$. Then $o(\alpha_Q)+\Lambda b+o(\alpha')+\mathfrak{a}=\Lambda$. Let $F_1, \ldots, F_s$ be the irreducible components of X, and choose $x_i\in F_i-\bigcup_{j\neq i}F_j$, $1\leq i\leq s$. Modulo the product of the x_i, Λ is a semi-local ring, so we can apply (6.4) to find $a_Q+a'+c\in o(\alpha_Q)+o(\alpha')+\mathfrak{a}$ such that the image of $b+a_Q+a'+c$ is a unit in Λ modulo the x_i, and hence already in Λ_{x_i}, $1\leq i\leq s$. Let g be an endomorphism of P such that $g(\beta')=0$, $g(P')\subset\beta'\Lambda$, and $g(\alpha')=\beta' a'$. The existence of g follows from the definition of $o(\alpha')$ (§ 6). If $\sigma=1_P+g$ then σ is an automorphism (since $g^2=0$) and $\sigma\alpha_P=\beta'(b+a')+\alpha'$. Setting $\beta=\sigma^{-1}(\beta')$ and $\alpha_1=\sigma^{-1}(\alpha')\in P_1=\sigma^{-1}(P')$ we have $P=\beta\Lambda\oplus P_1$ and $\alpha_P=\beta(b+a')+\alpha_1$. Now choose $f_1: Q\to\beta\Lambda\subset P$ such that $f_1(\alpha_Q)=\beta a_Q$; again, f_1 exists by the definition of $o(\alpha_Q)$. Putting $b_1=a_Q+b+a'$, then, we have

$$f_1(\alpha_Q)+\alpha_P=\beta b_1+\alpha_1, \tag{*}$$

and b_1+c is a unit in Λ_{x_i}, $1\leq i\leq s$. If $\dim X=0$ then $X=\{x_1, \ldots, x_s\}$, and we're done. In general, we can find a $t\in A$ belonging to none of the x_i such that $t\Lambda\subset\Lambda b_1+\mathfrak{a}$. (E. g. semi-localize at $x_1, \ldots, x_s$, solve $(b_1+c)z=1$, and clear denominators.)

Let $A^*=A/At$, $\Lambda^*=\Lambda/\Lambda t$, $\mathfrak{a}^*=$ image of $\mathfrak{a}$ in Λ^*, etc. Then $X^*=\max(A^*)$ is a closed subset of X containing no x_i, hence no F_i, so $\dim X^*<\dim X$. Hence, using (7.1) we have $f\text{-rank}_{\Lambda^*}P_1^*\geq f\text{-rank}_\Lambda P_1=f\text{-rank}_\Lambda P-1>\dim X-1\geq\dim X^*$. Consider $\gamma^*=\alpha_Q^*+\alpha_1^*\in Q^*\oplus P_1^*$. Since $\Lambda=o(\alpha_Q)+\Lambda b_1+o(\alpha_1)+\mathfrak{a}$ we have, over Λ^*, $o(\gamma^*)+\Lambda^* b_1^*+\mathfrak{a}^*=\Lambda^*$. Putting this together we are in a position to apply our induction hypothesis to $\gamma^*\in Q^*\oplus P_1^*$ and the left ideal $\Lambda^* b_1^*+\mathfrak{a}^*$, the result being a homomorphism $h^*: Q^*\to P_1^*$ such that $o(h^*\alpha_Q^*+\alpha_1^*)+\Lambda^* b_1^*+\mathfrak{a}^*=\Lambda^*$. Since Q is projective we can cover h^* with an $h: Q\to P_1\subset P$.

Now, for the theorem we take $f=f_1+h: Q\to\beta\Lambda\oplus P_1=P$. It remains to show that $\mathfrak{b}+\mathfrak{a}=\Lambda$, where $\mathfrak{b}=o(f\alpha_Q+\alpha_P)$. By (*) above we have

$$f\alpha_Q+\alpha_P=(h\alpha_Q+f_1\alpha_Q)+\alpha_P=h\alpha_Q+(\beta b_1+\alpha_1)=\beta b_1+(h\alpha_Q+\alpha_1)\in\beta\Lambda\oplus P_1.$$

It is thus evident that $\Lambda b_1\subset\mathfrak{b}$, and that $o(h\alpha_Q+\alpha_1)\subset\mathfrak{b}$. But since P_1 is projective it follows that the image of $o(h\alpha_Q+\alpha_1)$ in Λ^* is $o(h^*\alpha_Q^*+\alpha_1^*)$. By construction of h^* this together with $\Lambda^* b_1^*+\mathfrak{a}^*$ generates Λ^*. Back in Λ, then, $o(h\alpha_Q+\alpha_1)+\Lambda b_1+\mathfrak{a}+\Lambda t=\Lambda$. But $\Lambda t\subset\Lambda b_1+\mathfrak{a}$ so we have $\mathfrak{b}+\mathfrak{a}\supset o(h\alpha_Q+\alpha_1)+\Lambda b_1+\mathfrak{a}=\Lambda$, as required.

Let P be a Λ-module. Recall that an element $\alpha\in P$ is unimodular if there is an $f\in P^*=\mathrm{Hom}_\Lambda(P, \Lambda)$ for which $f\alpha=1$. We shall similarly call $f\in P^*$ *unimodular* if there is an $\alpha\in P$ for which $f\alpha=1$ (i.e. if f is surjective). Further, we shall find it useful sometimes to identify P with $\mathrm{Hom}_\Lambda(\Lambda, P)$; i.e. we identify $\alpha\in P$ with $g_\alpha: \Lambda\to P$ defined by $g_\alpha(a)=\alpha a$. Thus, if $f: P\to\Lambda$ we can compose, $\alpha f: P\to P$; αf is defined by $(\alpha f)\beta=\alpha(f\beta)$ for $\beta\in P$. An endomorphism τ of P will be called a *transvection* if $\tau=1_P+\alpha f$, where $\alpha\in P, f\in P^*, f\alpha=0$, and either α or f is unimodular. τ is then necessarily an automorphism, since $(\alpha f)^2 = 0$. If $\mathfrak{q}$ is an ideal in Λ we call τ a $\mathfrak{q}$-*transvection* if $\operatorname{Im}(\alpha f) \subset P\mathfrak{q}$.

Denote by $\mathbf{GL}(P)$ the group of Λ-automorphisms of P, and by $\mathbf{E}(P)$ the subgroup generated by the transvections. More generally, if $\mathfrak{q}$ is an ideal, let $\mathbf{E}(P, \mathfrak{q})$ be the subgroup generated by the $\mathfrak{q}$-transvections. Suppose $\tau = 1 + \alpha f$ is a $\mathfrak{q}$-transvection and $\sigma \in \mathbf{GL}(P)$. Then $\tau^\sigma = \sigma^{-1}\tau\sigma = 1 + (\sigma^{-1}\alpha)(f\sigma)$ is clearly again a $\mathfrak{q}$-transvection; hence $\mathbf{E}(P, \mathfrak{q})$ is a normal subgroup of $\mathbf{GL}(P)$. The next result should be compared with Theorem 4.2 *a)*.

Theorem **(9.2)**. — *Let* M *be a* Λ*-module which has a projective direct summand of* f-rank$> d+1$ ($d = \dim X$). *Then for any ideal* $\mathfrak{q}$ *in* Λ, *the orbits of* $\mathbf{E}(M, \mathfrak{q})$ *operating on the unimodular elements of* M *are precisely the congruence classes* mod $\mathfrak{q}$. *In particular,* $\mathbf{E}(M)$ *is transitive on the unimodular elements.*

Proof. — By hypothesis, $M = P' \oplus N$ with P' projective and f-rank $P' > d+1$. By Serre's theorem (twice), $P' = \beta\Lambda \oplus \gamma\Lambda \oplus P_1$ with β and γ unimodular.

Let $\mathfrak{q}$ be an ideal in Λ. We will first show that if α is a unimodular element of M and if $\alpha \equiv \beta \bmod M\mathfrak{q}$, then there is a $\tau \in \mathbf{E}(M, \mathfrak{q})$ such that $\tau\alpha = \beta$.

With $P = \beta\Lambda \oplus P_1$ we have $M = \gamma\Lambda \oplus P \oplus N$. Write $\alpha = \gamma q + \alpha_P + \alpha_N$. Since $\alpha \equiv \beta \in P \bmod \mathfrak{q}$, we have $q \in \mathfrak{q}$. There is an $h : M \to \Lambda$ with $h\alpha = 1$, by assumption, so we can write

$$(*) \qquad 1 = h\alpha = (h\gamma)q + h\alpha_P + h\alpha_N.$$

Let $\bar{\alpha} = \gamma r + \alpha_P + \alpha_N$ with $r = q(h\gamma)q$. To see that $\bar{\alpha}$ is also unimodular, we first note that $o(\bar{\alpha}) = \Lambda r + o(\alpha_P) + o(\alpha_N)$. Left multiplying (*) by q, then, we find $q \in o(\bar{\alpha})$, so that $o(\bar{\alpha}) \supset \Lambda q + o(\alpha_P) + o(\alpha_N) = o(\alpha) = \Lambda$.

Now we apply Theorem 9.1 above to $Q = \gamma\Lambda$, P, $\gamma r + \alpha_P \in Q \oplus P$, and $\mathfrak{a} = o(\alpha_N)$. The result is an $f : Q \to P$ such that $o((f\gamma)r + \alpha_P) + o(\alpha_N) = \Lambda$. Let $\delta_1 = (f\gamma)q(h\gamma) \in P\mathfrak{q}$, and let $g_1 : M \to \Lambda$ by $g_1(\gamma) = 1$ and $g_1(P \oplus N) = 0$. Then $\tau_1 = 1_M + \delta_1 g_1 \in \mathbf{E}(M, \mathfrak{q})$, and $\tau_1\alpha = \gamma q + \alpha'_P + \alpha_N \in \gamma\Lambda \oplus P \oplus N$, where $\alpha'_P = (f\gamma)r + \alpha_P$. We have arranged above that $o(\alpha'_P) + o(\alpha_N) = \Lambda$, so $\alpha'_P + \alpha_N$ is unimodular. Hence we can find an $h' : M \to \Lambda$ with $h'(\alpha'_P + \alpha_N) = 1$, and such that $h'(\gamma) = 0$.

Write $\alpha'_P = \beta b + \alpha_1 \in P = \beta\Lambda \oplus P_1$. Since $\tau_1\alpha \equiv \beta \bmod M\mathfrak{q}$ (recall $\tau_1 \equiv 1_M \bmod \mathfrak{q}$) we have

$$(**) \qquad \beta(b-1) + \alpha_1 + \alpha_N \in M\mathfrak{q},$$

and in particular, $1 - b \in \mathfrak{q}$. Thus, if $g_2 = ((1-b) - q)h' : M \to \Lambda$, $\operatorname{Im} g_2 \subset \mathfrak{q}$, so $\tau_2 = 1_M + \gamma g_2 \in \mathbf{E}(M, \mathfrak{q})$. $\tau_2\tau_1\alpha = \gamma(1-b) + \alpha'_P + \alpha_N$. Let $\sigma = 1_M + \beta g_1 \in \mathbf{E}(M)$; then

$$\sigma\tau_2\tau_1\alpha = \gamma(1-b) + (\alpha'_P + \beta(1-b)) + \alpha_N = \gamma(1-b) + \beta + \alpha_1 + \alpha_N \in \gamma\Lambda \oplus \beta\Lambda \oplus P_1 \oplus N.$$

Let $g_3 : M \to \Lambda$ by $g_3\beta = 1$ and $g_3(\gamma\Lambda \oplus P_1 \oplus N) = 0$, and let $\delta_3 = -(\gamma(1-b) + \alpha_1 + \alpha_N)$, which, by (**), is in $(\ker g_3)\mathfrak{q}$. Then $\tau_3 = 1_M + \delta_3 g_3 \in \mathbf{E}(M, \mathfrak{q})$ and $\tau_3\sigma\tau_2\tau_1\alpha = \beta$. The presence of σ, which need not belong to $\mathbf{E}(M, \mathfrak{q})$, is harmless because $\sigma^{-1}\beta = \beta - \beta g_1\beta = \beta$, so that $\tau = \sigma^{-1}\tau_3\sigma\tau_2\tau_1 \in \mathbf{E}(M, \mathfrak{q})$ solves our problem.

Now for the general case, we note first (taking $\mathfrak{q}=\Lambda$) that we have shown $\mathbf{E}(M)$ to be transitive on unimodular elements. Suppose given arbitrary unimodular elements α and α' in M with $\alpha\equiv\alpha' \bmod M\mathfrak{q}$. Choose $\sigma\in\mathbf{E}(M)$ with $\sigma\alpha'=\beta$ (β as above). Then $\sigma\alpha\equiv\sigma\alpha'=\beta \bmod M\mathfrak{q}$ so the argument above produces a $\tau\in\mathbf{E}(M, \mathfrak{q})$ with $\tau\sigma\alpha=\sigma\alpha'$. Finally, $\sigma^{-1}\tau\sigma\alpha=\alpha'$ does the trick.

Theorem **(9.3)** (" Cancellation "). — *Let* M *be a* Λ*-module which has a projective direct summand of* f-rank$>d=\dim X$, *and let* Q *be a finitely generated projective module. Then, if* M' *is another* Λ*-module,*

$$Q\oplus M\cong Q\oplus M' \Rightarrow M\cong M'.$$

Proof. — Since $Q\oplus Q'\cong\Lambda^n$ for some n and Q' we can reduce, by induction on n, to the case $Q=\Lambda$. Then using the given isomorphism to identify $\Lambda\oplus M$ with $\Lambda\oplus M'$ we can write $\beta\Lambda\oplus M=\alpha\Lambda\oplus M'$ with β and α unimodular. $\beta\Lambda\oplus M$ satisfies the hypothesis of (9.2), so there is a $\tau\in\mathbf{GL}(\beta\Lambda\oplus M)$ with $\tau\alpha=\beta$. Hence

$$M\cong(\beta\Lambda\oplus M)/\beta\Lambda=\tau(\alpha\Lambda\oplus M')/\tau(\alpha\Lambda)\cong(\alpha\Lambda\oplus M')/\alpha\Lambda\cong M'.$$

Remarks. — 1) If M satisfies the coherence condition in Serre's Theorem, and if f-rank $M>2d$, then M has a free direct summand $\cong\Lambda^{d+1}$ (by Serre's Theorem), and hence M fulfills the hypothesis of (9.3) above.

2) If $d=0$ then Λ is semi-local, and (6.6) gives the conclusion of (9.3) with no restrictions on M. If $d=1$ and $\Lambda=A$ is commutative, then for M projective no further hypothesis is needed. For one can reduce this case further so that spec(A) is connected and M is finitely generated (using [6]). Then one applies Serre's Theorem to make M and M' each a direct sum of a free module and a projective module of rank one. The desired isomorphism then follows by taking suitable exterior powers (see [29, no. 8]).

However, for $d=1$ and Λ non commutative, (9.3) gives the best possible result even for M projective and finitely generated (see Swan [33]). For $d>1$ (9.3) is best possible even with $\Lambda=A$ commutative. For if $A=\mathbf{R}[x, y, z]$, $x^2+y^2+z^2=1$, is the algebraic coordinate ring of the real 2-sphere, and if $P=A^3/(x, y, z)A$ is the projective A-module corresponding to the tangent bundle on S^2, then P is not free, whereas $A\oplus P\cong A\oplus A^2$ (see Swan [34, Example 1]).

§ 10. Stable isomorphism type.

Keep A, X, and Λ as above (see (8.1)). The Proposition which follows is merely a reformulation of a special case of the preceding results. We include it for the sake of putting in evidence the faithfulness of the analogy between the present theory, and its topological source (see Introduction).

Proposition **(10.1)**. — *Let* $P_n(\Lambda)$ *denote the isomorphism types of finitely generated projective* Λ*-modules* P, *such that* $P_x\cong\Lambda_x^n$, *for all* $x\in X$. *Let* $f_n : P_n(\Lambda)\to P_{n+1}(\Lambda)$ *be the map induced by* $\oplus\Lambda$. *Then, if* $\dim X=d$:

1. (Serre's Theorem) f_n *is surjective for* $n \geq d$.

2. (Cancellation) f_n *is injective for* $n > d$.

Of course our theorems are much more general. For example, one could formulate a similar result for all finitely presented modules, relative to f-rank. Thus, calling M and N " stably isomorphic " if $Q \oplus M \cong Q \oplus N$ for some finitely generated projective module Q, we see that stable isomorphism = isomorphism for semi-local rings (6.6), and more generally also for M projective of f-rank $> d$, or finitely presented of f-rank $> 2d$.

In a special case of some interest we can make a mild improvement in Serre's Theorem for non projective modules.

Proposition (**10.2**). — *Let* A *be a Dedekind ring of characteristic zero and* π *a finite group. If* M *is a finitely generated torsion free* $A\pi$*-module of* f-rank ≥ 1, *then* $M = P \oplus N$ *with* P *a projective* $A\pi$*-module locally free of rank one.*

Proof. — With L the field of quotients of A let Λ be a maximal order of $L\pi$ containing $A\pi$, and let $\mathfrak{a}$ be the annihilator of the A-module $\Lambda/A\pi$. Characteristic zero guarantees $\mathfrak{a} \neq 0$. If we semi-localize at the maximal ideals containing $\mathfrak{a}$ we obtain a free summand of M (Serre's Theorem in dimension zero) generated, say, by $\alpha \in M$. Let P be the A-pure submodule of M generated by $\alpha A\pi$; this is automatically an $A\pi$-submodule. Then $0 \to P \to M \to M/P \to 0$ is an exact sequence of torsion free $A\pi$-modules which splits at all maximal ideals containing $\mathfrak{a}$. At all others it splits automatically because $A\pi$ there agrees with the hereditary ring Λ. Hence the sequence splits — its is an element of $\mathrm{Ext}^1_{A\pi}(M/P, P)$ which vanishes at all localizations. P, being locally projective, is projective.

Finally, since $L \otimes_A P$ is $L\pi$-free of rank one, a theorem of Swan [32] (see also [5]) guarantees that P is locally free of rank one.

Remark. — Short of the last sentence, and its special conclusion, it is clear that we have invoked only very general properties of $A\pi$.

Corollary (**10.3**). — *With* A *and* π *as above, if* M *is a finitely generated torsion free* $A\pi$*-module of* f-rank ≥ 2, *and* Q *a finitely generated projective* $A\pi$*-module, then*

$$Q \oplus M \cong Q \oplus M' \Rightarrow M \cong M'.$$

This Corollary responds to a question of Swan [32] and Swan himself has shown it to be best possible [33]. He shows, moreover [33, Theorem 2], using a result of Eichler [19], that if Λ is a maximal order in a semi-simple algebra over a number field, then one can always cancel projectives unless $\mathbf{R} \otimes_{\mathbf{Z}} \Lambda$ has a quaternion factor.

§ 11. A stable range for GL(Λ), and a conjecture.

We keep A, X, and Λ fixed as in (8.1).

Theorem (**11.1**). — *If* $d = \dim X$, *then* $d+1$ *defines a stable range for* $\mathbf{GL}(\Lambda)$, *in the sense of* § 4. *Hence, the conclusions of Theorem* 4.2 *are valid for* Λ *with* $n = d+1$.

Proof. — We must show that if $r > d+1$ and if $\Sigma_{i=1}^{r} \Lambda a_i = \Lambda$, then there exist $b_1, \ldots, b_{r-1} \in \Lambda$ such that $\Sigma_{i=1}^{r-1} \Lambda(a_i + b_i a_r) = \Lambda$.

Note first that $\alpha=(a_1, \ldots, a_r)$ is unimodular in $\Lambda^r=\Lambda^{r-1}\oplus\beta\Lambda$, $\beta=(0, \ldots, 0, 1)$; $\alpha=\alpha'+\beta a_r$. Since f-rank $\Lambda^{r-1}=r-1\geq d+1>\dim X$ we can apply (9.1) and obtain $f:\beta\Lambda\to\Lambda^{r-1}$ such that $\alpha'+f(\beta a_r)$ is unimodular. Then $f\beta=(b_1, \ldots, b_{r-1}, 0)$ solves our problem.

Let $\mathfrak{q}$ be an ideal in Λ, and write, in the notation of Chapter I, § 1,

$$G_r(\mathfrak{q})=\mathbf{GL}(r, \Lambda, \mathfrak{q})/\mathbf{E}(r, \Lambda, \mathfrak{q}).$$

Let

$$f_r: G_r(\mathfrak{q})\to G_{r+1}(\mathfrak{q})$$

be induced by the inclusions. We obtain a direct sequence of sets with base points, and

$$\lim_{r\to\infty} G_r(\mathfrak{q})=\mathbf{GL}(\Lambda, \mathfrak{q})/\mathbf{E}(\Lambda, \mathfrak{q}),$$

which, by (3.1) *a)* is an abelian group.

With this notation we can now translate parts of (4.2) in such a way as to exhibit the (partial) analogy with Proposition 10.1.

Proposition **(11.2)**. — *Suppose* $\dim X=d$. *Then for all ideals* $\mathfrak{q}$, *we have:*

a) ((4.2) *c)*) $G_r(\mathfrak{q})$ *is a group for* $r>d+1$.

b) ((4.2) *b)*) $f_r: G_r(\mathfrak{q})\to G_{r+1}(\mathfrak{q})$ *is surjective for* $r\geq d+1$.

c) ((4.2) *f)*) $G_r(\mathfrak{q})$ *is an abelian group for* $r\geq 2(d+1)$ *and* ≥ 3.

The missing link here is an assertion that f_r becomes injective. Our prevailing topological analogy suggests quite explicitly in this regard, the following,

Conjecture. — Under the conditions of (11.2) $f_r: G_r(\mathfrak{q})\to G_{r+1}(\mathfrak{q})$ is injective for $r>d+1$. In terms of matrix groups this says, for $r>d+1$,

$$\mathbf{E}(r+1, \Lambda, \mathfrak{q})\cap\mathbf{GL}(r, \Lambda, \mathfrak{q})=\mathbf{E}(r, \Lambda, \mathfrak{q}).$$

When Λ is a division ring (so $d=0$) it is the affirmative solution of essentially this problem which constitutes Dieudonné's theory of non commutative determinants [17]. Klingenberg [24] has generalized his solution to local rings (still $d=0$), although Klingenberg's proof is not valid when $\mathfrak{q}=\Lambda$. On the other hand it works in any ring provided $\mathfrak{q}\subset\operatorname{rad}\Lambda$. This procedure of axiomatically constructing determinants (see Artin [1, Chap. V], for a good exposition) runs into severe computational difficulties if one tries to generalize it naively.

The interest in the conjecture above stems from more than a simple love of symmetry. One can consult § 20 below and [7, § 1] for some rather striking consequences of its affirmative solution.

CHAPTER III

THE FUNCTORS K

§ 12. $K^0(A)$ and $K^1(A, \mathfrak{q})$.

Let A be a ring and $\mathscr{P}=\mathscr{P}(A)$ the category of finitely generated projective right A-modules, and A-homomorphisms. Let

$$\gamma=\gamma_A : \text{obj}\,\mathscr{P}\to K^0(A)$$

solve the universal problem for maps into an abelian group which satisy

(A) (Additivity) If $0\to P'\to P\to P''\to 0$ is an exact sequence (as A-modules) then

$$\gamma P=\gamma P'+\gamma P''.$$

Uniqueness of $(\gamma_A, K^0(A))$ is the usual formality, and existence follows by reducing the free abelian group generated by isomorphism types of obj $\mathscr{P}$ by the relations dictated by (A).

Let T be an infinite cyclic group with generator t. We build now a new category, $\mathscr{P}[T]=\mathscr{P}[t, t^{-1}]$, whose objects are A-automorphisms, α, of modules $P=\text{dom}\,\alpha\in\text{obj}\,\mathscr{P}$. If $\alpha'\in\text{Aut}_A(P')$ is another, a morphism, $\alpha\to\alpha'$, is an A-homomorphism, $f: P\to P'$, such that $f\alpha=\alpha' f$.

If $\alpha\in\text{Aut}_A(P)$ then α defines an A-representation of T on P, t acting as α. In this sense we can think of $\mathscr{P}[T]$ as a category of $A[T]=A[t, t^{-1}]$-modules, and as such we may speak of " exact sequences " of α's.

Let $\mathfrak{q}$ be an ideal in A (possibly $\mathfrak{q}=A$) and let $\mathscr{P}_{\mathfrak{q}}[T]$ be the full subcategory of $\mathscr{P}[T]$ whose objects are those α for which " $\alpha\equiv 1 \bmod \mathfrak{q}$ "; i.e. if $P=\text{dom}\,\alpha$, we require that $\alpha\otimes_A A/\mathfrak{q}=1$ on $P\otimes_A A/\mathfrak{q}$.

We define the group $K^1(A, \mathfrak{q})$ by letting

$$W_{\mathfrak{q}} : \text{obj}\,\mathscr{P}_{\mathfrak{q}}[T]\to K^1(A, \mathfrak{q})$$

solve the universal problem for maps into an abelian group which satisfy

(A) (Additivity) If $0\to\alpha'\to\alpha\to\alpha''\to 0$ is exact then $W_{\mathfrak{q}}\alpha=W_{\mathfrak{q}}\alpha'+W_{\mathfrak{q}}\alpha''$.

(M) (Multiplicativity) If $\text{dom}\,\alpha=\text{dom}\,\beta$ then $W_{\mathfrak{q}}\alpha\beta=W_{\mathfrak{q}}\alpha+W_{\mathfrak{q}}\beta$.

Existence and uniqueness are clear by a remark analogous to that above for K^0. Although we have no need for this fact, the reader will be able to determine easily that K^1 is unaltered if we relax (A) to apply only to split exact sequences.

When $\mathfrak{q}=A$ we call $K^1(A)=K^1(A, A)$ the " Whitehead group " of A, and $W=W_A$ the " Whitehead determinant ".

Let α be a automorphism in $\mathscr{P}_{\mathfrak{q}}[T]$ and $P \in \operatorname{obj} \mathscr{P}$. Then (M) implies $W_{\mathfrak{q}} 1_P = 0$, so (A) further implies that $W_{\mathfrak{q}}(\alpha \oplus 1_P) = W_{\mathfrak{q}}\alpha$.

If $\alpha \in \mathbf{GL}(n, A, \mathfrak{q}) \subset \mathbf{GL}(n, A) = \operatorname{Aut}_A(A^n)$ then we can regard $\alpha \in \operatorname{obj} \mathscr{P}_{\mathfrak{q}}[T]$. With our convention, $\mathbf{GL}(n, A) \subset \mathbf{GL}(n+m, A)$, α is identified with $\alpha \oplus 1_{A^m}$. The last paragraph shows that $W_{\mathfrak{q}}$ respects this convention, and hence we have a map, also denoted $W_{\mathfrak{q}}$,

$$W_{\mathfrak{q}} : \mathbf{GL}(A, \mathfrak{q}) = \cup_n \mathbf{GL}(n, A, \mathfrak{q}) \to K^1(A, \mathfrak{q}),$$

which, by (M), is a homomorphism. Suppose $\alpha \in \mathbf{GL}(A, \mathfrak{q})$ and $\beta \in \mathbf{GL}(A)$. Then $\alpha \cong \beta^{-1}\alpha\beta$, so $W_{\mathfrak{q}}\alpha = W_{\mathfrak{q}}(\beta^{-1}\alpha\beta)$. This means that $[\mathbf{GL}(A), \mathbf{GL}(A, \mathfrak{q})] = \mathbf{E}(A, \mathfrak{q}) \subset \ker W_{\mathfrak{q}}$, so we have an induced homomorphism

$$f : \mathbf{GL}(A, \mathfrak{q})/\mathbf{E}(A, \mathfrak{q}) \to K^1(A, \mathfrak{q}).$$

Proposition (**12.1**). — *The inclusions* $\mathbf{GL}(n, A, \mathfrak{q}) \subset \operatorname{obj} \mathscr{P}_{\mathfrak{q}}[t, t^{-1}]$ *induce an isomorphism*

$$f : \mathbf{GL}(A, \mathfrak{q})/\mathbf{E}(A, \mathfrak{q}) \to K^1(A, \mathfrak{q}).$$

Proof. — Let $\alpha \in \operatorname{obj} \mathscr{P}_{\mathfrak{q}}[T]$, $P = \operatorname{dom} \alpha$. We can find a Q such that $P \oplus Q \cong A^n$ (some n). This isomorphism induces an isomorphism $\alpha \oplus 1_Q \cong \alpha_n \in \mathbf{GL}(n, A, \mathfrak{q})$. With Q fixed α_n varies in its conjugacy class in $\mathbf{GL}(n, A)$, so its image in $G = \mathbf{GL}(A, \mathfrak{q})/\mathbf{E}(A, \mathfrak{q})$ doesn't change. If we replace Q by $Q \oplus A^m$, α_n is replaced by a conjugate of $\alpha_n \oplus 1_{A^m}$, and again its image in G is unaffected. Finally, if $P \oplus Q' \cong A^{n'}$, then $Q \oplus A^{n'} \cong Q \oplus P \oplus Q' \cong A^n \oplus Q'$, so we see that the image of α_n in G is independent of Q. We define thus a map

$$g : \operatorname{obj} \mathscr{P}_{\mathfrak{q}}[T] \to G,$$

and we propose to show that g is additive and multiplicative. Once shown, the universality of $W_{\mathfrak{q}}$ produces a unique homomorphism $K^1(A, \mathfrak{q}) \to G$, which is manifestly an inverse for f.

g is multiplicative. For if $\operatorname{dom} \alpha = P = \operatorname{dom} \beta$ an isomorphism $P \oplus Q \cong A^n$ induces $\alpha \oplus 1_Q \cong \alpha_n$, $\beta \oplus 1_Q \cong \beta_n$, and $\alpha\beta \oplus 1_Q \cong \alpha_n\beta_n$.

g is additive. Let $0 \to \alpha' \to \alpha \to \alpha'' \to 0$ be an exact sequence, with domains P′, P, P″, respectively. Choose Q′ and Q″ with isomorphisms $P' \oplus Q' \cong A^n \cong P'' \oplus Q''$. Since the exact sequence induces $P \cong P' \oplus P''$, we can choose an isomorphism $P \oplus Q' \oplus Q'' \cong A^n \oplus A^n$ compatible with the direct sum of the given sequence with $0 \to 1_{Q'} \to 1_{Q'} \oplus 1_{Q''} \to 1_{Q''} \to 0$. With these isomorphisms, we have $\alpha' \oplus 1_{Q'} \cong \alpha'_n$, $\alpha'' \oplus 1_{Q''} \cong \alpha''_n$, and $\alpha \oplus 1_{Q' \oplus Q''} \cong \alpha_{2n} = \begin{vmatrix} \alpha'_n & q \\ 0 & \alpha''_n \end{vmatrix} = \begin{vmatrix} \alpha'_n & 0 \\ 0 & \alpha''_n \end{vmatrix} \begin{vmatrix} 1 & (\alpha'_n)^{-1}q \\ 0 & 1 \end{vmatrix}$.

Since the second factor is manifestly in $\mathbf{E}(2n, A, \mathfrak{q})$, and since

$$\begin{vmatrix} \alpha'_n & 0 \\ 0 & \alpha''_n \end{vmatrix} \equiv \begin{vmatrix} \alpha'_n\alpha''_n & 0 \\ 0 & 1 \end{vmatrix} \mod \mathbf{E}(2n, A, \mathfrak{q}),$$

by the Whitehead Lemma (1.7), we see that α_{2n} and $\alpha'_n\alpha''_n$ do indeed agree in G, as required.

Suppose H is a normal subgroup of $\mathbf{GL}(A)$. Then, by Theorem 3.1, $\mathbf{E}(A, \mathfrak{q}) \subset H \subset \mathbf{GL}(A, \mathfrak{q})$ for a unique $\mathfrak{q}$, and $H/\mathbf{E}(A, \mathfrak{q})$ is, via Proposition 12.1 above, a subgroup of $K^1(A, \mathfrak{q})$. Conversely any subgroup of $K^1(A, \mathfrak{q})$ defines in this way a normal subgroup of $\mathbf{GL}(A)$. Thus we see that a determination of all normal subgroups of $\mathbf{GL}(A)$ is equivalent to a determination of all subgroups of $K^1(A, \mathfrak{q})$, for all $\mathfrak{q}$.

Finally we note that K^0 and K^1 are functors. If $\varphi : A \to B$ is a ring homomorphism, then $\otimes_A B : \mathscr{P}(A) \to \mathscr{P}(B)$ induces

$$\varphi^0 : K^0(A) \to K^0(B).$$

If $\mathfrak{q}$ is an ideal in A and $\mathfrak{q}'$ an ideal of B containing $\varphi(\mathfrak{q})$, then $\otimes_A 1_B$ induces

$$\varphi^1 : K^1(A, \mathfrak{q}) \to K^1(B, \mathfrak{q}').$$

Note that the isomorphism of Proposition 12.1 is an isomorphism of functors.

§ 13. The exact sequence.

Let $\varphi : A \to B$ be a ring homomorphism. If P and f are a right A-module and A-homomorphism, we shall abbreviate $PB = P \otimes_A B$, and $fB = f \otimes_A 1_B$.

Our objective is to construct a relative group, $K^0(A, \varphi)$, which will fit into an exact sequence (Theorem 13.1, below). To this end we manufacture the category $\mathscr{C}(\mathscr{P})$ whose objects are triples, $\sigma = (P, \alpha, Q)$, with $P, Q \in \operatorname{obj} \mathscr{P}(A)$ (i.e. finitely generated projective right A-modules) and α a B-isomorphism, $PB \to QB$. If $\sigma' = (P', \alpha', Q')$ is another such triple, a morphism $\sigma' \to \sigma$ is a pair, (f, g), of A-homomorphisms, $f : P' \to P$ and $g : Q' \to Q$, making

$$\begin{array}{ccc} P'B & \xrightarrow{\alpha'} & Q'B \\ {\scriptstyle fB}\downarrow & & \downarrow{\scriptstyle gB} \\ PB & \xrightarrow{\alpha} & QB \end{array}$$

commutative. We call $\sigma' \xrightarrow{(f',g')} \sigma \xrightarrow{(f'',g'')} \sigma''$ "exact" if $P' \xrightarrow{f'} P \xrightarrow{f''} P''$ and $Q' \xrightarrow{g'} Q \xrightarrow{g''} Q''$ are both exact sequences of A-modules.

Note that the objects of $\mathscr{C}(\varphi)$ are a groupoid. Thus, if $\sigma = (P, \alpha, P')$ and $\sigma' = (P', \alpha', P'')$ then we write $\sigma'\sigma = (P, \alpha'\alpha, P'')$.

Now we define $K^0(A, \varphi)$ by letting

$$R : \operatorname{obj} \mathscr{C}(\varphi) \to K^0(A, \varphi)$$

solve the universal problem for maps into an abelian group which satisfy:

(A) (Additivity) If $0 \to \sigma' \to \sigma \to \sigma'' \to 0$ is exact then

$$R\sigma = R\sigma' + R\sigma''.$$

(M) (Multiplicativity) If $\sigma'\sigma$ is defined then

$$R\sigma'\sigma = R\sigma' + R\sigma.$$

We need to know $K^0(A, \varphi)$ in some detail, and it will be convient to introduce some provisional terminology for this purpose. A triple $\tau=(P, \beta, P)$ will be called an " automorphism ". Since $\beta \in \mathrm{Aut}_B(PB)$, $W_B\beta \in K^1(B)$ is defined, and we shall write $W_B\tau = W_B\beta$ in this case. $\tau_P = (P, 1_P, P)$ will be called an " identity ". Since $\tau_P\tau_P = \tau_P$, (M) implies

$$R\tau_P = 0. \tag{1}$$

If there is an exact sequence

$$0 \to \tau_P \to \sigma \to \tau_Q \to 0, \tag{2}$$

then (A) further implies $R\sigma = 0$.

Let $\beta \in \mathbf{GL}(B) = \bigcup_n \mathbf{GL}(n, B)$; say $\beta \in \mathbf{GL}(n, B) = \mathrm{Aut}_B(A^nB)$. Then

$$\sigma = (A^n, \beta, A^n) \in \mathrm{obj}\ \mathscr{C}(\varphi).$$

Viewing $\beta \in \mathbf{GL}(n+m, B)$ replaces σ by $\sigma \oplus 1_{A^m}$, so $R\sigma$ is unaltered. Hence we have a well defined map

$$\mathbf{GL}(B) \to K^0(A, \varphi), \tag{3}$$

which is a homomorphism, by (M). Now if β is elementary, then σ appears in an exact sequence of type (2), with $P = A^{n-1}$, $Q = A$. Hence $\mathbf{E}(B)$ is in the kernel of (3), so (3) induces a homomorphism

$$\delta : K^1(B) = \mathbf{GL}(B)/\mathbf{E}(B) \to K^0(A, \varphi).$$

Let $\tau = (P, \beta, P)$ be an automorphism. If $W_B\tau = 0$ then, by Proposition 12.1, we can find a Q so that $\tau \oplus \tau_Q \cong (A^n, \gamma, A^n)$, with $\gamma \in \mathbf{E}(n, B)$, and hence $R\tau = 0$. Now if σ, $\sigma' \in \mathrm{obj}\ \mathscr{C}(\varphi)$ write

$$\sigma \sim \sigma'$$

if there exist identities, τ and τ', and an automorphism ε with $W_B\varepsilon = 0$, such that $(\sigma \oplus \tau)\varepsilon \cong (\sigma' \oplus \tau')$. A tedious, but straightforward, exercise shows that $\sim$ is an equivalence relation. Our earlier remarks show that $\sigma \sim \sigma' \Rightarrow R\sigma = R\sigma'$. Hence, if

$$R' : \mathrm{obj}\ \mathscr{C}(\varphi) \to G = \mathrm{obj}\ \mathscr{C}(\varphi)/\sim$$

is the natural projection, then $R = hR'$ for a unique $h : G \to K^0(A, \varphi)$. An easy check shows that $\sim$ respects $\oplus$, so that $\oplus$ induces on G the structure of a monoid, with neutral element the class of the identities, and relative to which h is clearly a homomorphism.

We propose to show that *a)* G is a group, and that R' is *b)* additive and *c)* multiplicative. Once shown, the universality of R produces a homomorphism $h' : K^0(A, \varphi) \to G$ which is clearly an inverse for h. This isomorphism will permit us to conclude:

(4) Every element of $K^0(A, \varphi)$ has the form $R\sigma$, $\sigma \in \mathrm{obj}\ \mathscr{C}(\varphi)$. $R\sigma = 0 \Leftrightarrow \sigma \sim 0$.

a) G is a group. Given $\sigma = (P, \alpha, Q)$, let $\sigma' = (Q, -\alpha^{-1}, P)$. Then

$$\sigma \oplus \sigma' = \left(P \oplus Q, \begin{vmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{vmatrix}, P \oplus Q\right).$$

It follows from the Whitehead Lemma (1.7) (using the fact that $PB \cong QB$) that $W_B(\sigma \oplus \sigma') = 0$.

b) R' is additive. Let $0 \to \sigma' \to \sigma \to \sigma'' \to 0$ be exact. If $\sigma' = (P', \alpha', Q')$ and $\sigma'' = (P'', \alpha'', Q'')$, then $\sigma \cong \sigma_1 = (P' \oplus P'', \alpha, Q' \oplus Q'')$, where

$$\alpha = \begin{vmatrix} \alpha' & \gamma \\ 0 & \alpha'' \end{vmatrix} = \begin{vmatrix} \alpha' & 0 \\ 0 & \alpha'' \end{vmatrix} \begin{vmatrix} 1_{P'B} & (\alpha')^{-1}\gamma \\ 0 & 1_{P''B} \end{vmatrix} = (\alpha' \oplus \alpha'')\varepsilon.$$

Since $W_B \varepsilon = 0$ we have $\sigma_1 \sim \sigma' \oplus \sigma''$, as required.

c) R' is multiplicative. Suppose $\sigma = (P, \alpha, P')$ and $\sigma' = (P', \alpha', P'')$. We must show $\sigma'\sigma \sim \sigma \oplus \sigma'$. From the commutative diagram

$$\begin{array}{ccc} PB & \xrightarrow{\alpha} & P'B \\ {\scriptstyle 1_P B}\downarrow & & \downarrow{\scriptstyle (-1_P)B} \\ PB & \xrightarrow{-\alpha} & P'B \end{array}$$

we see that $\sigma \cong -\sigma = (P, -\alpha, P')$. Hence it will suffice for us to show $(\sigma'\sigma \oplus \tau_{P'}) \sim (-\sigma) \oplus \sigma'$.

$$\sigma'\sigma \oplus \tau_{P'} = (P \oplus P', \beta = \begin{vmatrix} \alpha'\alpha & 0 \\ 0 & 1_{P'B} \end{vmatrix}, P'' \oplus P')$$

and

$$(-\sigma) \oplus \sigma' = (P \oplus P', \gamma = \begin{vmatrix} 0 & \alpha' \\ -\alpha & 0 \end{vmatrix}, P'' \oplus P')$$

$$= (\sigma'\sigma \oplus \tau_{P'})\varepsilon,$$

where $\varepsilon = (P \oplus P', \gamma^{-1}\beta, P \oplus P')$. Since $PB \cong P'B$ it follows from the Whitehead Lemma (1.7) that $W_B \varepsilon = 0$, and this completes the proof of *c)*, hence also of (4).

Finally, we define a homomorphism

$$d : K^0(A, \varphi) \to K^0(A)$$

by $d(R(P, \alpha, Q)) = \gamma_A P - \gamma_A Q$. It is clear that d is well defined.

Theorem **(13.1)**. — *Let* $\varphi : A \to B$ *be a ring homomorphism. Then the sequence*

$$K^1(A) \xrightarrow{\varphi^1} K^1(B) \xrightarrow{\delta} K^0(A, \varphi) \xrightarrow{d} K^0(A) \xrightarrow{\varphi^0} K^0(B)$$

is exact. If $\varphi : A \to A/\mathfrak{q}$ *is the natural map, then*

$$K^1(A, \mathfrak{q}) \xrightarrow{d_1} K^1(A) \to K^1(A/\mathfrak{q}) \to K^0(A, \mathfrak{q}) \to K^0(A) \to K^0(A/\mathfrak{q})$$

is exact.

(In the second sequence, $K^0(A, \mathfrak{q}) = K^0(A, \varphi)$, and d_1 is induced by the inclusion $\mathbf{GL}(A, \mathfrak{q}) \subset \mathbf{GL}(A)$.)

Proof. — Exactness at $K^0(A)$.

$$\varphi^0 d(R(P, \alpha, Q)) = \varphi^0(\gamma_A P - \gamma_A Q) = \gamma_B PB - \gamma_B QB = 0$$

(since $\alpha : PB \cong QB$). Suppose $\varphi^0(\gamma_A P - \gamma_A Q) = \gamma_B PB - \gamma_B QB = 0$. Then there exists an $\alpha : PB \oplus B^n \cong QB \oplus B^n$ for some n, and hence $\gamma_A P - \gamma_A Q = d(R(P \oplus A^n, \alpha, Q \oplus A^n))$.

Exactness at $K^0(A, \varphi)$. If $\beta \in \mathbf{GL}(n, B)$ then

$$d\delta(W_B \beta) = d(R(A^n, \beta, A^n)) = \gamma_A A^n - \gamma_A A^n = 0.$$

Suppose $d(R(P, \alpha, Q)) = \gamma_A P - \gamma_A Q = 0$. Then $P \oplus P' \cong Q \oplus P'$ for some P', and we can even arrange $P \oplus P' \cong A^n$. Hence $(P \oplus P', \alpha \oplus 1_{P'B}, Q \oplus P') \cong (A^n, \beta, A^n)$ for some $\beta \in \mathbf{GL}(n, B)$ and $R(P, \alpha, Q) = \delta(W_B(\beta))$.

Exactness at $K^1(B)$. If $\alpha \in \mathbf{GL}(n, A)$, then $\delta\varphi^1(W_A \alpha) = \delta(W_B \varphi\alpha) = R(A^n, \alpha B, A^n)$ (note $\varphi\alpha = \alpha B$). But the commutative square

$$\begin{array}{ccc} A^n B & \xrightarrow{\alpha B} & A^n B \\ {\scriptstyle \alpha B}\downarrow & & \downarrow{\scriptstyle 1_{A^n B}} \\ A^n B & \xrightarrow{1_{A^n B}} & A^n B \end{array}$$

shows that $R(A^n, \alpha B, A^n) = 0$. Now suppose $\beta \in \mathbf{GL}(n, B)$ and $\delta(W_B \beta) = 0$. Then, by (4) above, $\sigma = (A^n, \beta, A^n) \sim 0$. This means that $\sigma \oplus \tau_P \cong \varepsilon$ for some automorphism ε, with $W_B \varepsilon = 0$. We can add an identity to both sides and further assume $P = A^m$ and $\varepsilon = (A^{n+m}, \gamma, A^{n+m})$. Then the isomorphism $\sigma \oplus \tau_{A^m} \cong \varepsilon$ is given by isomorphisms $f, g \in \mathbf{GL}(n+m, A)$ making

$$\begin{array}{ccc} B^{n+m} & \xrightarrow{\beta \oplus 1_{B^m}} & B^{n+m} \\ {\scriptstyle fB}\downarrow & & \downarrow{\scriptstyle gB} \\ B^{n+m} & \xrightarrow{\gamma} & B^{n+m} \end{array}$$

commutative. Hence, in $\mathbf{GL}(n+m, B)$, $\beta = (\varphi g)^{-1}\gamma(\varphi f)$. Since $W_B \gamma = 0$ we have $W_B \beta = W_B(\varphi(g^{-1}f)) = \varphi^1(W_A(g^{-1}f))$.

Now suppose $\varphi : A \to A/\mathfrak{q}$.

Exactness at $K^1(A)$. If $\alpha \in \mathbf{GL}(A, \mathfrak{q})$ then $\varphi^1 d_1(W_\mathfrak{q} \alpha) = \varphi^1(W_A \alpha) = W_{A/\mathfrak{q}}(\varphi\alpha) = 0$, since $\varphi\alpha = 1$. Suppose $\alpha \in \mathbf{GL}(A)$ and $\varphi^1(W_A \alpha) = W_{A/\mathfrak{q}}(\varphi\alpha) = 0$. Then $\varphi\alpha \in \mathbf{E}(A/\mathfrak{q})$, so it follows by Homotopy Extension (1.1) that there is an $\varepsilon \in \mathbf{E}(A)$ with $\varphi\varepsilon = \varphi\alpha$. Hence $\alpha\varepsilon^{-1} \in \mathbf{GL}(A, \mathfrak{q})$ and $W_A \alpha = W_A(\alpha\varepsilon^{-1}) = d_1(W_\mathfrak{q}(\alpha\varepsilon^{-1}))$.

If we knew how to define, and extend the exact sequence to, K^2 the next result would be an immediate corollary.

Proposition (**13.2**). — *Suppose* $A = A_0 \oplus \mathfrak{q}$ *(as abelian group) with* A_0 *a subring and* $\mathfrak{q}$ *an ideal. Then there are split exact sequences*

$$0 \to K^i(A, \mathfrak{q}) \to K^i(A) \to K^i(A_0) \to 0, \qquad i = 0, 1.$$

Proof. — Let $\varphi : A \to A_0$ be the retraction with kernel $\mathfrak{q}$. Since φ has a right inverse, so does φ^i, $i = 0, 1$. Hence the Proposition follows from (13.1) provided we show $K^1(A, \mathfrak{q}) \to K^1(A)$ has a left inverse. If $\alpha \in \mathbf{GL}(A)$, $\alpha = \alpha_0 \alpha_1$ where $\alpha_0 = \varphi\alpha \in \mathbf{GL}(A_0)$, and $\alpha_1 = (\varphi\alpha)^{-1}\alpha \in \mathbf{GL}(A, \mathfrak{q})$. If also $\beta \in \mathbf{GL}(A)$ then $(\alpha\beta)_1 = \varphi(\alpha\beta)^{-1}\alpha\beta = (\varphi\beta)^{-1}(\varphi\alpha)^{-1}\alpha\beta = (\varphi\beta)^{-1}\alpha_1\beta = [\varphi\beta, \alpha_1^{-1}]\alpha_1(\varphi\beta)\beta^{-1}\beta = [\beta_0, \alpha_1^{-1}]\alpha_1\beta_1$. Now $[\beta_0, \alpha_1^{-1}] \in [\mathbf{GL}(A), \mathbf{GL}(A, \mathfrak{q})] = \mathbf{E}(A, \mathfrak{q})$. Hence $W_{\mathfrak{q}}(\alpha\beta)_1 = W_{\mathfrak{q}}\alpha_1 + W_{\mathfrak{q}}\beta_1$, and we have constructed a homomorphism $\mathbf{GL}(A) \to K^1(A, \mathfrak{q})$ whose restriction to $\mathbf{GL}(A, \mathfrak{q})$ is $W_{\mathfrak{q}}$. This clearly induces the required retraction $K^1(A) \to K^1(A, \mathfrak{q})$.

Examples. — 1) If A is a local ring (e.g. a field) then $K^0(A) = \mathbf{Z}$, and $K^1(A)$ is the commutator quotient group of $A^* = \mathbf{GL}(1, A)$. The latter is due to Dieudonné [17] for division rings, and to Klingenberg [24] in general.

2) If $\mathfrak{q} \subset \text{rad}\, A$ (Jacobson radical) and $\varphi : A \to A/\mathfrak{q}$, then φ^1 is easily seen to be surjective, and φ^0 to be injective (see Lemma 18.1 below). Hence $K^0(A, \mathfrak{q}) = 0$. The methods of Klingenberg [24] adapt without essential change to compute $K^1(A, \mathfrak{q})$ also in this case. In case A is $\mathfrak{q}$-adic complete, or if φ has a right inverse, then φ^0 is even an isomorphism.

3) The following remark is often useful. Let A be semi-local and $P, Q \in \text{obj}\, \mathscr{P}(A)$. Then $\gamma_A P = \gamma_A Q \Rightarrow P \cong Q$. For $\gamma_A P = \gamma_A Q$ implies $P \oplus A^n \cong Q \oplus A^n$ for some n, so our conclusion follows from (6.6) by induction on n.

4) In § 16 we describe in detail the exact sequence associated with the embedding of a Dedeking ring in its field of quotients.

§ 14. Algebras.

Tensor products endow our functors with various ring and module structures, and it is convenient to record these circumstances now.

Let A be a commutative ring, Λ and Λ' A-algebras, and $\mathfrak{q}$ an ideal in Λ. If $P \in \mathscr{P}(\Lambda)$ and $P' \in \mathscr{P}(\Lambda')$ then $P' \otimes_A P \in \mathscr{P}(\Lambda' \otimes_A \Lambda)$, and this induces a pairing,

$$(1') \qquad K^0(\Lambda') \times K^0(\Lambda) \to K^0(\Lambda' \otimes_A \Lambda).$$

If $\alpha \in \mathscr{P}(\Lambda)_{\mathfrak{q}}[t, t^{-1}]$, then $1_{P'} \otimes \alpha \in \mathscr{P}(\Lambda' \otimes_A \Lambda)_{\Lambda' \otimes \mathfrak{q}}[t, t^{-1}]$ and this induces a pairing

$$(2') \qquad K^0(\Lambda') \times K^1(\Lambda, \mathfrak{q}) \to K^1(\Lambda' \otimes_A \Lambda, \Lambda' \otimes \mathfrak{q}).$$

Taking $\Lambda' = A = \Lambda$ in (1'), $K^0(A)$ becomes a ring. Then with $\Lambda' = A$ in (1') and (2'), $K^0(\Lambda)$ and $K^1(\Lambda, \mathfrak{q})$ become $K^0(A)$-modules. Moreover, the pairings are $K^0(A)$-bilinear, so they define

$$(1) \qquad K^0(\Lambda') \otimes_{K^0(A)} K^0(\Lambda) \to K^0(\Lambda' \otimes_A \Lambda)$$

and

$$(2) \qquad K^0(\Lambda') \otimes_{K^0(A)} K^1(\Lambda, \mathfrak{q}) \to K^1(\Lambda' \otimes_A \Lambda, \Lambda' \otimes \mathfrak{q}).$$

These structures have the obvious naturality properties with respect to A-algebra homomorphisms.

In order to treat K^0 and K^1 simultaneously we shall sometimes consider the following situation. Let $\mathfrak{q}$ be an ideal in A. For an A-algebra, Λ, write

$$K^*(\Lambda, \mathfrak{q}\Lambda) = K^0(\Lambda) \oplus K^1(\Lambda, \mathfrak{q}\Lambda).$$

$K^*(\Lambda, \mathfrak{q}\Lambda)$ is, as noted above, a $K^0(A)$-module. Moreover, (2) above gives us

$$K^1(A, \mathfrak{q}) \otimes_{K^0(A)} K^0(\Lambda) \to K^1(\Lambda, \mathfrak{q}\Lambda) = K^1(A \otimes_A \Lambda, \mathfrak{q} \otimes \Lambda).$$

Hence, if we view $K^*(A, \mathfrak{q})$ as a graded ring, zero in degrees >2, then $K^*(\Lambda, \mathfrak{q}\Lambda)$ is a graded $K^*(A, \mathfrak{q})$-module.

Finally, if $A \to B$ is a homomorphism of commutative rings, then $B \otimes_A$ induces $K^*(\Lambda, \mathfrak{q}) \to K^*(B \otimes_A \Lambda, B \otimes \mathfrak{q})$. In case B is finitely generated and projective as an A-module, then there is an obvious " restriction " functor

$$\mathscr{P}(B \otimes_A \Lambda)_{B \otimes \mathfrak{q}}[t, t^{-1}] \to \mathscr{P}(\Lambda)_{\mathfrak{q}}[t, t^{-1}],$$

and this induces a homomorphism $K^*(B \otimes_A \Lambda, B \otimes \mathfrak{q}) \to K^*(\Lambda, \mathfrak{q})$ which, following the one above, gives the homothetie of $K^*(\Lambda, \mathfrak{q})$ defined by $\gamma_A(B) \in K^0(A)$.

§ 15. A filtration on K^0.

There seem to be several " geometrically reasonable " filtrations on K^0 (under suitable circumstances). We choose one here with the properties needed for our applications.

Let A be a commutative *noetherian* ring, and $X = \max(A)$. All modules and A-algebras will be understood finitely generated as A-modules. Let Λ be an A-algebra. We shall consider complexes,

$$C: \ldots \to P_i \to P_{i-1} \to \ldots$$

of right Λ-modules which are finite and projective (i.e. P_i is projective, and $=0$ for almost all i). Then $\chi(C) = \chi_\Lambda(C) = \Sigma_i(-1)^i \gamma_\Lambda P_i \in K^0(\Lambda)$ is defined. Our finiteness conditions guarantee that $H(C)$ is a finitely generated A-module, so supp $H(C)$ is a closed subset of X. Since localization, being exact, commutes with homology, we see that

$$\operatorname{supp} H(C) = \{x \in X \mid H(C)_x \neq 0\}$$
$$= \{x \in X \mid C_x \text{ is not acyclic}\}.$$

Definition. — $K^0(\Lambda)_i$ consists of all $\xi \in K^0(\Lambda)$ with the following property: Given Y closed in X, there is a complex C, as above, such that $\chi(C) = \xi$, and $\operatorname{codim}_Y(Y \cap \operatorname{supp} H(C)) \geq i$.

Proposition **(15.1)**. — *The* $K^0(\Lambda)_i$ *are subgroups of* $K^0(\Lambda)$ *which satisfy*:

1) $K^0(\Lambda)_0 = K^0(\Lambda) \supset K^0(\Lambda)_1 \supset K^0(\Lambda)_2 \supset \ldots$

2) *If* Γ *is another* A-*algebra the pairing*

$$T: K^0(\Lambda) \otimes_{K^0(A)} K^0(\Gamma) \to K^0(\Lambda \otimes_A \Gamma) \qquad (\textit{see } \S\ 14)$$

induces

$$K^0(\Lambda)_i \otimes K^0(\Gamma)_j \to K^0(\Lambda \otimes_A \Gamma)_{i+j}.$$

In particular, $K^0(A)$ *is a filtered ring, and* $K^0(\Lambda)$ *a filtered* $K^0(A)$*-module.*

3) *The filtration is natural with respect to homomorphisms of* A*-algebras.*

4) $K^0(\Lambda)_i = 0$ *for* $i > \dim X$.

A very useful consequence of this Proposition is

Corollary **(15.2)**. — *If* $d = \dim X$, *then* $K^0(A)_1^{d+1} = 0$.

Proof of 15.1. — $K^0(\Lambda_i)$ is a subgroup because the support of a direct sum is the union of the supports, and the codimension of a (finite) union is the infimum of the codimensions.

1) follows from the definition.

2) Let $\xi \in K^0(\Lambda)_i$, $\eta \in K^0(\Gamma)_j$, and let Y be closed in X. Choose C′ over Λ with $\chi_\Lambda(C') = \xi$ and $\operatorname{codim}_Y(Y \cap \operatorname{supp} H(C')) \geq i$. Choose C″ over Γ with $\chi_\Gamma(C'') = \eta$ and $\operatorname{codim}_{Y \cap \operatorname{supp} H(C')}(Y \cap \operatorname{supp} H(C') \cap \operatorname{supp} H(C'')) \geq j$. Then, if $C = C' \otimes_A C''$ over $\Lambda \otimes_A \Gamma$, it is clear that $\chi_{\Lambda \otimes_A \Gamma}(C) = T(\xi \otimes \eta)$. Moreover the inequality above implies $\operatorname{codim}_Y(Y \cap \operatorname{supp} H(C') \cap \operatorname{supp} H(C'')) \geq i + j$. Hence we can conclude by showing that $\operatorname{supp} H(C) \subset \operatorname{supp} H(C') \cap \operatorname{supp} H(C'')$. But if, say, C'_x is acyclic, then it is homotopic to zero (a finite acyclic complex of projective modules), so likewise for $C'_x \otimes_{A_x} C''_x = C_x$.

3) Let $f : \Lambda \to \Gamma$ be a homomorphism of A-algebras, and $\xi \in K^0(\Lambda)_i$; we want $f^0\xi \in K^0(\Gamma)_i$. Given Y, choose C over Λ with $\chi_\Lambda(C) = \xi$ and $\operatorname{codim}_Y(Y \cap \operatorname{supp} H(C)) \geq i$. Since $f^0\xi = \chi_\Gamma(C \otimes_\Lambda \Gamma)$ it suffices to note that $\operatorname{supp} H(C \otimes_\Lambda \Gamma) \subset \operatorname{supp} H(C)$. But if C_x is acyclic it is homotopic to zero, so likewise for $C_x \otimes_{\Lambda_x} \Gamma_x = (C \otimes_\Lambda \Gamma)_x$.

4) Take $Y = X$ in the definition.

Proposition **(15.3)**. — $K^0(\Lambda)_1 = \bigcap_{x \in X} \ker(K^0(\Lambda) \to K^0(\Lambda_x))$.

Proof. — If $\xi \in K^0(\Lambda)_1$ and $Y = \{x\}$ then $\xi = \chi_\Lambda(C)$ with $\operatorname{codim}_Y(Y \cap \operatorname{supp} H(C)) \geq 1$. This implies $x \notin \operatorname{supp} H(C)$, so C_x is acyclic. But then the image of ξ in $K^0(\Lambda_x)$ is $\chi_{\Lambda_x}(C_x) = 0$.

Conversely, given $\xi = \gamma_\Lambda P - \gamma_\Lambda Q \in K^0(\Lambda)$, to belong to the right side of our equation means $\gamma_{\Lambda_x} P_x = \gamma_{\Lambda_x} Q_x$ for all x, and then by (6.6) (see Example 3) in § 14), $P_x \cong Q_x$. Now, given Y closed in X, let $Y_1, \ldots, Y_s$ be the irreducible components of Y, and choose $x_i \in Y_i - \bigcup_{j \neq i} Y_j$. Then, if we reduce modulo the product of the x_i, P and Q become isomorphic. Lift such an isomorphism to $f : Q \to P$; then f_{x_i} is an isomorphism, $1 \leq i \leq s$. Let C be the complex with f the differential in degree 1, and zero in all degrees $\neq 0$ or 1. Then $\chi(C) = (-1)^1 \gamma_\Lambda Q + (-1)^0 \gamma_\Lambda P = \xi$. Since $Y_i \not\subset \operatorname{supp} H(C)$, $1 \leq i \leq s$, $\operatorname{codim}_Y(Y \cap \operatorname{supp} H(C)) \geq 1$, as required.

Proposition 15.3 gives a description of the first term of the filtration which behaves well without any finiteness assumptions on A, as we shall see below.

Let A be now an arbitrary commutative ring, and let spec(A) denote its prime ideal spectrum (Zariski topology). If $P \in \text{obj}\,\mathscr{P}(A)$ and $x \in \text{spec}(A)$, then P_x is a free A_x-module of rank $\rho_P(x)$,

$$\rho_P : \text{spec}(A) \to \mathbf{Z}.$$

It is easy to see that ρ_P is continuous (for the discrete topology on $\mathbf{Z}$).

Let $C(A)$ denote the ring of all continuous functions from $\text{spec}(A)$ to $\mathbf{Z}$. Since ρ_P is (clearly) additive and multiplicative in P it induces a ring homomorphism

$$\rho : K^0(A) \to C(A).$$

ρ even has a right inverse φ. To define φ, suppose $f : \text{spec}(A) \to \mathbf{Z}$ is continuous, and let $X_n = f^{-1}(n)$. Since $\text{spec}(A)$ is quasi-compact, $X_n = \emptyset$ for almost all n. Now disjoint decompositions $\text{spec}(A) = \bigcup_n X_n$, all X_n open, correspond, bijectively, to decompositions $1 = \Sigma_n e_n$ of 1 as a sum of orthogonal idempotents, almost all zero (see e.g. [30, Chap. I]). Define $\varphi(f) = \Sigma_n f(n)\gamma_A(Ae_n)$; φ is clearly the desired right inverse to ρ.

We shall use φ to identify $C(A)$ with a subring of $K^0(A)$. Thus,

$$K^0(A) = C(A) \oplus J(A),$$

where $J(A) = \ker \rho$. Suppose $\xi = \gamma_A P - \gamma_A Q \in J(A)$. Then $\rho\xi = \rho_P - \rho_Q = 0$. Thus

$$P_x \cong Q_x$$

for all $x \in \text{spec}(A)$. Conversely, if $P_x \cong Q_x$ for all $x \in \max(A)$, then we see by localizing in two steps that $P_x \cong Q_x$ for all $x \in \text{spec}(A)$, so $\gamma_A P - \gamma_A Q \in J(A)$.

Let us summarize:

Proposition **(15.4)**. — *Let* A *be an arbitrary commutative ring and let*

$$J(A) = \bigcap_{x \in \max(A)} \ker(K^0(A) \to K^0(A_x)).$$

Then

$$K^0(A) = C(A) \oplus J(A),$$

where $C(A)$ *is isomorphic to the ring of all continuous functions from* $\text{spec}(A)$ *to* $\mathbf{Z}$. $J(A)$ *is both the nil and Jacobson radical of* $K^0(A)$.

Proof. — It remains to prove the last assertion, and it clearly suffices to show that every $\xi = \gamma P - \gamma Q \in J(A)$ is nilpotent. If A were noetherian and of finite Krull dimension this would follow from (15.2) and (15.3). In general, however, ξ is induced from a finitely generated subring of A, and such a subring is of the latter type. Hence our conclusion is a consequence of the following lemma:

Lemma. — *If* $f : B \to A$ *is a homomorphism of commutative rings, then* $(f^0)^{-1}(J(A)) \supset J(B)$, *with equality if* f *is injective.*

Proof. — If $\xi \in J(B)$ we want $f^0(\xi) \in J(A)$; i.e. if $x \in \text{spec}(A)$ we want $f^0(\xi)$ to go to 0 in $K^0(A_x)$. But $B \to A \to A_x$ is the same as $B \to B_y \to A_x$, where $y = f^{-1}(x) \in \text{spec}(B)$ and ξ goes to 0 in $K^0(B_y)$.

Conversely, if $\xi \in K^0(B)$ we want to show $f^0(\xi) \in J(A) \Rightarrow \xi \in J(B)$. Thus, given

$y \in \operatorname{spec}(B)$, we must show that ξ_y, the image of ξ in $K^0(B_y)$, is zero. Let $S = f(B - y)$; since f is injective, this multiplicative set in A does not contain zero. Moreover

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \downarrow & & \downarrow \\ B_y & \xrightarrow{S^{-1}f} & S^{-1}A \end{array}$$

is commutative. If $S^{-1}\xi$ is the image of $f^0(\xi)$ in $K^0(S^{-1}A)$, then $S^{-1}\xi = (S^{-1}f)^0(\xi_y)$. Since B_y is local, $\xi_y = n \in \mathbf{Z} = K^0(B_y)$, and hence also $S^{-1}\xi = n \in K^0(S^{-1}A)$. But our hypothesis implies that this n vanishes in K^0 of any localization of $S^{-1}A$, and hence $n = 0$ as desired.

Now for any $\xi \in K^0(A)$ we can write $\xi = \rho\xi + (\xi - \rho\xi)$, with $\rho\xi \in C(A)$, $\xi - \rho\xi \in J(A)$. We shall call $\rho\xi$ the *rank* of ξ, and often identify it with a function from spec(A) to $\mathbf{Z}$. In particular, $\rho(\gamma P) = \rho_P$ is the " rank " of the projective module, P. Expressions like " rank $\xi \geq r$ ", $r \in \mathbf{Z}$, make sense now, where we think of r as a constant function. If P is a projective A-module, rank $P \geq r$ is equivalent to " f-rank $P \geq r$ " in the sense of § 7.

Proposition **(15.5)**. — *Let* A *be a commutative ring and* P *a finitely generated projective* A-*module. Then the following conditions are equivalent:*

a) P *is faithful.*

b) rank $P \geq 1$.

c) *Every* $K^0(A)$-*module annihilated by* $\gamma_A P$ *is torsion.*

d) $1 \otimes \gamma_A P$ *generates the unit ideal in* $\mathbf{Q} \otimes_{\mathbf{Z}} K^0(A)$.

Proof. — *a)* $\Leftrightarrow$ *b)* because P is locally free, and a finitely generated projective module is faithful if, and only if, it is locally non zero.

b) $\Rightarrow$ *c)*. Since ρ_P is everywhere positive and bounded, there is a function $f \in C(A)$ with $f\rho_P = n > 0$. Hence $f\gamma P = n + j$, with $j \in J(A)$ nilpotent. Thus, modulo $\gamma P . K^0(A)$, n is nilpotent, so $n^m \in \gamma P . K^0(A)$ for some $m \geq 0$.

c) $\Rightarrow$ *b)*. If $x \in \operatorname{spec}(A)$ and $P_x = 0$, then $K^0(A_x) \cong \mathbf{Z}$ is a $K^0(A)$-module annihilated by γP.

c) $\Leftrightarrow$ *d)* is evident.

Proposition **(15.6)**. — *Let* A *be a commutative ring for which* max(A) *is a noetherian space of dimension* $\leq d$. *Then, if* $\xi \in K^0(A)$ *has rank* $\geq d$, $\xi = \gamma P$ *for some* P. *If* rank $P > d$ *and* $\gamma P = \gamma Q$, *then* $P \cong Q$.

Proof. — We can write $\xi = \gamma Q - \gamma A^n$. Then $\rho_Q = \rho\xi + n \geq d + n$. Hence, by Serre's Theorem (8.2), $Q \cong P \oplus A^n$, so $\xi = \gamma P$.

If $\gamma P = \gamma Q$ for some P and Q, then $P \oplus P' \cong Q \oplus P'$ for some projective P'. If rank $P > d$ we can invoke the Cancellation Theorem (9.3) and conclude that $P \cong Q$.

Chapter IV

APPLICATIONS

§ 16. Multiplicative inverses. Dedekind rings.

Throughout this section all modules are finitely generated right modules.

Proposition **(16.1)**. — *Let* A *be a commutative ring with* max(A) *a noetherian space of dimension* $d<\infty$, *and let* Λ *be a finite* A*-algebra. If* P *is a projective* Λ*-module such that* $1\otimes\gamma_\Lambda P$ *generates* $\mathbf{Q}\otimes_{\mathbf{Z}}K^0(\Lambda)$ *as a* $(\mathbf{Q}\otimes_{\mathbf{Z}}K^0(A))$*-module, then there is a projective* A*-module* Q, *such that* $Q\otimes_A P$ *is* Λ*-free.*

Proof. — $1\otimes\gamma_\Lambda\Lambda^n$ is a $\mathbf{Q}\otimes K^0(A)$ multiple of $1\otimes\gamma_\Lambda P$, and by choosing n large we can solve $1\otimes\gamma_\Lambda\Lambda^n=1\otimes\eta(\gamma_\Lambda P)$ with $\eta\in K^0(A)$. Hence $\gamma_\Lambda\Lambda^n-\eta.\gamma_\Lambda P$ is a torsion element in $K^0(\Lambda)$. If then we replace n by a multiple of n, we can achieve $\gamma_\Lambda\Lambda^n=\eta.\gamma_\Lambda P$. Let $x\in\max(A)$; $K^0(A_x)\cong\mathbf{Z}$ and the image there of η is $(\rho\eta)(x)$ (see § 15 for the definition of ρ). The equation above together with (6.6) (see also Example 3) in § 13) implies that $\Lambda_x^n\cong(\rho\eta)(x).P_x$, and this evidently implies $(\rho\eta)(x)>0$. Thus, if we replace n by a further multiple, if necessary, we can achieve $\rho\eta\geq d$. It then follows from (15.6) that $\eta=\gamma_A Q$, for some projective A-module Q. Our equation above then becomes $\gamma_\Lambda(\Lambda^n)=\gamma_\Lambda(Q\otimes_A P)$. Since, by this time, $n>d$, we can invoke the Cancellation Theorem (9.3) and conclude that $Q\otimes_A P\cong\Lambda^n$, as desired.

Corollary **(16.2)**. — *Let* A *be any commutative ring and* P *a faithful projective* A*-module. Then* $Q\otimes_A P$ *is* A*-free for some projective* A*-module* Q.

Proof. — Since P is induced from a finitely generated subring of A, it suffices to solve our problem there, so we may assume A a noetherian ring of finite Krull dimension. We can now invoke (16.1) provided $1\otimes\gamma P$ generates $\mathbf{Q}\otimes K^0(A)$, but the latter is a consequence of (15.4).

Let A be a commutative ring and $\mathfrak{m}$ the class of faithful projective A-modules. Write $P\sim Q$ if $P\otimes_A A^n\cong Q\otimes_A A^m$ for some $n, m>0$. Then $\sim$ is an equivalence relation respecting $\otimes_A$ so that

$$M(A)=\mathfrak{m}/\sim$$

is an abelian monoid with neutral element the class of the free modules. Corollary 16.2 says that M(A) is even a group.

Proposition **(16.3)**. — *Let* A *be a commutative ring for which* max(A) *is a finite dimensional noetherian space. Then*

$$M(A)=\mathbf{GL}(1, \mathbf{Q}\otimes_{\mathbf{Z}}K^0(A))/\mathbf{GL}(1, \mathbf{Q}).$$

Proof. — Let us write $B = \mathbf{Q}\otimes_{\mathbf{Z}} K^0(A)$, and $B^* = \mathbf{GL}(1, B)$. If $P \in \mathfrak{m}$, let μP denote its class in $M(A)$. By (15.4) $1\otimes\gamma P \in B^*$; let πP denote its class in $B^*/\mathbf{Q}^*$. Evidently $f: M(A) \to B^*/\mathbf{Q}^*$ by $f(\mu P) = \pi P$ is a well defined homomorphism. Suppose $\pi P = 1$; i.e. $1\otimes\gamma P \in \mathbf{Q}^*$. Since $K^0(A) = C(A)\oplus J(A)$ (15.4), we have $B = (\mathbf{Q}\otimes C(A))\oplus(\mathbf{Q}\otimes J(A))$, and writing $\gamma P = \rho_P + (\gamma P - \rho_P)$ we see that $1\otimes\gamma P = 1\otimes\rho_P \in \mathbf{Q}^*$. Hence $n(\gamma P - \rho_P) = 0$ for some $n>0$, so $n\gamma P = \gamma(A^n\otimes P) = m \in \mathbf{Z}$. Choosing n sufficiently large we conclude from (15.6) that $A^n\otimes P \cong A^m$, so $\mu P = 1$. This shows that f is a monomorphism.

To see that f is surjective consider an element of B^*. Modulo $\mathbf{Q}^*$ we can assume it has the form $1\otimes\xi$ with $\xi \in K^0(A)$ of rank $\geq \dim \max(A)$. Hence $\xi = \gamma P$ for some P by (15.6), and the class of $\xi \bmod \mathbf{Q}^*$ is $\pi P = f(\mu P)$.

The classical Steinitz-Chevalley theory [15] of modules over a Dedekind ring furnishes a familiar setting in which to illustrate the general shape of our theory.

We consider finitely generated torsion free (hence projective) modules P, over a Dedekind ring A. First $P \cong \mathfrak{a}\oplus F$, $\mathfrak{a}$ an ideal and F free — Serre's Theorem. If $P\oplus A \cong P'\oplus A$ then $P \cong P'$; if we required rank $P \geq 2$ this would be the Cancellation Theorem. The stronger conclusion here is possible only because of commutativity. If $\mathfrak{a}$ and $\mathfrak{b}$ are non zero ideals, $\mathfrak{a}\oplus\mathfrak{b} \cong A\oplus(\mathfrak{a}\mathfrak{b})$. As an equation in $K^0(A)$ we recover this from the nilpotency of $J(A)$ (Proposition 15.4): $(1-\gamma\mathfrak{a})(1-\gamma\mathfrak{b}) = 0$ in $K^0(A)$. We indicate below (Proposition 16.4) how to recover the actual isomorphism, as well as a variety of similar identities.

With these facts we see easily that, as a ring, $K^0(A) \cong \mathbf{Z}\oplus J(A)$ with $J(A)$ an ideal of square zero, additively isomorphic to the ideal class group G of A. Alternatively, if I is the augmentation ideal of the integral group ring $\mathbf{Z}G$, then $K^0(A) \cong \mathbf{Z}G/I^2$ as an augmented ring.

Let $\varphi: A \to L$ be the inclusion of A in its field of quotients. We propose now to interpret the exact sequence

$$K^1(A) \xrightarrow{\varphi^1} K^1(L) \to K^0(A, \varphi) \to K^0(A) \xrightarrow{\varphi^0} K^0(L).$$

The composite $K^0(A) \xrightarrow{\varphi^0} K^0(L) \underset{\cong}{\xrightarrow{\dim}} \mathbf{Z}$ is the augmentation, " rank ", with kernel $J(A) \cong G$. Next we recall that $K^0(A, \varphi)$ is built out of triples (P, α, Q) with P and Q A-projective, and $\alpha: L\otimes_A P \to L\otimes_A Q$ an L-isomorphism. Using the description of $K^0(A, \varphi)$ given in § 13 one can show easily that every element is represented by a triple $(\mathfrak{a}, u, A)$ with $\mathfrak{a}$ an ideal and $u \in L^*$, viewed as a homothetie of L. Moreover, the fractional ideal $\mathfrak{a}u$ is an invariant which defines an isomorphism of $K^0(A, \varphi)$ with the group of fractional ideals in L.

With this identification, $K^0(A, \varphi) \to G \cong \ker \varphi^0$ assigns to each ideal its class. Moreover, $K^1(L) \underset{\cong}{\xrightarrow{\det}} L^*$, and $K^1(L) \to K^0(A, \varphi)$ assigns to $u \in L^*$ the principal ideal Au. Thus, the kernel is $A^* = \mathbf{GL}(1, A) = \operatorname{Im} \varphi^1$. Finally, $\ker \varphi^1 = \mathbf{SL}(A)/\mathbf{E}(A)$ is the commutator quotient group of $\mathbf{SL}(A)$. We shall see in § 19 that if A is the ring of integers in a (finite) algebraic number field, then this group is finite.

Now let A be any commutative ring. If P is an A-module, and n a non negative integer, let nP denote a direct sum of n copies of P, and $P^{\otimes n}$ a tensor product of n copies of P. Also, if Q is another A-module, write $P+Q=P\oplus Q$ and $PQ=P\otimes_A Q$. With these conventions, if $f(T_1, \ldots, T_n)=\Sigma a_{i_1\ldots i_n} T_1^{i_1}\ldots T_n^{i_n}\in \mathbf{Z}[T_1, \ldots, T_n]$ is a polynomial with non negative coefficients, and if $P_1, \ldots, P_n$ are A-modules, we can write

$$f(P_1, \ldots, P_n)=\Sigma a_{i_1\ldots i_n} P_1^{\otimes i_1}\ldots P_n^{\otimes i_n}.$$

Proposition **(16.4)**. — *Let* A *be a commutative ring such that* max(A) *is a noetherian space of dimension* $<n$. *Then, if* $P_1, \ldots, P_n$ *are invertible* A*-modules and if* $S_i(T_1, \ldots, T_n)$ *is the* i^{th} *elementary symmetric function, we have*

$$\sum_{i \text{ even}} S_i(P_1, \ldots, P_n)=\sum_{i \text{ odd}} S_i(P_1, \ldots, P_n).$$

Proof. — $1-\gamma P_i\in J(A)$ for all i, so $\Pi_{i=1}^n(1-\gamma P_i)=0$ by (15.2). Thus the equation is valid after applying γ to it. Since the ranks of the two sides of the equation exceed dim max(A), (15.6) permits us to remove the γ.

§ 17. Some remarks on algebras.

For a ring Λ let Λ-mod and mod-Λ denote the categories of left, respectively, right, Λ-modules. A *generator* for such a category is a module whose homomorphic images suffice to generate any other module.

Let E be a right Λ-module and put $\Gamma=\mathrm{Hom}_\Lambda(E, E)$ and $E^*=\mathrm{Hom}_\Lambda(E, \Lambda)$. We are in the situation $({}_\Gamma E_\Lambda, {}_\Lambda E^*_\Gamma)$, and there are natural bimodule homomorphisms:

$$\begin{aligned}&E^*\otimes_\Gamma E\to\Lambda.\\ &E\otimes_\Lambda E^*\to\Gamma\end{aligned} \tag{1}$$

Moreover, we shall consider the functors:

$$\begin{aligned}&E^*\otimes_\Gamma \ : \Gamma\text{-mod}\to\Lambda\text{-mod}.\\ &E\otimes_\Lambda \ : \Lambda\text{-mod}\to\Gamma\text{-mod}\end{aligned} \tag{2}$$

The following basic result is due essentially to Morita [26], although the best exposition of these, and other matters in this section, is in Gabriel [21, Chapter V, § 1].

Theorem (Morita). — *Let* Λ *be a ring,* E *a right* Λ*-module,* $\Gamma=\mathrm{Hom}_\Lambda(E, E)$, *and* $E^*=\mathrm{Hom}_\Lambda(E, \Lambda)$. *Then the following conditions are equivalent:*

a) E *is a finitely generated projective generator for* mod-Λ.

b) *The homomorphisms* (1) *are isomorphisms.* (It suffices that they be epimorphisms.)

c) *The composites of the functors* (2) *are each naturally equivalent to the identity functors.*

In this case $\mathfrak{q}\leftrightarrow E\mathfrak{q}$ *defines a bijection between the two-sided ideals of* Λ *and the* Γ-Λ*-submodules of* E.

Conversely, if Λ *and* Γ *are rings and* Λ-mod *is equivalent to* Γ-mod, *then any such equivalence is isomorphic to one as in* (2) *above, with* E *determined up to* Γ-Λ*-isomorphism. Thus, the situation above is entirely symmetric with respect to* E *and* E^* *and to* Λ *and* Γ.

Now let A be a commutative ring and Λ an A-algebra. Then, in the above setting, Λ-mod is an " A-category " (i.e. all the Hom's are A-modules), likewise for the A-algebra Γ, and the functors in (2) are " A-functors " (i.e. induce A-homomorphisms on the Hom's). The converse part of the Morita Theorem remains true also in this sense, provided the given equivalence from Λ-mod to Γ-mod is assumed to be an A-functor. If Λ-mod and Γ-mod are A-equivalent we shall call Λ and Γ *Morita equivalent*, denoted $\Lambda \sim_M \Gamma$.

Let E be a finitely generated, projective right A-module. It is easy to see (e.g. using Proposition 15.5) that E is then a generator for mod-A $\Leftrightarrow$ E is faithful. Using the Morita Theorem we then see that, if $\Gamma = \mathrm{Hom}_A(E, E)$, $\Gamma \otimes_A \Lambda \cong \mathrm{Hom}_\Lambda(E \otimes_A \Lambda, E \otimes_A \Lambda) \sim_M \Lambda$. In this situation we shall write $\Lambda \sim_B (\Gamma \otimes_A \Lambda)$ and call the two algebras *Brauer equivalent.*

Suppose $\Lambda \sim_M \Lambda'$, say $\Lambda' = \mathrm{Hom}_\Lambda(P, P)$ with P a finitely generated projective generator for mod-Λ. Can we show that $\Lambda \sim_B \Lambda'$, i.e. that the two equivalence relations are the same? Suppose there is a faithful, finitely generated, projective A-module Q, such that $Q \otimes_A P \cong \Lambda^n \cong A^n \otimes_A \Lambda$. Then we have

$$\Lambda' \sim_B \mathrm{Hom}_A(Q, Q) \otimes_A \Lambda' \cong \mathrm{Hom}_\Lambda(Q \otimes_A P, Q \otimes_A P) \cong$$
$$\cong \mathrm{Hom}_\Lambda(\Lambda^n, \Lambda^n) \cong \mathrm{Hom}_A(A^n, A^n) \otimes_A \Lambda \sim_B \Lambda.$$

Thus, modulo the existence of Q, $\sim_M$ and $\sim_B$ agree. On the other hand, Proposition 16.1 gives a criterion for the existence of Q which we will verify under suitable conditions below.

Let Λ be an A-algebra, Λ^0 the opposite algebra, and $\Lambda^e = \Lambda \otimes_A \Lambda^0$. To avoid confusion we shall use E to denote Λ viewed as a right A-module. Λ is called an *Azumaya algebra* (= *central separable algebra* in [4], see also [21, Chapter V, § 1]) if it satisfies the following equivalent conditions:

(i) Λ is a projective generator for Λ^e-mod.

(ii) *a)* E is a faithful, finitely generated, projective A-module.

b) $\Lambda^e \cong \mathrm{Hom}_A(E, E)$ as A-algebras.

In this case every ideal of Λ has the form $\mathfrak{q}\Lambda$ for some ideal $\mathfrak{q}$ of A [4, Corollary 3.2].

We propose now to consider the $K^*(A, \mathfrak{q})$-module $K^*(\Lambda, \mathfrak{q}\Lambda)$, with Λ an Azumaya algebra. For this purpose we use the diagram of functors

$$\begin{array}{ccccc} & & \text{mod-}A & & \text{mod-}\Lambda \\ & & \uparrow {\scriptstyle \otimes_{\Lambda^e} E} & & \uparrow {\scriptstyle \otimes_{(\Lambda \otimes_A \Lambda^e)} (\Lambda \otimes_A E)} \\ \text{mod-}\Lambda & \xrightarrow{\otimes_A \Lambda^0} & \text{mod-}\Lambda^e & \xrightarrow{\otimes_A \Lambda} & \text{mod-}(\Lambda \otimes_A \Lambda^e), \end{array}$$

in which, by the Morita Theorem and condition (ii) *a)* above, the vertical arrows are equivalences. We obtain thus $K^*(A, \mathfrak{q})$-homomorphisms

$$(*) \qquad K^*(\Lambda, \mathfrak{q}\Lambda) \xrightarrow{f} K^*(A, \mathfrak{q}) \xrightarrow{g} K^*(\Lambda, \mathfrak{q}\Lambda),$$

To compute them, let M be a right Λ-module. Then

$$(M\otimes_A\Lambda^e)\otimes_{(\Lambda\otimes_A\Lambda^e)}(\Lambda\otimes_A E)\cong(M\otimes_\Lambda\Lambda)\otimes_\Lambda(\Lambda^e\otimes_{\Lambda^e}E)\cong M\otimes_\Lambda E.$$

It follows that gf is the homothetie defined by $\gamma_A(E)\in K^0(A)$. Next we note that $(M\otimes_A\Lambda^0)\otimes_{\Lambda^e}E\cong M\otimes_\Lambda(\Lambda\otimes_A\Lambda^0)\otimes_{\Lambda^e}E\cong M\otimes_\Lambda E\cong M$ (as A-modules), recalling that $E=\Lambda$ viewed as a left Λ^e-, right A-module. Hence f is the " restriction " map obtained by viewing Λ-modules as A-modules. In particular, $\gamma_A(E)\in \mathrm{Im}(f)$.

Now by Proposition 15.5, $1\otimes\gamma_A(P)$ is a unit in $\mathbf{Q}\otimes_{\mathbf{Z}}K^0(A)$ for any faithful, finitely generated, projective module P. Hence, using (ii) *a)*, we see that $\mathbf{Q}\otimes f$ and $\mathbf{Q}\otimes gf$ are isomorphisms, and we have proved:

Theorem **(17.1)**. — *Let* A *be a commutative ring and* Λ *an Azumaya* A-*algebra. Then for all ideals* $\mathfrak{q}$ *in* A, $\mathbf{Q}\otimes_{\mathbf{Z}}K^*(\Lambda, \mathfrak{q}\Lambda)$ *is a free* $\mathbf{Q}\otimes_{\mathbf{Z}}K^*(A, \mathfrak{q})$-*module generated by* $1\otimes\gamma_\Lambda P$ *for any finitely generated faithful projective* Λ-*module* P.

Using Proposition 16.1 and the discussion above we further obtain:

Corollary **(17.2)**. — *Let* A *be a commutative ring for which* max(A) *is a finite dimensional noetherian space, and let* Λ *be an Azumaya* A-*algebra. Then:*

1) *If* P *is a faithful, finitely generated, projective* Λ-*module, there is an* A-*module* Q *of the same type, for which* $P\otimes_A Q$ *is a free* Λ-*module.*

2) *The class of* Λ *in the Brauer group of* A (see [4]) *depends only on the* A-*category* Λ-mod.

Theorem 17.1 is not very useful for computing $K^0(\Lambda)$ since, in number theoretic contexts, all the interesting invariants are torsion. On the other hand the theorem is no longer true if we remove the " $\mathbf{Q}\otimes_{\mathbf{Z}}$ ". To see this, let $F=\mathbf{Q}(\sqrt{p})$ with p a prime $\equiv -1 \bmod 4$, let I denote the ideal group of F, P the subgroup of principal ideals, and $P^+\subset P$ those principal ideals generated by totally positive elements. If A is the ring of integers in F, then $\sqrt{p}A\in P, \notin P^+$. For if $\sqrt{p}A$ had a totally positive generator there would be a unit in A of norm -1, and by [23, p. 288] there is no such unit. Hence $[P:P^+]>1$, in fact, $=2$. Let Σ be the standard quaternion algebra over F and Λ a maximal order in Σ. Then Σ is unramified except at ∞, so Λ is an Azumaya A-algebra. On the other hand, Eichler [19] has completely determined $K^0(\Lambda)$ (see Swan [33, Theorem 2]). Namely, $K^0(\Lambda)\cong\mathbf{Z}\oplus I/P^+$. Hence $K^0(\Lambda)$ is not isomorphic to $K^0(A)\cong\mathbf{Z}\oplus I/P$ (see § 16). This example was pointed out by Serre.

A further amusing example in this connection is the following: Let Σ be the quaternion algebra over $\mathbf{Q}$ ramified at p and ∞, p a prime $\equiv -1 \bmod 4$ and sufficiently large (e.g. $p=11$ will do). Σ has a basis $1, i, j, k$ with $i^2=-1, j^2=k^2=-p$, and $ij=k=-ji$. The result of Eichler-Swan quoted above shows that, if Λ is a maximal order in Σ, then $\mathbf{M}(n, \Lambda)$ is a principal ideal ring for all $n\geq 2$. On the other hand, according to Eichler [18, Satz 2], the class number of Λ itself is >1.

Finally suppose A is a field and Λ is a central simple A-algebra. Then $K^0(\Lambda)\cong\mathbf{Z}$, so if B is any field extension of A, $K^0(\Lambda)\to K^0(B\otimes_A\Lambda)$ is a monomorphism. What about K^1? Dieudonné [17] shows that $K^1(\Lambda)$ is the commutator quotient group of $\Lambda^* = \mathbf{GL}(1, \Lambda)$. Therefore, if B is a splitting field for Λ, we have homomorphisms

$$\Lambda^* \to K^1(\Lambda) \to K^1(B \otimes_A \Lambda) \cong B^*,$$

and it is easy to see that the resulting map $\Lambda^* \to B^*$ is the reduced norm (see Bourbaki, [11, § 12]) and has image, therefore, in A^*. Combining these remarks with the fact that every field is contained in a splitting field, we have:

Proposition (**17.3**). — *If* Λ *is a central separable algebra over a field* A, $K^1(\Lambda) \to K^1(B \otimes_A \Lambda)$ *is a monomorphism for all field extensions* B *if, and only if, the commutator subgroup of* Λ^* *is the kernel of the reduced norm.*

By a theorem of Wang [35], this is the case if A is a number field.

§ 18. Finite generation of K.

We propose to show (Theorem 18.6 below) that if Λ is a finite **Z**-algebra, then $K^*(\Lambda, \mathfrak{q})$ is a finitely generated abelian group for all ideals $\mathfrak{q}$.

Lemma (**18.1**). — a) *Let* Λ *be a ring. If* N *is an ideal in* rad Λ, *then* $K^0(\Lambda) \to K^0(\Lambda/N)$ *is a monomorphism, and even an isomorphism if* Λ *is* N*-adic complete* (e.g. *if* N *is nilpotent*).

b) *If* Λ *is a semi-local ring,* $K^0(\Lambda)$ *is a free abelian group of finite rank.*

Proof. — *a)* Suppose $\gamma_\Lambda P - \gamma_\Lambda Q \in \ker(K^0(\Lambda) \to K^0(\Lambda/N))$. After adding a free module to P and Q we can assume that $P/PN \cong Q/QN$. Such an isomorphism lifts to a homomorphism $f: P \to Q$. f is surjective mod N, hence surjective, by Nakayama. Hence $\ker f$, being a direct summand of P, is finitely generated. Since $\ker f$ is zero mod N, it too is zero, by Nakayama.

Suppose now that Λ is N-adic complete, and let P be a projective (Λ/N)-module, say $P \oplus Q = (\Lambda/N)^n$. If $\Gamma = \mathrm{Hom}_\Lambda(\Lambda^n, \Lambda^n) = \mathbf{M}(n, \Lambda)$, then $\Gamma/N\Gamma = \mathbf{M}(n, \Lambda/N)$. Let $e \in \Gamma/N\Gamma$ be an idempotent projection onto P. Since Γ is $(N\Gamma)$-adic complete, e lifts to an idempotent $e' \in \Gamma$ (see e.g., [16, Lemma 77.4]). Now $P' = \mathrm{Im}\, e'$ is a direct summand of Λ^n covering P, so $\gamma_{\Lambda/N} P$ is the image of $\gamma_\Lambda P'$.

b) If Λ is semi-local, $\Lambda/\mathrm{rad}\,\Lambda$ is a finite product of simple Artin rings. Hence $K^0(\Lambda/\mathrm{rad}\,\Lambda)$ is free abelian, of rank equal to the number of simple factors of $\Lambda/\mathrm{rad}\,\Lambda$. By *a)* $K^0(\Lambda) \to K^0(\Lambda/\mathrm{rad}\,\Lambda)$ is injective, so our conclusion follows.

Proposition (**18.2**). — *Let* A *be a noetherian integral domain of Krull dimension one with field of quotients* L. *Let* Λ *be a finite* A*-algebra,* N *the nil radical of* Λ, *and* T *the torsion* A*-submodule of* Λ/N. *Then* N *is nilpotent,* T *is a semi-simple Artin ring, and* $\Lambda/N = T \times \Gamma$ *(product of rings), where* Γ *is an* A*-order in the semi-simple* L*-algebra* $L \otimes_A \Gamma$.

Proof. — $L \otimes_A N$ is a nil, hence nilpotent, ideal in the finite dimensional L-algebra $L \otimes_A \Lambda$. Hence some power of N lies in the torsion submodule of Λ and is therefore a nil ideal of finite length. Therefore some further power of N is zero.

For the rest we may assume $N = 0$, i.e. $\Lambda = \Lambda/N$. Regard T as an A-algebra, possibly without identity. If $J = \mathrm{rad}\, T$, J can be described as the intersection of all ker$f$, where f is a T-homomorphism into a simple right T-module S, such that $ST=S$. (Note that S must be an A-module, and f compatible with this structure, in particular.) Suppose $g: T \to T$ is a right T-endomorphism and $f: T \to S$ as above. Then $fg: T \to S$ so $J \subset \ker fg$; i.e. $g(J) \subset \ker f$. Letting f vary we see that $g(J) \subset J$. Now letting g be left multiplication by an element of Λ we see that J is a left Λ-ideal (using the obvious fact that T is an ideal of Λ). Similarly, J is a right ideal. However, T has finite length as an A-algebra, so J is nilpotent. But Λ now has no nilpotent ideals $\neq 0$, so $J=0$. Hence T is semi-simple, so it has an identity element, e. If $a \in \Lambda$, then $ae \in T$ so $ae = eae$. Similarly $ea = eae$, so e is central. Thus $\Lambda = T \times \Gamma$ where $\Gamma = (1-e)\Lambda$. Since Γ is torsion free with zero nil radical, $\Gamma \subset L \otimes_A \Gamma$, and $L \otimes_A \Gamma$ is a semi-simple L-algebra.

Lemma **(18.3)**. — *Let Λ be a ring and N a nilpotent ideal finitely generated as a* **Z**-*module. Then, for all $n \geq 1$, every subgroup of* $\mathbf{GL}(n, \Lambda, N)$ *is a finitely generated group.*

Proof. — Induction on m, where $N^m = 0$, reduces us immediately to the case $N^2 = 0$. Then $\mathbf{GL}(n, \Lambda, N)$ consists of all $1+a$ where a is an $n \times n$ matrix with coordinates in N. If $1+a'$ is another, then $(1+a)(1+a') = 1+(a+a')$, so $\mathbf{GL}(n, \Lambda, N) \cong N$ (the additive group).

Theorem 18.7 below will be proved by a reduction to the following classical results:

Lemma **(18.4)**. — *Let Λ be an order in a finite semi-simple* **Q**-*algebra* $\Sigma = \mathbf{Q} \otimes_{\mathbf{Z}} \Lambda$.

0) (Jordan-Zassenhaus, see [37]). *If* M *is a finitely generated Σ-module, there are only finitely many isomorphism types of finitely generated Λ-submodules of* M.

1) (See Siegel [31] or Borel-Harish-Chandra [9]) $\mathbf{GL}(n, \Lambda)$ *is finitely generated for all $n \geq 1$. Moreover, if Σ is simple, the subgroup of elements of reduced norm* 1 *in* $\mathbf{GL}(n, \Lambda)$ *is likewise finitely generated.*

Proposition **(18.5)**. — *Let Λ be a finite* **Z**-*algebra, and let* $\mathfrak{q}$ *be an ideal in Λ. Then* $\mathbf{GL}(n, \Lambda, \mathfrak{q})$ *is finitely generated for all $n \geq 1$.*

Proof. — If N is the nil radical of Λ, then $\mathbf{GL}(n, \Lambda, \mathfrak{q}) \to \mathbf{GL}(n, \Lambda/N, \mathfrak{q}(\Lambda/N))$ is surjective, and $\mathbf{GL}(n, \Lambda, \mathfrak{q}) \cap \mathbf{GL}(n, \Lambda, N)$ is finitely generated by (18.3). Hence we can reduce to the case $N=0$. Then $\Lambda = T \times \Gamma$ as in (18.2) and $\mathbf{GL}(n, \Lambda, \mathfrak{q})$ splits likewise into a product. It suffices then to treat T and Γ separately, and T, being finite, causes no problem. The result for Γ is a consequence of Siegel's theorem above (see Lemma 19.4 below).

Theorem **(18.6)**. — *Let Λ be a finite* **Z**-*algebra and* $\mathfrak{q}$ *an ideal in Λ. Then* $K^0(\Lambda)$ *and* $K^1(\Lambda, \mathfrak{q})$ *are finitely generated abelian groups.*

Proof. — Since $\dim \max(\mathbf{Z}) = 1$ it follows from (11.2) *b)* that $K^1(\Lambda, \mathfrak{q})$ is a homomorphic image of $\mathbf{GL}(2, \Lambda, \mathfrak{q})$, and (18.5) says the latter is finitely generated. Now for K^0:

Let N be the nil radical of Λ and write $\Lambda/N = T \times \Gamma$ as in (18.2). Then, by (18.1), $K^0(\Lambda) \to K^0(\Lambda/N) = K^0(T) \oplus K^0(\Gamma)$ is an isomorphism, and $K^0(T)$ is free abelian of finite rank. It remains to show $K^0(\Gamma)$ finitely generated. Let Γ' be a maximal order in $\mathbf{Q} \otimes \Gamma$ containing Γ. Then Γ' is hereditary [3], so every projective Γ'-module is isomorphic to a direct sum of right ideals [14, Chap. I, Theorem 5.3]. By Jordan-Zassenhaus (18.4, 0)) there are only finitely many of these, up to isomorphism, so $K^0(\Gamma')$ is finitely generated. Now Γ has finite index, say m, in Γ'. Let S be the multiplicative set of integers prime to m. Then $S^{-1}\Gamma$ is semi-local, so $K^0(S^{-1}\Gamma)$ is free abelian of finite rank (18.1, *b*)). It will be sufficient for the theorem, therefore, to show that the homomorphism

$$K^0(\Gamma)\to K^0(\Gamma')\oplus K^0(S^{-1}\Gamma),$$

induced by the inclusions, has finite kernel H. Let $\gamma_\Gamma P-\gamma_\Gamma\Gamma^n\in H$; we may assume $n\geq 2$. Then $\gamma_{\Gamma'}(\Gamma'\otimes_\Gamma P)=\gamma_{\Gamma'}(\Gamma'^n)$ and $\gamma_{S^{-1}\Gamma}(S^{-1}P)=\gamma_{S^{-1}\Gamma}((S^{-1}\Gamma)^n)$. Since Γ' and $S^{-1}\Gamma$ are algebras over rings with maximal spectra of dimensions 1 and 0, respectively, and since $n\geq 2$, (15.6) tells us that $P\otimes_\Gamma\Gamma'\cong(\Gamma')^n$ and $S^{-1}P\cong(S^{-1}\Gamma)^n$.

Let $x\in\max(\mathbf{Z})$. If $m\in x$ then P_x is a localization of $S^{-1}P$, hence free. If $m\notin x$ then P_x is a localization of $P\otimes_\Gamma\Gamma'$, hence free. Thus P_x is free of rank n for all $x\in\max(\mathbf{Z})$, so we can apply Serre's Theorem (8.2) and write $P\cong Q\oplus\Gamma^{n-1}$, with Q locally free of rank one, by (6.6). But then Q is isomorphic to an ideal in Γ, and again, by Jordan-Zassenhaus (18.4, 0)), there are only finitely many such Q up to isomorphism. Since $\gamma_\Gamma P-\gamma_\Gamma\Gamma^n=\gamma_\Gamma Q-\gamma_\Gamma\Gamma$, this proves H is finite, as claimed.

§ 19. A finiteness theorem for SL (n, Λ).

Let $\Sigma=\Pi_i\Sigma_i$ with Σ_i a central simple algebra over a finite algebraic number field C_i. The reduced norms (see [11, § 12]) give homomorphisms $\mathbf{GL}(n,\Sigma_i)\to C_i^*=\mathbf{GL}(1,C_i)$ (compatible with the inclusions $\mathbf{GL}(n)\subset\mathbf{GL}(n+1)$), and their product defines a homomorphism $\mathbf{GL}(n,\Sigma)\to C^*$, where $C=\Pi_iC_i=\text{center }\Sigma$. We shall call this also the reduced norm, and denote its kernel by $\mathbf{SL}(n,\Sigma)$. If Λ is an order in Σ and $\mathfrak{q}$ an ideal in Λ we shall write $\mathbf{SL}(n,\Lambda)=\mathbf{GL}(n,\Lambda)\cap\mathbf{SL}(n,\Sigma)$, and $\mathbf{SL}(n,\Lambda,\mathfrak{q})=\mathbf{GL}(n,\Lambda,\mathfrak{q})\cap\mathbf{SL}(n,\Sigma)$.

Theorem (**19.1**). — *Let Σ be a semi-simple algebra finite over $\mathbf{Q}$, let Λ be an order in Σ, and let $\mathbf{SL}(n,\Lambda)$ denote the elements of reduced norm one in $\mathbf{GL}(n,\Lambda)$ (in the sense defined above). Then there is an integer $n_0=n_0(\Sigma)$ such that, for all $n\geq n_0$ and for all ideals $\mathfrak{q}$ in Λ, $\mathbf{SL}(n,\Lambda,\mathfrak{q})/\mathbf{E}(n,\Lambda,\mathfrak{q})$ is finite.*

We shall begin by deriving a reformulation of this theorem which will be useful in its proof. The next two sections are devoted to some of its applications.

Lemma (**19.2**). — *If $\mathfrak{q}$ is an ideal in Λ, there is another ideal $\mathfrak{q}'$ for which $\mathfrak{q}\cap\mathfrak{q}'=0$ and $\Lambda/(\mathfrak{q}+\mathfrak{q}')$ is finite.*

Proof. — $\Sigma=\mathfrak{q}\Sigma\oplus\Sigma'$, being semi-simple, and $\mathfrak{q}'=\Sigma'\cap\Lambda$ clearly serves our purpose.

Lemma (**19.3**). — *If $\mathfrak{q}$ and $\mathfrak{q}'$ are ideals with $\mathfrak{q}\cap\mathfrak{q}'=0$, then*

$$\mathbf{GL}(n,\Lambda,\mathfrak{q}+\mathfrak{q}')=\mathbf{GL}(n,\Lambda,\mathfrak{q})\times\mathbf{GL}(n,\Lambda,\mathfrak{q}')$$

(direct product), and similarly for $\mathbf{SL}(n,\Lambda,\mathfrak{q}+\mathfrak{q}')$ and $\mathbf{E}(n,\Lambda,\mathfrak{q}+\mathfrak{q}')$. In particular, $K^1(\Lambda,\mathfrak{q}+\mathfrak{q}')=K^1(\Lambda,\mathfrak{q})\oplus K^1(\Lambda,\mathfrak{q}')$.

Proof. — If $1+q+q' \in \mathbf{GL}(n, \Lambda, \mathfrak{q}+\mathfrak{q}')$, where q and q' have coordinates in $\mathfrak{q}$ and $\mathfrak{q}'$, respectively, then $1+q+q'=(1+q)(1+q') \in \mathbf{GL}(n, \Lambda, \mathfrak{q}) \times \mathbf{GL}(n, \Lambda, \mathfrak{q}')$, since $qq'=0=q'q$. The conclusion for **SL** follows from the factorwise definition of **SL**, and for **E** it follows by applying the reasoning above to its generators.

Lemma (**19.4**). — $\mathbf{GL}(n, \Lambda, \mathfrak{q})$ *and* $\mathbf{SL}(n, \Lambda, \mathfrak{q})$ *are finitely generated groups for all* n *and* $\mathfrak{q}$.

Proof. — Lemmas 19.2 and 19.3 permit us to assume $\Lambda/\mathfrak{q}$ is finite, in which case $\mathbf{GL}(n, \Lambda, \mathfrak{q})$ has finite index in $\mathbf{GL}(n, \Lambda)$, so it suffices to show the latter finitely generated. If Γ is a maximal order containing Λ, then $m\Gamma \subset \Lambda$ for some $m>0$, so $\mathbf{GL}(n, \Gamma, m\Gamma) \subset \mathbf{GL}(n, \Lambda)$, showing that $\mathbf{GL}(n, \Lambda)$ has finite index in $\mathbf{GL}(n, \Gamma)$. Since Γ is a product of maximal orders in the simple factors of Σ, finite generation of $\mathbf{GL}(n, \Gamma)$ follows from Siegel's theorem (Lemma 18.4, 1)). Exactly the same proof applies to **SL**.

Now consider the direct system.

$$\ldots \mathbf{SL}(n, \Lambda, \mathfrak{q})/\mathbf{E}(n, \Lambda, \mathfrak{q}) \to \mathbf{SL}(n+1, \Lambda, \mathfrak{q})/\mathbf{E}(n+1, \Lambda, \mathfrak{q}) \to \ldots \quad \text{with limit} \quad \mathbf{SL}(\Lambda, \mathfrak{q})/\mathbf{E}(\Lambda, \mathfrak{q}).$$

Thanks to Theorem 11.1 we can apply Theorem 4.2 to Λ with $n=2$. Hence we know from (4.2, *b*)) that the maps above are surjective for $n \geq 2$, and, from (4.2, *f*)) and (19.4), that the terms are finitely generated abelian groups for $n \geq 4$. Consequently, since finitely generated abelian groups are noetherian, the system stabilizes; i.e. the maps are eventually all isomorphisms. (Indeed, the conjecture of § 11 alleges they are isomorphisms already for $n \geq 3$. If true, one could take $n_0=3$ in Theorem 19.1, as the proof will show.)

By the theorem of Wang [35] (see Proposition 17.3) $\mathbf{SL}(n, \Sigma)$ is the commutator subgroup of $\mathbf{GL}(n, \Sigma)$, and, by Dieudonné [17] (see Proposition 5.1, *b*)) the latter is just $\mathbf{E}(n, \Sigma)$ for $n \geq 2$. Thus the reduced norm induces a *monomorphism*

$$K^1(\Sigma) = \mathbf{GL}(\Sigma)/\mathbf{E}(\Sigma) \to C^*.$$

Moreover, the inclusion $\mathbf{GL}(\Lambda, \mathfrak{q}) \subset \mathbf{GL}(\Sigma)$ induces an exact sequence

$$\begin{array}{ccccccc} 0 \to \mathbf{SL}(\Lambda, \mathfrak{q})/\mathbf{E}(\Lambda, \mathfrak{q}) & \to & \mathbf{GL}(\Lambda, \mathfrak{q})/\mathbf{E}(\Lambda, \mathfrak{q}) & \to & \mathbf{GL}(\Sigma)/\mathbf{E}(\Sigma) \\ & & \| & & \| \\ & & K^1(\Lambda, \mathfrak{q}) & \longrightarrow & K^1(\Sigma) \end{array}$$

The next corollary summarizes some of these remarks:

Corollary (**19.5**). — *The following conditions on* Λ *and* $\mathfrak{q}$ *are equivalent:*

(i) *There exists an* $n_0 \geq 2$ *such that* $\mathbf{SL}(n_0, \Lambda, \mathfrak{q})/\mathbf{E}(n_0, \Lambda, \mathfrak{q})$ *is finite* (resp. *trivial*).

(ii) *For the same* n_0, $\mathbf{SL}(n, \Lambda, \mathfrak{q})/\mathbf{E}(n, \Lambda, \mathfrak{q})$ *is finite* (resp. *trivial*) *for all* $n \geq n_0$.

(iii) $\mathbf{SL}(\Lambda, \mathfrak{q})/\mathbf{E}(\Lambda, \mathfrak{q})$ *is finite* (resp. *trivial*).

(iv) $K^1(\Lambda, \mathfrak{q}) \to K^1(\Sigma)$ *has finite* (resp. *trivial*) *kernel.*

Remark. — As noted above, the conjecture of § 11 asserts that $n_0=3$ already suffices in the corollary.

Except for the dependence of n_0 only on Σ, Theorem 19.1 is now seen to be contained in the following result, of which the last assertion has already been noted above.

Theorem (**19.6**). — *Let* Λ *be an order a semi-simple algebra* Σ, *finite, over* $\mathbf{Q}$, *with center* C. *Then, for any ideal* $\mathfrak{q}$ *in* Λ,

$$K^1(\Lambda, \mathfrak{q}) \to K^1(\Sigma)$$

has finite kernel, and its image is isomorphic to the image of $\mathbf{GL}(2, \Lambda, \mathfrak{q})$ *in* C^* *under the reduced norm.*

The following sequence of lemmas will permit various reductions in the proof of this theorem.

Lemma (**19.7**). — *Let* G *be a group and* R *a normal subgroup. Then, if* $\pi = G/R$ *and* $G/[G, G]$ *are finite, so also is* $G/[G, R]$.

Proof. — $R/[G, R] \to G/[G, G]$ has finite image, so it suffices to see that it has finite kernel. But this is just the second map in the exact sequence

$$H_2(\pi) \to H_0(\pi, H_1(R)) \to H_1(G)$$

which comes from the Hochschild-Serre spectral sequence in homology with integral coefficients (see [14, XVI, § 6, (4 *a*)]) for the group extension $1 \to R \to G \to \pi \to 1$. Since π is finite, so is $H_2(\pi)$, and this proves the lemma.

Corollary (**19.8**). — *If* $\Lambda/\mathfrak{q}$ *is finite, then so also is* $\mathbf{E}(n, \Lambda)/\mathbf{E}(n, \Lambda, \mathfrak{q})$ *for all* $n \geq 3$.

Proof. — Put $G = \mathbf{E}(n, \Lambda)$ and $R = \mathbf{GL}(n, \Lambda, \mathfrak{q}) \cap \mathbf{E}(n, \Lambda)$. Then G/R is finite, and $G = [G, G]$, by (1.5, (i)). By (4.2, *d*)) $\mathbf{E}(n, \Lambda, \mathfrak{q}) = [G, R]$, so the corollary now follows from (19.7).

Corollary (**19.9**). — *If, for some* $n_0 \geq 3$, $\mathbf{SL}(n_0, \Lambda)/\mathbf{E}(n_0, \Lambda)$ *is finite, then* $\mathbf{SL}(n, \Lambda, \mathfrak{q})/\mathbf{E}(n, \Lambda, \mathfrak{q})$ *is finite for all* $\mathfrak{q}$ *and all* $n \geq n_0$.

Proof. — By (19.2) and (19.3) the conclusion above for all $\mathfrak{q}$ follows once we know it for $\mathfrak{q}$ with $\Lambda/\mathfrak{q}$ finite, so we now assume this. If $n \geq n_0$, then the finiteness of $\mathbf{SL}(n, \Lambda)/\mathbf{E}(n, \Lambda)$ follows from our hypothesis and (19.5), and that of $\mathbf{E}(n, \Lambda)/\mathbf{E}(n, \Lambda, \mathfrak{q})$ from (19.8) above. Hence $\mathbf{SL}(n, \Lambda)/\mathbf{E}(n, \Lambda, \mathfrak{q})$ is finite, and this proves the corollary.

Lemma (**19.10**). — *Let* Λ *and* Λ' *be two orders in* Σ, *and let* $\mathfrak{q}$ *be an ideal in* Λ *for which* $\Lambda/\mathfrak{q}$ *is finite. Then there is an ideal* $\mathfrak{q}'$ *in* Λ' *with* $\Lambda'/\mathfrak{q}'$ *finite such that, for all* $n \geq 4$, $\mathbf{E}(n, \Lambda', \mathfrak{q}') \subset \mathbf{E}(n, \Lambda, \mathfrak{q})$.

Proof. — $m\Lambda' \subset \Lambda$ for some $m > 0$, so $\mathfrak{q}_1 = \Lambda' m \mathfrak{q} m \Lambda'$ is a Λ' ideal contained in $\mathfrak{q}$, and clearly $\Lambda'/\mathfrak{q}_1$ is finite. Let $H = \mathbf{GL}(n, \Lambda, \mathfrak{q}_1) = \mathbf{GL}(n, \Lambda', \mathfrak{q}_1)$. Then from (1.3) (using $n \geq 3$) and (4.2, *f*)) (using $n \geq 4$) we have:

$$\mathbf{E}(n, \Lambda', \mathfrak{q}_1^2) \subset [\mathbf{E}(n, \Lambda', \mathfrak{q}_1), \mathbf{E}(n, \Lambda', \mathfrak{q}_1)] \subset [H, H]$$
$$\subset \mathbf{E}(n, \Lambda, \mathfrak{q}_1) \subset \mathbf{E}(n, \Lambda, \mathfrak{q}).$$

Hence $\mathfrak{q}' = (\mathfrak{q}_1)^2$ serves our purpose.

Corollary (**19.11**). — *If, for some order* Λ' *in* Σ, $K^1(\Lambda') \to K^1(\Sigma)$ *has finite kernel, then there is an* n_0 *such that, for all* $n \geq n_0$, *for all orders* Λ, *and for all ideals* $\mathfrak{q}$ *in* Λ, $\mathbf{SL}(n, \Lambda, \mathfrak{q})/\mathbf{E}(n, \Lambda, \mathfrak{q})$ *is finite.*

Proof. — Our hypothesis and (19.5) imply $\mathbf{SL}(n_0, \Lambda')/\mathbf{E}(n_0, \Lambda')$ is finite for some $n_0 \geq 3$, so our conclusions for Λ' follow from (19.9). Taking $n_0 \geq 4$, we can apply Lemma 19.10 to any other Λ, and conclude that $\mathbf{E}(n_0, \Lambda', \mathfrak{q}') \subset \mathbf{E}(n_0, \Lambda)$ for some $\mathfrak{q}'$ with $\Lambda'/\mathfrak{q}'$ finite. Since $\mathbf{SL}(n_0, \Lambda')/\mathbf{E}(n_0, \Lambda, \mathfrak{q}')$ is finite, $\mathbf{E}(n_0, \Lambda) \cap \mathbf{SL}(n_0, \Lambda')$ has finite index in $\mathbf{SL}(n_0, \Lambda) \cap \mathbf{SL}(n_0, \Lambda')$. But the latter contains $\mathbf{SL}(n_0, \Lambda \cap \Lambda')$, which has finite index in $\mathbf{SL}(n_0, \Lambda)$. Therefore $\mathbf{SL}(n_0, \Lambda)/\mathbf{E}(n_0, \Lambda)$ is finite, and the corollary now follows from (19.9).

Corollary 19.11 reduces Theorems 19.1 and 19.6 to showing that $K^1(\Lambda) \to K^1(\Sigma)$ has a finite kernel for some Λ. Since $K^1(\Lambda)$ is a finitely generated abelian group (Theorem 18.6), we need only show the kernel is torsion, and for this the following criterion is useful. It is here that K^1 effectively intervenes in the proof.

Proposition **(19.12)**. — *Let* $A \subset B$ *be commutative rings with* B *finitely generated and projective as an* A*-module. Then, if* Λ *is a finite* A*-algebra, the kernel of* $K^1(\Lambda) \to K^1(B \otimes_A \Lambda)$ *is torsion.*

Proof. — The nature of B provides us with a homomorphism $K^1(B \otimes_A \Lambda) \to K^1(\Lambda)$ (see § 14) whose composite with the one above is the homothetie of the $K^0(A)$-module $K^1(\Lambda)$, defined by $\gamma_A(B) \in K^0(A)$. The Proposition now results from (15.6), which tells us that anything killed by $\gamma_A(B)$ is torsion.

We come now to the proof that $\ker(K^1(\Lambda) \to K^1(\Sigma))$ is torsion. Passing to $A \otimes_{\mathbf{Z}} \Lambda \subset L \otimes_{\mathbf{Q}} \Sigma$, where A is the ring of integers in a splitting field L for Σ, we can reduce, thanks to (19.12) above, to the case where Σ is split. By (19.11), moreover, we may take for Λ a maximal order. But then Λ is a product of maximal orders in the simple factors of Σ, and everything decomposes accordingly, so we reduce further to the case $\Sigma = \mathrm{End}_L(V)$, V a vector space over the number field L, and then (see [3] or [15]) $\Lambda = \mathrm{End}_A(P)$, with P a projective module over the ring A of integers in L. Now the Morita theorem (§ 17) gives us equivalences from the categories of A-modules to Λ-modules ($\otimes_A P$), and from L-modules to Σ-modules ($\otimes_L V$), which commute with the passages from A to L and Λ to Σ, respectively. Thus we have

$$\begin{array}{ccc} K^1(A) & \to & K^1(L) \\ \wr\| & & \wr\| \\ K^1(\Lambda) & \to & K^1(\Sigma) \end{array}$$

commutative, and it suffices, finally, to show that $\ker(K^1(A) \to K^1(L))$ is torsion. Using (19.12) again, we see that this is a consequence of the following proposition:

Proposition **(19.13)**. — *Let* A *be the ring of integers in a finite extension* L *of* $\mathbf{Q}$, *and let* $\xi \in \ker(K^1(A) \to K^1(L))$. *Then there is a finite solvable extension* F *of* L, *such that* $\xi \in \ker(K^1(A) \to K^1(B))$, *where* B *is the ring of integers in* F.

Proof. — By (11.2, *b)*) $\xi = W_A \alpha$ with α an automorphism of A^2, and $\xi \in \ker(K^1(A) \to K^1(L))$ simply means $\det \alpha = 1$. Passing to a quadratic extension F_0 of L, with integers B_0, we can give $\alpha_0 = 1_{B_0} \otimes_A \alpha$ an eigenvalue. As an automorphism of F_0^2, α thus has a one dimensional invariant subspace, and since B_0 is a Dedekind ring, the latter contracts to a direct summand P_0 of B_0^2. P_0 is invariant under α_0, and, having rank one, $P_0 \cong \mathfrak{a}$, $\mathfrak{a}$ an ideal in B_0. Since the class group of B_0 is torsion (even finite) $\mathfrak{a}^h = (a)$ is principal for some $h > 0$. Let $F = F_0(\sqrt[h]{a})$ have integers B. Then $P = B \otimes_{B_0} P_0 \cong \mathfrak{a}B \cong B$, and P is invariant under $\beta = 1_B \otimes_{B_0} \alpha_0 = 1_B \otimes_A \alpha$. If we choose a basis for B^2 the first member of which generates P, then β is represented by a matrix of the form

$$\begin{vmatrix} u & x \\ 0 & v \end{vmatrix} = \begin{vmatrix} u & 0 \\ 0 & v \end{vmatrix} \begin{vmatrix} 1 & u^{-1}x \\ 0 & 1 \end{vmatrix}.$$

The second factor is manifestly in $\mathbf{E}(2, B)$. Since $\det \beta = 1$, we have $v = u^{-1}$, so the first factor lies also in $\mathbf{E}(2, B)$ by the Whitehead lemma (1.7). Therefore $W_B(1_B \otimes_A \alpha) = W_B(\beta) = 0$, as required.

Remarks. — 1) Theorems 19.1 and 19.6 are probably valid also for semi-simple algebras over a function field in one variable over a finite field. The proof above has two ingredients which are not known, to my knowledge, in that case. One is Wang's theorem. However, this can be circumvented easily since the discrepancy between $\mathbf{E}(n, \Sigma)$ and $\mathbf{SL}(n, \Sigma)$ is easily shown to be a torsion group for semi-simple algebras over any field. The second point, which I don't know how to supply or outmaneuver, is the finite generation of $\mathbf{SL}(n, \Lambda)$, say for $n \geq 3$ (1). Similarly, this is the only point requiring attention if one works throughout, say, with orders in Σ over a ring of the form $\mathbf{Z}[n^{-1}]$, for some $n \in \mathbf{Z}$.

2) Theorem 19.6 suggests an obvious analogue for K^0. Namely, one can ask that $K^0(\Lambda) \to K^0(\Sigma)$ have finite kernel. Jan Strooker (Utrecht thesis) has pointed out that a necessary and sufficient condition for this is that every projective Λ-module P, for which $\mathbf{Q} \otimes_{\mathbf{Z}} P$ is Σ-free, be locally free. He gives examples for which this fails.

§ 20. Groups of simple homotopy types.

Theorem **(20.1)**. — *Let Σ be a finite semi-simple $\mathbf{Q}$-algebra with q simple factors, and suppose $\mathbf{R} \otimes_{\mathbf{Q}} \Sigma$ has r simple factors. Then, if Λ is an order in Σ and $\mathfrak{q}$ is an ideal in Λ, $K^1(\Lambda, \mathfrak{q})$ is a finitely generated abelian group of rank $\leq r - q$, and $= r - q$ if $\Lambda/\mathfrak{q}$ is finite.*

Theorem **(20.2)**. — *In the above setting the following conditions are equivalent:*

1) $K^1(\Lambda)$ *is finite.*

2) $K^1(\Lambda, \mathfrak{q})$ *is finite for all* $\mathfrak{q}$.

3) *An irreducible Σ-module remains irreducible under scalar extension from* $\mathbf{Q}$ *to* $\mathbf{R}$.

4) *The center of each simple factor of Σ is either* $\mathbf{Q}$ *or an imaginary quadratic extension of* $\mathbf{Q}$.

Proof of (20.1). — By (19.2) and (19.3) we can assume $\Lambda/\mathfrak{q}$ is finite. Let Γ be a maximal order containing Λ. Then $\mathbf{GL}(n, \Lambda, \mathfrak{q}) \subset \mathbf{GL}(n, \Lambda) \subset \mathbf{GL}(n, \Gamma)$ are both

(1) This has recently been established by O'Meara (On the finite generation of linear groups over Hasse domains, to appear) for commutative Λ.

subgroups of finite index, for all n. Thus $K^1(\Lambda, \mathfrak{q}) \to K^1(\Gamma)$ has finite cokernel. By (19.6), moreover, the maps $K^1(\Lambda, \mathfrak{q}) \to K^1(\Gamma) \to K^1(\Sigma)$ both have finite kernel. Hence rank $K^1(\Lambda, \mathfrak{q}) = \text{rank } K^1(\Gamma)$.

Now Γ is a product of maximal orders in the simple factors of Σ, and $K^1(\Gamma)$ splits accordingly. Since the function $r-q$ likewise adds over the simple factors we can reduce to the case where Σ is simple (i.e. $q=1$), say with center L. $\mathbf{R}\otimes_{\mathbf{Q}} L = \text{center } \mathbf{R}\otimes_{\mathbf{Q}}\Sigma$ has the same number r of simple factors as $\mathbf{R}\otimes_{\mathbf{Q}}\Sigma$, and we want to show that rank $K^1(\Gamma) = r-1$. We know from (19.6) that rank $K^1(\Gamma)$ is the rank of the image $U \subset L^*$ of $\mathbf{GL}(\Gamma)$ under the reduced norm. If A denotes the integers in L, then Γ being integral over A implies $U \subset A^*$. On the other hand $A^* \subset \Gamma^* = \mathbf{GL}(1, \Gamma)$, so $(A^*)^n \subset U$, where $[\Sigma : L] = n^2$. Hence rank $U = \text{rank } A^*$. By the Dirichlet Unit Theorem, rank $A^* = r-1$, and this completes the proof.

Proof of (20.2). — The equivalence of 1), 2), and 3) is an immediate consequence of the above theorem, and that of 3) and 4) is trivial.

If π is a finite group then $\mathbf{Z}\pi$ is an order in the semi-simple algebra $\mathbf{Q}\pi$, so we may apply the preceding results. Viewing $\pm\pi \subset \mathbf{GL}(1, \mathbf{Z}\pi) \subset \mathbf{GL}(\mathbf{Z}\pi)$, it makes sense to write $K^1(\mathbf{Z}\pi)/\pm\pi$, with a minor abuse of notation. J. H. C. Whitehead showed [36] that if X and Y are finite simplicial complexes of the same homotopy type and fundamental group π, then the simplicial homotopy equivalences from X to Y, modulo the simple homotopy equivalences, are classified by invariants which live in $K^1(\mathbf{Z}\pi)/\pm\pi$. Herein lies the principal interest of the next result, which elaborates on some earlier work of G. Higman [38]:

Corollary **(20.3)**. — *Let π be a finite group, r the number of irreducible real representations of π, and q the number of irreducible rational representations of π. Then the commutator quotient group of* $\mathbf{GL}(\mathbf{Z}\pi)$ *is a finitely generated abelian group of rank $r-q$.*

There are well known group theoretic interpretations of r and q : q is the number of conjugacy classes of cyclic subgroups of π (Artin). Write $a \sim b$ in π if a is conjugate to $b^{\pm 1}$. Then r is the number of $\sim$ classes (Berman-Witt). Both of these results can be found in Curtis-Reiner [16, Theorem, 42.8].

Examples. — 1) If π is abelian, then $r=q$ if, and only if, π has exponent 4 or 6. For each simple factor of $\mathbf{Q}\pi$ is a cyclotomic field of n^{th} roots of unity, where $n \mid \exp \pi$. These fields are either $\mathbf{Q}$ itself or totally imaginary. They have degree ≤ 2 precisely when $n \mid 4$ or 6.

2) The rationals are a splitting field for the symmetric groups and $\mathbf{Q}(\sqrt{-1})$ for the quaternions. Hence the Whitehead group is finite in these cases. For groups with this property the results of the next section can be used to give a crude bound on its order. It is not inconceivable that it even be trivial.

3) If π is cyclic of order n, then π has $\delta(n)$ irreducible $\mathbf{Q}$ representations, $\delta(n) =$ the number of divisors of n. π has $[n/2]+1$ irreducible $\mathbf{R}$ representations, where $[x] =$ the integral part of x. Hence the Whitehead group has rank $[n/2]+1-\delta(n)$ in this case.

4) In [7] it is shown that the Whitehead group is trivial when π is free abelian. Milnor has asked whether it is always a finitely generated abelian group if π is. It seems reasonable, though difficult to show, that $K^1(A)$ is finitely generated for A any finitely generated commutative ring over **Z** with no nilpotent elements. The same statement for K^0 would generalize the Mordell-Weil Theorem.

§ 21. Subgroups of finite index in SL(n, A).

Does every subgroup of finite index in $\mathbf{SL}(n, \mathbf{Z})$ contain a congruence subgroup, $SL(n, \mathbf{Z}, q\mathbf{Z})$, for some $q>0$? The answer is easily seen to be " no " for $n=2$, as was already known to Klein. For $n\geq 3$, however, the solution is affirmative; a proof is outlined in [40]. The method consists of an application of the present results to reduce the problem to a (rather formidable) cohomological calculation. The latter, in turn, depends heavily on some recent results of Lazard on analytic groups over p-adic fields.

I shall summarize here, in a form adapted to this method, the information provided by the present material.

Let A be an order in a simple algebra Σ, finite over **Q**. We introduce the following abbreviations in our notation:

$$S = \mathbf{SL}(n, A) \qquad E = \mathbf{E}(n, A);$$

for each ideal $\mathfrak{q}$,

$$S_{\mathfrak{q}} = \mathbf{SL}(n, A, \mathfrak{q}), \; E_{\mathfrak{q}} = \mathbf{E}(n, A, \mathfrak{q}), \text{ and } F_{\mathfrak{q}} = \mathbf{E} \cap S_{\mathfrak{q}}.$$

Theorem **(21.1)**. — *For* $n\geq 2$ center S = center E *is isomorphic to the (cyclic) group of* n^{th} *roots of unity in the center of* A. *For* $n\geq 3$, *a non central subgroup of* S *normalized by* E *contains* $E_{\mathfrak{q}}$ *for some* $\mathfrak{q}\neq 0$, *and* $E/E_{\mathfrak{q}}$ *is finite. Hence a normal subgroup of* E *is either finite or of finite index, and the same is true of* S *as soon as* S/E *is finite. The latter holds for all sufficiently large n.*

Proof. — By (2.4) an element of $\mathbf{GL}(n, A)$, $n\geq 2$, centralized by E, has the form $u.1$, with $u\in$ center A. Being in S means $u^n=1$, and it then follows from (1.7) that $u.1\in E$. Since center A $\subset$ center Σ, a field, the n^{th} roots of unity form a cyclic group.

The rest of the theorem is an immediate consequence of (4.2, *e)*), (19.8) and (19.1).

To avoid some technical difficulties we shall henceforth assume A *is commutative*, i.e. Σ is a number field.

We shall be speaking of " profinite " (= compact, totally disconnected) groups, and their cohomology, for which we give Serre's notes [39] as a general reference. If H is any group we denote by $\hat{H}$ its completion in the topology defined by all subgroups of finite index. Since each of the latter contains a normal subgroup of finite index we can describe $\hat{H}$ by

$$\hat{H} = \varprojlim_{H/H_\alpha \text{finite}} H/H_\alpha.$$

This defines a functor from groups (and homomorphisms) to profinite groups (and continuous homomorphisms) which evidently preserves epimorphisms.

On the other hand, S and E above can be completed also in the " congruence topology " defined by taking the S_q, resp. F_q, $q \neq 0$, as a basis for neighborhoods of the identity. By Corollary 5.2 the inclusion $E \subset S$ induces isomorphisms

$$E/F_q \cong S/S_q \cong \mathbf{SL}(n, A/q),$$

for each $q \neq 0$. Since this group splits uniquely according to the primary decomposition of q, and since $\varprojlim_m \mathbf{SL}(n, A/q^m) = \mathbf{SL}(n, \hat{A}_q)$ for q prime, we conclude that S and E have the same congruence completion, $\Pi = \prod_{q \text{ prime}, \neq 0} \mathbf{SL}(n, \hat{A}_q)$. We shall write

$$C = \ker(\hat{S} \to \Pi) \quad \text{and} \quad C_0 = \ker(\hat{E} \to \Pi).$$

The question discussed above asks whether the congruence and profinite topologies in S coincide, i. e. whether $C = 0$.

Theorem **(21.2)**. — (i) *There is a commutative diagram with exact rows,*

$$\begin{array}{ccccccccc} 1 & \to & C & \to & \hat{S} & \to & \Pi & \to & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \to & C_0 & \to & \hat{E} & \to & \Pi & \to & 1. \end{array}$$

Here $\Pi = \varprojlim_{q \neq 0} E/F_q = \varprojlim_{q \neq 0} S/S_q = \prod_{q \text{ prime}, \neq 0} \mathbf{SL}(n, \hat{A}_q)$.

(ii) *For* $n \geq 3$

$$C_0 = \varprojlim_{q \neq 0} F_q/E_q, \quad \textit{and} \quad C = \varprojlim_{q \neq 0} \widehat{(S_q/E_q)},$$

and the maps in both of these projective systems are all surjective.

(iii) *For* $n \geq 3$, $C_0 \subset \text{center } \hat{E}$. *For* $n \geq 4$, $C \subset \text{center } \hat{S}$, *and* $\hat{E} \to \hat{S}$ *is a monomorphism.*

(iv) *Consider the following conditions:*

a) $S_q/E_q = \{1\}$ *for all* q.

b) *A non central normal subgroup of* S *contains* S_q *for some* $q \neq 0$.

c) *A subgroup of finite index in* S *contains* S_q *for some* $q \neq 0$.

d) $C = \{1\}$.

e) $\widehat{(S_q/E_q)} = \{1\}$ *for all* q.

For $n \geq 3$ *we have* a) $\Leftrightarrow$ b) $\Rightarrow$ c) $\Leftrightarrow$ d) $\Leftrightarrow$ e), *and for* $n \geq 4$ *they are all equivalent. Similarly, they are all equivalent for* $n \geq 3$ *if we substitue* E, F_q, *and* C_0 *for* S, S_q, *and* C, *respectively.*

(v) *Writing* $S = S(n)$, $C = C(n)$, *etc., to denote their dependence on* n, *the inclusions* $S(n-1) \subset S(n)$ *and* $E(n-1) \subset E(n)$ *induce homomorphisms* $C(n-1) \to C(n)$ *and* $C_0(n-1) \to C_0(n)$ *which are surjective for* $n \geq 3$.

Proof. — Part (i) is contained in the remarks preceding the theorem.

It is immediate from (21.1) above that

$$(*) \qquad \hat{E} = \varprojlim_{q \neq 0} E/E_q \qquad \text{for } n \geq 3,$$

and this makes it evident that $C_0 = \varprojlim_{q \neq 0} F_q/E_q$. $S \to S/E_q$ induces $\hat{S} \to \widehat{(S/E_q)} \to 1$, and hence $\hat{S} \to \varprojlim_{q \neq 0} \widehat{(S/E_q)}$. It follows from (21.1) again that this is injective. It is surjective since the image is (clearly) dense, and $\hat{S}$ is compact. Since S/S_q is finite for $q \neq 0$ it is now evident that

$$C = \ker(\varprojlim_{q \neq 0} \widehat{(S/E_q)} \to \varprojlim_{q \neq 0} S/S_q)$$

admits the description in (ii). The last part of (ii) follows once we show that, if $0 \neq q \subset q'$, then $F_q/E_q \to F_{q'}/E_{q'}$ and $S_q/E_q \to S_{q'}/E_{q'}$ are surjective. This means simply that $F_q E_{q'} = F_{q'}$ and $S_q E_{q'} = S_{q'}$. The first of these equations is a consequence of the second, and the second is contained in Corollary 5.2.

For $n \geq 3$ we know from (4.2, *d)*) that $[E, F_q] = E_q$, so it follows from (ii) that $C_0 \subset \text{center } \hat{E}$. If $n \geq 4$ then from (4.2, *f)*) we have $[S, S_q] = E_q$, so it follows similarly from (ii) that $C \subset \text{center } \hat{S}$. To show that $\hat{E} \to \hat{S}$ is a monomorphism it suffices, by (*) above, to show that S/E_q is separated in its profinite topology. But for $n \geq 4$, S/E_q is a central extension of a finite group S/S_q by a finitely generated abelian group S_q/E_q, using (4.2, *f)*) and (19.4), and such a group is clearly separated. This proves (iii).

Now for (iv). Assume $n \geq 3$:

a) $\Leftrightarrow$ *b)* follows from (21.1) and (ii).

b) $\Rightarrow$ *c)* since a subgroup of finite index contains a normal subgroup of finite index, and the center of S is finite.

c) $\Leftrightarrow$ *d)* since *c)* asserts the coincidence of the profinite and congruence topologies.

d) $\Leftrightarrow$ *e)* follows from (ii).

If $n \geq 4$, then S_q/E_q is a finitely generated abelian group, as already noted in the last paragraph, so *d)* $\Rightarrow$ *a)*.

The proof for E is parallel, but the last point is simplified since F_q/E_q is even finite already for $n \geq 3$.

For part (v) it suffices, by compactness, to show that $C(n-1) \to C(n)$ has a dense image for $n \geq 3$, and similarly for C_0. Denseness means that $C(n-1)$ projects onto every finite quotient of $C(n)$. But it follows from (ii) that every finite quotient of $C(n)$ has the form $S_q(n)/H$ with $E_q(n) \subset H$. By (4.2, *b)*), $S_q(n) = S_q(n-1)E_q(n) = S_q(n-1)H$, so $S_q(n-1)/H \cap S_q(n-1)$ coming from $C(n-1)$ maps onto $S_q(n)/H$, as required. The proof for C_0 is identical, after replacing S and S_q by E and F_q, respectively.

In [40] it is shown that, when $A = \mathbf{Z}$, $H^2(\Pi(2), \mathbf{Q}/\mathbf{Z}) = 0$ (cohomology in the sense of [39]), and on this basis that $C(n) = 0$ for $n \geq 3$. (By virtue of (21.2, (iv)) this is equivalent to $E_{(q)} = S_{(q)}$ for all $q > 0$. For $q \leq 5$ this had been shown by Brenner [13] by direct calculation.)

In the general case one knows only the following result, which Serre has proved using recent results of Lazard and of Steinberg (Colloque de Bruxelles, 1962).

Theorem **(21.3)** (Serre, unpublished) $H^2(\Pi, \mathbf{Q}/\mathbf{Z})$ *is finite.*

Plugging this into the argument of [40], and using the information in (21.1) and (21.2) above, one obtains:

Corollary **(21.4)**. — C_0 *is a finite group for* $n \geq 3$, *and* C *is finite for n large enough, so that* S/E *is finite.*

§ 22. Some remarks on polynomial rings.

Let A be commutative and noetherian, and let $B=A[t_1, \dots, t_n]$ with $t_1, \dots, t_n$ $(n \geq 1)$ indeterminates. Grothendieck has shown that, if A is regular, the homomorphism $K^0(A) \to K^0(B)$ is an isomorphism (see [29] or [7]). It follows that if P is a projective B-module, then $\gamma_B P = \gamma_B(B \otimes_A Q)$ for the projective A-module $Q = P/(t_1, \dots, t_n)P$. Now this equation in $K^0(B)$ can be replaced by the isomorphism $P \cong B \otimes_A Q$, provided $\operatorname{rank} P > \dim \max(B)$ (Proposition 15.6). At this point the unpleasant fact emerges that $\dim \max(B) = \dim \operatorname{spec}(B) = n + \dim \operatorname{spec}(A)$. Thus, for example, if A is local (so $\dim \max(A) = 0$), $\dim \max(A[t])$ can be arbitrarily large. In any event, we can record the following conclusion, using the fact that:

A is regular and $\dim \operatorname{spec}(A) = d \Leftrightarrow$ global dim $A = d$.

Theorem **(22.1)**. — *Let* A *be a commutative noetherian ring of global dimension d, and let* $B = A[t_1, \dots, t_n]$, *the* t_i *being indeterminates. Then a projective* B*-module of rank* $> d+n$ *has the form* $B \otimes_A Q$ *for some projective* A*-module* Q.

If A is a field we see that projective B-modules of $\operatorname{rank} > n$ are free, but we can't conclude this if A is only local. However, we can make a very small compensation in this case (Corollary 22.3 below). For here, in the equation $\gamma_B P = \gamma_B(B \otimes_A Q)$, Q will be free, so we can conclude that $P \oplus B^s \cong B^r$, for some r, s; we want P to be free. If we write $(P \oplus B^{s-1}) \oplus B \cong B^{r-1} \oplus B$ and apply induction, we are reduced to showing, under suitable hypotheses, that $P \oplus B \cong B^r \Rightarrow P \cong B^{r-1}$. It is easy to see (cf. proof of the Cancellation Theorem, 9.3) that this conclusion is equivalent to the assertion that $\operatorname{Aut}_B(B^r)$ is transitive on the unimodular elements of B^r.

Proposition **(22.2)**. — *Let* A *be commutative and noetherian and suppose*

$$d = \dim \operatorname{spec}(A) > \dim \operatorname{spec}(A/\operatorname{rad} A).$$

Then, if $B = A[t_1, \dots, t_n]$, t_i *indeterminates,* $\mathbf{E}(r, B)$ *is transitive on the unimodular elements of* B^r *for* $r > d+n$.

Remark. — If we replace $d+n$ above by $d+n+1$ then this Proposition is contained in Theorems 11.1 and 4.1, *a)*, since $\dim \max(B) = d+n$.

Proof. — Suppose $\alpha = (a_1, \dots, a_r) \in B^r$ is unimodular; we seek $\varepsilon \in \mathbf{E}(r, B)$ such that $\varepsilon\alpha = (1, 0, \dots, 0)$. The remark above, together with our hypothesis, shows that this can be done if we replace A by A/rad A. It follows, using Lemma 1.1, that we can find $\varepsilon_1 \in \mathbf{E}(r, B)$ so that $\varepsilon_1 \alpha = (a'_1, \dots, a'_r) \equiv (1, 0, \dots, 0) \bmod \operatorname{rad} A.B^r$. Since $a'_r \equiv 1 \bmod \operatorname{rad} A.B$, a maximal ideal of B containing a'_1 cannot contract to a maximal ideal of A. It follows that $\dim \max(B/a'_1 B) < n+d-1$. Hence we can again apply the remark above, this time to the (unimodular) image of $(a'_2, \dots, a'_r)$ in $(B/a'_1 B)^{r-1}$, and transform this image into $(1, 0, \dots, 0)$ with $\varepsilon'_2 \in \mathbf{E}(r-1, B/a'_r B)$. By Lemma 1.1 again, ε'_2 lifts to $\varepsilon''_2 \in \mathbf{E}(r-1, B)$, and we set $\varepsilon_2 = \begin{vmatrix} 1 & 0 \\ 0 & \varepsilon''_2 \end{vmatrix} \in \mathbf{E}(r, B)$. Then $\varepsilon_2 \varepsilon_1 \alpha$ has the form $(a'_1, 1 + b_2 a'_1, b_3 a'_1, \dots, b_r a'_1)$, and it is now clear how to finish with elementary transformations.

From the discussion preceding this proposition we derive the following corollary:

Corollary **(22.3)**. — *If, in Proposition* 22.2, A *is a regular ring for which* $K^0(A) \cong \mathbf{Z}$, *then a projective* B*-module of rank* $\geq d+n$ *is free.*

As a special case, we have the following corollary:

Corollary **(22.4)**. — *If* A *is a semi-local principal ideal domain (so* $d \geq 1$*) then a projective* $A[t_1, \ldots, t_n]$*-module (*t_i *indeterminates) of rank* $> n$ *is free.*

This last corollary has recently been strengthened, for $n=2$, by S. Endô [20], who shows in this case that all projective modules are free. This generalizes the theorem of Seshadri [27].

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