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THEORY OF SPHERICAL FUNCTIONS ON REDUCTIVE ALGEBRAIC GROUPS OVER $\mathfrak{p}$-ADIC FIELDS

by Ichiro SATAKE

Dedicated to Professor Y. Akizuki on his Sixtieth Birthday.

CONTENTS

INTRODUCTION

The theory of spherical functions on semi-simple Lie groups has been developed by Gelfand, Neumark, Harish-Chandra, Godement and others. Among them, Harish-Chandra [13] proved that the totality of zonal spherical functions on a (connected) non-compact semi-simple Lie group G with finite center, relative to a maximal compact subgroup U, can be canonically identified with a quotient space of the form $\mathbf{C}^\nu/W$, where ν denotes the " rank " of G, i.e. the dimension of a maximal vector part of a Cartan subgroup of G and W the restricted Weyl group of G. Recently the theory has been extended to the case of some classical groups over p-adic fields by Mautner [17], Tamagawa [23] and Bruhat [4], [5]. The main purpose of this paper is to show that the principal part of the theory, including the above-mentioned theorem of Harish-Chandra, holds for a wider class of reductive algebraic groups over p-adic fields, containing all simple classical groups without center.

To be more precise, let k be a local field, G a Zariski-connected reductive algebraic subgroup of $GL(n, k)$, A a maximal k-trivial torus in G and N a maximal k-closed unipotent subgroup of G, normalized by A (G, A, N, ... being understood as to represent the groups of k-rational points); the pair (A, N) is then unique up to inner automorphisms of G. Put $\dim A = \nu$ and denote by A^u the unique maximal compact subgroup of A. Then the restricted Weyl group of G relative to A, $W = N(A)/Z(A)$, operates in a natural way on A as a group of automorphisms, and hence also on the character group (in the algebraic sense) of A, $Y = X(A)(\cong \mathbf{Z}^\nu)$, and on the group of quasi-characters (in the topological sense) of A/A^u, $\mathrm{Hom}\,(A/A^u, \mathbf{C}^*) = Y \otimes \mathbf{C}(\cong \mathbf{C}^\nu)$. Now let $k = \mathbf{R}$ or $\mathbf{C}$, and let U be a maximal compact subgroup of G; the quotient space $S = U\backslash G$ is then the associated symmetric space. A C^∞-function ω on G is called a zonal spherical function (or elementary spherical function) on G relative to U, if it satisfies the following conditions

(i) $\omega(ugu') = \omega(g)$ for all $g \in G$, $u, u' \in U$,

(ii) $\omega(1) = 1$,

(iii) ω, considered as a function on S, is an eigen-function for all invariant differential operators on S.

As is well-known, the algebra (over $\mathbf{C}$) of all invariant differential operators on S is canonically isomorphic to the algebra of all W-invariant polynomial functions on the dual of the Lie algebra of A, or, what is the same, on $Y \otimes \mathbf{C}$, so that it is actually a polynomial algebra of ν variables over $\mathbf{C}$. (Thus the condition (iii) is reduced to the ν conditions corresponding to the generators of this algebra. Furthermore, it is also known that the condition (iii) may be replaced by a similar condition for invariant integral operators.) These being said, the theorem of Harish-Chandra asserts that the set of all zonal spherical functions on G relative to U is identified with the quotient space $Y \otimes \mathbf{C}/W$. The proof for this consists essentially in showing that, U, A, N being taken suitably so that we have the decompositions

$$(*) \qquad G = U.A.U = U.AN,$$

the Fourier transformation with respect to zonal spherical functions (parametrized by $Y \otimes \mathbf{C}$) gives actually an isomorphism from the algebra of the invariant differential operators onto that of the W-invariant polynomial functions on $Y \otimes \mathbf{C}$.

The first difficulty, in translating these results from real to p-adic, arises from the difference of the nature of maximal compact subgroups. Whereas they are " algebraic " and mutually conjugate in the real case, they have no more such a property in the p-adic case. As a matter of fact, we have to say that, for the time being, our knowledge on this subject is still very poor. Therefore, in this paper, we will make certain assumptions on G, assuring the existence of a favorable maximal compact subgroup U (the assumptions (I), (II) in § 3). These assumptions are nothing but p-adic analogues of some well-known properties of semi-simple Lie groups; in particular, the condition (I) implies the possibility of a decomposition like (*), but A should now be replaced by a bigger subgroup H, such that $A \subset H \subset Z(A)$, and W-invariant. On the other hand, as we shall show in Chapter III, these conditions are satisfied by " all " known examples of classical groups, by virtue of the theory of elementary divisors. Thus one may hope to find a unified *proof* for these assumptions.

Now, under these assumptions, let $\mathscr{L}(G, U)$ denote the algebra (over $\mathbf{C}$) of all invariant integral operators on $U\backslash G$, whose kernel is given by a function on G with compact carrier. One defines a zonal spherical function as a function on G satisfying (i), (ii) and the condition (iii) stated in terms of $\mathscr{L}(G, U)$; then a zonal spherical function determines a homomorphism (of algebras over $\mathbf{C}$) from $\mathscr{L}(G, U)$ onto $\mathbf{C}$ and *vice versa.* On the other hand, call H^u the unique maximal compact subgroup of H and put $M = H/H^u (\cong \mathbf{Z}^\nu)$. Then our main theorem (Th. 3 in § 6) asserts that $\mathscr{L}(G, U)$ is isomorphic to the algebra of all W-invariant polynomial functions on $\mathrm{Hom}(M, \mathbf{C}^*) (\cong \mathbf{C}^\nu)$, allowing this time negative powers in an obvious sense; thus $\mathscr{L}(G, U)$ is an affine algebra of (algebraic) dimension ν over $\mathbf{C}$. From this follows immediately the analogue of the theorem of Harish-Chandra asserting that the totality of zonal spherical functions on G relative to U is canonically identified with a quotient space of the form $(\mathbf{C}^*)^\nu/W$ (Th. 2 in § 5). As examples, it will be shown that, in case G is a simple classical group without center and U a maximal compact subgroup of G defined by a " maximal lattice ", the algebra $\mathscr{L}(G, U)$ is actually a polynomial algebra of ν variables over $\mathbf{C}$ (Th. 7, 9 in §§ 8, 9). More precisely, the so-called (local) " Hecke ring " $\mathscr{L}(G, U)_{\mathbf{Z}}$ is a polynomial ring of ν variables over $\mathbf{Z}$.

These theorems are proved by the usual method of Fourier transformation and, in fact, rather simply, compared with the real case. But to determine the explicit form of zonal spherical functions and the Plancherel measure, it seems necessary to know the (infinite) matrix of this Fourier transformation more explicitly. This has been done by Mautner [17] for $PL(2, k)$, but is still an open problem for the general case. As a partial result in this direction, we will calculate in Appendix I (local) Hecke series and especially ζ-functions attached to $GL(n, \mathfrak{K})$, where $\mathfrak{K}$ is a central division algebra over k, and to the group of symplectic similitudes.

Besides these, we will analyze in § 7 the behavior of zonal spherical functions under a homomorphism $\lambda : G \rightarrow G'$, and especially under a k-isogeny (Th. 4). Here we do not assume *a priori* the conditions (I), (II) on G, G′ in full, and will see how (parts of) the conditions on the one of G, G′ imply the corresponding conditions on the other. As another application, we will determine in Appendix II all zonal spherical functions of positive type, or, what amounts to the same, all unitary equivalence classes of irreducible unitary representations of the first kind, of $PL(2, \mathfrak{K})$, obtaining again a result quite analogous to the real case.

A part of results of this paper (N° 7.3) has been announced in a short note [**19**], which will also serve as an introduction to this paper.

The author has much profited by seminars on *Spherical functions*, organized during the period of 1960-62, by Professor Y. Akizuki, to whom this paper is dedicated with sincere gratitude and respects.

Notations and Conventions. Throughout this paper, k denotes a $\mathfrak{p}$-adic number field, i.e. a finite extension of the p-adic number field $\mathbf{Q}_p$. The valuation-ring in k and its (unique) prime ideal are denoted by $\mathfrak{o}$, $\mathfrak{p}=(\pi)$, respectively, π denoting a prime element. We denote by $|\ |_\mathfrak{p}$ (or simply by $|\ |$) the normalized valuation of k, i.e.

$$|\xi|_\mathfrak{p}=q^{-\mathrm{ord}_\mathfrak{p}\xi} \qquad \text{for } \xi\in k,$$

q denoting the number of elements in the residue class field $\mathfrak{o}/\mathfrak{p}$.

All algebraic groups we consider are supposed to be *affine*, so that they are realized as groups of matrices. Thanks to a result of Rosenlicht (*Annali di Matematica*, vol. **43** (1957), p. 44), in any such group, defined over k, the subgroup formed of k-rational points is everywhere dense in the sense of the Zariski topology, provided all the connected components contain a k-rational point. Hence, in this paper, we will understand by an " algebraic group over k " the group formed of k-rational points of an algebraic group (in the sense of algebraic geometry) defined over k. If G is a (Zariski-) connected algebraic group over k and if K is an overfield of k, the group formed of K-rational points in the same algebraic group will be denoted by G^K.

In an algebraic group G over k, one can consider two kinds of topologies, i.e. the $\mathfrak{p}$-adic topology and the Zariski topology. Without any specific reference, the words " closed " (or " k-closed "), " connected " will be used exclusively in the sense of the Zariski topology, while the words " open ", " compact " will always be understood in the sense of the $\mathfrak{p}$-adic topology. " Closure " (in the sense of the Zariski topology) of M is denoted by cl(M). When G, G′ are algebraic groups over k and φ a k-morphism (i.e. a rational homomorphism defined over k) from G into G′, the symbols $\mathrm{Im}\,\varphi=\varphi(G)$, $\mathrm{Ker}\,\varphi=\varphi^{-1}(1)$ are used in the set-theoretical sense; thus $\varphi^{-1}(1)$ is k-closed but $\varphi(G)$ is not, and (in case G is connected) the image of φ in the algebraic sense is $\mathrm{cl}(\varphi(G))$. (More generally, the similar convention is made for any rational map defined over k.) In particular, for a k-closed normal subgroup H of G, we denote (by abuse of notation) the factor group in the algebraic sense by cl(G/H).

As usual, for any ring R with the unit element 1, R^* stands for the multiplicative group of regular elements in R, and $M_n(R)$ for the ring of all $n\times n$ matrices with coefficients in R. The unit matrix of degree n is denoted by 1_n (or simply by 1).

For $X_i \in M_{n_i}(R)$ $(1 \leqslant i \leqslant r)$, the symbol diag. $(X_1, \ldots, X_r)$ will represent a matrix of degree $n = \sum_i n_i$ of the following form:

$$\begin{pmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_r \end{pmatrix}.$$

Z, **Q**, **R**, **C** denote, respectively, the ring of rational integers, the rational number field, the real number field and the complex number field; the real and imaginary parts of $s \in \mathbf{C}$ are denoted by Re s, Im s, respectively. For a finite set M, the symbol $\# M$ represents the number of elements in M. For a map φ defined on a set M and for a subset M_1 of M, the symbol $\varphi | M_1$ stands for the restriction of φ to M_1.

Chapter I

REDUCTIVE ALGEBRAIC GROUPS OVER p-ADIC FIELDS (¹)

§ 1. k-Borel subgroups.

1.1. *k-Borel pairs.* Let k be a p-adic number field. An algebraic group G over k is called a torus, if it is connected, commutative and consisting only of semi-simple elements. For a torus G over k, there exists a finite extension K/k such that G^K is K-isomorphic (i.e. birationally isomorphic over K) to $(K^*)^l$; such a field K is called a " splitting field " of G. In case this isomorphism is obtained in k, i.e. in case G is k-isomorphic to $(k^*)^l$, G is called a *k-trivial torus.* An algebraic group G over k is called unipotent, if it only contains unipotent elements; a unipotent algebraic group G over k is always connected and nilpotent.

Let G be an algebraic group over k. A maximal k-trivial torus (resp. maximal k-unipotent subgroup) in G is a k-trivial torus (resp. k-closed unipotent subgroup) in G which is maximal with respect to this property. A pair (A, N) of a maximal k-trivial torus A and a maximal k-unipotent subgroup N in G such that A normalizes N is called a *k-Borel pair.* For such a pair (A, N), AN becomes a k-closed subgroup of G, called a *k-Borel subgroup* of G, which is a semi-direct product over k of A and N. A typical example of a k-Borel pair is

$G = GL(n, k)$,

$A = D(n, k)$ (the group of all diagonal matrices in $GL(n, k)$),

$N = T^u(n, k)$ (the group of all upper unipotent matrices, i.e. matrices $x = (\xi_{ij})$ with $\xi_{ii} = 1$, $\xi_{ij} = 0$ for $i > j$, in $GL(n, k)$).

It is known (Borel) that, for any pair (A, N) of a k-trivial torus A and a k-closed unipotent subgroup N in G such that A normalizes N, there exists always a k-Borel pair (A', N') with $A' \supset A$, $N' \supset N$ and that all k-Borel pairs in G are conjugate to each other with respect to the inner automorphisms of G [12]. It follows that, for any Borel pair (A, N) in G, we can transform G by an inner automorphism of the " ambient group " $GL(n, k)$ in such a way that A coincides with the connected component of the neutral element of $G \cap D(n, k)$ and N coincides with $G \cap T^u(n, k)$.

(¹) For the fundamental concepts on algebraic groups, see [1]; especially on algebraic toruses, see [18]. Cf. also [7], [12], [20].

1.2. The following proposition gives a characterization of k-Borel subgroups in terms of the $\mathfrak{p}$-adic topology.

PROPOSITION 1.1. *Let* G *be an algebraic group over a* $\mathfrak{p}$*-adic number field* k. *Then, for a* k*-closed subgroup* H *of* G, *the homogeneous space* G/H *is compact (in the* $\mathfrak{p}$*-adic topology), if and only if* H *contains a* k*-Borel subgroup of* G.

Proof. The proof is obviously reduced to the case where G is connected. Hence, assuming G to be connected, let H be a k-Borel subgroup of G. Then since G/H is identified (birationally) with the set of k-rational points in a projective variety defined over k [**12**], it is compact; whence follows the " if " part of the Proposition. Conversely, let H′ be a k-closed subgroup of G such that G/H′ is compact. One can take a k-Borel subgroup H of G containing a k-Borel subgroup of H′. Then, from what we have proved above, $H'/(H\cap H')$ is compact. Therefore HH′ is closed in the $\mathfrak{p}$-adic topology, so that $H/(H\cap H')\cong HH'/H'$ is compact by the assumption. But, since H is a k-Borel subgroup, it has a composition series (as algebraic group over k): $H=H_0\supset H_1\supset\ldots\supset H_r=\{1\}$ such that the factor groups H_{i-1}/H_i are isomorphic either to k^* or to k; and it is then a trivial matter to prove inductively that $H_i(H\cap H')=H$ $(1\leqslant i\leqslant r)$. Thus we get $H=H\cap H'$, i.e. $H'\supset H$, q.e.d.

COROLLARY. *An algebraic group* G *over* k *is compact (in the* $\mathfrak{p}$*-adic topology) if and only if* k*-Borel subgroups of* G *reduce to the neutral element.*

In view of a decomposition theorem of Chevalley ([**7**], p. 144) it follows that a compact algebraic group over k is necessarily reductive, i.e. isogeneous to the direct product of a semi-simple algebraic group and a torus.

Remark. Proposition 1.1 and its Corollary are also valid for algebraic groups over **R** or **C**. (The same proof!)

1.3. *Maximal compact subgroups.*

PROPOSITION 1.2. *If an algebraic group* G *over* k *has a maximal compact subgroup,* G *is reductive.*

Proof. Suppose that G is not reductive. Then, by virtue of the decomposition theorem of Chevalley, G has a k-closed unipotent normal subgroup $N_1\neq\{1\}$. The center N_2 of N_1, being a unipotent commutative group, is k-isomorphic to a vector space over k, i.e. we have a k-isomorphism $f: N_2\rightarrow k^m$ $(m>0)$. Then the inner automorphism I_g defined by $g\in G$ induces an automorphism of N_2, which, by f, corresponds to a linear transformation, denoted by ρ_g, of k^m. Let U be a maximal compact subgroup of G and let $L_1=\mathfrak{o}^m\subset k^m$. Then, $U_1=\{u\in U\,|\,\rho_u L_1=L_1\}$ is an open subgroup of U and hence is of finite index in U. Put

$$U=\bigcup_i u_i U_1, \qquad L=\bigcap_i \rho_{u_i} L_1.$$

Then L is an " $\mathfrak{o}$-lattice " in k^m (see N° 8.1) invariant under ρ_u $(u\in U)$. It follows that

$$U^{(i)}=U.f^{-1}(\pi^{-i}L) \qquad (i=1,2,\ldots)$$

are all compact subgroups of G containing U, so that, by the maximality of U, one gets $f^{-1}(\pi^{-i}L) \subset U$ $(i=1, 2, \ldots)$. This implies that $N_2 \subset U$, which is a contradiction, q.e.d.

Now, for a reductive algebraic group G over **R**, it is well-known that G has always a maximal compact subgroup, which is **R**-closed, and that all maximal compact subgroups of G are conjugate to each other with respect to the inner automorphisms of G. Moreover, if (A, N) is an **R**-Borel pair in G and if A_+ denotes the connected component (in the sense of the usual topology) of the neutral element of A, we have

$$G = U.A_+N, \qquad U \cap (A_+N) = \{1\} \tag{1.1}$$

for all maximal compact subgroup U, and

$$G = UA_+U \tag{1.2}$$

for a suitable U.

Unfortunately, in the p-adic case, we are not yet in possession of such a general result. Since, however, the existence of a maximal compact subgroup with properties similar to (1.1), (1.2) is indispensable to the theory of spherical functions, we will assume it in § 3; for classical groups, our assumptions will be verified case by case in Chapter III. On the other hand, it should be noted that, in the p-adic case, the maximal compact subgroups of a reductive algebraic group G are, if they exist, not necessarily conjugate to each other with respect to the automorphisms of G and that they are not necessarily corresponding in one-to-one way under a *k*-isogeny.

§ 2. Reductive algebraic groups.

2.1. In this section, G denotes a connected reductive algebraic group over a p-adic number field *k*. Let A be a maximal *k*-trivial torus in G, Y the character module of A and let

$$\dim A = \operatorname{rank} Y = \nu.$$

If $\{\eta_1, \ldots, \eta_\nu\}$ is a system of generators of Y, the correspondence

$$A \ni a \longrightarrow (\eta_1(a), \ldots, \eta_\nu(a)) \in (k^*)^\nu \tag{2.1}$$

gives a *k*-isomorphism $A \cong (k^*)^\nu$.

Let $\mathfrak{g}$, $\mathfrak{a}$ be the Lie algebras of G, A, respectively. It is easy to see that, for any *k*-homomorphism ρ from A into another algebraic group over *k*, the closure of the image $\rho(A)$ is again a *k*-trivial torus. Applying this to ρ = adjoint representation of G, we have

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\gamma \in \bar{\mathfrak{r}}} \mathfrak{g}_\gamma, \tag{2.2}$$

where $\bar{\mathfrak{r}} \subset Y$ is the " restricted root system " relative to A [**20**] and

$$\begin{aligned} \mathfrak{g}_\gamma &= \{x \in \mathfrak{g} \mid \mathrm{Ad}(a)x = \gamma(a)x \text{ for all } a \in A\}, \\ \mathfrak{g}_0 &= \{x \in \mathfrak{g} \mid \mathrm{Ad}(a)x = 0 \quad \text{for all } a \in A\}. \end{aligned} \tag{2.3}$$

It is clear that $\mathfrak{g}_0$ is the Lie algebra of Z(A) (= the centralizer of A). On the other hand, for any linear order in Y, put

$$\mathfrak{n} = \sum_{\gamma > 0} \mathfrak{g}_\gamma \tag{2.4}$$

and call N the corresponding connected subgroup of G. Then we have [**20**]:

PROPOSITION 2.1 *The notations being as above,* Z(A) *is a connected reductive algebraic group over k consisting of only semi-simple elements,* N *is a maximal k-unipotent subgroup of* G, *normalized by* Z(A), *and* Z(A).N *is a semi-direct product of* Z(A) *and* N *over k.*

2.2. For any connected reductive algebraic group G over k, we denote by X(G) the module of all k-rational characters (i.e. k-morphisms of G into k^*). Furthermore we put

$$G^1 = \{g \in G \mid \chi(g) = 1 \text{ for all } \chi \in X(G)\}. \tag{2.5}$$

Then G^1 is a connected k-closed normal subgroup of G and there exists a k-trivial torus A′ contained in the center of G such that

$$G = \mathrm{cl}(G^1.A'), \qquad G^1 \cap A' = \text{finite};$$

in other words, the natural homomorphism gives a k-isogeny: $G^1 \times A' \to G$. To see this, let S, T the semi-simple and the torus parts of G, respectively. Since $S \subset G^1$, it is clear that the closure of the canonical image of G^1 in cl(G/S) is equal to $\mathrm{cl}(G/S)^1$. On the other hand, for a torus, our assertion is known ([**20**], Prop. 1), i.e. if A′ denotes the (unique) maximal k-trivial torus in T, we have a k-isogeny $T^1 \times A' \to T$. It follows also that $\mathrm{cl}(G/S)^1$, and consequently G^1 is connected. Under the k-isogeny $T \to \mathrm{cl}(G/S)$, induced by the canonical homomorphism, T^1 corresponds to $\mathrm{cl}(G/S)^1$, whence one gets $G^1 = \mathrm{cl}(T^1.S)$. Thus one has

$$G \sim T \times S \sim T^1 \times A' \times S \sim G^1 \times A',$$

where $G \sim G'$ means that G is isogeneous to G′. This proves our assertion. It follows that the homomorphism $X(G) \to X(A')$ defined by the restriction is injective and has a finite cokernel.

In particular, if G consists only of semi-simple elements (which implies necessarily that G is reductive), it follows from Proposition 2.1 that G = Z(A) (i.e. A = A′ in the above notation), and hence by Corollary to Proposition 1.1 that G^1 is compact. Thus we obtain the following

PROPOSITION 2.2 *Let* H *be a connected algebraic group over k consisting only of semi-simple elements and* A *the (unique) maximal k-trivial torus in* H. *Then* H^1 *is compact and* H *is k-isogeneous to the direct product of* H^1 *and* A. *Moreover,* X(H) *may be identified with a submodule of* X(A) = Y *with finite index.*

Let $\{\chi_1, \ldots, \chi_\nu\}$ be a system of independent generators of X(H). Then the mapping

$$\Phi : H \ni h \longrightarrow (\chi_1(h), \ldots, \chi_\nu(h)) \in (k^*)^\nu \tag{2.6}$$

defines an injective homomorphism of H/H^1 into $(k^*)^\nu$. In case this mapping is surjective, we say that H *satisfies the condition* (N).

2.3. We denote by $\mathfrak{u}$ the unit group in k, i.e. $\mathfrak{u}=\mathfrak{o}^*$. Let H be a connected algebraic group consisting only of semi-simple elements and put

(2.7) $$H^{\mathfrak{u}}=\{h\in H \mid \chi(h)\in\mathfrak{u} \text{ for all } \chi\in X(H)\}.$$

Then we have the following

PROPOSITION 2.3. *The notations being as above,* $H^{\mathfrak{u}}$ *is the unique maximal compact subgroup of* H; *it is a normal subgroup of* H, *containing* H^1. *Moreover, there exists a subgroup* D *of* H, *isomorphic to* $\mathbf{Z}^\nu$ ($\nu=\operatorname{rank} X(H)$), *such that*

(2.8) $$H=D.H^{\mathfrak{u}}, \quad D\cap H^{\mathfrak{u}}=\{1\}.$$

Proof. Let $\Phi: H\to(k^*)^\nu$ be as defined by (2.6). Then, since $[Y:X(H)]<\infty$ (Proposition 2.2), the restriction of Φ on A is a k-isogeny. Therefore, as is well-known, $\Phi(A)$ is an open subgroup (in the sense of the $\mathfrak{p}$-adic topology) of $(k^*)^\nu$, of finite index, and so is also $\Phi(H)$. Now it is clear that $H^{\mathfrak{u}}$ contains H^1, which is compact. Since $H^{\mathfrak{u}}/H^1\cong\Phi(H^{\mathfrak{u}})=\Phi(H)\cap\mathfrak{u}^\nu$, we see that $H^{\mathfrak{u}}$ is compact. Since H/H^1 is commutative, $H^{\mathfrak{u}}$ is a normal subgroup of H. On the other hand, $H/H^{\mathfrak{u}}\cong\Phi(H)/\Phi(H^{\mathfrak{u}})$, being isomorphic to $\mathbf{Z}^\nu$, does not contain any compact subgroup. Therefore $H^{\mathfrak{u}}$ is a maximal compact subgroup and, since it is normal, it is the unique maximal compact subgroup. The existence of the subgroup D is obvious, q.e.d.

COROLLARY. $A^{\mathfrak{u}}=A\cap H^{\mathfrak{u}}$ *is the unique maximal compact subgroup of* A.

We put
$$\hat{X}(H)=\operatorname{Hom}(X(H),\mathbf{Z}).$$

For every $h\in H$, the correspondence $l_h: X(H)\to\mathbf{Z}$ defined by

(2.9) $$l_h(\chi)=\operatorname{ord}_{\mathfrak{p}}\chi(h) \qquad \text{for } \chi\in X(H)$$

is an element of $\hat{X}(H)$, and the correspondence $h\to l_h$ is a homomorphism from H into $\hat{X}(H)$, whose kernel is equal to $H^{\mathfrak{u}}$. Thus, denoting by M the image of this homomorphism, one has

(2.10) $$H/H^{\mathfrak{u}}\cong M.$$

When the decomposition (2.8) is fixed once for all, the homomorphism $h\to l_h$ induces an isomorphism of D onto M. Hence, when $l_d=\mathbf{m}$ with $d\in D$, $\mathbf{m}\in M$, one writes

(2.11) $$d=\pi^{\mathbf{m}};$$

by definition, one has

(2.12) $$|\chi(\pi^{\mathbf{m}})|_{\mathfrak{p}}=q^{-\langle\chi,\mathbf{m}\rangle} \qquad \text{for all } \chi\in X(H), \mathbf{m}\in M,$$

$\langle\ \rangle$ denoting the pairing of $X(H)$ and $\hat{X}(H)$.

If, in particular, H satisfies the condition (N), one has $M=\hat{X}(H)$. This is surely the case for A. Thus we obtain the following commutative diagram

(2.13)
$$\begin{array}{ccc} A/A^u & \xrightarrow{\text{inj.}} & H/H^u \\ {\scriptstyle \text{isom.}}\downarrow & & \downarrow{\scriptstyle \text{inj.}} \\ \hat{Y} & \xrightarrow{\text{inj.}} & \hat{X}(H) \end{array}$$

which allows us to consider that $\hat{Y}\subset M\subset\hat{X}(H)$.

§ 3. Fundamental assumptions.

3.1. *Assumption* (I). Let G be an algebraic group over k. We shall now make two fundamental assumptions (I), (II) on G. In the first place, we assume

(I) *There exist, in* G, *an open compact subgroup* U, *a connected (reductive) k-closed subgroup* H *consisting only of semi-simple elements and a k-unipotent subgroup* N *normalized by* H *such that the following conditions are satisfied*:

(3.1) $$G=U.HN=U.H.U,$$

(3.2) $$U\supset H^u.$$

Let A be the unique maximal k-trivial torus in H; then we have

(3.3) $$A\subset H\subset Z(A).$$

Since AN is a k-closed subgroup of G such that G/AN is compact (by (3.1) and Proposition 2.2), it follows from Proposition 1.1 that (A, N) is a k-Borel pair in G. Let D be a subgroup of H as described in Proposition 2.3. Then by (3.1), (3.2) one has

(3.1)′ $$G=U.DN=U.D.U.$$

From this one concludes at once that U is a maximal compact subgroup of G and so, by Proposition 1.2, that G is reductive.

Now HN is clearly a semi-direct product of H, N over k. Moreover one has

(3.4) $$HN\cap U=H^u.(N\cap U).$$

In fact, for $u\in HN\cap U$, write $u=hn$ with $h\in H, n\in N$. Then the correspondence $u\to h$ being a continuous homomorphism (with respect to the p-adic topology), one concludes that its image is compact and so contained in H^u, by Proposition 2.3.

LEMMA 3.1. *Under the assumption* (I), *suppose further that* G *is connected, that* H *satisfies the condition* (N) *and that* $H\cap G^1$ *is connected,* G^1 *being defined by* (2.5). *Then we have*

(3.5) $$G=H.G^1,\quad U=H^u.U^1,$$

where $U^1=U\cap G^1$.

Proof. Since H contains the unique maximal k-trivial torus A′ in the center of G and since $H\cap G^1$ is connected, it follows from what we stated in n° 2.2

that $H \sim (H \cap G^1) \times A'$ and $G = cl(H.G^1)$. Hence the restriction homomorphism $X(G) \to X(H)$ is injective and, if one identifies $X(G)$ with its image (i.e. the annihilator in $X(H)$ of $H \cap G^1$, which is connected), $X(H)/X(G)$ has no torsion [**18**]. Therefore one may take a system of (independent) generators $\{\chi_1, \ldots, \chi_\nu\}$ of $X(H)$ in such a way that $\{\chi_1, \ldots, \chi_{\nu'}\}$ ($\nu' = \text{rank } X(G)$) forms a system of generators of $X(G)$. Since H satisfies the condition (N), one concludes from this immediately that for any $g \in G$ there exists $h \in H$ such that $\chi(g) = \chi(h)$ for all $\chi \in X(G)$, i.e. $h^{-1}g \in G^1$, which proves the first equality (3.5). If, in the above, $g \in U$, one has $\chi(g) \in \mathfrak{u}$ for all $\chi \in X(G)$, so that one may choose the above h so as to belong to $H^{\mathfrak{u}}$. This proves the second equality (3.5), q.e.d.

Remark. In case $H = Z(A)$, the third assumption in Lemma 3.1 is surely satisfied. In fact, call A″ the connected component of the neutral element of $A \cap G^1$. Then clearly $A = cl(A'.A'')$ and one sees that $Z(A) \cap G^1$ is equal to the centralizer of A″ in G^1. Hence $Z(A) \cap G^1$ is connected ([**1**], Prop. 18.4).

It follows that, under the assumptions of Lemma 3.1, one may replace U by U^1 in (3.1), i.e.

$$(3.1)'' \qquad G = U^1.HN = U^1.H.U^1.$$

Furthermore, since $N \subset G^1$, one gets also

$$(3.6) \qquad G^1 = U^1.(H \cap G^1)N = U^1.(H \cap G^1).U^1.$$

This shows that $G^1, U^1, H \cap G^1, N$ also satisfy assumption (I).

3.2. We give here several procedures which allow us to construct groups satisfying assumption (I), starting from other such groups.

PROPOSITION 3.1. *If* G_i, U_i, H_i, N_i $(i = 1, 2)$ *satisfy* (I), *so do also* $G_1 \times G_2$, $U_1 \times U_2$, $H_1 \times H_2$, $N_1 \times N_2$.

Trivial.

PROPOSITION 3.2. *Let* G, U, H, N *satisfy* (I) *and assume further that* G, H *satisfy the conditions stated in Lemma* 3.1. *Let* $X = X(G)$ *and* X_1 *a submodule of* X *such that* X/X_1 *has no torsion. Put*

$$G^* = \{g \in G \mid \chi(g) = 1 \text{ for all } \chi \in X_1\}.$$

Then, $G^*, U^* = U \cap G^*, H^* = H \cap G^*, N$ *also satisfy assumption* (I).

Proof. One identifies X with the character module of the k-trivial torus $cl(G/G^1)$. Then, G^* is the (complete) inverse image, under the canonical homomorphism $G \to cl(G/G^1)$, of the annihilator of X_1 in $cl(G/G^1)$, which is a subtorus by the assumption [**18**]. Therefore G^* is connected. As we have done in the proof of Lemma 3.1, one may consider that $X \subset X(H)$; then, by our assumptions, $X(H)/X_1$ also has no torsion. Hence, by the same reason as above, H^* is connected. Now, since G^* contains U^1 and N, it follows from (3.1)″ that

$$G^* = U^1.H^*N = U^1.H^*.U^1$$

and *a fortiori* (3.1) for G^*, U^*, H^*, N. One has also $H^{*\mathfrak{u}} = H^{\mathfrak{u}} \cap G^* \subset U^*$, i.e. (3.2), q.e.d.

PROPOSITION 3.3. *Let* G, U, H, N *satisfy* (I). *Let* Z *be a k-trivial torus contained in the center of* G. *Put*

$$(3.7) \qquad H_0=\{h\in H \mid h^m\in ZH^u \ \textit{for some}\ m\in \mathbf{Z}\}$$

and suppose that H_0 *normalizes* U. *Then* $\bar{G}=G/Z$, $\bar{U}=(H_0U)/Z$, $\bar{H}=H/Z$, $\bar{N}=NZ/Z$ *also satisfy* (I).

Proof. First we note that, since Z is a k-trivial torus, the canonical homomorphism $G\rightarrow cl(G/Z)$ is surjective, i.e. $cl(G/Z)=G/Z$. Since Z is contained in any maximal k-trivial torus, we have $Z\subset H$. Hence, from the definition, it follows that H_0 is an open subgroup of H such that $H_0/(ZH^u)$ is the torsion part of $H/(ZH^u)$. H_0/Z is therefore the unique maximal compact subgroup of $\bar{H}=H/Z$. Furthermore, by the assumption, one sees that $(H_0U)/Z=(H_0/Z).(UZ/Z)$ is an open compact subgroup of $\bar{G}=G/Z$. Our Proposition is now obvious, q.e.d.

3.3. *Weyl groups.* Let the notations be as in N° 3.1. For $s\in N(H)$ (=the normalizer of H), the inner automorphism I_s defined by s induces an automorphism of H, and hence that of X(H), which we call w_s, by the formula

$$(3.8) \qquad (w_s\chi)(shs^{-1})=\chi(h) \qquad \text{for all } h\in H,\ \chi\in X(H).$$

Since A is the unique maximal k-trivial torus contained in the center of H, I_s leaves A invariant. Therefore, w_s can be extended to a (uniquely determined) automorphism of $Y=X(A)$, which we denote again by w_s, by

$$(3.9) \qquad (w_s\eta)(sas^{-1})=\eta(a) \qquad \text{for all } a\in A,\ \eta\in Y.$$

The group formed of all w_s $(s\in N(H))$ is called the (restricted) *Weyl group* of G relative to H and is denoted by W_H. The kernel of the homomorphism $s\rightarrow w_s$ being given by $N(H)\cap Z(A)$, one has

$$(3.10) \qquad W_H\cong N(H)/(N(H)\cap Z(A)).$$

As stated above, W_H may be regarded as a subgroup of the Weyl group W_A of G relative to A. In case G is connected, W_A is the Weyl group of the restricted root system $\bar{\mathfrak{r}}$ in the usual sense [**20**].

W_H also operates on $\hat{X}(H)=\mathrm{Hom}(X(H),\mathbf{Z})$ in a natural manner, i.e. by

$$(3.11) \qquad \langle w\omega, w\chi\rangle=\langle\omega,\chi\rangle \qquad \text{for } \chi\in X(H),\ \omega\in\hat{X}(H).$$

Then (in the notation of n° 2.3), for $s\in N(H)$, $h\in H$, one gets from (2.9), (3.8), (3.11) the relation

$$(3.12) \qquad w_s l_h=l_{shs^{-1}}.$$

Thus W_H leaves $M\subset\hat{X}(H)$ invariant.

3.4. *Assumption* (II). Suppose that there is given a subgroup W of W_H such that every $w\in W$ can be written in the form $w=w_u$ with $u\in N(H)\cap U$. As we have seen in n° 3.3, W operates on M. Taking a linear order in M, put

$$(3.13) \qquad \Lambda=\{\mathbf{m}\in M \mid w\mathbf{m}\leqslant\mathbf{m} \ \text{for all}\ w\in W\}.$$

Then it is clear that Λ is a " fundamental domain " of W in M, i.e. every $\mathbf{m}\in M$ is equivalent, under W, to one and only one element in Λ. From (3.1)′, (3.12) and from our assumption on W, it follows that

$$G = \bigcup_{\mathbf{r}\in\Lambda} U\pi^{\mathbf{r}}U. \tag{3.14}$$

Now we state our second assumption:

(II) *The notations being as defined in n^os 2.3, 3.3, there exists a subgroup* W *of* W_H *such that every* $w\in W$ *can be written in the form* $w = w_u$ *with* $u\in N(H)\cap U$ *and a linear order in* M *satisfying the following property: If* $\pi^{\mathbf{m}}N\cap U\pi^{\mathbf{r}}U \neq \emptyset$ *with* $\mathbf{m}\in M, \mathbf{r}\in\Lambda$, *we have* $\mathbf{m}\leqslant\mathbf{r}$, *where* Λ *is a fundamental domain of* W *in* M *defined by* (3.13).

This implies the following weaker condition:

(II_1) *The notations being as above, the double cosets* $U\pi^{\mathbf{r}}U$ ($\mathbf{r}\in\Lambda$) *are mutually distinct.* (In other words, (3.14) is a disjoint union.)

In fact, let $U\pi^{\mathbf{r}}U = U\pi^{\mathbf{r}'}U$ with $\mathbf{r}, \mathbf{r}'\in\Lambda$. Since we have $\pi^{\mathbf{r}'}\in U\pi^{\mathbf{r}}U$, it follows from (II) that $\mathbf{r}'\leqslant\mathbf{r}$. Similarly we have $\mathbf{r}\leqslant\mathbf{r}'$ and so $\mathbf{r}=\mathbf{r}'$.

Under the assumption (II_1), every $g\in G$ can be expressed in the form

$$g = u\pi^{\mathbf{r}}u', \qquad u, u'\in U$$

with a uniquely determined $\mathbf{r}\in\Lambda$; therefore one puts $\mathbf{r} = r(g)$. Then the function $r: G\rightarrow\Lambda$ is characterized by the following properties

$$\begin{aligned} r(ugu') &= r(g) && \text{for all } g\in G,\ u, u'\in U, \\ r(\pi^{\mathbf{r}}) &= \mathbf{r} && \text{for all } \mathbf{r}\in\Lambda; \end{aligned} \tag{3.15}$$

if moreover (II) is satisfied, we have

$$r(\pi^{\mathbf{m}}n) \geqslant \mathbf{m} \qquad \text{for all } \mathbf{m}\in M,\ n\in N. \tag{3.16}$$

The existence of the function r satisfying (3.15), (3.16) (resp. (3.15)) is equivalent to (II) (resp. (II_1)).

We list below some direct consequences of the assumption (II).

1° *If* $\pi^{\mathbf{m}}N\cap U\neq\emptyset$, *we have* $\mathbf{m}=0$. In fact, it follows from (II) that $\mathbf{m}\leqslant 0$. Since we have also $\pi^{-\mathbf{m}}N\cap U = (\pi^{\mathbf{m}}N\cap U)^{-1}\neq\emptyset$, we have $\mathbf{m}\geqslant 0$; hence $\mathbf{m}=0$.

2° For $h\in H$, one has

$$h \equiv u\pi^{r(h)}u^{-1} \pmod{H^{u}} \qquad \text{with } u\in N(H)\cap U. \tag{3.17}$$

It follows that, for $h, h'\in H$, one has $UhU = Uh'U$ if and only if

$$h' \equiv uhu^{-1} \pmod{H^{u}} \qquad \text{with } u\in N(H)\cap U.$$

(Note that this is a consequence of only (II_1).)

3° *$r(h)$ for $h\in H$ is invariant under the inner automorphisms of* G, i.e. *if* $h, h'\in H$ *and* $h' = ghg^{-1}$ *with* $g\in G$, *we have* $r(h') = r(h)$. In fact, it is clear that, in replacing h by uhu^{-1} with $u\in U$, if necessary, we may assume, without any loss of generality, that $l_h \in \Lambda$, i.e. $l_h = r(h)$ (and similarly that $l_{h'} = r(h')$). Now, let $g = uh_1n$ with $u \in \mathrm{U}$, $h_1 \in \mathrm{H}$, $n \in \mathrm{N}$. Then one has

$$h' = ghg^{-1} = uh_1nhn^{-1}h_1^{-1}u^{-1} = u(h_1hh_1^{-1})\,n'u^{-1}$$

with $n' \in \mathrm{N}$. Hence, by (3.15), (3.16), one gets

$$r(h') = r((h_1hh_1^{-1})n') \geqslant l_{h_1hh_1^{-1}} = l_h = r(h).$$

Similarly, one gets $r(h) \geqslant r(h')$; hence $r(h') = r(h)$, as desired.

4° *We have* $\mathrm{W} = \mathrm{W}_{\mathrm{H}} = \mathrm{W}_{\mathrm{A}}$. In fact, if $\mathrm{W} \neq \mathrm{W}_{\mathrm{A}}$, there would be $\mathbf{r} \neq \mathbf{r}'$ in $\Lambda \cap \hat{\mathrm{Y}}$ such that $r' = wr$ with $w \in \mathrm{W}_{\mathrm{A}}$, because $\hat{\mathrm{Y}}$ is of finite index in M and $\Lambda \cap \hat{\mathrm{Y}}$ is a fundamental domain of W in $\hat{\mathrm{Y}}$. Then we would have $\pi^{\mathbf{r}'} \equiv s\pi^{\mathbf{r}}s^{-1}$ (mod. H^{u}) with $s \in \mathrm{N(A)}$, which contradicts 3°.

§ 4. Haar measures.

4.1. In this section, G denotes an algebraic group over k satisfying the assumption (I) with respect to U, H, N. The groups G, U, H, N are then all " unimodular ", i.e. their left-invariant Haar measures are also right-invariant. We denote by dg, du, dh, dn the volume-elements of the (both-sides-invariant) Haar measures of G, U, H, N, respectively, normalized as follows:

$$(4.1)\qquad \int_{\mathrm{U}} dg = \int_{\mathrm{U}} du = \int_{\mathrm{H}^{\mathrm{u}}} dh = \int_{\mathrm{N}\cap\mathrm{U}} dn = 1.$$

Then the left- and right-invariant Haar measures of HN are given by

$$(4.2)\qquad d_l(hn) = dh.dn, \qquad d_r(hn) = \delta(h)dh.dn,$$

δ being a positive quasi-character of H (i.e. a continuous homomorphism of H into the multiplicative group of positive real numbers with respect to the $\mathfrak{p}$-adic topology) defined by

$$(4.3)\qquad d(hnh^{-1}) = \delta(h)dn.$$

For any integrable function f on G, one has

$$\int_{\mathrm{G}} f(g)dg = \int_{\mathrm{U}} \int_{\mathrm{HN}} f(uhn)du.d_r(hn),$$

or symbolically

$$(4.4)\qquad dg = du.d_r(hn) = du.\delta(h)dh\,dn.$$

We need in Chapter II, § 5 the following transformation formula of the relatively invariant measure on $\mathrm{U}/(\mathrm{U}\cap\mathrm{HN})$.

LEMMA 4.1. *Let* $g_0 \in \mathrm{G}$. *For* $u \in \mathrm{U}$, *write* $g_0^{-1}u = u'h'n'$ *with* $u' \in \mathrm{U}, h' \in \mathrm{H}, n' \in \mathrm{N}$. *Then the cosets* $u'(\mathrm{U}\cap\mathrm{HN})$, $h'\mathrm{H}^{\mathrm{u}}$ *are uniquely determined by* g_0 *and* $u(\mathrm{U}\cap\mathrm{HN})$. *Denoting by* $d\bar{u}$ *the volume-element of a relatively invariant measure on* $\mathrm{U}/(\mathrm{U}\cap\mathrm{HN})$, *we have*

$$(4.5)\qquad d\bar{u} = \delta(h')d\bar{u}'.$$

(Note that, since $\delta(\mathrm{H}^{\mathrm{u}}) = 1$, $\delta(h')$ depends only on the coset $h'\mathrm{H}^{\mathrm{u}}$.)

Proof. By (3.4), the first statement is obvious. To prove the second, let h, n be " generic " elements in H, N, respectively, and put $g = uhn$. Then $g_0^{-1}g = u'h'n'hn = u'(h'h)(h^{-1}n'hn)$. Hence, by (4.4) and by the invariance of the Haar measures, one has

$$\begin{aligned} d(g_0^{-1}g) &= du'.\delta(h'h)d(h'h)d(h^{-1}n'hn) \\ &= du'.\delta(h')\delta(h)dh\,dn \\ &= dg = du.\delta(h)dh\,dn, \end{aligned}$$

whence follows (4.5), q.e.d.

4.2. *An integral formula.* Let

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\gamma \in \mathfrak{r}} \mathfrak{g}_\gamma$$

be the decomposition of $\mathfrak{g}$ given in N° 2.1. Since $H \subset Z(A)$, all the subspaces $\mathfrak{g}_\gamma$ $(\gamma \in \mathfrak{r})$ are invariant under Ad h $(h \in H)$, Ad denoting the adjoint representation of G. One denotes by $R_\gamma(h)$ the restriction of Ad h to $\mathfrak{g}_\gamma$. Then $\det(R_\gamma(h))$ is a k-rational character of H, whose restriction to A is equal to $d_\gamma.\gamma$ (in the additive notation), d_γ denoting the dimension of $\mathfrak{g}_\gamma$. Thus, identifying X(H) with a submodule of $Y = X(A)$, one gets $d_\gamma.\gamma \in X(H)$ and

$$\det(R_\gamma(h)) = (d_\gamma.\gamma)(h) \qquad \text{for } h \in H. \tag{4.6}$$

Moreover, taking a so-called Weyl basis of $\mathfrak{g}^{\bar{k}} = \mathfrak{g} \otimes \bar{k}$ ($\bar{k}$=algebraic closure of k), one sees immediately that $\mathfrak{g}_{-\gamma}$ may be identified with the dual of $\mathfrak{g}_\gamma$ with respect to the inner product induced by the Killing form. Since this inner product is invariant under Ad h $(h \in H)$, one has

$$R_{-\gamma} \text{ is equivalent to } {}^tR_\gamma^{-1}. \tag{4.7}$$

Now, for $h \in H$, denote by $\mathrm{Ad}_\mathfrak{n}(h)$ the restriction of Ad h on $\mathfrak{n} = \sum_{\gamma > 0} \mathfrak{g}_\gamma$, and put

$$\begin{aligned} \Delta(h) &= |\det(\mathrm{Ad}_\mathfrak{n}(h) - 1_\mathfrak{n})|_\mathfrak{p} \\ &= \prod_{\gamma > 0} |\det(R_\gamma(h) - 1_\gamma)|_\mathfrak{p}, \end{aligned} \tag{4.8}$$

$1_\mathfrak{n}$, 1_γ denoting the identity transformations on $\mathfrak{n}$, $\mathfrak{g}_\gamma$, respectively. Then we have

LEMMA 4.2. *For* $h \in H$ *with* $\Delta(h) \neq 0$, *the mapping*

$$\Psi_h : N \ni n \longrightarrow n' = hnh^{-1}n^{-1}$$

is an injective rational mapping from N *into itself, the image* $\Psi_h(N)$ *contains a Zariski open set in* N *and one has*

$$dn' = \Delta(h)dn. \tag{4.9}$$

Proof. Every $n \in N$ can be written uniquely in the form

$$n = \exp x = \exp(\sum_{\gamma > 0} x_\gamma) \qquad \text{with } x = \sum_{\gamma > 0} x_\gamma \in \mathfrak{n},\ x_\gamma \in \mathfrak{g}_\gamma$$

and one has, for $h \in H$,

$$hnh^{-1} = \exp(\mathrm{Ad}(h)x) = \exp(\sum_{\gamma > 0} R_\gamma(h)x_\gamma).$$

Therefore one has $hnh^{-1}=n$ if and only if $R_\gamma(h)x_\gamma=x_\gamma$ for all $\gamma>0$; in particular, if $\Delta(h)\neq 0$, the latter condition implies that $x=0$, i.e. $n=1$. Thus, for $h\in H$ with $\Delta(h)\neq 0$, Ψ_h is an injective rational mapping from N into itself. Applying the same consideration to $N^{\bar{k}}$, $\bar{k}$ denoting the algebraic closure of k, one sees that $\Psi_h(N^{\bar{k}})$ contains a Zariski open set in $N^{\bar{k}}$ ([**16**], p. 88, Prop. 4). Now, let $n\in N^{\bar{k}}$. Since one has $\Psi_h(n^\sigma)=\Psi_h(n)^\sigma$ for all automorphisms σ of $\bar{k}$ over k, it follows from the injectivity of Ψ_h that, if $\Psi_h(n)\in N$, one has $n^\sigma=n$ for all σ, i.e. $n\in N$. In other words, one has $\Psi_h(N)=\Psi_h(N^{\bar{k}})\cap N$, which proves that $\Psi_h(N)$ contains a Zariski open set in N.

Now to prove the last assertion, we regard $x\in N$ as a left-invariant vector-field on N (in the algebraic sense) and denote by x_n the tangent vector at $n\in N$ determined by x. Then one has

$$d\Psi_h(x_n)=(\mathrm{Ad}(n).(\mathrm{Ad}(h)-1)x)_{hnh^{-1}n^{-1}}, \tag{4.10}$$

$d\Psi_h$ denoting the " differential " of the rational mapping Ψ_h. In fact, by definitions, one has for any rational function f on N, defined over k and regular at n_0,

$$x_{n_0}(f)=\left[\frac{d}{d\lambda}f(n_0.\exp(\lambda x))\right]_{\lambda=0}$$

and therefore

$$\begin{aligned}(d\Psi_h.x_{n_0})(f)&=x_{n_0}(f\circ\Psi_h)\\&=\left[\frac{d}{d\lambda}f(hn_0.\exp(\lambda x)h^{-1}.\exp(-\lambda x)n_0^{-1})\right]_{\lambda=0}\\&=\left[\frac{d}{d\lambda}f(hn_0h^{-1}n_0^{-1}.(n_0\exp(\lambda\,\mathrm{Ad}(h)x)\exp(-\lambda x)n_0^{-1})\right]_{\lambda=0}\\&=\left[\frac{d}{d\lambda}f(hn_0h^{-1}n_0^{-1}.\exp(\lambda\,\mathrm{Ad}(n_0)(\mathrm{Ad}(h)-1)x)\right]_{\lambda=0}\end{aligned}$$

which proves (4.10). Now if we denote by ω an invariant differential form of the highest degree on N and by ${}^t d\Psi_h$ the linear mapping on the space of differential forms on N extending the dual of $d\Psi_h$, it follows from (4.10) and from the fact that $\det(\mathrm{Ad}_{\mathfrak{n}}(n_0))=1$ that

$${}^t d\Psi_h.\omega_{hnh^{-1}n^{-1}}=\det(\mathrm{Ad}_{\mathfrak{n}}(h)-1_{\mathfrak{n}})\omega_n.$$

Since we have symbolically $dn=|\omega_n|_{\mathfrak{p}}$, up to a constant multiple ([**26**], 2.2), we obtain (4.9), q.e.d.

By the similar argument as above, we get also

$$\begin{aligned}\delta(h)&=|\det(\mathrm{Ad}_{\mathfrak{n}}(h))|_{\mathfrak{p}}\\&=\prod_{\gamma>0}|\det(R_\gamma(h))|_{\mathfrak{p}},\end{aligned}$$

or by (4.6)

$$\delta=\prod_{\gamma>0}|\gamma^{d_\gamma}|_{\mathfrak{p}}\quad\text{(in the multiplicative notation).}\tag{4.11}$$

Put further

$$(4.12) \qquad D(h) = \prod_{\gamma \in \bar{\mathfrak{r}}} |\det(R_\gamma(h) - 1_\gamma)|_{\mathfrak{p}}^{1/2}$$

Then from the definitions and from (4.7) one gets easily the following relations

$$(4.13) \qquad D(h^{-1}) = D(h),$$

$$(4.14) \qquad D(h) = \delta^{-\frac{1}{2}}(h)\Delta(h).$$

It should be also noted that one has

$$(4.15) \qquad \Delta(h_1 h h_1^{-1}) = \Delta(h) \qquad \text{for all } h, h_1 \in H.$$

LEMMA 4.3. *Let f be a (complex-valued) function on G with a compact carrier, satisfying the relation*

$$f(ugu') = f(g) \qquad \textit{for all } g \in G,\ u, u' \in U.$$

Then, for $h \in H$ with $\Delta(h) \neq 0$, we have

$$(4.16) \qquad D(h) \int_{G/A} f(ghg^{-1})d\bar{g} = \delta^{\frac{1}{2}}(h) \int_N f(hn)dn,$$

$d\bar{g}$ denoting the volume-element of a (suitably normalized) relatively invariant measure on G/A (1).

Proof. Since one has $dg = du.dn.dh$ for $g = unh$, one has symbolically $d\bar{g} = du.dn.d\bar{h}$, $d\bar{h}$ denoting the volume-element of a relatively invariant measure on H/A; here we normalize $d\bar{h}$ in such a way that $\int_{H/A} d\bar{h} = 1$. Then the left-hand side of (4.16) is equal to

$$D(h) \int_U \int_N \int_{H/A} f(unh_1 h h_1^{-1} n^{-1} u^{-1})\, du\, dn\, d\bar{h}_1$$

$$= D(h) \int_{H/A} \left(\int_N f(nh_1 h h_1^{-1} n^{-1})\, dn \right) d\bar{h}_1 \qquad \text{(by the assumption)}$$

$$= D(h)\Delta(h^{-1})^{-1} \int_{H/A} \left(\int_N f(h_1 h h_1^{-1} n')\, dn' \right) d\bar{h}_1 \qquad \text{(by (4.9), (4.15))}.$$

Since $h_1 h h_1^{-1} \equiv h \pmod{H^u}$, one gets from the assumption $f(h_1 h h_1^{-1} n') = f(hn')$; therefore, by (4.13), (4.14), this last expression is equal to

$$= \delta^{\frac{1}{2}}(h) \int_N f(hn)\, dn, \text{ q.e.d.}$$

Since $D(h)$ is invariant under the inner automorphisms defined by elements in $N(H)$, this Lemma implies that, if one puts

$$\tilde{f}(h) = \delta^{\frac{1}{2}}(h) \int_N f(hn)\, dn,$$

$\tilde{f}(\pi^{\mathbf{m}})$, viewed as a function of $\mathbf{m} \in M$, is invariant under the operation of the Weyl group W_H.

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(1) This is an analogue of an integral formula of Harish-Chandra ([**13**], p. 261).

CHAPTER II

THEORY OF SPHERICAL FUNCTIONS

§ 5. Zonal spherical functions (1).

5.1. *The algebra* $\mathscr{L}(G, U)$. Let G be a unimodular locally compact group and U a compact subgroup of G. We denote by $\mathscr{L}=\mathscr{L}(G, U)$ the algebra over $\mathbf{C}$ formed of all complex-valued continuous functions φ on G with compact carrier and satisfying the condition

$$(5.1) \qquad \varphi(ugu') = \varphi(g) \qquad \text{for all } g \in G,\ u, u' \in U,$$

the product in $\mathscr{L}$ being defined by the " convolution "

$$\varphi * \psi(g) = \int_G \varphi(gg_1^{-1})\psi(g_1)dg_1.$$

If U is an open subgroup and if the Haar measure of G is normalized in such a way that $\int_U dg = 1$, the characteristic function c_0 of U is the unit element of the algebra $\mathscr{L}$. Moreover, in this case, all $\mathbf{Z}$-valued functions in $\mathscr{L}$ forms a subring, which we denote by $\mathscr{L}(G, U)_{\mathbf{Z}}$. From the arithmetical point of view, it is important to consider this ring.

The following theorem will be proved:

THEOREM 1. *Let* G *be a connected algebraic group over a* $\mathfrak{p}$*-adic number field* k *and* U *an open compact subgroup of* G, *satisfying the assumptions* (I), (II) (*see* § 3), *and let* ν *be the dimension of a maximal* k*-trivial torus in* G. *Then the algebra* $\mathscr{L}(G, U)$ *is an affine algebra of (algebraic) dimension* ν *over* $\mathbf{C}$, i.e. *a (commutative) integral domain with unit element, which is finitely generated over* $\mathbf{C}$ *and of transcendence degree* ν *over* $\mathbf{C}$. *Moreover, if* Λ (*see* n° 3.4) *is generated (as semi-group) by* $\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(l)}$ *and if* $c^{(i)}$ *denotes the characteristic function of* $U\pi^{\mathbf{r}^{(i)}}U$, $\mathscr{L}(G, U)$ *is generated over* $\mathbf{C}$ *by* $c^{(i)}$ $(1 \leqslant i \leqslant l)$.

Remark. If, besides the above assumptions, an additional condition that

$$(5.2) \qquad U\pi^{\mathbf{r}}N \cap U\pi^{\mathbf{r}}U = U\pi^{\mathbf{r}} \qquad \text{for all } \mathbf{r} \in \Lambda$$

is satisfied, we shall see that $\mathscr{L}(G, U)_{\mathbf{Z}}$ is generated (over $\mathbf{Z}$) by $c^{(i)}$ $(1 \leqslant i \leqslant l)$.

(1) For the fundamental concepts on spherical functions, see [11], [22].

5.2. G, U being as above, a complex-valued continuous function ω on G in called a *zonal spherical function* (abbreviated in the following as z.s.f.) on G relative to U, if the following conditions are satisfied:

(i) $\omega(ugu') = \omega(g)$ *for all* $g \in G$, $u, u' \in U$.

(ii) $\omega(1) = 1$.

(iii) *For every* $\varphi \in \mathscr{L}$, ω *is an eigen-function of the integral operator defined by* φ, i.e. *we have*

$$\varphi * \omega = \lambda_\varphi \omega \tag{5.3}$$

with $\lambda_\varphi \in \mathbf{C}$.

We denote by $\Omega = \Omega(G, U)$ the totality of z.s.f. on G relative to U. For $\omega \in \Omega$, we denote λ_φ in (5.3) by $\hat{\omega}(\varphi)$, i.e. we put

$$\hat{\omega}(\varphi) = \int_G \varphi(g)\omega(g^{-1})dg. \tag{5.4}$$

Then it is clear that $\hat{\omega}$ is a homomorphism (of algebras over $\mathbf{C}$) from $\mathscr{L}$ onto $\mathbf{C}$. Conversely, if U is an open subgroup of G, it is easy to see that any non-trivial homomorphism $\omega : \mathscr{L} \to \mathbf{C}$ comes in this way from a uniquely determined z.s.f. ω ([**22**], pp. 366-367). Thus, under the assumptions of Theorem 1, Ω may be viewed as a model of the affine algebra $\mathscr{L}$. More precisely, the correspondence

$$\Omega \ni \omega \longleftrightarrow (\hat{\omega}(c^{(1)}), \ldots, \hat{\omega}(c^{(l)}))$$

gives a bijection of Ω onto an affine variety associated with $\mathscr{L} = \mathbf{C}[c^{(1)}, \ldots, c^{(l)}]$.

5.3. *Construction of z.s.f.* From now on, until the end of § 6, we assume that G is an algebraic group over k satisfying the assumption (I) with respect to the subgroups U, H, N and that the Haar measures are normalized as stated in N° 4.1.

In order to construct z.s.f. depending on complex parameters, we make use of the representations of the " principal series " of G. Namely, let α be a quasi-character of H (i.e. a continuous homomorphism of H into $\mathbf{C}^*$ with respect to the $\mathfrak{p}$-adic topology) and call $\mathscr{H}^\alpha$ the Hilbert space formed of all complex-valued measurable functions f on G satisfying the following conditions

$$f(ghn) = \alpha(h)f(g) \qquad \text{for all } g \in G,\ h \in H,\ n \in N, \tag{5.5}$$

$$\|f\|^2 = \int_U |f(u)|^2 du < \infty, \tag{5.6}$$

the inner product in $\mathscr{H}^\alpha$ being defined by

$$\langle f_1, f_2 \rangle = \int_U \overline{f_1(u)} f_2(u) du.$$

For $g_0 \in G$, $f \in \mathscr{H}^\alpha$, we put

$$(T^\alpha_{g_0} f)(g) = f(g_0^{-1} g). \tag{5.7}$$

Then we have:

PROPOSITION 5.1. *In the above notations, for every* $g_0 \in G$, $f \in \mathscr{H}^\alpha$, *we have* $T^\alpha_{g_0} f \in \mathscr{H}^\alpha$, *and* $T^\alpha_{g_0}$ *is a bounded operator on* $\mathscr{H}^\alpha$. *The correspondence* $g_0 \to T^\alpha_{g_0}$ *is a strongly continuous representation of* G *(with respect to the* $\mathfrak{p}$*-adic topology) by bounded operators on* $\mathscr{H}^\alpha$.

Proof. It is clear that $T^\alpha_{g_0} f$ satisfies the condition (5.5). If we put $\bar{U} = U/(U \cap HN)$, it follows from Lemma 4.1 that

$$\begin{aligned} \|T^\alpha_{g_0} f\|^2 &= \int_{\bar{U}} |f(g_0^{-1}u)|^2 \, d\bar{u} \\ &= \int_{\bar{U}} |f(u'h'n')|^2 \, \delta(h') d\bar{u}' \\ &= \int_{\bar{U}} |f(u')|^2 \, |\alpha(h')|^2 \, \delta(h') d\bar{u}'. \end{aligned}$$

Since $|\alpha(h')|^2 \delta(h')$ is a continuous function of $u(U \cap HN) \in \bar{U}$, we have $|\alpha(h')|^2 \delta(h') \leqslant C$ with a positive constant C. Therefore we have $\|T^\alpha_{g_0} f\|^2 \leqslant C\|f\|^2$, i.e. $T^\alpha_{g_0}$ is a bounded operator on $\mathscr{H}^\alpha$. Moreover, if $g_0 \in U$, we may take as $u' = g_0^{-1}u$, $h' = n' = 1$, so that $T^\alpha_{g_0}$ $(g_0 \in U)$ is a unitary operator. Now the correspondence $g_0 \to T^\alpha_{g_0}$ is clearly a representation; to prove that it is strongly continuous, it is enough to show that, for a fixed $f \in \mathscr{H}^\alpha$, $g_0 \to T^\alpha_{g_0} f$ is strongly continuous at $g_0 = 1$. Since f can be approximated by a continuous function in $\mathscr{H}^\alpha$ as closely as we wish (with respect to the norm in $\mathscr{H}^\alpha$), we may assume that f is continuous. Then f is uniformly continuous on U and our assertion is obvious, q.e.d.

From the above proof, we obtain.

COROLLARY 1. *The representation* T^α *is unitary, if and only if*

$$|\alpha|^2 \delta = 1. \tag{5.8}$$

The representation T^α is called " of the first kind " (relative to U), if there exists an element $\psi \neq 0$ in $\mathscr{H}^\alpha$ such that $T^\alpha_u \psi = \psi$ for all $u \in U$.

COROLLARY 2. *The representation* T^α *is of the first kind, if and only if*

$$\alpha(H^u) = 1. \tag{5.9}$$

Proof. If there exists ψ as stated above, one has $\psi(uhn) = \alpha(h)\psi(1)$, so that $\psi(1) \neq 0$, $\alpha(H^u) = 1$. Conversely if $\alpha(H^u) = 1$, one can define ψ_α satisfying the above condition by putting

$$\psi_\alpha(uhn) = \alpha(h), \quad \text{q.e.d.} \tag{5.10}$$

In case α satisfies condition (5.9), one sees readily that

$$\omega^\alpha(g) = \langle \psi_\alpha, T^\alpha_g \psi_\alpha \rangle = \int_U \psi_\alpha(g^{-1}u) du \tag{5.11}$$

is a z.s.f. on G relative to U ([**22**], p. 370). In particular, if α is a restriction on H of a quasi-character of G, denoted also by α, which is trivial on U and N, we have

$$\omega^\alpha(g) = \alpha^{-1}(g). \tag{5.12}$$

5.4. *Parametrization of z.s.f.* Now we introduce complex parameters in $\{\omega^\alpha\} \subset \Omega$ as follows. If α is a quasi-character of H satisfying (5.9), α is uniquely determined by $\alpha(\pi^{\mathbf{m}})$ ($\mathbf{m}\in\mathrm{M}$) (see N° 2.3), and the correspondence $\mathbf{m}\rightarrow\log_q\alpha(\pi^{\mathbf{m}})$ is a homomorphism from M into $\mathbf{C}$. Therefore one can find $\mathbf{s}\in\mathrm{X(H)}\otimes\mathbf{C}$ such that

$$(5.13)\qquad \alpha(\pi^{\mathbf{m}})=q^{-\mathbf{m}.\mathbf{s}},$$

$\mathbf{m}.\mathbf{s}$ denoting the natural pairing of $\mathbf{m}\in\mathrm{M}\subset\hat{\mathrm{X}}(\mathrm{H})$ and $\mathbf{s}\in\mathrm{X(H)}\otimes\mathrm{C}$; such $\mathbf{s}$ is uniquely determined modulo $\frac{2\pi i}{\log q}\hat{\mathrm{M}}$, where

$$(5.14)\qquad \hat{\mathrm{M}}=\{\mathbf{s}\in\mathrm{X(H)}\otimes\mathbf{C}\,|\,\mathbf{m}.\mathbf{s}\in\mathbf{Z}\ \text{ for all }\ \mathbf{m}\in\mathrm{M}\}.$$

Thus we obtain a canonical isomorphism

$$(5.15)\qquad \mathrm{Hom}(\mathrm{H/H^u},\mathbf{C}^*)\cong\mathrm{X(H)}\otimes\mathbf{C}\Big/\left(\frac{2\pi i}{\log q}\hat{\mathrm{M}}\right).$$

If $\{\chi_1,\ldots,\chi_\nu\}$ is a system of generators of X(H), our correspondence can also be defined by the relation

$$\alpha(h)=\prod_i|\chi_i(h)|_{\mathfrak{p}}^{s_i}\Leftrightarrow\mathbf{s}=\sum_i s_i\chi_i \qquad \text{with } s_i\in\mathbf{C}.$$

When (5.13) holds, we write $\alpha\leftrightarrow\mathbf{s}$. When $\delta^{\frac{1}{2}}\alpha\leftrightarrow\mathbf{s}$, we put $\omega^\alpha=\omega_{\mathbf{s}}$.

Now the operation of $w\in\mathrm{W}=\mathrm{W_H}$ on X(H) can be canonically extended to a $\mathbf{C}$-linear transformation of $\mathrm{X(H)}\otimes\mathbf{C}$, leaving $\hat{\mathrm{M}}$ invariant. Then we have

PROPOSITION 5.2. *We have*

$$(5.16)\qquad \omega_{-\mathbf{s}}(g)=\omega_{\mathbf{s}}(g^{-1}),$$

$$(5.17)\qquad \omega_{w\mathbf{s}}=\omega_{\mathbf{s}} \qquad \textit{for all } w\in\mathrm{W_H}.$$

Proof. As in Lemma 4.1, put $g_0^{-1}u=u'h'n'$. Then $g_0u'=u(h'n')^{-1}=uh'^{-1}n''$ with $n''\in\mathrm{N}$. Hence from the definition and Lemma 4.1, we have

$$\omega_{\mathbf{s}}(g_0^{-1})=\int_{\bar{\mathrm{U}}}\psi_\alpha(g_0u')d\bar{u}'=\int_{\bar{\mathrm{U}}}\alpha(h')^{-1}\delta(h')^{-1}du$$
$$=\int_{\bar{\mathrm{U}}}\psi_{\alpha^{-1}\delta^{-1}}(g_0^{-1}u)d\bar{u}=\omega_{-\mathbf{s}}(g_0),$$

because, if $\delta^{\frac{1}{2}}\alpha\leftrightarrow\mathbf{s}$, one has $\delta^{\frac{1}{2}}(\delta^{-1}\alpha^{-1})\leftrightarrow-\mathbf{s}$. This proves (5.16). The proof of (5.17) (depending on Lemma 4.3) will be given in N° 6.1, q.e.d.

In the following, we denote by $\mathrm{W}.\left(\frac{2\pi i}{\log q}\hat{\mathrm{M}}\right)$ the group of "affine transformations" of $\mathrm{X(H)}\otimes\mathbf{C}$ generated by W and by the group of translations defined by $\frac{2\pi i}{\log q}\hat{\mathrm{M}}$. Since W leaves $\hat{\mathrm{M}}$ invariant, $\mathrm{W}.\left(\frac{2\pi i}{\log q}\hat{\mathrm{M}}\right)$ is actually a semi-direct product of these two groups. From (5.17) it follows that two parameters $\mathbf{s},\mathbf{s}'\in\mathrm{X(H)}\otimes\mathbf{C}$ which are equivalent with respect to $\mathrm{W}.\left(\frac{2\pi i}{\log q}\hat{\mathrm{M}}\right)$ give one and the same z.s.f. Actually we will prove the following.

THEOREM 2. *The assumptions being as in Theorem 1, all z.s.f.* ω *on* G *relative to* U *can be written in the form* $\omega=\omega_{\mathbf{s}}$ *with* $\mathbf{s}\in X(H)\otimes\mathbf{C}$; *and we have* $\omega_{\mathbf{s}}=\omega_{\mathbf{s}'}$ *for* $\mathbf{s}, \mathbf{s}'\in X(H)\otimes\mathbf{C}$, *if and only if* $\mathbf{s}, \mathbf{s}'$ *are equivalent with respect to* $W.\left(\frac{2\pi i}{\log q}\hat{M}\right)$.

Thus $\Omega=\Omega(G, U)$ is analytically isomorphic to the quotient space

$$X(H)\otimes\mathbf{C}/W.\left(\frac{2\pi i}{\log q}\hat{M}\right).$$

§ 6. Proofs of the main results.

6.1. *Fourier transform.* We keep the notations in § 5. For $\varphi\in\mathscr{L}$, we define its *Fourier transform* $\hat{\varphi}$ by

$$\hat{\varphi}(\mathbf{s})=\hat{\omega}_{\mathbf{s}}(\varphi)=\int_G\varphi(g)\omega_{\mathbf{s}}(g^{-1})dg. \tag{6.1}$$

Then we have, if $\delta^{\frac{1}{2}}\alpha\leftrightarrow\mathbf{s}$,

$$\begin{aligned}\hat{\varphi}(\mathbf{s})&=\int_G\int_U\varphi(g)\psi_\alpha(gu)dg\,du && \text{(by (5.11))}\\ &=\int_G\varphi(g)\psi_\alpha(g)dg && \text{(by } (g,u)\to(gu^{-1},u))\\ &=\int_U\int_H\int_N\varphi(uhn)\psi_\alpha(uhn)du.\delta(h)dh\,dn && \text{(by (4.4))}\\ &=\int_H\int_N\varphi(hn)\alpha(h)\delta(h)dh\,dn && \text{(by (5.10)).}\end{aligned}$$

Hence, putting

$$\tilde{\varphi}(h)=\delta^{\frac{1}{2}}(h)\int_N\varphi(hn)dn, \tag{6.2}$$

which depends actually only on the class of h (mod. H^u), we have

$$\hat{\varphi}(\mathbf{s})=\sum_{\mathbf{m}\in M}\tilde{\varphi}(\pi^{\mathbf{m}})\delta^{\frac{1}{2}}\alpha(\pi^{\mathbf{m}})=\sum_{\mathbf{m}\in M}\tilde{\varphi}(\pi^{\mathbf{m}})q^{-\mathbf{m}.\mathbf{s}}. \tag{6.3}$$

Since φ has a compact carrier, one has $\tilde{\varphi}(\pi^{\mathbf{m}})\neq0$ for only finitely many $\mathbf{m}\in M$. Incidentally we notice that (5.17) is equivalent to the fact that the Fourier coefficients $\tilde{\varphi}(\pi^{\mathbf{m}})$ ($\mathbf{m}\in M$) are invariant under the operation of the Weyl group W, which was already established by Lemma 4.3. This remark completes the proof of Proposition 5.2. Now by virtue of (5.17), we can further transform (6.3) in the form

$$\hat{\varphi}(\mathbf{s})=\sum_{\mathbf{r}\in\Lambda}\tilde{\varphi}(\pi^{\mathbf{r}}).\sum_{w:W/W_{\mathbf{r}}}q^{-w\mathbf{r}.\mathbf{s}}, \tag{6.4}$$

the second summation being taken over a complete set of representatives of $W/W_{\mathbf{r}}$, $W_{\mathbf{r}}$ denoting the subgroup of W consisting of all the $w\in W$ leaving $\mathbf{r}$ invariant. Thus, if we denote by $\{\mathbf{m}^{(1)},\ldots,\mathbf{m}^{(\nu)}\}$ a system of generators of M, $\hat{\varphi}$ is a Fourier polynomial (allowing negative powers) in $q^{\mathbf{m}^{(i)}.\mathbf{s}}$ ($1\leqslant i\leqslant\nu$) invariant under W. We denote the totality of such Fourier polynomials by $\mathbf{C}[q^{M.\mathbf{s}}]^W$.

6.2. Up to now, we have not used the connectedness of G and the assumption (II). Now we shall show that, under these assumptions, the correspondence $\varphi \to \hat{\varphi}$ gives actually an isomorphism of $\mathscr{L}(G, U)$ onto $\mathbf{C}[q^{M.s}]^W$.

For $\mathbf{r} \in M$, we denote by $c_\mathbf{r}$ the characteristic function of the double coset $U\pi^\mathbf{r}U$. Then, by assumption (II_1), $\{c_\mathbf{r}\,(\mathbf{r}\in\Lambda)\}$ forms a basis of $\mathscr{L}(G, U)$ (over $\mathbf{C}$). On the other hand, if we put

$$F_\mathbf{r}(s) = \sum_{w:W/W_\mathbf{r}} q^{-w\mathbf{r}.\mathbf{s}}, \tag{6.5}$$

$\{F_\mathbf{r}\,(\mathbf{r}\in\Lambda)\}$ forms a basis of $\mathbf{C}[q^{M.s}]^W$. By (6.4), we have

$$\hat{c}_\mathbf{r} = \sum_{\mathbf{r}'\in\Lambda} \tilde{c}_\mathbf{r}(\pi^{\mathbf{r}'})F_{\mathbf{r}'}. \tag{6.6}$$

Here we have, for every $\mathbf{r}, \mathbf{m} \in M$,

$$\begin{aligned} \tilde{c}_\mathbf{r}(\pi^\mathbf{m}) &= \delta^{\frac{1}{2}}(\pi^\mathbf{m}) \int_N c_\mathbf{r}(\pi^\mathbf{m} n)\,dn \\ &= \delta^{\frac{1}{2}}(\pi^\mathbf{m})\,.\,\text{meas. of } ((\pi^{-\mathbf{m}}U\pi^\mathbf{r}U)\cap N). \end{aligned} \tag{6.7}$$

Hence it follows from (II) that we have $\tilde{c}_\mathbf{r}(\pi^{\mathbf{r}'}) = 0$ for $\mathbf{r}, \mathbf{r}' \in \Lambda$, $\mathbf{r} < \mathbf{r}'$, and it is clear from the definition that $\tilde{c}_\mathbf{r}(\pi^\mathbf{r}) \neq 0$ for all $\mathbf{r} \in \Lambda$. These mean that the infinite matrix $(\tilde{c}_\mathbf{r}(\pi^{\mathbf{r}'}))$ with the indices $\mathbf{r}, \mathbf{r}' \in \Lambda$ arranged in the linear order $<$ is of the lower triangular form with non-zero diagonal elements.

We shall now show that the matrix $(\tilde{c}_\mathbf{r}(\pi^{\mathbf{r}'}))$ has actually an inverse matrix. For that purpose, let T (resp. S) be the torus (resp. semi-simple) part of G, let A′ (resp. A″) be a maximal k-trivial torus in T (resp. S) such that $A = \mathrm{cl}(A'.A'')$, and put $Y_\mathbf{Q} = X(A)\otimes\mathbf{Q}$, $Y'_\mathbf{Q} = X(A')\otimes\mathbf{Q}$, $Y''_\mathbf{Q} = X(A'')\otimes\mathbf{Q}$ and $\hat{Y}_\mathbf{Q} = \hat{X}(A)\otimes\mathbf{Q}$; then $Y_\mathbf{Q}$ is identified with the direct sum of $Y'_\mathbf{Q}$ and $Y''_\mathbf{Q}$, and $\hat{Y}_\mathbf{Q}$ with the dual space of $Y_\mathbf{Q}$ over $\mathbf{Q}$. Moreover, call M″ the intersection of $M \subset \hat{Y}_\mathbf{Q}$ with the annihilator of $Y'_\mathbf{Q}$. From what we have stated in n° 2.2, M″ can be also defined as the submodule of M formed of all $\mathbf{m} \in M$ such that $|\chi(\pi^\mathbf{m})|_\mathfrak{p} = 1$ for all $\chi \in X(G)$. Then we have

Lemma 5.1. *If* $\pi^\mathbf{m}N \cap U\pi^\mathbf{r}U \neq \emptyset$ *with* $\mathbf{m}, \mathbf{r} \in M$, *we have* $\mathbf{m} \equiv \mathbf{r} \pmod{M''}$.

Proof. Let χ be any k-rational character of G. Then, since N has no non-trivial k-rational character, we have $\chi(N) = 1$. On the other hand, we have clearly $|\chi(u)|_\mathfrak{p} = 1$ for all $u \in U$. Therefore, if $g \in \pi^\mathbf{m}N \cap U\pi^\mathbf{r}U$, we have $|\chi(g)|_\mathfrak{p} = |\chi(\pi^\mathbf{m})|_\mathfrak{p} = |\chi(\pi^\mathbf{r})|_\mathfrak{p}$ and so $|\chi(\pi^{\mathbf{m}-\mathbf{r}})|_\mathfrak{p} = 1$. As this holds for all $\chi \in X(G)$, we have $\mathbf{m} - \mathbf{r} \in M''$, q.e.d.

Lemma 5.2. *For every* $\mathbf{r} \in \Lambda$, *the set of* $\mathbf{r}' \in \Lambda$ *such that* $\mathbf{r} \equiv \mathbf{r}' \pmod{M''}$ *and* $\mathbf{r}' < \mathbf{r}$ *is finite in number.*

Proof. We extend the linear order in M to that in $\hat{Y}_\mathbf{Q}$ in a natural manner. Then there exists a $\mathbf{Q}$-linear form L on $\hat{Y}_\mathbf{Q}$, not identically zero, such that, for $x \in \hat{Y}_\mathbf{Q}$, $L(x) > 0$ implies $x > 0$ (and hence that $x > 0$ implies $L(x) \geqslant 0$). For each $w \in W$, $w \neq 1$, we can define an order in $\hat{Y}_\mathbf{Q}$ by

$$x \underset{w}{>} 0 \Leftrightarrow (1 - w)x > 0,$$

which induces a linear order on the factor space of $\hat{Y}_Q$ modulo the subspace formed of all $x \in \hat{Y}_Q$ such that $wx = x$. Hence, similarly as above, there exists a non-trivial linear form L_w on $\hat{Y}_Q$ having the property that $L_w(x) > 0$ implies $x > wx$ (and hence that $x > wx$ implies $L_w(x) \geqslant 0$ and that $x = wx$ implies $L_w(x) = 0$). Therefore, if we put

$$\Lambda_Q = \{x \in \hat{Y}_Q \mid x \geqslant wx \quad \text{for all} \quad w \in W\},$$
$$\Lambda'_Q = \{x \in \hat{Y}_Q \mid L_w(x) \geqslant 0 \quad \text{for all} \quad w \in W\},$$

Λ_Q is contained in Λ'_Q and the interior of Λ'_Q is contained in Λ_Q. Since $W = W_A$ is the Weyl group of the restricted root system of G relative to A, we can conclude from this that we have $\Lambda_Q = \Lambda'_Q$ and that Λ_Q is a (closed) " Weyl chamber " of W. Thus Λ_Q is actually a closed cone defined by $\nu''(=\text{rank } Y'')$ linear inequalities $L_i(x) = L_{w_i}(x) \geqslant 0$ $(1 \leqslant i \leqslant \nu'')$.

Now consider first the case where $X(G) = 1$, i.e. $Y_Q = Y''_Q$. The L_i's being linearly independent, we can write as $L = \Sigma \lambda_i L_i$ with $\lambda_i \in \mathbf{Q}$. Here we assert that all λ_i are > 0. Clearly, it is enough to show that, if $L_i(x) \geqslant 0$ for all i (i.e. $x \in \Lambda_Q$) and $x \neq 0$, then $L(x) > 0$. From the assumption, it follows that $x \geqslant wx$ and so $L(x) \geqslant L(wx)$ for all $w \in W$. But, since $Y'_Q = \{0\}$, we have $\sum_{w \in W} wx = 0$. Hence, if $L(x) \leqslant 0$, we would have $L(wx) = 0$ for all $w \in W$. But this is impossible, because for any $x \neq 0$, the set $\{wx \mid w \in W\}$ contains always ν'' linear independent vectors. It follows that, for any $\mathbf{r} \in \Lambda$, the set $\{x \in \Lambda_Q \mid x \leqslant \mathbf{r}\}$ is bounded, and therefore that $\{\mathbf{r}' \in \Lambda \mid \mathbf{r}' < \mathbf{r}\}$ is finite.

In the general case, we have, from what we have proved above, $L \equiv \Sigma \lambda_i L_i$ (mod. Y'_Q) with $\lambda_i > 0$. Hence, if $x - \mathbf{r}$ is in the annihilator of Y'_Q and $x \leqslant \mathbf{r}$, we have $L(\mathbf{r} - x) = \Sigma \lambda_i L_i(\mathbf{r} - x) \geqslant 0$. Therefore the set formed of all $x \in \Lambda_Q$ satisfying these conditions is bounded, and so the intersection of it with Λ is finite, q.e.d.

From Lemmas 5.1, 2, we see that the matrix $(\widetilde{c}_{\mathbf{r}}(\pi^{\mathbf{r}'}))(\mathbf{r}, \mathbf{r}' \in \Lambda)$ can be decomposed into the direct sum of the (countably many) matrices $(\widetilde{c}_{\mathbf{r}}(\pi^{\mathbf{r}'}))(\mathbf{r}, \mathbf{r}' \in \Lambda, \mathbf{r} \equiv \mathbf{r}' (\text{mod. } M''))$, each of which is of the lower triangular form with respect to a set of indices isomorphic to $\{1, 2, \ldots\}$. Hence $(\widetilde{c}_{\mathbf{r}}(\pi^{\mathbf{r}'}))$ has an inverse matrix of the same form. Thus we conclude that the mapping $\varphi \to \hat{\varphi}$ is an isomorphism (of vector spaces over $\mathbf{C}$) from $\mathscr{L}(G, U)$ onto $\mathbf{C}[q^{M \cdot s}]^W$. Since it is also a homomorphism of algebras over $\mathbf{C}$, we have proved the following theorem:

THEOREM 3. *The assumptions being as in Theorem* 1, *the Fourier transformation* $\varphi \to \hat{\varphi}$ *gives an isomorphism* (*of algebras over* $\mathbf{C}$) *from* $\mathscr{L}(G, U)$ *onto* $\mathbf{C}[q^{M \cdot s}]^W$, *the algebra of all* W-*invariant Fourier polynomials in* $q^{\pm \mathbf{m}^{(i)} \cdot s}$ *with coefficients in* $\mathbf{C}$, $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(\nu)}\}$ *denoting a system of (independent) generators of* M.

Remark 1. The connectedness of G was needed essentially only in the proof of Lemma 5.1. (We used it also in the proof of Lemma 5.2, but this is not indispensable.) As is seen from that proof, this condition may be weakened, without changing the conclusion, into any one of the following conditions, where G^0 denotes the connected component of the neutral element of G :

(O_1) The rank of the character module $X(G)$ is equal to that of $X(G^0)$.

(O_2) In the notation of the assumption (I), we have $G^0=(U\cap G^0)H(U\cap G^0)$.

On the other hand, it follows from Lemma 5.1 that, if the condition (II) is satisfied for a linear order in M, the same is also true for any linear order in M inducing the same order in M''.

Remark 2. The formula (6.7) can be further transformed in the following form:

$$(6.8)\qquad \tilde{c}_{\mathbf{r}}(\pi^{\mathbf{m}})=\delta^{-\frac{1}{2}}(\pi^{\mathbf{m}})\gamma_{\mathbf{rm}},\quad \gamma_{\mathbf{rm}}=\#(U\backslash(U\pi^{\mathbf{m}}N\cap U\pi^{\mathbf{r}}U)).$$

In fact, one first observes that by (4.3)

$$\text{meas. of } (N\cap\pi^{-\mathbf{m}}U\pi^{\mathbf{m}})=\delta(\pi^{\mathbf{m}})^{-1}.$$

On the other hand, it is easy to see that the cosets in $(N\cap\pi^{-\mathbf{m}}U\pi^{\mathbf{m}})\backslash(N\cap\pi^{-\mathbf{m}}U\pi^{\mathbf{r}}U)$ are in one-to-one correspondence with those in $U\backslash(U\pi^{\mathbf{m}}N\cap U\pi^{\mathbf{r}}U)$. From these follows (6.8). Now, if the additional condition (5.2) is satisfied, we have $\gamma_{\mathbf{rr}}=1$ for all $\mathbf{r}\in\Lambda$. Therefore, in that case, the matrix $(\gamma_{\mathbf{rr}'})$ $(\mathbf{r},\mathbf{r}'\in\Lambda)$ and its inverse are *integral*, so that the Fourier transformation $\varphi\to\hat{\varphi}$ gives actually an isomorphism (of rings) of the subring $\mathscr{L}(G,U)_{\mathbf{Z}}=\sum_{\mathbf{r}\in\Lambda}c_{\mathbf{r}}\mathbf{Z}$ of $\mathscr{L}(G,U)$ onto the subring $\sum_{\mathbf{r}\in\Lambda}(\delta^{-\frac{1}{2}}(\pi^{\mathbf{r}})F_{\mathbf{r}})\mathbf{Z}$ of $\mathbf{C}[q^{M.\mathbf{s}}]^W$.

Remark 3. From the definitions, we have

$$\hat{c}_{\mathbf{r}}(\mathbf{s})=\#(U\backslash U\pi^{\mathbf{r}}U)\omega_{\mathbf{s}}(\pi^{-\mathbf{r}}).$$

Therefore we have

$$(6.9)\qquad \omega_{\mathbf{s}}(\pi^{-\mathbf{r}})=\#(U\backslash U\pi^{\mathbf{r}}U)^{-1}\sum_{\substack{\mathbf{r}'\in\Lambda\\ \mathbf{r}'\leqslant\mathbf{r}}}\gamma_{\mathbf{rr}'}.\delta^{-\frac{1}{2}}(\pi^{\mathbf{r}'})F_{\mathbf{r}'},$$

which shows that $\omega_{\mathbf{s}}(\pi^{-\mathbf{r}})$ is a W-invariant Fourier polynomial in $q^{\pm\mathbf{m}^{(i)}.\mathbf{s}}$. But it seems far more difficult to describe how $\omega_{\mathbf{s}}(\pi^{-\mathbf{r}})$ depends on $\mathbf{r}\in\Lambda$. In any case, it is one of fundamental problems in our theory to obtain a handy expression (analogous to the " character formula "?) for $\omega_{\mathbf{s}}(\pi^{-\mathbf{r}})$ or for $\gamma_{\mathbf{rr}'}$.

6.3. *Proofs of Theorems 1, 2.* The first half of Theorem 1 follows from Theorem 3 immediately. To prove the second half, let $\{\mathbf{r}^{(1)},\ldots,\mathbf{r}^{(l)}\}$ be a set of generators of Λ (as semi-group) and put $c^{(i)}=c_{\mathbf{r}^{(i)}}$. (Note that Λ, being defined by a finite number of linear inequalities with integral coefficients, has clearly a finite set of generators.) The notations being as in the preceding paragraph, let Λ_* denote an intersection of Λ with a coset modulo M''; by Lemma 5.2, Λ_* is isomorphic (as ordered set) with the set of natural numbers. We prove, by induction on $\mathbf{r}\in\Lambda_*$, that $F_{\mathbf{r}}$ can be expressed as polynomial in $\hat{c}^{(1)},\ldots,\hat{c}^{(l)}$ with coefficients in $\mathbf{C}$.

For that purpose, denote, for every $\mathbf{r}\in\Lambda$, by $\mathfrak{M}_{\mathbf{r}}$ the vector space over $\mathbf{C}$ generated by $F_{\mathbf{r}'}$ with $\mathbf{r}'\in\Lambda$, $\mathbf{r}'<\mathbf{r}$, $\mathbf{r}\equiv\mathbf{r}'(\text{mod. } M'')$. Then it is clear that we have

$$(6.10)\qquad F_{\mathbf{r}}.F_{\mathbf{r}'}\equiv F_{\mathbf{r}+\mathbf{r}'}\qquad(\text{mod. }\mathfrak{M}_{\mathbf{r}+\mathbf{r}'})$$

for every $\mathbf{r},\mathbf{r}'\in\Lambda$. It follows that

$$(6.11)\qquad F_{\mathbf{r}}.\mathfrak{M}_{\mathbf{r}'}\subset\mathfrak{M}_{\mathbf{r}+\mathbf{r}'},\quad \mathfrak{M}_{\mathbf{r}}.\mathfrak{M}_{\mathbf{r}'}\subset\mathfrak{M}_{\mathbf{r}+\mathbf{r}'}.$$

On the other hand, by n° 5.2, we have

$$\hat{c}_{\mathbf{r}} \equiv \lambda_{\mathbf{r}} F_{\mathbf{r}} \qquad (\text{mod. } \mathfrak{M}_{\mathbf{r}}) \tag{6.12}$$

with $\lambda_{\mathbf{r}} = \gamma_{\mathbf{rr}} \delta^{-\frac{1}{2}}(\pi^{\mathbf{r}}) \neq 0$.

Now let $\mathbf{r} \in \Lambda_*$ and suppose that our assertion is true for all $\mathbf{r}' \in \Lambda_*$, $\mathbf{r}' < \mathbf{r}$. Since $\{\mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(l)}\}$ is a set of generators of Λ, $\mathbf{r}$ can be expressed in the form $\mathbf{r} = \sum_{i=1}^{l} n_i \mathbf{r}^{(i)}$ with $n_i \in \mathbf{Z}$, $n_i \geqslant 0$. Then from (6.10), (6.11), (6.12) we have

$$\prod_i \hat{c}^{(i) n_i} \equiv \prod_i (\lambda_{\mathbf{r}^{(i)}} F_{\mathbf{r}^{(i)}})^{n_i} \equiv (\prod_i \lambda_{\mathbf{r}^{(i)}}^{n_i}) F_{\mathbf{r}} \qquad (\text{mod. } \mathfrak{M}_{\mathbf{r}}).$$

Hence, applying the induction assumption on $F_{\mathbf{r}'} \in \mathfrak{M}_{\mathbf{r}}$, we conclude that $F_{\mathbf{r}}$ is a polynomial in $\hat{c}^{(i)}$'s with coefficients in $\mathbf{C}$. It follows then by Theorem 3 that $\mathscr{L}(G, U)$ is generated over $\mathbf{C}$ by $c^{(i)}$ $(1 \leqslant i \leqslant l)$.

(By the same arguments, replacing $F_{\mathbf{r}}$ by $\delta^{-\frac{1}{2}}(\pi^{\mathbf{r}}) F_{\mathbf{r}}$ and $\mathbf{C}$ by $\mathbf{Z}$ in the above proof, we can also conclude that, under the condition (5.2), $\mathscr{L}(G, U)_{\mathbf{Z}}$ is generated (over $\mathbf{Z}$) by $c^{(i)}$ $(1 \leqslant i \leqslant l)$.)

As to Theorem 2, we have to prove the following two statements:

1° Every $\omega \in \Omega$ can be written in the form $\omega = \omega_{\mathbf{s}}$ with $\mathbf{s} \in X(H) \otimes \mathbf{C}$.

2° We have $\omega_{\mathbf{s}} = \omega_{\mathbf{s}'}$, if and only if

$$\mathbf{s}' \equiv w\mathbf{s} \left(\text{mod. } \frac{2\pi i}{\log q} \hat{M}\right) \qquad \text{with } w \in W.$$

To prove 1°, let ω be any z.s.f. on G relative to U. Let $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(\nu)}\}$ be a system of generators of M and put $X_i = q^{\mathbf{m}^{(i)} \cdot \mathbf{s}}$. Then, by Theorem 3, $\mathscr{L}$ can be identified with the subalgebra of $\mathbf{C}[X_1^{\pm 1}, \ldots, X_\nu^{\pm 1}]$ formed of W-invariant elements. Since $\mathbf{C}[X_1^{\pm 1}, \ldots, X_\nu^{\pm 1}]$ is integral over $\mathscr{L}$, the homomorphism $\hat{\omega} : \mathscr{L} \to \mathbf{C}$ can be extended to a homomorphism, denoted again by $\hat{\omega}$, from $\mathbf{C}[X_1^{\pm 1}, \ldots, X_\nu^{\pm 1}]$ onto $\mathbf{C}$ ([**16**], p. 8, Th. 1, p. 12, Prop. 4). Since $\hat{\omega}(X_i) \neq 0$, one can put $\hat{\omega}(X_i) = q^{s_i}$ with $s_i \in \mathbf{C}$. Take $\mathbf{s} \in X(H) \otimes \mathbf{C}$ such that $\mathbf{m}^{(i)} . \mathbf{s} = s_i$ $(1 \leqslant i \leqslant \nu)$. Then we have $\hat{\omega} = \hat{\omega}_{\mathbf{s}}$, so that $\omega = \omega_{\mathbf{s}}$.

Proof of 2°. For $\mathbf{s}, \mathbf{s}' \in X(H) \otimes \mathbf{C}$, we have $\omega_{\mathbf{s}} = \omega_{\mathbf{s}'}$ if and only if $\hat{\varphi}(\mathbf{s}) = \hat{\varphi}(\mathbf{s}')$ for all $\varphi \in \mathscr{L}$, or, what amounts to the same by Theorem 3, $\hat{\varphi}(\mathbf{s}) = \hat{\varphi}(\mathbf{s}')$ for all $\hat{\varphi} \in \mathbf{C}[q^{\mathrm{M} . \mathbf{s}}]^{\mathrm{W}}$. But this last condition is clearly equivalent to saying that $\mathbf{s}' \equiv w\mathbf{s} \left(\text{mod. } \frac{2\pi i}{\log q} \hat{M}\right)$ with $w \in W$.

§ 7. Homomorphisms.

7.1. Let G, G′ be two algebraic groups satisfying the assumption (I) with respect to U, H, N and to U′, H′, N′, respectively. We suppose further that there are given $W \subset W_H$, resp. $W' \subset W_{H'}$, and their fundamental domains Λ, resp. Λ' (defined by (3.13)) and that G′ satisfies assumption (II_1) with respect to Λ'. Let λ be a homomorphism from G into G′ satisfying the following conditions (i), (ii), (iii). (In this paragraph, it is enough to assume that λ is continuous in the sense of the p-adic topology.)

(i) $\lambda(\mathrm{H})\subset\mathrm{H}'$, $\lambda(\mathrm{N})\subset\mathrm{N}'$, $\lambda(\mathrm{U})\subset\mathrm{U}'$.

Then, since $\lambda(\mathrm{H}^{\mathrm{u}})\subset\mathrm{H}'^{\mathrm{u}}$, λ induces a homomorphism

$$\mathrm{M}(\cong\mathrm{H}/\mathrm{H}^{\mathrm{u}})\to\mathrm{M}'(\cong\mathrm{H}'/\mathrm{H}'^{\mathrm{u}}),$$

which we denote again by λ, such that $\lambda(l_h)=l_{\lambda(h)}$ for all $h\in\mathrm{H}$.

(ii) $\lambda(\Lambda)\subset\Lambda'$,

(iii) U *is normal in* $\lambda^{-1}(\mathrm{U}')$.

It follows from (i), (iii) that

$$\lambda^{-1}(\mathrm{U}')=\bigcup_{\mathbf{m}\in\mathrm{M}_\lambda}\pi^{\mathbf{m}}\mathrm{U}, \qquad \pi^{\mathbf{m}}\mathrm{U}=\mathrm{U}\pi^{\mathbf{m}} \quad \text{for all } \mathbf{m}\in\mathrm{M}_\lambda, \tag{7.1}$$

where M_λ denotes the kernel of $\lambda:\mathrm{M}\to\mathrm{M}'$. More generally, one obtains

LEMMA 7.1. *We have*

$$\lambda^{-1}(\mathrm{U}'\lambda(g)\mathrm{U}')=\mathrm{U}g\lambda^{-1}(\mathrm{U}')=\bigcup_{\mathbf{m}\in\mathrm{M}_\lambda}\mathrm{U}g\pi^{\mathbf{m}}\mathrm{U} =\lambda^{-1}(\mathrm{U}')g\mathrm{U}=\bigcup_{\mathbf{m}\in\mathrm{M}_\lambda}\mathrm{U}\pi^{\mathbf{m}}g\mathrm{U} \quad \textit{(disjoint union)}. \tag{7.2}$$

In other words, we have $\mathrm{U}'\lambda(g_1)\mathrm{U}'=\mathrm{U}'\lambda(g)\mathrm{U}'$ *if and only if we have*

$$\mathrm{U}g_1\mathrm{U}=\mathrm{U}g\pi^{\mathbf{m}}\mathrm{U}(=\mathrm{U}\pi^{\mathbf{m}}g\mathrm{U})$$

with $\mathbf{m}\in\mathrm{M}_\lambda$, *where* $\mathbf{m}$ *is uniquely determined by* g, g_1.

Proof. According to (3.14), one puts $g=u_1\pi^{\mathbf{r}}u_2$, $g_1=u_3\pi^{\mathbf{r}_1}u_4$, $\lambda(g_1)=u_1'\lambda(g)u_2'$ with $u_1,\ldots,u_4\in\mathrm{U}$, $u_1',u_2'\in\mathrm{U}'$, $\mathbf{r},\mathbf{r}_1\in\Lambda$; then one has $\lambda(u_3)\lambda(\pi^{\mathbf{r}_1})\lambda(u_4)=u_1'\lambda(u_1)\lambda(\pi^{\mathbf{r}})\lambda(u_2)u_2'$. Hence from (i), (ii) and (II_1) for G′, one gets $\lambda(\mathbf{r}_1)=\lambda(\mathbf{r})$, so that $\mathbf{r}_1=\mathbf{r}+\mathbf{m}$ with $\mathbf{m}\in\mathrm{M}_\lambda$. Then $g_1=u_3\pi^{\mathbf{r}}\pi^{\mathbf{m}}u_4=(u_3u_1^{-1})g(u_2^{-1}\pi^{\mathbf{m}}u_4)\in\mathrm{U}g\pi^{\mathbf{m}}\mathrm{U}$ by (7.1) (and similarly $g_1\in\mathrm{U}\pi^{\mathbf{m}}g\mathrm{U}$). To show the uniqueness of $\mathbf{m}$, let $\mathrm{U}g\pi^{\mathbf{m}}\mathrm{U}=\mathrm{U}g\pi^{\mathbf{m}'}\mathrm{U}$ with $\mathbf{m},\mathbf{m}'\in\mathrm{M}_\lambda$. Then we have $\mathrm{U}g\pi^{\mathbf{m}}\mathrm{U}=\mathrm{U}g\pi^{\mathbf{m}}\mathrm{U}\pi^{\mathbf{m}'-\mathbf{m}}$, so that $g\pi^{\mathbf{m}+l(\mathbf{m}'-\mathbf{m})}\in\mathrm{U}g\pi^{\mathbf{m}}\mathrm{U}$ for all $l\in\mathbf{Z}$. But this is impossible unless $\mathbf{m}=\mathbf{m}'$, because $\mathrm{U}g\pi^{\mathbf{m}}\mathrm{U}$ is compact and $\{g\pi^{\mathbf{m}_1}\mid\mathbf{m}_1\in\mathrm{M}_\lambda\}$ is discrete, q.e.d.

It follows, in particular, that in the above notation, if $g=g_1$, we have $\mathbf{r}=\mathbf{r}_1$, i.e. G also satisfies (II_1).

Now, let $\mathscr{L}=\mathscr{L}(\mathrm{G},\mathrm{U})$, $\mathscr{L}'=\mathscr{L}(\mathrm{G}',\mathrm{U}')$ and, for $\varphi\in\mathscr{L}$, define $\varphi'\in\mathscr{L}'$ by

$$\varphi'(g')=\begin{cases}\int_{\lambda^{-1}(\mathrm{U}')}\varphi(gv)\,dv=\sum_{\mathbf{m}\in\mathrm{M}_\lambda}\varphi(g\pi^{\mathbf{m}}) \\ =\int_{\lambda^{-1}(\mathrm{U}')}\varphi(vg)\,dv=\sum_{\mathbf{m}\in\mathrm{M}_\lambda}\varphi(\pi^{\mathbf{m}}g) & \text{if } g'\in\mathrm{U}'\lambda(g)\mathrm{U}', \\ 0 & \text{if } g'\notin\mathrm{U}'\lambda(\mathrm{G})\mathrm{U}'.\end{cases} \tag{7.3}$$

From Lemma 7.1 it is clear that φ' is well-defined and belongs to $\mathscr{L}'$. We put $\varphi'=\lambda^*(\varphi)$. The mapping λ^* can also be defined as a linear mapping $\mathscr{L}\to\mathscr{L}'$ such that

$$\lambda^*(c_{\mathbf{r}})=c_{\lambda(\mathbf{r})} \qquad \text{for all } \mathbf{r}\in\Lambda. \tag{7.4}$$

We now assume further that λ satisfies the condition

(iv) $\lambda(G).U'=U'.\lambda(G)$.

Proposition 7.1. *Let* G, G′ *be two algebraic groups over k satisfying* (I), (II_1) *and let* λ *be a homomorphism from* G *into* G′ *satisfying* (i), (ii), (iii), (iv). *Then*

a) λ^* *is a homomorphism (of algebras over* $\mathbf{C}$*) from* $\mathscr{L}(G, U)$ *into* $\mathscr{L}(G', U')$.

b) *For* $\omega'\in\Omega(G', U')$, *we have* $\omega=\omega'\circ\lambda\in\Omega(G, U)$ *and*

$$\hat{\omega}(\varphi)=\hat{\omega}'(\lambda^*(\varphi)) \qquad \text{for all } \varphi\in\mathscr{L}(G, U). \tag{7.5}$$

Proof. *a)* If $\lambda^*(\varphi_i)=\varphi_i'$ $(i=1, 2)$, we have

$$\begin{aligned}(\varphi_1'*\varphi_2')(\lambda(g)) &= \int_{G'}\varphi_1'(\lambda(g)g_1')\varphi_2'(g_1'^{-1})dg_1' \\ &= \int_{\lambda(G)}\int_{U'}\varphi_1'(\lambda(g)\lambda(g_1)u')\varphi_2'(u'^{-1}\lambda(g_1)^{-1})d\lambda(g_1)du' \qquad \text{(by (iv))} \\ &= \int_{\lambda(G)}\varphi_1'(\lambda(gg_1))\varphi_2'(\lambda(g_1)^{-1})d\lambda(g_1) \\ &= \int_{G/\lambda^{-1}(1)}d\lambda(g_1)\int_{\lambda^{-1}(U')}\varphi_1(gg_1v_1)dv_1\int_{\lambda^{-1}(U')}\varphi_2(v_1^{-1}g_1^{-1}v_2)dv_2 \\ & \qquad \text{(by (7.3))},\end{aligned}$$

where the measure on $\lambda(G)\cong G/\lambda^{-1}(1)$ is normalized in such a way that the total measure of the open subgroup $\lambda(G)\cap U'\cong\lambda^{-1}(U')/\lambda^{-1}(1)$ is equal to 1. Hence the last expression is equal to

$$\begin{aligned}&= \int_G\int_{\lambda^{-1}(U')}\varphi_1(gg_2)\varphi_2(g_2^{-1}v_2)dg_2dv_2 \\ &= \lambda^*(\varphi_1*\varphi_2)\lambda(g).\end{aligned}$$

On the other hand, if $g'\notin\lambda(G)U'$, we have clearly

$$(\varphi_1'*\varphi_2')(g')=\lambda^*(\varphi_1*\varphi_2)(g')=0.$$

b) It is clear that $\omega'\circ\lambda$ satisfies the conditions (i), (ii) of z.s.f. If $\lambda^*(\varphi)=\varphi'$, we get quite similarly as above

$$\begin{aligned}(\varphi'*\omega')(\lambda(g)) &= \int_{G'}\varphi'(\lambda(g)g_1')\omega'(g_1'^{-1})dg_1' \\ &= \int_{\lambda(G)}\varphi'(\lambda(gg_1))\omega'(\lambda(g_1)^{-1})d\lambda(g_1) \\ &= \int_{G/\lambda^{-1}(1)}\int_{\lambda^{-1}(U')}\varphi(gg_1v)\omega'\circ\lambda(g_1^{-1})d\lambda(g_1)dv \\ &= \int_G\varphi(gg_2)\omega'\circ\lambda(g_2^{-1})dg_2 \\ &= (\varphi*(\omega'\circ\lambda))(g).\end{aligned}$$

Thus we have

$$\varphi * (\omega' \circ \lambda)(g) = (\varphi' * \omega')(\lambda(g)) = \hat{\omega}'(\varphi') \, . \, \omega'(\lambda(g)),$$

which proves (iii) (in N° 5.1). At the same time, (7.5) is proved, q.e.d.

7.2. *Isogenies.* We first apply the above considerations to the case of a k-isogeny. Let, as in N° 7.1, G, G′ be algebraic groups over k satisfying (I) and suppose that G′ satisfies (II_1); we suppose futher that G, G′ are connected and that $W = W_A, W' = W_{A'}$, where A, A′ denote the unique maximal k-trivial toruses in H, H′, respectively. Let $\lambda : G \rightarrow G'$ be a k-isogeny satisfying (i). Then, A, N and A′, N′ being the subgroups of G, G′, respectively, corresponding under λ, we have

$$\text{(7.6)} \qquad \begin{aligned} &A = \text{the connected component of } \lambda^{-1}(A'), \quad A' = \mathrm{cl}(\lambda(A)), \\ &N = \text{the connected component of } \lambda^{-1}(N'), \quad N' = \lambda(N). \end{aligned}$$

From the maximality of U, we also have

$$\text{(7.7)} \qquad U = \lambda^{-1}(U').$$

Thus (iii) is trivially satisfied. Here $\lambda : M \rightarrow M'$ is injective, and, by our assumption, W, W′ may be identified with each other canonically. Therefore, taking a linear order in M induced from that in M′, we have

$$\text{(7.8)} \qquad \Lambda = \lambda^{-1}(\Lambda'),$$

or $\Lambda = \Lambda' \cap M$, when we consider that $M \subset M'$. Thus (ii) is also satisfied. Therefore, as we stated in 7.1, G satisfies (II_1). Furthermore, it is clear that if G′ satisfies (II), so does also G.

To proceed further, let us first note that λ is not necessarily surjective. In fact, it is known that λ(G) *is a normal subgroup of* G′ *and that* G′/λ(G) *is finite, commutative.* (This is true for any isogeny between connected algebraic groups over a p-adic field. See [**15**], Prop. 3.) It follows, in particular, that λ satisfies the condition (iv).

PROPOSITION 7.2. *The assumptions being as above,* G′/(λ(G)U′) *is canonically isomorphic to* M′/λ(M).

Proof. Since λ(G)U′ is a normal subgroup of G′ containing $N' = \lambda(N)$, we have, by (3.1), $G' = H'.\lambda(G)U'$. Next, we assert that

$$H' \cap (\lambda(G)U') = \lambda(H)H'^{u}.$$

In fact, let $H' \cap (\lambda(G)U') \ni h' = \lambda(g)u'$ with $g \in G$, $u' \in U'$ and put $g = u_1 h u_2$ with $h \in H$, $u_1, u_2 \in U$. Replacing h by uhu^{-1} with $u \in N(H) \cap U$, if necessary, we may assume that $l_{\lambda(h)}$, $l_{h'}$, belong to one and the same fundamental domain $w'\Lambda'$ ($w' \in W'$). Then, from $h' = \lambda(u_1)\lambda(h)\lambda(u_2)u'$, we have $r(h') = r(\lambda(h))$ and so $l_{h'} = l_{\lambda(h)}$, i.e. we obtain $h' = \lambda(h)u''$ with $u'' \in H'^{u}$. This proves the inclusion ⊂; the inverse inclusion is trivial. It follows that

$$\text{(7.8)} \qquad G'/(\lambda(G)U') \cong H'/(\lambda(H)H'^{u}).$$

On the other hand, since $M \cong H/H^u$, $M' \cong H'/H'^u$ and $\lambda(H^u) = \lambda(H) \cap H'^u$, we have

$$H'/(\lambda(H)H'^u) \cong M'/\lambda(M). \tag{7.9}$$

From these, we conclude the Proposition, q.e.d.

We denote by Ξ the character group of $G'/(\lambda(G)U')$ and consider $\xi \in \Xi$ as a function on G'. Then Ξ operates on $\mathscr{L}' = \mathscr{L}(G', U')$ by

$$(\xi.\varphi')(g') = \xi(g').\varphi'(g') \qquad \text{for } \xi \in \Xi,\ \varphi' \in \mathscr{L}'. \tag{7.10}$$

The correspondence $\varphi' \to \xi\varphi'$ is clearly an automorphism of $\mathscr{L}'$ (as algebra over $\mathbf{C}$). We denote by $\mathscr{L}(G', U')^{\Xi}$ the subalgebra of $\mathscr{L}'$ formed of all Ξ-invariant elements. Then we have

PROPOSITION 7.3. *The assumptions being as above, λ^* is an injective homomorphism (of algebras over $\mathbf{C}$) from $\mathscr{L}(G, U)$ into $\mathscr{L}(G', U')$ and its image is equal to $\mathscr{L}(G', U')^{\Xi}$.*

Proof. Since λ satisfies the conditions (i)$\sim$(iv), it follows from Proposition 7.1 that λ^* is a homomorphism. Since $\lambda : M \to M'$ is injective, it follows from (7.4) that λ^* is injective. Finally, it is clear that an element φ' of $\mathscr{L}(G', U')$ belongs to the image of λ^*, if and only if its carrier is contained in $U'\lambda(G)$, and this latter condition is equivalent to saying that φ' is invariant under Ξ, q.e.d.

Remark. It is easy to see from the definition that

$$\lambda^*(\mathscr{L}(G, U)_{\mathbf{Z}}) = \lambda^*(\mathscr{L}(G, U)) \cap \mathscr{L}(G', U')_{\mathbf{Z}} = (\mathscr{L}(G', U')_{\mathbf{Z}})^{\Xi}.$$

On the other hand, Ξ operates also on $\Omega' = \Omega(G', U')$. Namely, as is easily seen, for every $\xi \in \Xi$, $\omega' \in \Omega'$, we have $\xi.\omega' \in \Omega'$ ([**22**], Prop. 5) (in particular, putting $\omega' = 1$ (constant), we have $\xi \in \Omega'$). From the definitions, it is clear that

$$\hat{\omega}'(\xi\varphi') = \widehat{\xi^{-1}\omega'}(\varphi'). \tag{7.11}$$

Now, as stated in N^{os} 5.3, 5.4, a part of Ω' is parametrized by $X(H') \otimes \mathbf{C}/W'.\left(\frac{2\pi i}{\log q}\hat{M}'\right)$; for $\mathbf{s} \in X(H') \otimes \mathbf{C}$, denote by $\omega'_{\mathbf{s}}$ the corresponding z.s.f. on G'. Then we have the following

LEMMA 7.2. *If $\xi | H' \leftrightarrow \mathbf{s}_\xi \left(\mathrm{mod.}\ \frac{2\pi i}{\log q}\hat{M}'\right)$ in the sense of N° 5.4, we have*

$$\xi^{-1}.\omega'_{\mathbf{s}} = \omega'_{\mathbf{s}+\mathbf{s}_\xi}. \tag{7.12}$$

Remark. This Lemma is valid for any quasi-character ξ of G' which is trivial on U' and N'.

Proof. If $\delta^{\frac{1}{2}}\alpha \leftrightarrow \mathbf{s}$, one has

$$\omega'_{\mathbf{s}}(g') = \int_{U'} \psi_\alpha(g'^{-1}u')du'$$

Hence one obtains

$$\begin{aligned} \xi^{-1}.\omega'_{\mathbf{s}}(g') &= \xi(g'^{-1}) \int_{U'} \psi_\alpha(g'^{-1}u')du' \\ &= \int_{U'} \psi_{\alpha\xi}(g'^{-1}u')du', \end{aligned}$$

because $\xi(U')=\xi(N')=1$. As $\xi\leftrightarrow \mathbf{s}_\xi$, one has $\delta^{\frac{1}{2}}\alpha\xi\leftrightarrow \mathbf{s}+\mathbf{s}_\xi$. Hence the last expression is equal to $\omega'_{\mathbf{s}+\mathbf{s}_\xi}(g')$, q.e.d.

By (7.11), (7.12) is equivalent to

$$\widehat{\xi.\varphi'}(\mathbf{s})=\hat{\varphi}'(\mathbf{s}+\mathbf{s}_\xi). \tag{7.13}$$

Next, let us consider the correspondence $\Omega'\ni\omega'\to\omega=\omega'\circ\lambda\in\Omega$ more closely. Let $\lambda_H: H\to H'$ be the restriction of λ on H and ${}^t\lambda_H: X(H')\to X(H)$ its "dual". Then ${}^t\lambda_H$ is injective, with finite cokernel, so that it can be extended canonically to an isomorphism

$$X(H')\otimes\mathbf{C}\to X(H)\otimes\mathbf{C},$$

which we denote again by ${}^t\lambda_H$. Since we have ${}^t\lambda_H(\hat{M}')\subset\hat{M}$, it induces a homomorphism from $X(H')\otimes\mathbf{C}/W'.\left(\frac{2\pi i}{\log q}\hat{M}'\right)$ onto $X(H)\otimes\mathbf{C}/W.\left(\frac{2\pi i}{\log q}\hat{M}\right)$. We now assert that the following diagram is commutative:

$$\begin{array}{ccc} \Omega & \longleftarrow & \Omega' \\ \uparrow & & \uparrow \\ X(H)\otimes\mathbf{C}/W.\left(\frac{2\pi i}{\log q}\hat{M}\right) & \leftarrow & X(H')\otimes\mathbf{C}/W'.\left(\frac{2\pi i}{\log q}\hat{M}'\right) \end{array} \tag{7.14}$$

Namely we shall prove the following

Lemma 7.3. *If* $\mathbf{s}\in X(H)\otimes\mathbf{C}$, $\mathbf{s}'\in X(H')\otimes\mathbf{C}$ *are such that* $\mathbf{s}={}^t\lambda_H(\mathbf{s}')$, *we have*

$$\omega_{\mathbf{s}}=\omega'_{\mathbf{s}'}\circ\lambda, \tag{7.15}$$

or equivalently, by (7.5),

$$\hat{\varphi}(\mathbf{s})=\widehat{\lambda^*\varphi}(\mathbf{s}') \qquad \textit{for all } \varphi\in\mathscr{L}(G, U). \tag{7.16}$$

Proof. In view of (6.3) and the relation $\mathbf{m}.\mathbf{s}=\lambda(\mathbf{m}).\mathbf{s}'$ $(\mathbf{m}\in M)$, it is enough to show that we have

$$\tilde{c}_{\mathbf{r}}(\pi^{\mathbf{m}})=\tilde{c}_{\lambda(\mathbf{r})}(\pi^{\lambda(\mathbf{m})}) \qquad \text{for all } \mathbf{r},\mathbf{m}\in M.$$

By (6.8) and by the coincidence of δ for G and G′, this is equivalent to

$$\#(U\backslash(U\pi^{\mathbf{r}}U\cap U\pi^{\mathbf{m}}N))=\#(U'\backslash(U'\pi^{\lambda(\mathbf{r})}U'\cap U'\pi^{\lambda(\mathbf{m})}N')),$$

which can be proved as follows. From Lemma 7.1, we have $\lambda^{-1}(U'\pi^{\lambda(\mathbf{r})}U')=U\pi^{\mathbf{r}}U$. Since $N'=\lambda(N)\subset\lambda(G)$, it follows that

$$(\lambda(\pi^{\mathbf{m}})N')\cap U'\pi^{\lambda(\mathbf{r})}U'=\lambda(\pi^{\mathbf{m}}N\cap U\pi^{\mathbf{r}}U).$$

This shows that the mapping

$$U\backslash(U\pi^{r}U\cap U\pi^{m}N) \rightarrow U'\backslash(U'\pi^{\lambda(r)}U'\cap U'\pi^{\lambda(m)}N')$$

defined by $Ug \rightarrow U'\lambda(g)$ $(g\in\pi^{m}N\cap U\pi^{r}U)$ is surjective. It is also injective because of the relation $(\lambda(\pi^{m})N')\cap U'=\lambda(\pi^{m}N\cap U)$. This proves our assertion, q.e.d.

The formula (7.16) implies the commutativity of the diagram

$$(7.17)\qquad \begin{array}{ccc} \mathscr{L}(G, U) & \xrightarrow{\lambda^{*}} & \mathscr{L}(G', U') \\ \downarrow & & \downarrow \\ \mathbf{C}[q^{M.s}]^{W} & \longrightarrow & \mathbf{C}[q^{M'.s}]^{W'}, \end{array}$$

where the second horizontal homomorphism is defined by the correspondence $q^{\mathbf{m}.\mathbf{s}} \rightarrow q^{\lambda(\mathbf{m}).\mathbf{s}}$.

Finally, let us suppose that G′ (and hence also G) satisfies the assumption (II). Then it follows that the mapping $\Omega \leftarrow \Omega'$ in (7.14) is surjective and that we have $\omega_1'\circ\lambda=\omega_2'\circ\lambda$ for $\omega_1', \omega_2'\in\Omega'$ if and only if $\omega_1'=\xi.\omega_2'$ with $\xi\in\Xi$. In fact, the "if" part being trivial, suppose that $\omega_{\mathbf{s}_1}'\circ\lambda=\omega_{\mathbf{s}_2}'\circ\lambda$ with $\mathbf{s}_1, \mathbf{s}_2\in X(H')\otimes\mathbf{C}$. Then, by Lemma 7.3, we have ${}^t\lambda_H(\mathbf{s}_1)\equiv{}^t\lambda_H(\mathbf{s}_2)\left(\text{mod. } W.\frac{2\pi i}{\log q}\hat{M}\right)$, and so $\mathbf{s}_1\equiv\mathbf{s}_2\left(\text{mod. } W'.\frac{2\pi i}{\log q}{}^t\lambda_H^{-1}(\hat{M})\right)$. On the other hand, by virtue of Proposition 7.2, we have

$$(7.18)\qquad {}^t\lambda_H^{-1}(\hat{M})/\hat{M}'=\left\{\frac{\log q}{2\pi i}\,\mathbf{s}_\xi(\text{mod. }\hat{M}')\,|\,\xi\in\Xi\right\}.$$

Hence it follows that $\mathbf{s}_2\equiv\mathbf{s}_1+\mathbf{s}_\xi\left(\text{mod. } W'.\frac{2\pi i}{\log q}\hat{M}'\right)$ with $\xi\in\Xi$, and so, by Lemma 7.2, $\omega_{\mathbf{s}_2}'=\xi^{-1}.\omega_{\mathbf{s}_1}'$, which proves the "only if" part. Thus Ω can be identified with Ω'/Ξ. Summing up, we obtain

THEOREM 4. *Let* G, G′ *be two connected algebraic groups over k satisfying assumption* (I) *with respect to* U, H, N *and to* U′, H′, N′, *respectively, and let* λ *be a k-isogeny from* G *to* G′ *such that* $\lambda(H)\subset H'$, $\lambda(N)\subset N'$, $\lambda(U)\subset U'$; *suppose further that* G′ *satisfies* (II) (*with respect to* $W'=W_{A'}$). *Let* Ξ *be the character group of* $G'/\lambda(G)U'$ (*which is a finite commutative group*). *Then,* G *satisfies also* (II) *with respect to* $W=W_A$ *and the linear order in* M *induced from that in* M′; *and*

a) Ξ *operates on* $\mathscr{L}(G', U')$ *by* (7.10), *and* $\mathscr{L}(G, U)$ *can be identified with* $\mathscr{L}(G', U')^{\Xi}$ *by the mapping* λ^{*} *defined by* (7.3). *Moreover the diagram* (7.17) *is commutative.*

b) Ξ *operates on* $\Omega(G', U')$ *similarly, and* $\Omega(G, U)$ *can be identified with* $\Omega(G', U')/\Xi$ *by the mapping* $\Omega(G', U')\ni\omega'\rightarrow\omega'\circ\lambda\in\Omega(G, U)$. *Moreover the diagram* (7.14) *is commutative.*

Remark. Supposing only that (II_1) and Theorem 3 (hence also Theorem 2) hold for G′, U′, instead of assuming the assumption (II) for G′, in the above theorem, we can conclude that Theorem 3 for G, U, as well as the statements *a)*, *b)*, hold. In fact, in view of the diagram (7.17) and Proposition 7.3, it is enough to show that $\mathscr{L}(G', U')^{\Xi}$ and $\mathbf{C}[q^{\lambda(M).s}]^{W'}$ are corresponding under the isomorphism $\mathscr{L}(G', U') \cong \mathbf{C}[q^{M'.s}]^{W'}$. But this follows immediately from (7.13) and (7.18).

7.3. As a second application of N° 7.1, we consider the case where λ is injective. Namely, let G be a k-closed subgroup of G′ (both satisfying (I)), and suppose that λ = identity satisfies the conditions (i), (ii). Then, again from the maximality of A, N, U, we get

$$(7.19)\qquad \begin{aligned} &A = \text{the connected component of } A' \cap G,\\ &N = N' \cap G, \quad U = U' \cap G. \end{aligned}$$

Thus (iii) is trivially satisfied. It follows that, if G′ satisfies (II_1), so does also G. On the other hand, since $H^u = H \cap H'^u$, $\lambda : M \to M'$ is injective, and by what we have stated in N° 3.4, 3°, it can readily be seen that

$$(7.20)\qquad \Lambda = \Lambda' \cap M.$$

Therefore, it is clear that, if G′ satisfies (II), so does also G (with respect to the induced linear order in M).

Now, it is known (cf. N° 8.2) that $G' = GL(n, k)$ satisfies the conditions (I), (II) with respect to

$$\begin{aligned} &H' = A' = D(n, k), \quad N' = T^u(n, k),\\ &U' = GL(n, \mathfrak{o}),\\ &W' = \mathfrak{S}_n \text{ (symmetric group of } n \text{ letters)}; \end{aligned}$$

in this case, M′ is canonically identified with $\mathbf{Z}^n$, and, taking the lexicographical linear order in M′, we have

$$\Lambda' = \{\mathbf{m} = (m_i) \in M' \mid m_1 \geqslant \ldots \geqslant m_n\}.$$

For $G \subset GL(n, k)$, put $U = G \cap GL(n, \mathfrak{o})$. Then, the conditions (I) for G, U and (i), (ii) for λ = identity can be stated as follows:

(I^*) *There exist, in* G, *a connected k-closed subgroup* A *contained in* $D(n, k)$ *and a k-closed subgroup* N *contained in* $T^u(n, k)$, *normalized by* A, *such that we have* $G = U.AN = U.A.U$.

(II^*) *There exists a subgroup* W *of* W_A *such that every* $w \in W$ *can be written in the form* $w = w_u$ *with* $u \in N(A) \cap U$, *and that, for* $\mathbf{m} = (m_i) \in M\ (\subset \mathbf{Z}^n)$ *with* $w\mathbf{m} \leqslant \mathbf{m}$ *for all* $w \in W$ (*with respect to the lexicographical order in* $\mathbf{Z}^n$), *we have* $m_1 \geqslant \ldots \geqslant m_n$.

Therefore, from what we have stated above, we obtain the following:

THEOREM 5 **[19]**. *Let* G *be a (connected) k-closed subgroup of* $GL(n, k)$, $U = G \cap GL(n, \mathfrak{o})$, *and suppose that* G, U *satisfy the conditions* (I^*), (II^*). *Then Theorems* 1, 2, 3 *hold for* G, U.

As is seen from (7.20) and N° 6.2, the connectedness assumption on G is unnecessary. Theorem 5 can be applied, for instance, to $SL(n, k)$, $Sp(n, k)$, $SO(n, k, S)$, taken in a suitable matricial expression.

Example. SO(n, k, S). By definition,

$$SO(n, k, S) = \{g \in GL(n, k) \mid {}^t g S g = S, \det g = 1\},$$

where S is a non-singular symmetric matrix of degree n. By a theorem of Witt, S can be transformed into the following form

$$S = \begin{pmatrix} 0 & 0 & \iota \\ 0 & S_0 & 0 \\ \iota & 0 & 0 \end{pmatrix} \begin{matrix} \}\nu \\ \}n_0, \\ \}\nu \end{matrix} \qquad n = n_0 + 2\nu,$$

$\iota = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$, S_0 = symmetric matrix of degree n_0 corresponding to an " anisotropic " quadratic form (i.e. for $x \in k^{n_0}$, ${}^t x S_0 x = 0$ implies $x = 0$).

Then, from the theory of maximal lattices (cf. § 9), it can be verified that $G = SO(n, k, S)$ satisfies (I*), (II*). But, if we consider the group of similitudes with respect to S instead, it appears that (I*) is not always satisfied.

Remark. Proposition 7.1 cannot be applied, unless the condition (iv) is satisfied. It can surely be applied to SL(n, k), and more generally to the situation considered in Proposition 3.2. In these cases, Lemma 7.3 (i.e. commutativity of the diagram (7.14)) can also be proved.

7.4. Finally we apply our considerations in N° 7.1 to the situation considered in Proposition 3.3. Namely, let G be an algebraic group over k satisfying (I), (II_1) with respect to U, H, N, $W \subset W_H$, let Z be a k-trivial torus contained in the center of G and put

$$(7.21) \qquad \bar{G} = G/Z, \quad \bar{U} = (H_0 U)/Z, \quad \bar{H} = H/Z, \quad \bar{N} = ZN/Z,$$

H_0 being defined by (3.7). Suppose that H_0 normalizes U. Then $\bar{G}$ satisfies (I) (Proposition 3.3). We now prove that $\bar{G}$ satisfies also (II_1). In fact, put

$$(7.22) \qquad M_0 = \{\mathbf{m} \in M \mid \varkappa \mathbf{m} = l_z \ \text{ for some } \ \varkappa \in \mathbf{Z}, z \in Z\}.$$

Then, $M_0 \cong H_0/H^u$ and so $\bar{M} \cong \bar{H}/\bar{H}^u \cong H/H_0 \cong M/M_0$. Since W operates trivially on M_0, the fundamental domain Λ consists of cosets modulo M_0, and $\bar{\Lambda} = \Lambda/M_0$ is clearly a fundamental domain, in $\bar{M}$, of the group $\bar{W} \subset W_H$ ($\bar{W} \cong W$) determined by W in a natural way. (More precisely, we may assume, without any loss of generality, that the linear order in M defining Λ is " adapted " to M_0, i.e. satisfies the condition that, if $\mathbf{m}, \mathbf{m}' \in M$, $\mathbf{m} \notin M_0$, $\mathbf{m} > 0$, $\mathbf{m} \equiv \mathbf{m}' (\text{mod. } M_0)$, then $\mathbf{m}' > 0$. Then, the linear order in M induces in a natural way that in $\bar{M} = M/M_0$, and the fundamental domain of $\bar{W}$ defined by the latter is precisely $\bar{\Lambda}$.) Then, it is clear that

$$\bar{G} = \bigcup_{\mathbf{r} \in \bar{\Lambda}} \bar{U} \pi^{\mathbf{r}} \bar{U} \qquad \text{(disjoint union)},$$

which proves our assertion. It is also clear that, if G satisfies (II) (with respect to a linear order in M adapted to M_0), so does also $\bar{G}$ (with respect to the " induced " linear order in $\bar{M}$).

Now, call λ the canonical homomorphism $G \to \bar{G} = G/Z$. Then, by our assumptions, λ satisfies the condition (i) $\sim$ (iv) (U', H', N', Λ' being replaced by $\bar{U}$, $\bar{H}$, $\bar{N}$, $\bar{\Lambda}$, respectively). We note that, in the notation of N° 7.1, we have $\lambda^{-1}(U') = H_0 U$, $M_\lambda = M_0$.

PROPOSITION 7.4. *The notations and assumptions being as above,* λ^* *is a surjective homomorphism (of algebra over* $\mathbf{C}$*) from* $\mathscr{L}(G, U)$ *onto* $\mathscr{L}(\bar{G}, \bar{U})$, *and its kernel is equal to the ideal generated by* $c_0 - c_{\mathbf{m}}$ $(\mathbf{m} \in M_0)$, *or, what amounts to the same, by* $c_0 - c_{\mathbf{m}^{(i)}}$ $(1 \leq i \leq \nu_0)$, $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(\nu_0)}\}$ $(\nu_0 = \operatorname{rank} M_0)$ *being a system of generators of* M_0.

Remark. It is easy to see that $\lambda^*(\mathscr{L}(G, U)_{\mathbf{Z}}) = \mathscr{L}(\bar{G}, \bar{U})_{\mathbf{Z}}$. Therefore, by this proposition, $\mathscr{L}(\bar{G}, \bar{U})_{\mathbf{Z}}$ can be identified with the factor ring of $\mathscr{L}(G, U)_{\mathbf{Z}}$ by the ideal generated by $c_0 - c_{\mathbf{m}}$ $(\mathbf{m} \in M_0)$.

Proof. The first statement is a direct consequence of Proposition 7.1, (7.4) and what we have stated above. To prove the second, choose a system of representatives $\{\mathbf{r}\}$ of Λ/M_0. Then every $\varphi \in \mathscr{L}(G, U)$ can be written uniquely in the form

$$\varphi = \sum_{\mathbf{r} : \Lambda/M_0} \sum_{\mathbf{m} \in M_0} \lambda_{\mathbf{r},\mathbf{m}} c_{\mathbf{r}+\mathbf{m}} \tag{7.23}$$

with $\lambda_{\mathbf{r},\mathbf{m}} \in \mathbf{C}$, and we have

$$\lambda^*(\varphi) = \sum_{\mathbf{r} : \Lambda/M_0} \Big(\sum_{\mathbf{m} \in M_0} \lambda_{\mathbf{r},\mathbf{m}} \Big) c_{\lambda(\mathbf{r})}.$$

Therefore, we have $\lambda^*(\varphi) = 0$, if and only if

$$\sum_{\mathbf{m} \in M_0} \lambda_{\mathbf{r},\mathbf{m}} = 0 \tag{7.24}$$

for each $\mathbf{r}$. On the other hand, from our assumption on H_0, we have

$$c_{\mathbf{r}} * c_{\mathbf{m}} = c_{\mathbf{m}} * c_{\mathbf{r}} = c_{\mathbf{r}+\mathbf{m}} \qquad \text{for all } \mathbf{r} \in M, \mathbf{m} \in M_0.$$

Hence, putting $c^{(i)} = c_{\mathbf{m}^{(i)}}$, we have, for $\mathbf{m} = \sum_i n_i \mathbf{m}^{(i)}$,

$$c_{\mathbf{r}+\mathbf{m}} = (c^{(1)})^{n_1} * \ldots * (c^{(\nu_0)})^{n_{\nu_0}} * c_{\mathbf{r}},$$

$(c^{(i)})^{n_i}$ denoting the n_i-th power of $c^{(i)}$ with respect to the convolution. Thus our assertion on the kernel of λ^* is reduced to the following, easy, purely algebraic lemma:

LEMMA 7.4. *Let*

$$\varphi(X) = \sum_{n_1, \ldots, n_\nu = -\infty}^{\infty} \lambda_{n_1, \ldots, n_\nu} X_1^{n_1} \ldots X_\nu^{n_\nu} \in \mathbf{C}[X_1^{\pm 1}, \ldots, X_\nu^{\pm 1}].$$

Then, we have $\sum_{n_1, \ldots, n_\nu} \lambda_{n_1, \ldots, n_\nu} = 0$, *if and only if* φ *belongs to the ideal generated by* $1 - X_i$ $(1 \leq i \leq \nu)$.

Now assume that G is connected and satisfies (II) and consider the relation between $\Omega = \Omega(G, U)$ and $\bar{\Omega} = \Omega(\bar{G}, \bar{U})$. By Proposition 7.4, $\bar{\Omega}$ can be identified with the subset of Ω formed of all $\omega \in \Omega$ satisfying the condition

$$\hat{\omega}(c_0 - c_{\mathbf{m}}) = 0 \qquad \text{for all } \mathbf{m} \in M_0,$$

or equivalently, if we write $\omega = \omega_{\mathbf{s}}$ with $\mathbf{s} \in X(H) \otimes \mathbf{C}$,

$$\hat{c}_{\mathbf{m}}(\mathbf{s}) = 1 \qquad \text{for all } \mathbf{m} \in M_0.$$

Now, from (6.3), we have

$$\hat{c}_{\mathbf{m}}(\mathbf{s}) = q^{-\mathbf{m}.\mathbf{s}} \qquad \text{for } \mathbf{m} \in \mathrm{M}_0.$$

In fact, from the definitions, we have $\delta^{\frac{1}{2}}(\pi^{\mathbf{m}}) = 1$ for $\mathbf{m} \in \mathrm{M}_0$; and if $\pi^{\mathbf{m}'}\mathrm{N} \cap \mathrm{U}\pi^{\mathbf{m}}\mathrm{U} \neq \emptyset$ for $\mathbf{m} \in \mathrm{M}_0$, $\mathbf{m}' \in \mathrm{M}$, we have $(\pi^{\mathbf{m}'-\mathbf{m}}\mathrm{N}) \cap \mathrm{U} \neq \emptyset$ and so $\mathbf{m}' = \mathbf{m}$ (N° 3.4, 1°). Hence by (6.7) we have

$$\tilde{c}_{\mathbf{m}}(\pi^{\mathbf{m}'}) = \begin{cases} 1 & \text{if } \mathbf{m}' = \mathbf{m}, \\ 0 & \text{if } \mathbf{m}' \neq \mathbf{m}, \end{cases}$$

which proves our assertion. It follows that, for $\omega_{\mathbf{s}} \in \Omega$, we have $\omega_{\mathbf{s}} \in \bar{\Omega}$, if and only if

$$\text{(7.25)} \qquad \mathbf{m}.\mathbf{s} \equiv 0 \left(\text{mod.} \frac{2\pi i}{\log q} \mathbf{Z}\right) \qquad \text{for all } \mathbf{m} \in \mathrm{M}_0.$$

This result can also be obtained from the commutativity of the following diagram:

$$\text{(7.26)} \qquad \begin{array}{ccc} \Omega & \longleftarrow & \bar{\Omega} \\ \uparrow & & \uparrow \\ \mathrm{X}(\mathrm{H}) \otimes \mathbf{C}/\mathrm{W}.\left(\frac{2\pi i}{\log q}\hat{\mathrm{M}}\right) & \leftarrow & \mathrm{X}(\bar{\mathrm{H}}) \otimes \mathbf{C}/\bar{\mathrm{W}}.\left(\frac{2\pi i}{\log q}\hat{\bar{\mathrm{M}}}\right), \end{array}$$

which can be proved quite similarly as Lemma 7.3. Thus we obtain

PROPOSITION 7.5. *Let the notations and assumptions be as stated at the beginning of the paragraph, assume further that* G *is connected and satisfies* (II) *with respect to* $\mathrm{W} \subset \mathrm{W}_{\mathrm{H}}$ *and a linear order in* M *adapted to* M_0. *Then,* $\bar{\mathrm{G}} = \mathrm{G}/\mathrm{Z}$ *satisfies also* (II) *with respect to* $\bar{\mathrm{W}} = \mathrm{W}_{\bar{\mathrm{A}}}$ *and the induced linear order in* $\bar{\mathrm{M}} = \mathrm{M}/\mathrm{M}_0$; *and* $\bar{\Omega}$ *can be identified with the subset of* Ω *formed of all* $\omega_{\mathbf{s}}$ *such that*

$$\mathbf{m}.\mathbf{s} \equiv 0 \left(\text{mod.} \frac{2\pi i}{\log q} \mathbf{Z}\right) \qquad \textit{for all } \mathbf{m} \in \mathrm{M}_0.$$

Moreover, the diagram (7.26) *is commutative.*

Chapter III

CASES OF CLASSICAL GROUPS

§ 8. Cases of $GL(n, \mathfrak{K})$, $SL(n, \mathfrak{K})$ and $PL(n, \mathfrak{K})$.

8.1. *Theory of elementary divisors.* Let $\mathfrak{K}$ be a central division algebra of degree d over k. The unique maximal order and its unique (two-sided) prime ideal are denoted by $\mathfrak{O}$, $\mathfrak{P}=(\Pi)$, respectively, Π denoting a generator of $\mathfrak{P}$. As is well-known, we have $\mathfrak{p}\mathfrak{O}=\mathfrak{P}^d$, $N(\mathfrak{P})=\mathfrak{p}$, N denoting the reduced norm of $\mathfrak{K}/k$.

Let V be an n-dimensional right vector space over $\mathfrak{K}$. A subset L of V is called an $\mathfrak{O}$-*lattice*, if L is a finitely generated (right) $\mathfrak{O}$-submodule of V such that $L\mathfrak{K}=V$. For any subset X and an $\mathfrak{O}$-lattice L in V, we put

$$(L:X)=\mathrm{Min}\{\mathrm{ord}_{\mathfrak{P}}\xi \mid \xi\in\mathfrak{K},\ X\xi\subset L\}.$$

An element x of V is called " primitive " in L, if $(L:x)=0$.

Then the following statements are fundamental (see, for instance, [**6**]):

(E_1) *For any $\mathfrak{O}$-lattice L in V, there exists a basis $(e_1, \ldots, e_n)$ such that*

$$L=\sum_{i=1}^{n} e_i\mathfrak{O}. \tag{8.1}$$

Remark. As e_1, we may take any primitive vector in L. It follows that the group of units U (relative to L) operates transitively on the set of all primitive vectors in L.

(E_2) *Let* L, L′ *be two $\mathfrak{O}$-lattices in* V. *Then there exists a basis $(e_1, \ldots, e_n)$ of* V *such that*

$$L=\sum_i e_i\mathfrak{O},\qquad L'=\sum_i e_i\mathfrak{P}^{r_i} \tag{8.2}$$

with $r_1\geqslant\ldots\geqslant r_n$. The ordered set of integers $(r_1, \ldots, r_n)$ is uniquely determined, independently of the choice of the basis $(e_1, \ldots, e_n)$.

We call $(r_1, \ldots, r_n)$ the (exponential) *elementary divisors* of L′ relative to L and denote it by $e(L':L)$:

$$e(L':L)=(r_1, \ldots, r_n).$$

Remark 1. We have

$$(L':L)=r_1,\qquad (L:L')=-r_n. \tag{8.3}$$

Remark 2. As e_1, we may take any vector such that $(L:e_1)=0$, $(L':e_1)=r_1$.

8.2. *Proofs of* (I), (II) *for* GL(n, $\mathfrak{K}$). We now fix an $\mathfrak{O}$-lattice L in V and a basis $(e_1, \ldots, e_n)$ of V satisfying (8.1) once for all, and set as follows:

$$\begin{aligned} G &= \text{the group of all (non-singular) } \mathfrak{K}\text{-linear transformations of V} \\ &= GL(n, \mathfrak{K}), \\ U &= \{u \in G \mid uL = L\} = GL(n, \mathfrak{O}), \\ H &= \{h \in G \mid he_i = e_i\xi_i \text{ with } \xi_i \in \mathfrak{K}\ (1 \leqslant i \leqslant n)\} \\ &= D(n, \mathfrak{K}) \cong (\mathfrak{K}^*)^n, \\ N &= \{n \in G \mid ne_i = e_i + \sum_{j<i} e_j\xi_{ji} \text{ with } \xi_{ji} \in \mathfrak{K}\ (1 \leqslant i \leqslant n)\} \\ &= T^u(n, \mathfrak{K}). \end{aligned}$$

Then, it is clear that G may be viewed as a connected algebraic group over k, of which U is an open compact subgroup, H a connected k-closed subgroup consisting of only semi-simple elements and N a k-unipotent subgroup normalized by H. Moreover, the center of H is equal to $A = D(n, k)$ and we have $H = Z(A)$. The condition (3.2) is obviously satisfied.

Now systems of generators of $Y = X(A)$ and of X(H) are given, respectively, by

$$\eta_i : A \ni a = \text{diag.}(\xi_i) \longrightarrow \xi_i \qquad (1 \leqslant i \leqslant n) \tag{8.4}$$

and by

$$\chi_i : H \ni h = \text{diag.}(\xi_i) \longrightarrow N\xi_i \qquad (1 \leqslant i \leqslant n), \tag{8.5}$$

N denoting the reduced norm of $\mathfrak{K}/k$. Here N is surjective, so that H satisfies the condition (N). We have $\chi_i = d\eta_i$ on A. Hence, if we identify $\hat{Y}$ with $\mathbf{Z}^n$ by means of $(\eta_1, \ldots, \eta_n)$ and if we consider as $\hat{Y} \subset M \subset X(H) \subset \mathbf{Q}^n$, we have

$$M = \hat{X}(H) = \frac{1}{d}\mathbf{Z}^n. \tag{8.6}$$

For $\mathbf{m} = (m_1/d, \ldots, m_n/d) \in M$, we put

$$\pi^{\mathbf{m}} = \text{diag.}(\Pi^{m_1}, \ldots, \Pi^{m_n}). \tag{8.7}$$

This notation is concordant with that introduced in N° 2.3.

It is clear that (in the notation of N° 3.4) we may take, as a group W, the symmetric group $\mathfrak{S}_n$ of n letters, operating on M as a group of permutations of the coordinates. Then, taking a lexicographical linear order in $M \subset \mathbf{Q}^n$, we have

$$\Lambda = \{\mathbf{m} \in M \mid m_1 \geqslant \ldots \geqslant m_n\}. \tag{8.8}$$

Applying (E_2) to $L' = gL$, we see that we have $g \in U\pi^{\mathbf{r}}U$ with $\mathbf{r} \in \Lambda$, if and only if $e(gL : L) = d\mathbf{r}$. ($e(gL : L)$ is called the " elementary divisors " of g relative to L). Therefore we have

$$G = \bigcup_{\mathbf{r} \in \Lambda} U\pi^{\mathbf{r}}U \qquad \text{(disjoint union).}$$

This proves $G = UHU$ and (II_1) with

$$r(g) = \frac{1}{d} e(gL : L). \tag{8.9}$$

We now give proofs for the remaining parts of the assumptions (I), (II).

Proof of $G = U.HN$. Let $g \in G$ and put $(L : ge_1) = -m_1$. Since U is transitive on the set of all primitive elements in L, there exists $u \in U$ such that $ge_1 \Pi^{-m_1} = ue_1$. This means that we have

$$g = u \begin{pmatrix} \Pi^{m_1} & * \\ 0 & g_1 \end{pmatrix}$$

where g_1 is (a matrix of) a linear transformation induced by g on $V/e_1\mathfrak{K}$ (with respect to the basis $(e_2, \ldots, e_n) (\mathrm{mod.}\ e_1\mathfrak{K})$). Therefore, by induction on n, we get the assertion.

Proof of (3.16). Let $g = \pi^{\mathbf{m}} n$ with $\mathbf{m} = (m_1/d, \ldots, m_n/d)$, $n \in N$, and let $\mathbf{r} = r(g) = (r_1/d, \ldots, r_n/d)$. We have to prove that $\mathbf{r} \geqslant \mathbf{m}$. First, since $(gL : L) = r_1$, we have $e_1 \Pi^{r_1} \in gL$; on the other hand, $ge_1 = e_1 \Pi^{m_1}$ is primitive in gL. Hence one gets $r_1 \geqslant m_1$. If $r_1 = m_1$, it follows from Remark 2 to (E_2) that there exists a basis $(e_1, e'_2, \ldots, e'_n)$ of V such that

$$L = e_1 \mathfrak{O} + \sum_{i=2}^{n} e'_i \mathfrak{O},$$

$$gL = e_1 \mathfrak{P}^{r_1} + \sum_{i=2}^{n} e'_i \mathfrak{P}^{r_i}.$$

This implies that, if we put $g = \begin{pmatrix} \Pi^{r_1} & * \\ 0 & g_1 \end{pmatrix}$, the elementary divisors of g_1 (relative to $\bar{L} = L/e_1\mathfrak{O}$) is given by $(r_2, \ldots, r_n)$. Hence, if one proceeds by induction on n, one gets $(r_2, \ldots, r_n) \geqslant (m_2, \ldots, m_n)$ by the induction assumption, q.e.d.

Remark. If $\mathbf{r} = \mathbf{m}$, we see from the above proof that the e'_i may be chosen in such a way that $e'_i \equiv e_i (\mathrm{mod.} \sum_{j=1}^{i-1} e_j \mathfrak{O})$. It follows that we may write $g = u\pi^{\mathbf{r}}$ with $u \in U \cap N$. In other words, we get

$$\pi^{\mathbf{r}} N \cap U \pi^{\mathbf{r}} U = (U \cap N) \pi^{\mathbf{r}}.$$

This proves that the additional condition (5.2) is also satisfied. (Hence the same is also true for any algebraic group $G \subset GL(n, k)$ satisfying (I^*), (II^*) stated in N° 7.3.)

8.3. It is now established that all results in §§ 5-6 can be applied to $G = GL(n, \mathfrak{K})$, $U = GL(n, \mathfrak{O})$. We shall determine here the isomorphism $\mathscr{L}(G, U) \cong \mathbf{C}[q^{M.s}]^W$ more explicitly. We identify $X(H) \otimes \mathbf{C}$ with $\mathbf{C}^n$ by means of the basis $(\chi_1, \ldots, \chi_n)$. Then it is clear from the definition that for $\mathbf{m} = (m_1/d, \ldots, m_n/d) \in M$, $\mathbf{s} = (s_1, \ldots, s_n) \in X(H) \otimes \mathbf{C}$, we have

$$\mathbf{m}.\mathbf{s} = \sum_{i=1}^{n} m_i s_i. \tag{8.10}$$

On the other hand, it can readily be seen that for $h = \mathrm{diag.}(\xi_i)$

$$\delta(h) = \prod_{i=1}^{n} |N(\xi_i)|_{\mathfrak{p}}^{(n+1-2i)d}. \tag{8.11}$$

Therefore, if $\delta^{\frac{1}{2}}\alpha \leftrightarrow \mathbf{s}$ in the sense of N° 5.4, we have

$$\alpha(h) = \prod_{i=1}^{n} |\mathrm{N}(\xi_i)|_{\mathfrak{p}}^{s_i+(i-\frac{n+1}{2})d},$$

or equivalently,

$$\alpha(\pi^{\mathbf{m}}) = q^{-\sum_i m_i(s_i+(i-\frac{n+1}{2})d)}. \tag{8.12}$$

Now, $\mathbf{C}[q^{\mathrm{M}\cdot\mathbf{s}}]^{\mathrm{W}}$ is the algebra of all symmetric Fourier polynomials in $q^{\pm s_1}, \ldots, q^{\pm s_n}$. It is therefore generated over $\mathbf{C}$ by the fundamental symmetric polynomials

$$\mathrm{F}_{\mathbf{r}^{(\varkappa)}}(\mathbf{s}) = \sum_{i_1<\ldots<i_\varkappa} q^{-(s_{i_1}+\ldots+s_{i_\varkappa})} \qquad (1 \leqslant \varkappa \leqslant n)$$

and $\mathrm{F}_{\mathbf{r}^{(n)}}(\mathbf{s})^{-1} = q^{\Sigma s_i}$, where

$$\mathbf{r}^{(\varkappa)} = \Big(\overbrace{\frac{1}{d}, \ldots, \frac{1}{d}}^{\varkappa}, 0, \ldots, 0\Big). \tag{8.13}$$

Putting $c^{(\varkappa)} = c_{\mathbf{r}^{(\varkappa)}}$, we have the following relation

$$\hat{c}^{(\varkappa)} = q^{\frac{1}{2}\varkappa(n-\varkappa)d}\,\mathrm{F}_{\mathbf{r}^{(\varkappa)}}. \tag{8.14}$$

In fact, if in (6.6) $\tilde{c}^{(\varkappa)}(\pi^{\mathbf{r}}) \neq 0$ with $\mathbf{r} = (r_i/d) \in \Lambda$, we have by N° 6.2, $\mathbf{r} \leqslant \mathbf{r}^{(\varkappa)}$, $\Sigma r_i = \varkappa$, but this is possible only for $\mathbf{r} = \mathbf{r}^{(\varkappa)}$. Hence by (6.8), (5.2), (8.11), we have

$$\tilde{c}^{(\varkappa)}(\pi^{\mathbf{r}}) = \begin{cases} \delta^{-\frac{1}{2}}(\pi^{\mathbf{r}^{(\varkappa)}}) = q^{\frac{1}{2}\varkappa(n-\varkappa)d} & \text{for } \mathbf{r} = \mathbf{r}^{(\varkappa)}, \\ 0 & \text{for } \mathbf{r} \neq \mathbf{r}^{(\varkappa)}, \end{cases}$$

which proves our assertion. Thus, as a special case of Theorem 1, we obtain the following

THEOREM 6 (1). *Let* $\mathrm{G} = \mathrm{GL}(n, \mathfrak{K})$, $\mathrm{U} = \mathrm{GL}(n, \mathfrak{O})$, *where* $\mathfrak{K}$ *is a central division algebra over* k *and* $\mathfrak{O}$ *is the maximal order in* $\mathfrak{K}$, *and let* $c^{(i)}$ *be as defined above. Then* $\mathscr{L}(\mathrm{G}, \mathrm{U})$ *is isomorphic to the polynomial algebra* $\mathbf{C}[\mathrm{X}_1, \ldots, \mathrm{X}_{n-1}, \mathrm{X}_n^{\pm 1}]$ *by the correspondence* $c^{(i)} \to \mathrm{X}_i$ $(1 \leqslant i \leqslant n)$.

As remarked after Theorem 1, we can also conclude that $\mathscr{L}(\mathrm{G}, \mathrm{U})_{\mathbf{Z}}$ is a polynomial ring $\mathbf{Z}[c^{(1)}, \ldots, c^{(n-1)}, c^{(n)\pm 1}]$.

8.4. *Groups isogeneous to* $\mathrm{SL}(n, \mathfrak{K})$. The notations being as before, we consider first the case of

$$\bar{\mathrm{G}} = \mathrm{PL}(n, \mathfrak{K}) = \mathrm{GL}(n, \mathfrak{K})/\mathrm{Z},$$

where

$$\mathrm{Z} = \text{the center of } \mathrm{GL}(n, \mathfrak{K}) = \{\xi 1_n \mid \xi \in k^*\}.$$

In the notation of Proposition 3.3 and N° 7.4, one has

$$\mathrm{H}_0 = \{\mathrm{diag.}(\xi_1, \ldots, \xi_n) \mid \xi_i \in \mathfrak{K}^*, \mathrm{ord}_{\mathfrak{P}}\xi_1 = \ldots = \mathrm{ord}_{\mathfrak{P}}\xi_n\}, \tag{8.15}$$

which surely normalizes U. Therefore, by Proposition 3.3, we see that $\bar{\mathrm{G}}$ satisfies the assumption (I) with respect to

$$\bar{\mathrm{U}} = (\mathrm{H}_0\mathrm{U})/\mathrm{Z}, \quad \bar{\mathrm{H}} = \mathrm{H}/\mathrm{Z}, \quad \bar{\mathrm{N}} = \mathrm{NZ}/\mathrm{Z}. \tag{8.16}$$

(1) This result was first obtained by Tamagawa [23] by a different method.

Moreover, denoting always the things relative to $\bar{G}$ by the corresponding symbols with bar, we have

$$\begin{aligned} \bar{A} &= A/Z, \\ \bar{Y} &= X(\bar{A}) = \{\sum_i m_i\eta_i \mid \sum_i m_i = 0\} \subset Y, \\ X(\bar{H}) &= \{\sum_i m_i\chi_i \mid \sum_i m_i = 0\} \subset X(H), \\ \bar{M} &= M/M_0, \quad M_0 = \left\{\frac{1}{d}(m, \ldots, m) \mid m \in \mathbf{Z}\right\}. \end{aligned} \tag{8.17}$$

If we take, as $\bar{W}$, the group induced by $W = \mathfrak{S}_n$ on $\bar{M}$, and, as the linear order in $\bar{M}$, that induced from the lexicographical order in M, we have

$$\bar{\Lambda} = \Lambda/M_0.$$

Thus the assumption (II) is also satisfied.

Moreover, $X(\bar{H}) \otimes \mathbf{C}$ may be identified with the linear subspace

$$\{\mathbf{s} = (s_1, \ldots, s_n) \mid \sum_i s_i = 0\}$$

of $X(H) \otimes \mathbf{C} = \mathbf{C}^n$, and if we denote by $\bar{\mathbf{m}}$ the class of $\mathbf{m} \in M = \frac{1}{d}\mathbf{Z}^n$ modulo M_0, $\bar{\mathbf{m}}.\mathbf{s}$ ($\bar{\mathbf{m}} \in \bar{M}$, $\mathbf{s} \in X(\bar{H}) \otimes \mathbf{C}$) is given by the same formula as (8.12). On the other hand, we have $\bar{\delta}(\pi^{\bar{\mathbf{m}}}) = \delta(\pi^{\mathbf{m}})$, and $\alpha(\pi^{\bar{\mathbf{m}}}) = \alpha(\pi^{\mathbf{m}})$ for all $\alpha \in \mathrm{Hom}(\bar{H}/\bar{H}^u, \mathbf{C}^*) = \mathrm{Hom}(H/H_0, \mathbf{C}^*)$.

Now, by Proposition 7.4, we have

$$\mathscr{L}(\bar{G}, \bar{U}) = \mathscr{L}(G, U)/(c_0 - c^{(n)}); \tag{8.18}$$

by Theorem 6, this is isomorphic to

$$\begin{aligned} &\mathbf{C}[X_1, \ldots, X_{n-1}, X_n^{\pm 1}]/(1 - X_n) \\ &\cong \mathbf{C}[X_1, \ldots, X_{n-1}] \end{aligned}$$

by the correspondence $\bar{c}^{(i)} = c_{\mathbf{r}(i)} \to X_i$ $(1 \leqslant i \leqslant n-1)$.

Now we have $X(G) = \{\tilde{N}\}$, $\tilde{N}$ denoting the reduced norm of $M_n(\mathfrak{K})/k$, and it is known that all (connected) algebraic groups isogeneous to $PL(n, \mathfrak{K})$ are given by

$$G^{(r)} = \{(g, \xi) \in GL(n, \mathfrak{K}) \times k^* \mid \tilde{N}(g) = \xi^{nd/r}\}/\{(\xi 1_n, \xi^r) \mid \xi \in k^*\} \tag{8.19}$$

where r is a positive integer dividing nd. Applying Propositions 3.1, 3.2, 3.3 and N° 7.2, one sees at once that all these groups satisfy (I), (II) with respect to the maximal compact subgroups

$$U^{(r)} = \{(\xi 1_n.u, \eta) \mid \xi \in \mathfrak{K}^*, u \in U, \eta \in k^*, \mathrm{ord}_{\mathfrak{P}}\xi = \frac{d}{r}\mathrm{ord}_{\mathfrak{p}}\eta\}/\{(\xi 1_n, \xi^r) \mid \xi \in k^*\}. \tag{8.20}$$

In particular, we have

$$G^{(1)} = G^1 = SL(n, \mathfrak{K}) = \{g \in GL(n, \mathfrak{K}) \mid \tilde{N}(g) = 1\}.$$

Denoting always the things relative to $G^{(1)}$ by the corresponding symbols with superscript$^{(1)}$, we have

$$
\begin{aligned}
&H^{(1)} = H \cap G^{(1)} = \{h = \mathrm{diag.}(\xi_i) \mid \prod_i N(\xi_i) = 1\},\\
&A^{(1)} = (\text{the connected component of } A \cap G^{(1)}) = \{a = \mathrm{diag.}(\xi_i) \mid \prod_i \xi_i = 1\},\\
&Y^{(1)} = X(A^{(1)}) = Y/\mathbf{Z}(\textstyle\sum_i \eta_i),\\
(8.21) \qquad &X(H^{(1)}) = X(H)/\mathbf{Z}(\textstyle\sum_i \chi_i),\\
&M^{(1)} = \{\mathbf{m} = (m_i/d) \in M \mid \textstyle\sum_i m_i = 0\},\\
&W^{(1)} = (\text{restriction of } W = \mathfrak{S}_n \text{ on } M^{(1)}),\\
&\Lambda^{(1)} = \Lambda \cap M^{(1)}.
\end{aligned}
$$

$X(H^{(1)}) \otimes \mathbf{C}$ may be identified with the factor space

$$X(H) \otimes \mathbf{C}/(\textstyle\sum_i \chi_i)\mathbf{C} = \mathbf{C}^n/\{(s, \ldots, s) \mid s \in \mathbf{C}\};$$

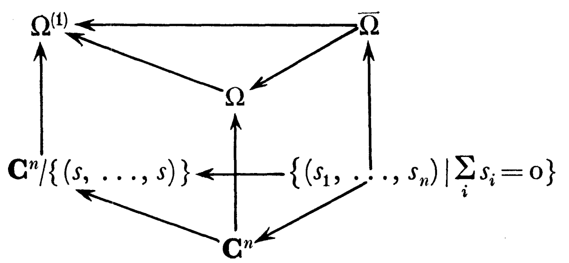
and if we denote by $\bar{\mathbf{s}}$ the class of $\mathbf{s} \in X(H) \otimes \mathbf{C}$ modulo $(\Sigma\chi_i)\mathbf{C}$,

$$\mathbf{m}.\bar{\mathbf{s}} \quad (\mathbf{m} \in M^{(1)},\ \bar{\mathbf{s}} \in X(H^{(1)}) \otimes \mathbf{C})$$

is given by the same formula as (8.10), and formulas (8.11), (8.12) remain true for $h, \pi^{\mathbf{m}} \in H^{(1)}$.

From the results in § 7 (Th. 4, Rem. in N° 7.3, Prop. 7.5), the following diagram is commutative

(8.22)



Finally, let λ be the canonical isogeny $G^{(r)} \to \bar{G}$, and put

$$\Xi_r = \text{character group of } \bar{G}/\lambda(G^{(r)})\bar{U}.$$

We have $\lambda(G^{(r)})\bar{U} = (\tilde{N}^{-1}(k^{*\frac{d}{r}}).H_0U)/Z$, and it is easy to see that, for $g \in G$, we have $g \in \tilde{N}^{-1}(k^{*\frac{nd}{r}}).H_0U$, if and only if $\mathrm{ord}_{\mathfrak{p}}\tilde{N}(g) \equiv 0 (\mathrm{mod}.\ \varkappa_r)$, where $\varkappa_r = (n, nd/r)$. It follows that Ξ_r is a cyclic group of order $\varkappa_r$.

By Theorem 4, we see that the natural mapping $\Omega(G^{(r)}, U^{(r)}) \leftarrow \Omega(\bar{G}, \bar{U})$ is $1 : \varkappa^r$ and that $\mathscr{L}(G^{(r)}, U^{(r)})$ can be identified with the subalgebra of $\mathscr{L}(\bar{G}, \bar{U})$ formed of all Ξ_r-invariant elements. Under the Fourier transformation, this subalgebra corresponds to the algebra formed of all symmetric Fourier polynomials in $q^{\pm s_i}$ consisting of only terms of degree divisible by $\varkappa_r$. This means that, under the isomorphism $\mathscr{L}(\overline{G}, \overline{U}) \cong \mathbf{C}[X_1, \ldots, X_{n-1}]$ stated above, $\mathscr{L}(G^{(r)}, U^{(r)})$ corresponds to the subalgebra formed of all polynomials in $X_1, \ldots, X_{n-1}$ consisting of only terms of weight divisible by $\varkappa_r$. Thus we have proved

THEOREM 7. *The notations being as above, $\mathscr{L}(\overline{G}, \overline{U})$ is isomorphic to the polynomial algebra $\mathbf{C}[X_1, \ldots, X_{n-1}]$ by the correspondence $\bar{c}^{(i)} \to X_i$. Under this isomorphism, $\mathscr{L}(G^{(r)}, U^{(r)})$ corresponds to the subalgebra formed of all polynomials having only terms of weight divisible by the greatest common divisor of n, nd/r.*

In view of Remarks after Propositions 7.3, 7.4, we see also that $\mathscr{L}(\overline{G}, \overline{U})_{\mathbf{Z}} = \mathbf{Z}[\bar{c}^{(1)}, \ldots, \bar{c}^{(n-1)}]$ and that $L(G^{(r)}, U^{(r)})_{\mathbf{Z}}$ corresponds to the subring of $\mathscr{L}(\overline{G}, \overline{U})_{\mathbf{Z}}$ formed of the polynomials of the type described in the above theorem.

§ 9. Cases of groups of similitudes (1).

9.1. *Theory of elementary divisors.* We treat the following five cases simultaneously:

(O) the case concerning a quadratic form,
(Sp) the case concerning an alternating form,
(U) the case concerning a hermitian form,
(U^+) the case concerning a quaternionic hermitian form,
(U^-) the case concerning a quaternionic anti-hermitian form.

For proofs of the results in this paragraph, see **[9]**, Ch. II, § 9 for (O), **[21]**, § 1 or **[5]** for (Sp), **[5]** for (U), **[21]**, § 3 for (U^+), **[24]** for (U^-) (2).

Let K be equal to k in the cases (O), (Sp), a quadratic extension field of k, of ramification exponent e, in the case (U) and a (unique) central quaternion division algebra over k in the cases (U^+), (U^-). We denote the (unique) maximal order in K and its prime ideal by $\mathfrak{O}$, $\mathfrak{P} = (\Pi)$, respectively.

Let V be an n-dimensional (right) vector space over K with a non-degenerate bilinear (resp. sesquilinear) form $< >$ of the following type:

symmetric bilinear form in the case (O),
alternating bilinear form in the case (Sp),
hermitian sesquilinear form in the cases (U), (U^+),
anti-hermitian sesquilinear form in the case (U^-).

Let ν be the Witt index of V and put $n = n_0 + 2\nu$. A system of vectors $\{e_i, e'_i \ (1 \leqslant i \leqslant \nu)\}$ in V is called " canonical ", if the following conditions are satisfied

$$<e_i, e'_j> = \delta_{ij}, \quad <e_i, e_j> = <e'_i, e'_j> = 0 \qquad \text{for all } i, j.$$

(1) For the fundamental concepts on quadratic forms, hermitian forms, etc., and the corresponding classical groups, see [2], [8], [9]. Cf. also [25].

(2) Though these references still do not cover the results in full generality, it is not difficult to complete them, e.g. by generalizing the method in [9]. The author should also mention that the main idea in this section has been given by Tamagawa through several lectures at University of Tokyo in 1960.

For an $\mathfrak{O}$-lattice L in V, we define its (exponential) *norm* $n(\mathrm{L})$ as follows:

(9.1)

$$n(\mathrm{L}) = \begin{cases} \mathrm{Min}\left\{\mathrm{ord}_{\mathfrak{p}}\left(\frac{1}{2}<x, x>\right) \mid x \in \mathrm{L}\right\} & \text{in the case (O),} \\ \mathrm{Min}\{\mathrm{ord}_{\mathfrak{p}}<x, y> \mid x, y \in \mathrm{L}\} & \text{in the case (Sp),} \\ \mathrm{Max}\{l \mid <x, x> \in \mathfrak{p}^{\left[\frac{l+\delta}{e}\right]}, <x, y> \in \mathfrak{P}^l \text{ for all } x, y \in \mathrm{L}\} & \text{in the case (U),} \\ \mathrm{Min}\{\mathrm{ord}_{\mathfrak{P}}<x, y> \mid x, y \in \mathrm{L}\} & \text{in the case (U}^+\text{),} \\ \mathrm{Max}\{l \mid <x, x> \in (\mathfrak{P}^l)^- \quad \text{for all} \quad x \in \mathrm{L}\} & \text{in the case (U}^-\text{),} \end{cases}$$

where $\mathfrak{P}^{\delta}$ is the different of K/k in the case (U) and $(\mathfrak{P}^l)^- = \{\xi - \bar{\xi} \mid \xi \in \mathfrak{P}^l\}$ ($\bar{\xi}$ = conjugate of ξ) in the case (U⁻). (Note that, in the case (U), if $e = 1$, the conditions " $<x, x> \in \mathfrak{p}^{\left[\frac{l+\delta}{e}\right]}$ for all $x \in \mathrm{L}$ " and " $<x, y> \in \mathfrak{P}^l$ for all $x, y \in \mathrm{L}$ " are equivalent, while, if $e = 2$, $\delta = 1$, the second condition implies the first.) An $\mathfrak{O}$-lattice L is called *maximal*, if it is maximal among the $\mathfrak{O}$-lattices in V with the same norm.

Now the following results are known:

(E_1) *For any maximal $\mathfrak{O}$-lattice* L *of norm l in* V, *there exists a canonical system* $\{e_i, e_i' \ (1 \leqslant i \leqslant \nu)\}$ *such that*

$$\mathrm{L} = \sum_{i=1}^{\nu} e_i \mathfrak{O} + \sum_{i=1}^{\nu} e_i' \mathfrak{P}^l + \mathrm{L}_0^{(l)}, \tag{9.2}$$

where $\mathrm{L}_0^{(l)}$ *is the unique maximal $\mathfrak{O}$-lattice in* $\mathrm{V}_0 = (\sum_i e_i \mathrm{K} + \sum_i e_i' \mathrm{K})^{\perp}$ ($\perp$ *denoting the orthogonal complement*) *of the smallest possible norm* $\geqslant l$. ($n(\mathrm{L}_0^{(l)}) = l$ *or* $l + 1$.)

Remark 1. We have $\dim \mathrm{V}_0 = n_0 \leqslant 4$ for (O), $= 0$ for (Sp), $\leqslant 2$ for (U), $\leqslant 1$ for (U⁺) and $\leqslant 3$ for (U⁻); and $\mathrm{L}_0^{(l)}$ is defined as follows:

$$\mathrm{L}_0^{(l)} = \{x \in \mathrm{V}_0 \mid \tfrac{1}{2}<x, x> \in \mathfrak{p}^l, \qquad \text{for (O)} \tag{9.3}$$

$$\text{resp.} <x, x> \in \mathfrak{p}^{\left[\frac{l+\delta}{e}\right]}, \qquad \text{for (U)}$$

$$\text{resp.} <x, x> \in \mathfrak{p}^{\left[\frac{l+1}{2}\right]}, \qquad \text{for (U}^+\text{)}$$

$$\text{resp.} <x, x> \in (\mathfrak{P}^l)^-\} \qquad \text{for (U}^-\text{).}$$

Remark 2. As e_1 (resp. e_1, e_1'), we may take any primitive isotropic vector in L (resp. any pair of isotropic vectors e_1, e_1' such that $<e_1, e_1'> = 1$, $(\mathrm{L} : e_1) = 0$, $(\mathrm{L} : e_1') = l$). It follows that the group of units (group of linear transformations leaving $< >$ and L invariant) operates transitively on the set of all primitive isotropic vectors (resp. of all pairs of isotropic vectors satisfying the above conditions).

(E_2) *Let* L, L′ *be two maximal $\mathfrak{O}$-lattices in* V, *of norm l, l', respectively. Then there exists a canonical system* $\{e_i, e_i' \ (1 \leqslant i \leqslant \nu)\}$ *in* V *such that*

$$\mathrm{L} = \sum_{i=1}^{\nu} e_i \mathfrak{O} + \sum_{i=1}^{\nu} e_i' \mathfrak{P}^l + \mathrm{L}_0^{(l)}, \tag{9.4}$$

$$\mathrm{L}' = \sum_{i=1}^{\nu} e_i \mathfrak{P}^{r_i} + \sum_{i=1}^{\nu} e_i' \mathfrak{P}^{l' - r_i} + \mathrm{L}_0^{(l')}$$

with $r_1 \geqslant \ldots \geqslant r_\nu \geqslant \frac{1}{2}(l'-l)$. *The ordered set of integers* $(r_1, \ldots, r_\nu)$ *is uniquely determined, independently of the choice of the canonical system* $\{e_i, e'_i\}$.

We call $(r_1, \ldots, r_\nu)$ the *elementary divisors* of L′ relative to L and denote it by $\varepsilon(L' : L)$.

Remark 1. We have

$$(L' : L) = r_1, \quad (L : L') = r_1 + l - l'. \tag{9.5}$$

Remark 2. As e_1 (resp. e_1, e'_1), we may take any isotropic vector such that $(L : e_1) = 0$, $(L' : e_1) = r_1$ (resp. any pair of isotropic vectors e_1, e'_1 such that $\langle e_1, e'_1 \rangle = 1$, $(L : e_1) = 0$, $(L' : e_1) = r_1$, $(L : e'_1) = l$, $(L' : e'_1) = l' - r_1$).

9.2. *Proofs of* (I), (II) *for groups of similitudes.* We now fix any maximal $\mathfrak{O}$-lattice L in V and a canonical system $\{e_i, e'_i\ (1 \leqslant i \leqslant \nu)\}$ satisfying (9.2) once for all; we take furthermore a basis $(f_1, \ldots, f_{n_0})$ of $L_0^{(l)}$ and understand that a K-linear transformation of V is represented by a matrix, whenever necessary, with respect to the following basis of V:

$$(e_1, \ldots, e_\nu, f_1, \ldots, f_{n_0}, e'_\nu, \ldots, e'_1).$$

In the following, we make a convention that the index i always ranges over $1, \ldots, \nu$.

Let us set as follows:

G = the group of all " proper " similitudes of V with respect to $\langle\ \rangle$
= the connected component of $\{g \in GL(n, K) \mid \langle gx, gy \rangle = \mu(g)\langle x, y\rangle \text{ for all } x, y \in V\}$
($\mu : G \to k^*$ is the " multiplicator ").

Remark. The group of all similitudes is already connected except for (O) with n even and for (U^-). For these cases, proper similitudes are defined by the condition $\det(g) = \mu(g)^{\frac{n}{2}}$, resp. $\widetilde{N}(g) = \mu(g)^n$, $\widetilde{N}$ denoting the reduced norm of $M_n(K)/k$.

(9.6)

$$U = \{u \in G \mid uL = L\} = \left\{ u \in G \mid u, u^{-1} \in \begin{pmatrix} \mathfrak{O} & \mathfrak{O} & \mathfrak{P}^{-l} \\ \mathfrak{O} & \mathfrak{O} & \mathfrak{P}^{-l} \\ \mathfrak{P}^{l} & \mathfrak{P}^{l} & \mathfrak{O} \end{pmatrix} \right\},$$

$$\begin{aligned} H &= \{h \in G \mid he_i = e_i\xi_i,\ he'_i = e'_i\xi'_i \ \text{ with } \ \xi_i, \xi'_i \in K\} \\ &= \{h = \text{diag.}(\xi_1, \ldots, \xi_\nu, h_0, \xi_0\bar{\xi}_\nu^{-1}, \ldots, \xi_0\bar{\xi}_1^{-1}) \mid \xi_i \in K^*, \xi_0 \in k^*, h_0 \in G_0 \text{ with } \mu_0(h_0) = \xi_0\} \\ &\cong \begin{cases} (K^*)^\nu \times k^* & \text{if } n_0 = 0, \\ (K^*)^\nu \times G_0 & \text{if } n_0 > 0, \end{cases} \end{aligned}$$

$$\begin{pmatrix} G_0 = \text{the group of proper similitudes of } V_0 \\ \mu_0 = \text{multiplicator of } G_0 \end{pmatrix}$$

$$N = G \cap T^u(n, K) = \left\{ n = \left(\begin{array}{cc|c|c} 1 & * & * & * \\ 0 & 1 & & \\ \hline 0 & & 1_{n_0} & * \\ \hline 0 & & 0 & \begin{matrix} 1 & * \\ 0 & 1 \end{matrix} \end{array} \right) \in G \right\}.$$

It is clear that U is an open compact subgroup of G, H a connected k-closed subgroup of G consisting of only semi-simple elements and N a k-unipotent subgroup normalized by H. Moreover, the center of H is equal to

$$(9.7)\qquad \begin{aligned} A &= \text{the connected component of } G\cap D(n, k) \\ &= \{a=\mathrm{diag.}(\xi_1, \ldots, \xi_\nu, \xi_0 1_{n_0}, \xi_0^{(2)}\xi_\nu^{-1}, \ldots, \xi_0^{(2)}\xi_1^{-1}) \mid \xi_i, \xi_0\in k^*\}, \end{aligned}$$

where (and henceforth) we put

$$(2)=\begin{cases} 1 & \text{if } n_0=0, \\ 2 & \text{if } n_0>0; \end{cases}$$

we have also $H=Z(A)$.

Now systems of generators of $Y=X(A)$ and of $X(H)$ are given, respectively, by

$$(9.8)\qquad \begin{aligned} &\eta_i : A\ni a \longrightarrow \xi_i \qquad (1\leqslant i\leqslant \nu), \\ &\eta_0=\frac{1}{(2)}\mu : a\longrightarrow \xi_0 \end{aligned}$$

and by

$$(9.9)\qquad \begin{aligned} &\chi_i : H\ni h \longrightarrow N\xi_i \qquad (1\leqslant i\leqslant \nu), \\ &\chi_0=\begin{cases} \mu : h\longrightarrow \xi_0 & \text{except for (O), } n: \text{odd}, \\ \frac{1}{2}\mu & \text{for (O), } n: \text{odd}, \end{cases} \end{aligned}$$

N denoting the reduced norm of K/k.

Proof of (3.2). One has $H^u\cong \mathfrak{U}^\nu\times G_0^u$, $\mathfrak{U}$ denoting the group of units in K^* and $G_0^u=\{h_0\in G_0 \mid \mathrm{ord}_p\mu(h_0)=0\}$. From the uniqueness of maximal lattice of a given norm in V_0, it follows that $h_0\in G_0^u$ leaves $L_0^{(l)}$ invariant, so that we have $H^u\subset U$, as desired.

We denote by d and e the degree and the ramification exponent of K/k, respectively, i.e.

$$\begin{aligned} &d=e=1 && \text{for (O), (Sp)}, \\ &d=2, \quad e=1 \text{ or } 2 && \text{for (U)}, \\ &d=e=2 && \text{for } (U^+), (U^-); \end{aligned}$$

and define d_0, e_0 (e_0 dividing d_0) as follows

$$(9.10)\qquad \begin{aligned} &d_0=\begin{cases} (2) & \text{except for (O), } n: \text{odd}, \\ 1 & \text{for (O), } n: \text{odd}, \end{cases} \\ &\mathrm{ord}_p\mu(G)=\mathrm{ord}_p\mu_0(G_0)=\frac{(2)}{e_0}\mathbf{Z}. \end{aligned}$$

Since we have $\chi_i=d\eta_i$ $(1\leqslant i\leqslant \nu)$ on A, it follows that, if we identify $\hat{Y}$ with $\mathbf{Z}^{\nu+1}$ by means of $(\eta_1, \ldots, \eta_\nu, \eta_0)$ and if we consider as $\hat{Y}\subset M\subset \hat{X}(H)\subset \mathbf{Q}^{\nu+1}$, we have

$$(9.11)\qquad \begin{aligned} M &= \frac{1}{e}\mathbf{Z}^\nu\times\frac{1}{e_0}\mathbf{Z}, \\ \hat{X}(H) &= \frac{1}{d}\mathbf{Z}^\nu\times\frac{1}{d_0}\mathbf{Z}. \end{aligned}$$

For $\mathbf{m}=(m_1/e, \ldots, m_\nu/e, m_0/e_0)\in M$, we put

$$(9.12)\qquad \pi^{\mathbf{m}}=\begin{cases}\text{diag.}(\Pi^{m_1}, \ldots, \Pi^{m_\nu}, \pi^{m_0}\overline{\Pi}^{-m_\nu}, \ldots, \pi^{m_0}\overline{\Pi}^{-m_1}) & \text{if } n_0=0,\\ \text{diag.}(\Pi^{m_1}, \ldots, \Pi^{m_\nu}, \varpi^{m_0}, \mu_0(\varpi)^{m_0}\overline{\Pi}^{-m_\nu}, \ldots, \mu_0(\varpi)^{m_0}\overline{\Pi}^{-m_1}) & \text{if } n_0>0,\end{cases}$$

where ϖ denotes an element of G_0 such that $\text{ord}_p\mu_0(\varpi)=2/e_0$. This notation is concordant with that introduced in N° 2.3.

Now, except for the case (O) with $n=2\nu$ (which will be treated separately in N° 9.5), it can readily be seen that one may take, as W (operating on $M=\{\mathbf{m}=(m_i/e, m_0/e_0)\}$), the group generated by all permutations of $(m_1, \ldots, m_\nu)$ and by the automorphism $w^{(i)}$ $(1\leqslant i\leqslant \nu)$ defined as follows:

$$(9.13)\qquad w^{(i)}:\begin{cases} m_i \longrightarrow -m_i+(2)\dfrac{e}{e_0}m_0\\ m_j\longrightarrow m_j \qquad (j\neq i)\end{cases}$$

Then, taking a linear order in M induced from the lexicographical linear order in $\mathbf{Q}^{2\nu}$ through the injection

$$M\ni\mathbf{m}\rightarrow\left(\frac{m_1}{e}, \ldots, \frac{m_\nu}{e}, \frac{(2)m_0}{e_0}-\frac{m_\nu}{e}, \ldots, \frac{(2)m_0}{e_0}-\frac{m_1}{e}\right)\in\mathbf{Q}^{2\nu},$$

we have

$$(9.14)\qquad \Lambda=\left\{\mathbf{r}=\left(\frac{r_i}{e}, \frac{r_0}{e_0}\right)\,|\, r_i, r_0\in\mathbf{Z}, r_1\geqslant\ldots\geqslant r_\nu\geqslant\frac{(2)e}{2\,e_0}r_0\right\}.$$

Now applying (E_2) to $L'=gL$, we see that, except for the case (O) with $n=2\nu$, we have $g\in U\pi^{\mathbf{r}}U$ with $\mathbf{r}=(r_i/e, r_0/e_0)\in\Lambda$, if and only if $\varepsilon(gL:L)=(r_1, \ldots, r_\nu)$ and $\text{ord}_p\mu(g)=(2)/e_0\cdot r_0$.
Therefore we have

$$G=\bigcup_{\mathbf{r}\in\Lambda}U\pi^{\mathbf{r}}U \qquad \text{(disjoint union)}.$$

This proves $G=UHU$ and (II_1) with

$$(9.15)\qquad r(g)=\left(\frac{1}{e}\varepsilon(gL:L), \frac{1}{(2)}\text{ord}_p\mu(g)\right).$$

Proof of $G=U.HN$. If $\nu=0$, we have $G=H$, and our assertion is trivial. Let $\nu\geqslant 1$. Let $g\in G$ and let $(L:ge_1)=-m_1$. Then $ge_1\Pi^{-m_1}$ being a primitive isotropic vector in L, one can find $u\in U$ with $\mu(u)=1$ such that $ge_1\Pi^{-m_1}=ue_1$. Then, since we have $\langle u^{-1}gx, e_1\rangle=\mu(g)\langle x, e_1\rangle\Pi^{-m_1}=0$ for all $x\in(e_1K)^\perp$ (=orthogonal complement of e_1K), we see that $u^{-1}g$ is of the following form

$$u^{-1}g=\begin{pmatrix}\Pi^{m_1} & * & *\\ 0 & g_1 & *\\ 0 & 0 & \mu(g)\overline{\Pi}^{-m_1}\end{pmatrix}$$

where g_1 is a similitude induced by g on $(e_1K)^\perp/e_1K$ (with respect to the basis $(e_2, \ldots, e_2')$ (mod. e_1K)). Hence our assertion follows from an induction on ν.

Proof of (3.16). Let $g=\pi^{\mathbf{m}}n$ with $\mathbf{m}=(m_i/e, m_0/e_0)\in M$, $n\in N$ and let

$$\mathbf{r}=r(g)=(r_i/e, r_0/e_0).$$

We have to prove $\mathbf{r}\geqslant\mathbf{m}$. First it is clear that $r_0=(e_0/(2))\mathrm{ord}_{\mathfrak{p}}\mu(g)=m_0$. Hence, if $\nu=0$, the assertion is trivial. Let $\nu\geqslant 1$. Since $(gL:L)=r_1$, we have $e_1\Pi^{r_1}\in gL$; on the other hand, $ge_1=e_1\Pi^{m_1}$ is primitive in gL. Hence we get $r_1\geqslant m_1$. If $r_1=m_1$, we have $(L:e_1)=0$, $(gL:e_1)=r_1$, and hence, by Remark 2 after (E_2), there exists an isotropic vector e_1'' such that $<e_1, e_1''>=1$, $(L:e_1'')=l$, $(gL:e_1'')=l+e\ \mathrm{ord}_{\mathfrak{p}}\mu(g)-r_1$. We now assert that, for any isotropic vector e_1'' with the properties $<e_1, e_1''>=1$, $(L:e_1'')=l$, one can find $u\in U\cap N$ of the form

$$u=\begin{pmatrix} 1 & * & * \\ 0 & 1_{n-2} & * \\ 0 & 0 & 1 \end{pmatrix}$$

such that $ue_1=e_1$, $ue_1'=e_1''$. In fact, put

$$e_1''=e_1\alpha+e_1'\beta+w \qquad \text{with } \alpha, \beta\in K,\ w\in(e_1K+e_1'K)^{\perp}.$$

Then, from the conditions on e_1'', we have $\beta=1$ and $\bar{\alpha}\pm\alpha+<w, w>=0$. It follows that the K-linear transformation u defined by

$$u:\begin{cases} e_1\to e_1 \\ e_1'\to e_1'' \\ x\to -e_1\overline{<x, w>}+x \end{cases}$$

meets all our requirements. Now, if we put $e_1'''=g^{-1}e_1''\mu(g)\bar{\Pi}^{-r_1}$, we have $<e_1, e_1'''>=<ge_1, e_1''>\bar{\Pi}^{-r_1}=1$ and $(L:e_1''')=(gL:e_1'')+r_1-e.\mathrm{ord}_{\mathfrak{p}}\mu(g)=l$, so that, by the same reason as above, one can find $u'\in U\cap N$ of the same form as u such that

$$\begin{cases} u'e_1=e_1=g^{-1}e_1\Pi^{r_1}, \\ u'e_1'=e_1'''=g^{-1}e_1''\mu(g)\bar{\Pi}^{-r_1}. \end{cases}$$

Therefore, if we put $g=\begin{pmatrix} \Pi^{r_1} & * & * \\ 0 & g_1 & * \\ 0 & 0 & \mu(g)\bar{\Pi}^{-r_1} \end{pmatrix}$, we have

$$u^{-1}gu'=\begin{pmatrix} \Pi^{r_1} & 0 & 0 \\ 0 & g_1 & 0 \\ 0 & 0 & \mu(g)\bar{\Pi}^{-r_1} \end{pmatrix}.$$

It follows that $r(g)=r(u^{-1}gu')=(r_1/e, r(g_1))$, and, by the induction assumption, that $r(g_1)=(r_2/e, \ldots, r_\nu/e, r_0/e_0)\geqslant(m_2/e, \ldots, m_\nu/e, m_0/e_0)$, which completes the proof.

Remark. If $\mathbf{m}=\mathbf{r}$, an easy induction on ν shows that g can be written in the form $g=u_1\pi^{\mathbf{r}}u_2$ with $u_1, u_2 \in \mathrm{U}\cap\mathrm{N}$. But, since $\pi^{\mathbf{r}}u_2\pi^{-\mathbf{r}}\in\mathrm{U}\cap\mathrm{N}$, this implies that

$$\pi^{\mathbf{r}}\mathrm{N}\cap\mathrm{U}\pi^{\mathbf{r}}\mathrm{U}=(\mathrm{U}\cap\mathrm{N})\pi^{\mathbf{r}}.$$

Thus the additional condition (5.2) is again satisfied.

9.3. The notations being as before, and still excepting the case (O), $n=2\nu$, we can now apply the results of Chapter II to determine $\mathscr{L}(\mathrm{G}, \mathrm{U})\cong\mathbf{C}[q^{\mathrm{M.s}}]^{\mathrm{W}}$ more explicitly. It is convenient to distinguish the following three cases:

1° $(2)e/e_0=1$, that is, $e=1$, $e_0=(2)$,

2° $(2)e/e_0=2$,

3° $(2)e/e_0=4$, that is, $e=2$, $e_0=1$, $(2)=2$.

From (9.14), it can readily be seen that Λ is generated (as semi-group) by the following $(\nu+2)$ vectors:

$$\pm\mathbf{r}^{(0)}, \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(\nu-1)}, \mathbf{r}^{(\nu)'} \qquad \text{in Case } 1^\circ,$$
$$\pm\mathbf{r}^{(0)}, \mathbf{r}^{(1)}, \ldots, \mathbf{r}^{(\nu)} \qquad \text{in Cases } 2^\circ, 3^\circ,$$

where

$$(9.16)\qquad \begin{aligned} \mathbf{r}^{(i)} &= \Big(\overbrace{\frac{1}{e}, \ldots, \frac{1}{e}}^{i}, 0, \ldots, 0, 0\Big) && (1\leqslant i\leqslant\nu),\\ \mathbf{r}^{(\nu)'} &= \Big(1, \ldots, 1, \frac{1}{(2)}\Big) && (\text{Case } 1^\circ)\\ \mathbf{r}^{(0)} &= \begin{cases} \Big(1, \ldots, 1, \dfrac{2}{(2)}\Big) & \text{in Case } 1^\circ,\\ \Big(\dfrac{1}{e}, \ldots, \dfrac{1}{e}, \dfrac{1}{e_0}\Big) & \text{in Case } 2^\circ,\\ (1, \ldots, 1, 1) & \text{in Case } 3^\circ. \end{cases} \end{aligned}$$

We put $c^{(i)}=c_{\mathbf{r}^{(i)}}$, $c^{(\nu)'}=c_{\mathbf{r}^{(\nu)'}}$. Then, from Theorem 1, we obtain the following

THEOREM 8. *Let* G *be the group of proper similitudes,* U *a maximal compact subgroup of* G *defined by a maximal* $\mathfrak{O}$*-lattice, and let* $c^{(i)}$ $(0\leqslant i\leqslant\nu)$, $c^{(\nu)'}$ *be as defined above. Then, except for the case* (O), $n=2\nu$, $\mathscr{L}(\mathrm{G}, \mathrm{U})$ *is a polynomial algebra* $\mathbf{C}[c^{(0)\pm1}, c^{(1)}, \ldots, c^{(\nu-1)}, c^{(\nu)'}]$ *in Case* 1° *and* $\mathbf{C}[c^{(0)\pm1}, c^{(1)}, \ldots, c^{(\nu)}]$ *in the other cases.*

Remark. We obtain also a more precise result that

$$\mathscr{L}(\mathrm{G}, \mathrm{U})_{\mathbf{Z}}=\mathbf{Z}[c^{(0)\pm1}, c^{(1)}, \ldots, c^{(\nu-1)}, c^{(\nu)'}] \quad\text{or}\quad \mathbf{Z}[c^{(0)\pm1}, c^{(1)}, \ldots, c^{(\nu)}].$$

The Fourier transformation of these $c^{(i)}$ are described as follows. We identify $\mathrm{X}(\mathrm{H})\otimes\mathbf{C}$ with $\mathbf{C}^{\nu+1}$ by means of the basis $(\chi_1, \ldots, \chi_\nu, \chi_0)$. Then, for $\mathbf{m}=(m_i/e, m_0/e_0)\in\mathrm{M}$, $\mathbf{s}=(s_i, s_0)\in\mathrm{X}(\mathrm{H})\otimes\mathbf{C}$, we have

$$\mathbf{m}.\mathbf{s}=\frac{d}{e}\sum_{i=1}^{\nu}m_is_i+\frac{d_0}{e_0}m_0s_0,$$

or

$$(9.17) \qquad q^{-\mathbf{m}.\mathbf{s}} = q_1^{-\sum_i m_i s_i} q_0^{-m_0 s_0}$$

where

$$q_1 = q^{\frac{d}{e}}, \qquad q_0 = q^{\frac{d_0}{e_0}}.$$

We put

$$(9.18) \qquad \begin{aligned} X_\varkappa &= F_{\mathbf{r}^{(\varkappa)}}(\mathbf{s}) = \sum_{i_1<\dots<i_\varkappa} q_1^{\pm s_{i_1} \pm \dots \pm s_{i_\varkappa}} & (1 \leqslant \varkappa \leqslant \nu), \\ X'_\nu &= F_{\mathbf{r}^{(\nu)'}}(\mathbf{s}) = q_0^{-s_0} \sum_{\substack{i_1<\dots<i_\varkappa \\ 0 \leqslant \varkappa \leqslant \nu}} q_1^{-s_{i_1} - \dots - s_{i_\varkappa}} \\ &= q_0^{-s_0} q_1^{-\frac{1}{2}\sum_1^\nu s_i} \prod_{i=1}^{\nu} (q_1^{\frac{s_i}{2}} + q_1^{-\frac{s_i}{2}}), \\ X_0 &= F_{\mathbf{r}^{(0)}}(\mathbf{s}) = \begin{cases} q_0^{-2s_0} q_1^{-\sum_1^\nu s_i} & \text{in Case } 1^\circ, \\ q_0^{-s_0} q_1^{-\sum_1^\nu s_i} & \text{in Case } 2^\circ, \\ q_0^{-s_0} q^{-2\sum_1^\nu s_i} & \text{in Case } 3^\circ. \end{cases} \end{aligned}$$

Then, by N° 6.2, we obtain the following relation:

$$(9.19) \qquad \begin{aligned} \hat{c}^{(0)} &= X_0, \\ \hat{c}^{(i)} &= \lambda_{i0} + \lambda_{i1} X_1 + \dots + \lambda_{ii} X_i & (1 \leqslant i \leqslant \nu), \\ \hat{c}^{(\nu)'} &= \lambda'_\nu X'_\nu & (\text{in Case } 1^\circ), \end{aligned}$$

where

$$\lambda_{ij} = \gamma_{\mathbf{r}^{(i)}, \mathbf{r}^{(j)}} \delta^{-\frac{1}{2}}(\pi^{\mathbf{r}^{(j)}}), \qquad \lambda'_\nu = \delta^{-\frac{1}{2}}(\pi^{\mathbf{r}^{(\nu)'}}).$$

9.4. The notations being as before, let us now consider the group $\bar{G} = G/Z$, Z being the center of G. We exclude here the cases (O), $n=2$, and (U^-), $n=1$, where G is commutative and Theorem 9 below holds trivially. Then, except for the case (U), we have $Z = \{\xi 1_n \mid \xi \in k^*\}$. In the notation of Proposition 3.3, one has

$$(9.20) \qquad H_0 = \{h = \text{diag.}((\xi_i), h_0, (\xi_0 \bar{\xi}_{\nu-i+1}^{-1})) \mid \frac{2}{e} \text{ord}_{\mathfrak{P}} \xi_i = \text{ord}_{\mathfrak{p}} \xi_0\}.$$

We shall show that H_0 actually normalizes U. In fact, let $h \in H_0$, written as above, and let $\text{ord}_{\mathfrak{P}} \xi_i = r$. Then, $\text{ord}_{\mathfrak{p}} \mu_0(h_0) = 2r/e$, and we see, from the uniqueness of the maximal lattice of a given norm in V_0, that $h_0 L_0^{(l)} = L_0^{(l+2r)} = L_0^{(l)} \mathfrak{P}^r$. It follows that $hL = \Sigma e_i \xi_i \mathfrak{O} + \Sigma e'_i \xi_0 \bar{\xi}_i^{-1} \mathfrak{P}^l + h_0 L_0^{(l)} = L\mathfrak{P}^r$. Therefore, for every $u \in U$, we have $uhL = hL$, i.e. $h^{-1}uh \in U$, as desired. Thus, by Proposition 3.3 and N° 7.4, we see that $\bar{G}$ satisfies the assumptions (I), (II). Since

$$(9.21) \qquad M_0 = \mathbf{r}^{(0)} \mathbf{Z},$$

we conclude by Proposition 7.4, Theorem 8 (or directly by Theorem 1) that

$$\begin{aligned}\mathscr{L}(\bar{G}, \bar{U}) &= \mathscr{L}(G, U)/(c_0 - c^{(0)}) \\ &= \mathbf{C}[\bar{c}^{(1)}, \ldots, \bar{c}^{(\nu)'}] \text{ or } \mathbf{C}[\bar{c}^{(1)}, \ldots, \bar{c}^{(\nu)}]\end{aligned}$$

where $\bar{c}^{(i)} = c_{\bar{\mathbf{r}}(i)}$, $\bar{\mathbf{r}}^{(i)} = (\mathbf{r}^{(i)} \text{mod. } M_0)$.

In the case (U), we have $Z = \{\xi 1_n | \xi \in K^*\}$. Though Z is a non-trivial torus, the first Galois cohomology in Z being trivial, we still have $\bar{G} = G/Z = \mathrm{cl}(G/Z)$. Then one sees immediately that all conclusions of Proposition 3.3 and N° 7.4 remain true with $H_0 = ZH^u$. Thus we obtain again the same result.

THEOREM 9. *The notations being as above and excepting the case* (O) *with* $n = 2\nu$, $\mathscr{L}(\bar{G}, \bar{U})$ *is a polynomial algebra* $\mathbf{C}[\bar{c}^{(1)}, \ldots, \bar{c}^{(\nu-1)}, \bar{c}^{(\nu)'}]$ *in Case* 1° *and* $\mathbf{C}[\bar{c}^{(1)}, \ldots, \bar{c}^{(\nu)}]$ *in the other cases.*

Now, except for the case (U), the semi-simple part of G is G^1. We note that $X(G) = \{\chi_0\}$ with χ_0 given in (9.9). Though H does not always satisfy (N), it follows easily from a theorem of Witt that we have $G = HG^1$ and $U = H^u U^1$. Hence, we can still apply Proposition 3.2 and N° 7.3, concluding that G^1 satisfies the assumptions (I), (II). Denoting by λ the canonical isogeny $G^1 \to \bar{G}$, we see that, for $g \in G$, one has $g(\text{mod. } Z) \in \lambda(G^1)\bar{U}$ if and only if $\mathrm{ord}_p \mu(g) \equiv 0 \left(\text{mod. } \frac{2}{e}\right)$. Hence $[\bar{G} : \lambda(G^1)\bar{U}] = 2$ in Case 1° and $\bar{G} = \lambda(G^1)\bar{U}$ in the other cases.

In the case (U), we have to consider the group of all K-rational characters (instead of X(G)), which is generated by μ and det. The semi-simple part of G is

$$G^{(1)} = \{g \in G \mid \mu(g) = \det(g) = 1\};$$

and, more generally, all (connected) algebraic groups isogeneous to $\bar{G}$ are given by

$$(9.22) \qquad G^{(r)} = \{(g, \xi) \in G \times K^* \mid \det(g) = \xi^{n/r}, \mu(g)^r = \xi\bar{\xi}\} / \{(\xi 1_n, \xi^r) \mid \xi \in K^*\}.$$

where r is a positive integer dividing n. Since we have again the relations $G = HG^{(1)}$, $U = H^u(U \cap G^{(1)})$ (1), it is easy to extend the considerations in N^os^ 3.2, 7.2 to see that all these groups satisfy (I), (II) with respect to the maximal compact subgroups

$$(9.23) \quad U^{(r)} = \{(\xi 1_n . u, \eta) \mid \xi \in K^*, u \in U, \eta \in K^*, r.\mathrm{ord}_{\mathfrak{P}}\xi = \mathrm{ord}_{\mathfrak{P}}\eta\} / \{(\xi 1_n, \xi^r) \mid \xi \in K^*\}$$

Moreover, denontig by λ the canonical isogeny $G^{(r)} \to \bar{G}$, we see that, for $g \in G$, one has $g(\text{mod. } Z) \in \lambda(G^{(r)})\bar{U}$ if and only if $\mathrm{ord}_p \mu(g) \equiv 0 \left(\text{mod. } \frac{2}{er}\right)$. Hence $[\bar{G} : \lambda(G^{(r)})\bar{U}] = 2$ in Case 1°, r : odd, and $\bar{G} = \lambda(G^{(r)})\bar{U}$ for all the other cases. (Note that in Case 1° n is necessarily even.)

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(1) In order to have $U = H^u(U \cap G^{(1)})$ in the case (U) with $n = 2\nu$, $e = 2$, it is necessary to make a restriction that $n(L) \equiv \mathrm{ord}_{\mathfrak{l}}\alpha$ (mod. 2), where α is an element of k such that $K = k(\sqrt{\alpha})$. In the contrary case, $G^{(1)}$, $U^{(1)}$ do *not* satisfy condition (I).

9.5. *The case* (O) *with* $n=2\nu$. We have $e=e_0=(2)=1$ and $H=A$. In this case, one takes as W the group generated by all permutations of $(m_1, \ldots, m_\nu)$ and by $w^{(i_1)}w^{(i_2)}$ $(i_1<i_2)$ with $w^{(i)}$ defined by (9.13). Then, taking the same linear order in M as before, one has

$$(9.24)\qquad \Lambda=\{\mathbf{r}=(r_i, r_0)\,|\,r_i, r_0\in\mathbf{Z}, r_1\geqslant\ldots\geqslant r_{\nu-1}\geqslant \operatorname{Max}\{r_\nu, r_0-r_\nu\}\}.$$

It is not hard to prove I), II) in modifying the proofs given in N° 9.2. The only points to be noted are the following:

1° Let G be the group of all similitudes and $\widetilde{U}=\{g\in\widetilde{G}\,|\,gL=L\}$. Then the arguments in N° 9.2 can be applied to $\widetilde{G}$, $\widetilde{U}$, A, N. It follows, in particular, that $\widetilde{G}=\widetilde{U}.AN$, which implies $G=U.AN$.

2° For $\mathbf{r}=(r_i, r_0)\in\Lambda$, we have

$$(9.25)\qquad \begin{aligned} &(\widetilde{U}\pi^{\mathbf{r}}\widetilde{U})\cap G=U\pi^{\mathbf{r}}U && \text{if } r_\nu=\frac{r_0}{2},\\ &(\widetilde{U}\pi^{\mathbf{r}}\widetilde{U})\cap G=U\pi^{\mathbf{r}}U\cup U\pi^{w^{(\nu)}\mathbf{r}}U && \text{(disjoint union) if } r_\nu\neq\frac{r_0}{2}. \end{aligned}$$

This implies $G=UAU$ and (II_1).

3° In the proof of (3.16), the induction argument should be applied for $\nu\geqslant 2$, the cases $\nu=0, 1$ being trivial.

Proof of 2°. We have $[\widetilde{U}:U]=2$ and, as a representative of the coset $\widetilde{U}-U$, one may take $u_0\in\widetilde{U}$ such that $u_0\pi^{\mathbf{m}}u_0^{-1}=\pi^{w^{(\nu)}\mathbf{m}}$ for all $\mathbf{m}\in M$. Hence our assertions are obvious, except for the disjointness of $U\pi^{\mathbf{r}}U$, $U\pi^{w^{(\nu)}\mathbf{r}}U$ for $\mathbf{r}$ with $r_\nu\neq\frac{r_0}{2}$. To prove this last point, suppose that it were not true. Then one can find $u\in\widetilde{U}-U$ such that $\pi^{\mathbf{r}}u\pi^{-\mathbf{r}}\in\widetilde{U}$. We may assume, without any loss of generality, that $n(L)=0$ and $r_\nu>\frac{r_0}{2}$. Then, denoting u in the from $\begin{pmatrix} u_1 & u_{12}\\ u_{21} & u_2\end{pmatrix}$ with $u_1, u_{12}, u_{21}, u_2\in M_\nu(\mathfrak{o})$, one sees that $u_{21}\equiv 0(\mathrm{mod.}\ \mathfrak{p})$; consequently, u_1, u_2 are non-singular. Putting $u_{21}=xu_1$, $u_{12}=yu_2$, one has from the relation ${}^tu\begin{pmatrix} 0 & \iota\\ \iota & 0\end{pmatrix}u=\mu(u)\begin{pmatrix} 0 & \iota\\ \iota & 0\end{pmatrix}$ that ιx, ιy are skew symmetric and ${}^tu_1(\iota+{}^tx\iota y)u_2=\mu(u)\iota$. It follows that

$$\begin{aligned}\det(u)&=\det\begin{pmatrix} 1 & y\\ x & 1\end{pmatrix}.\det\begin{pmatrix} u_1 & 0\\ 0 & u_2\end{pmatrix}\\ &=\det(1-xy)\det(u_1)\det(u_2)\\ &=\mu(u)^\nu,\end{aligned}$$

i.e. $u\in U$, which is a contradiction, q.e.d.

Now Λ is generated by the following vectors

$$\begin{aligned} \mathbf{r}^{(i)} &= (\overbrace{1, \ldots, 1}^{i}, 0, \ldots, 0) \qquad (1 \leqslant i \leqslant \nu-2), \\ \mathbf{r}^{(\nu)''} &= (1, \ldots, \ldots, 1, 0, 1), \\ \mathbf{r}^{(\nu)'} &= (1, \ldots, \ldots, 1, 1, 1), \\ \pm \mathbf{r}^{(0)} &= \pm (1, \ldots, \ldots, 1, 1, 2). \end{aligned} \tag{9.26}$$

Therefore, putting $c^{(i)} = c_{\mathbf{r}^{(i)}}$, $c^{(\nu)'} = c_{\mathbf{r}^{(\nu)'}}$, $c^{(\nu)''} = c_{\mathbf{r}^{(\nu)''}}$, we conclude:

THEOREM 8 add. *In the case* (O) *with* $n = 2\nu$, $\nu \geqslant 2$, $\mathscr{L}(G, U)$ *is a polynomial algebra* $\mathbf{C}[c^{(0)\pm 1}, c^{(1)}, \ldots, c^{(\nu-2)}, c^{(\nu)'}, c^{(\nu)''}]$.

Similarly, for $\overline{G} = G/Z$, $\overline{U} = UZ/Z$, we obtain

THEOREM 9 add. *In the case* (O) *with* $n = 2\nu$, $\nu \geqslant 2$, $\mathscr{L}(\overline{G}, \overline{U})$ *is a polynomial algebra* $\mathbf{C}[\bar{c}^{(1)}, \ldots, \bar{c}^{(\nu-2)}, \bar{c}^{(\nu)'}, \bar{c}^{(\nu)''}]$.

It should be noted that by means of the mapping

$$\mathscr{L}(\widetilde{G}, \widetilde{U}) \ni \varphi \rightarrow \varphi \,|\, G \in \mathscr{L}(G, U) \tag{9.27}$$

$\mathscr{L}(\widetilde{G}, \widetilde{U})$ can be identified with a subalgebra of $\mathscr{L}(G, U)$ consisting of elements invariant under the outer automorphism of G defined by u_0. On the other hand, we have $[\overline{G} : \lambda(G^1)\overline{U}] = 2$ and $\mathscr{L}(G^1, U^1)$ can be identified with a subalgebra $C[\bar{c}^{(1)}, \ldots, \bar{c}^{(\nu-2)}, \bar{c}^{(\nu)'2}, \bar{c}^{(\nu)''2}, \bar{c}^{(\nu)'}\bar{c}^{(\nu)''}]$ of $\mathscr{L}(\overline{G}, \overline{U})$.

Remark. We can consider $\mathscr{L}(\widetilde{G}, \widetilde{U})$, in general, for the case (O) with n even. But, except for the above case, the correspondence (9.27) is an isomorphism of $\mathscr{L}(\widetilde{G}, \widetilde{U})$ onto $\mathscr{L}(G, U)$.

Appendix I

CALCULATION OF SOME (LOCAL) HECKE SERIES AND ζ-FUNCTIONS ATTACHED TO CLASSICAL GROUPS

1. *Case of* $\mathrm{GL}(n, \mathfrak{K})$. Let $\mathfrak{K}$ be a central division algebra over k, $\mathrm{G} = \mathrm{GL}(n, \mathfrak{K})$ and let the notations be as introduced in nº 8.2.

For a non-negative integer m_0, put

$$(1) \qquad \mathfrak{X}_{m_0} = \{g \in \mathrm{GL}(n, \mathfrak{K}) \cap \mathrm{M}_n(\mathfrak{O}) \mid \mathrm{ord}_{\mathrm{P}} \widetilde{\mathrm{N}}(g) = m_0\}.$$

Then we have

$$\mathfrak{X}_{m_0} = \bigcup_{\mathbf{r} \in \Lambda^+_{m_0}} \mathrm{U}\pi^{\mathbf{r}}\mathrm{U},$$

where

$$(2) \qquad \Lambda^+_{m_0} = \{\mathbf{r} = (r_i/d) \in \Lambda \mid r_i \geqslant 0, \sum_{i=1}^{n} r_i = m_0\}$$

$$= \{\mathbf{r} = (r_i/d) \in \mathrm{M} \mid r_1 \geqslant \ldots \geqslant r_n \geqslant 0, \sum_{i=1}^{n} r_i = m_0\}.$$

Therefore, denoting by τ_{m_0} the characteristic function of $\mathfrak{X}_{m_0}$, we have

$$(3) \qquad \tau_{m_0} = \sum_{\mathbf{r} \in \Lambda^+_{m_0}} c_{\mathbf{r}}.$$

The (local) "Hecke series" attached to G is, by definition, a formal power series with coefficients in $\mathscr{L}(\mathrm{G}, \mathrm{U})_{\mathbf{Z}}$ defined as follows:

$$(4) \qquad \tau(\mathrm{Y}) = \sum_{m_0=0}^{\infty} \tau_{m_0} \mathrm{Y}^{m_0}.$$

Our purpose here is to obtain the "Euler expression" for this $\tau(\mathrm{Y})$. By virtue of our main results (Th. 3, Nº 8.3), $\tau(\mathrm{Y})$ may be equivalently replaced by its Fourier transform

$$(5) \qquad \hat{\tau}(\mathbf{s}, \mathrm{Y}) = \sum_{m_0=0}^{\infty} \hat{\tau}_{m_0}(\mathbf{s}) \mathrm{Y}^{m_0},$$

which is a formal power series with coefficients in $\mathbf{C}[q^{\pm s_1}, \ldots, q^{\pm s_n}]^{\mathfrak{S}_n}$.

Now, by (6.3), (6.2), one has

$$\hat{\tau}_{m_0}(\mathbf{s}) = \sum_{\mathbf{m} \in \mathrm{M}} \widetilde{\tau}_{m_0}(\pi^{\mathbf{m}}) q^{-\mathbf{m}.\mathbf{s}},$$

$$\widetilde{\tau}_{m_0}(\pi^{\mathbf{m}}) = \delta^{\frac{1}{2}}(\pi^{\mathbf{m}}) \cdot \int_{\mathrm{N}} \tau_{m_0}(\pi^{\mathbf{m}} n) dn,$$

where

$$\delta^{\frac{1}{2}}(\pi^{\mathbf{m}}) = q^{-d\sum_i\left(\frac{n+1}{2}-i\right)m_i}, \quad q^{-\mathbf{m}.\mathbf{s}} = q^{-\sum_i m_i s_i}$$

$$\int_{\mathrm{N}} \tau_{m_0}(\pi^{\mathbf{m}} n) dn = \begin{cases} \text{meas. of } (\mathrm{N} \cap \pi^{-\mathbf{m}} \mathrm{M}_n(\mathfrak{O})) = q^{d\sum_i (n-i)m_i} & \text{if } m_i \geqslant 0, \sum_i m_i = m_0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

(6)
$$\hat{\tau}(\mathbf{s}, Y) = \sum_{m_0=0}^{\infty} \left(\sum_{\substack{m_i \geqslant 0 \\ \Sigma m_i = m_0}} q^{d\frac{n-1}{2}m_0 - \sum_i m_i s_i} \right) Y^{m_0}$$
$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \prod_{i=1}^{n} \left(q^{d\frac{n-1}{2} - s_i} Y \right)^{m_i}$$
$$= \prod_{i=1}^{n} \left(1 - q^{d\frac{n-1}{2} - s_i} Y \right)^{-1}.$$

This shows that $\hat{\tau}(s, Y)$, hence also $\tau(Y)$, is actually a rational function in Y. By (8.14), $\hat{\tau}(s, Y)$ may also be written in the form

$$\hat{\tau}(\mathbf{s}, Y) = \left\{ 1 + \sum_{\varkappa=1}^{n} (-1)^{\varkappa} q^{d\frac{\varkappa(n-1)}{2}} \left(\sum_{i_1 < \ldots < i_\varkappa} q^{-s_{i_1} - \ldots - s_{i_\varkappa}} \right) Y^{\varkappa} \right\}^{-1}$$
$$= \left(1 + \sum_{\varkappa=1}^{n} (-1)^{\varkappa} q^{d\frac{\varkappa(\varkappa-1)}{2}} \hat{c}^{(\varkappa)} Y^{\varkappa} \right)^{-1},$$

or equivalently,

(7)
$$\tau(Y) = \left(1 + \sum_{\varkappa=1}^{n} (-1)^{\varkappa} q^{d\frac{\varkappa(\varkappa-1)}{2}} c^{(\varkappa)} Y^{\varkappa} \right)^{-1}.$$

This formula was first obtained by Tamagawa **[23]**.

Now the quasi-character $|\widetilde{N}(g)|_{\mathfrak{p}}^{-\lambda}$ of G is considered as a z.s.f. and, in fact, from (5.12) and (8.12) we have

$$|\widetilde{N}(g)|_{\mathfrak{p}}^{-\lambda} = \omega_{s_\lambda}(g)$$

with

$$\mathbf{s}_\lambda = \left(\lambda + \frac{n-1}{2}d,\ \lambda + \frac{n-3}{2}d,\ \ldots,\ \lambda - \frac{n-1}{2}d \right).$$

The corresponding $\hat{\tau}(\mathbf{s}_\lambda, 1)$ is nothing other than the usual " ζ-function " of G:

(8)
$$\zeta(\lambda) = \hat{\tau}(\mathbf{s}_\lambda, 1) = \prod_{i=1}^{n} (1 - q^{(i-1)d - \lambda})^{-1}.$$

2. As another example, we consider in the rest of this Appendix the case where G is the group of similitudes with respect to an alternating bilinear form, whose matrix we suppose to be of the following form:

$$\begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix} \begin{matrix} \}\nu \\ \}\nu \end{matrix}, \qquad \iota = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

In this case, we put

(9)
$$\mathfrak{X}_{m_0} = \{ g \in G \cap M_n(\mathfrak{O}) \mid \mathrm{ord}_{\mathfrak{p}} \mu(g) = m_0 \}.$$

Then we have

$$\mathfrak{X}_{m_0} = \bigcup_{\mathbf{r} \in \Lambda^+_{m_0}} U \pi^{\mathbf{r}} U$$

with

(10)
$$\Lambda^+_{m_0} = \{ \mathbf{r} = (r_i, r_0) \in \Lambda \mid r_i \geqslant 0,\ r_0 - r_i \geqslant 0,\ r_0 = m_0 \}$$
$$= \{ \mathbf{r} = (r_i, m_0) \in M \mid m_0 \geqslant r_1 \geqslant \ldots \geqslant r_\nu \geqslant m_0/2 \}.$$

The series $\tau(Y)$, $\hat{\tau}(\mathbf{s}, Y)$ are defined in the same way as in N° 1.

Now, for $\mathbf{m}=(m_i, m_0)\in M$, we have

(11) $$\delta^{\frac{1}{2}}(\pi^{\mathbf{m}})=q^{\frac{\nu(\nu+1)}{4}m_0-\sum_i(\nu-i+1)m_i},$$

(12) $$\widetilde{\tau}_{m_0'}(\pi^{\mathbf{m}})=\begin{cases}\sum_{m_0\geq r_1\geq\ldots\geq r_\nu\geq 0}\gamma_{(r_i)(m_i)}q^{\frac{\nu(\nu+1)}{4}m_0+\sum_i i(r_i-m_i)} & \text{if } m_0=m_0' \text{ and } 0\leq m_i\leq m_0 (1\leq i\leq\nu),\\ 0 & \text{otherwise,}\end{cases}$$

where $\gamma_{(r_i)(m_i)}$ is the γ for $GL(\nu, k)$, i.e.

$$\gamma_{(r_i)(m_i)}=\#(U_1\backslash(U_1\pi^{(r_i)}U_1\cap U_1\pi^{(m_i)}N_1))$$

with $U_1=GL(\nu, \mathfrak{o})$, $N_1=T^u(\nu, k)$.

Proof of (11). First an easy computation shows that $n\in N$ can be written uniquely in the form

(13) $$n=\begin{pmatrix}X_1 & 0\\ 0 & \iota{}^tX_1^{-1}\iota\end{pmatrix}\begin{pmatrix}1_\nu & Y_{12}\iota\\ 0 & 1_\nu\end{pmatrix}$$

with

$$X_1=\begin{pmatrix}1 & & x_{ij}\\ & \ddots & \\ 0 & & 1\end{pmatrix}\in T^u(\nu, k),\quad Y_{12}=(y_{ij}) : \text{symmetric}.$$

Hence for $h=\begin{pmatrix}h_1 & 0\\ 0 & \xi_0\iota h_1^{-1}\iota\end{pmatrix}$ with $h_1=\text{diag.}(\xi_i)$, we have

$$hnh^{-1}=\begin{pmatrix}h_1X_1h_1^{-1} & 0\\ 0 & \iota h_1^{-1}{}^tX_1^{-1}h_1\iota\end{pmatrix}\begin{pmatrix}1_\nu & \xi_0^{-1}h_1Y_{12}h_1\iota\\ 0 & 1_\nu\end{pmatrix}.$$

Thus the transformation $n\to hnh^{-1}$ induces the following transformations:

$$X_1\to h_1X_1h_1^{-1} \quad\text{or}\quad x_{ij}\to\xi_i x_{ij}\xi_j^{-1},$$
$$Y_{12}\to\xi_0^{-1}h_1Y_{12}h_1 \quad\text{or}\quad y_{ij}\to\xi_0^{-1}\xi_i y_{ij}\xi_j.$$

Since $dn=\prod_{i<j}dx_{ij}\prod_{i\leq j}dy_{ij}$, we have

$$\delta(h)=\prod_{i<j}|\xi_i||\xi_j|^{-1}\cdot|\xi_0|^{-\frac{\nu(\nu+1)}{2}}\prod_{i\leq j}|\xi_i||\xi_j|$$
$$=|\xi_0|^{-\frac{\nu(\nu+1)}{2}}\prod_{i=1}^{\nu}|\xi_i|^{2(\nu-i+1)},\qquad\text{q.e.d.}$$

Proof of (12). One has

$$\widetilde{\tau}_{m_0'}(\pi^{\mathbf{m}})=\delta^{\frac{1}{2}}(\pi^{\mathbf{m}})\int_N\tau_{m_0'}(\pi^{\mathbf{m}}n)dn$$

$\tau_{m_0'}$ being the characteristic function of $\mathfrak{X}_{m_0'}$ defined by (9). Hence we have $\widetilde{\tau}_{m_0'}(\pi^{\mathbf{m}})\neq 0$, only if $m_0=m_0'$ and $0\leq m_i\leq m_0$ $(1\leq i\leq\nu)$. Hence, assuming this, let us consider the condition for $\pi^{\mathbf{m}}n$ $(n\in N)$ to be integral. Writing $n\in N$ in the form (13), we see that $\pi^{\mathbf{m}}n$ is integral, if and only if

$$\pi^{(m_i)}X_1,\quad \pi^{m_0}(\pi^{(m_i)}X_1)^{-1},\quad \pi^{(m_i)}X_1Y_{12}$$

are integral. If we call (r_i) the elementary divisors of $\pi^{(m_i)}X_1$, the integrity condition for the first two is equivalent to saying that (r_i) satisfies the condition

(14) $$m_0\geq r_1\geq\ldots\geq r_\nu\geq 0.$$

Moreover, writing $\pi^{(m_i)}X_1 = u_1\pi^{(r_i)}u_2$, $Y_{12} = u_2^{-1}Y'_{12}{}^t u_2^{-1}$ with $u_1, u_2 \in U_1$, we see that $\pi^{(m_i)}X_1Y_{12}$ is integral, if and only if $\pi^{(r_i)}Y'_{12}$ is integral, and that, for a fixed X_1, the measure of such Y'_{12}'s is equal to $q^{\sum_1^\nu ir_i}$. Thus we conclude that the measure of the n's for which the elementary divisors of $\pi^{(m_i)}X_1$ are (r_i) is equal to

$$\int_{N_1} c_{(r_i)}(\pi^{(m_i)}X_1)\,dX_1 \cdot q^{\sum_1^\nu ir_i} = \delta^{-1}(\pi^{(m_i)})\,\gamma_{(r_i)(m_i)} \cdot q^{\sum_1^\nu ir_i},$$

where δ, γ are the ones for $GL(\nu, k)$, i.e.

$$\delta(\pi^{(m_i)}) = q^{-\sum_i(\nu+1-2i)m_i}.$$

From these follows (12), q.e.d.

From (12) we obtain the following formula:

$$\hat{\tau}(s, Y) = \sum_{m_0=0}^{\infty}\left(\sum_{0\leqslant m_i\leqslant m_0} \beta_{\mathbf{m}}\, q^{-\sum_i m_i s_i}\right) Y_0^{m_0}, \tag{15}$$

where we put

$$\beta_{\mathbf{m}} = \sum_{(r_i)} \gamma_{(r_i)(m_i)}\, q^{\sum_i i(r_i - m_i)}, \quad Y_0 = q^{\frac{\nu(\nu+1)}{4} - s_0} Y, \tag{16}$$

the summation on (r_i) being taken over all (r_i) satisfying the condition (14). (Note that we have $\beta_{\mathbf{m}} = 0$, unless $0 \leqslant m_i \leqslant m_0$.)

3. We can calculate $\beta_{\mathbf{m}}$ explicitly for $\nu = 1, 2$. For $\nu = 1$, it is clear that

$$\beta_{m_1, m_0} = \begin{cases} 1 & \text{if } 0 \leqslant m_1 \leqslant m_0, \\ 0 & \text{otherwise,} \end{cases}$$

so that we have

$$\hat{\tau}(s, Y) = \frac{1}{(1 - Y_0)(1 - q^{-s_1}Y_0)}.$$

(Since $G \cong GL(2, k)$, this is, of course, a special case of (6).)

For $\nu = 2$, we know that $\gamma_{(r_i)(m_i)}$ is given explicitly as follows:

$$\gamma_{(r_i)(m_i)} = \begin{cases} q^{r_1 - m_1} & \text{if } r_1 = \operatorname{Max}\{m_1, m_2\},\ r_1 + r\ = m_1 + m_2, \\ q^{r_1 - m_1}(1 - q^{-1}) & \text{if } r_1 > \operatorname{Max}\{m_1, m_2\},\ r_1 + r_2 = m_1 + m_2, \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

whence we obtain immediately

$$\beta_{m_1, m_2, m_0} = \begin{cases} 1 + \operatorname{Min}\{m_1, m_2, m_0 - m_1, m_0 - m_2\}(1 - q^{-1}) & \text{if } 0 \leqslant m_i \leqslant m_0, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

$$\begin{aligned} \hat{\tau}(s, Y) &= \sum_{m_0=0}^{\infty}\left(\sum_{0\leqslant m_i\leqslant m_0} \beta_{\mathbf{m}}\, q^{-m_1 s_1 - m_2 s_2}\right) Y_0^{m_0} \\ &= \frac{1 - q^{-(1+s_1+s_2)}Y_0^2}{(1 - Y_0)(1 - q^{-s_1}Y_0)(1 - q^{-s_2}Y_0)(1 - q^{-s_1 - s_2}Y_0)}. \end{aligned} \tag{19}$$

In terms of the notations introduced in N° 9.3 (Case 1°), we may write

$$\hat{\tau}(s, Y) = \frac{1 - q^2 X_0 Y^2}{1 - q^{\frac{3}{2}}X'_2 Y + q^3 X_0(2 + X_1)Y^2 - q^{\frac{9}{2}} X_0 X'_2 Y^3 + q^6 X_0^2 Y^4}.$$

On the other hand, the transformation formula (9.19) can also be calculated explicitly as follows:

$$\hat{c}^{(0)} = X_0,$$
(20) $$\hat{c}^{(1)} = (q^2 - 1) + q^2 X_1,$$
$$\hat{c}^{(2)'} = q^{\frac{3}{2}} X_2'.$$

Thus we get the following formula which was obtained also by Shimura:

(21) $$\tau(Y) = \frac{1 - q^2 c^{(0)} Y^2}{1 - c^{(2)'} Y + q c^{(0)}((1+q^2) + c^{(1)}) Y^2 - q^3 c^{(0)} c^{(2)'} Y^3 + q^6 c^{(0)2} Y^4}.$$

In general, it might be conjectured that $\hat{\tau}(s, Y)$ is a rational function in Y_0 of the following form:

$$\hat{\tau}(s, Y) = \frac{\text{polynomial of degree } 2^\nu - 2 \text{ in } Y_0}{\prod\limits_{\substack{i_1 < \dots < i_\varkappa \\ 0 \leqslant \varkappa \leqslant \nu}} (1 - q^{-s_{i_1} - \dots - s_{i_\varkappa}} Y_0)} \quad (?)$$

4. *Calculation of ζ-function.* From (5.12) and (11), we have

$$|\mu(g)|_p^{-\lambda_0} = \omega_{s_{\lambda_0}}(g)$$

with

$$s_{\lambda_0} = \left(\nu, \nu - 1, \dots, 1, \lambda_0 - \frac{\nu(\nu+1)}{4}\right).$$

Therefore

$$\zeta(\lambda_0) = \hat{\tau}(s_{\lambda_0}, 1) = \int_{G \cap M_n(\mathfrak{o})} |\mu(g)|_p^{\lambda_0} dg$$

is the usual ζ-function.

More generally, we shall put

(22) $$s_{\lambda, \lambda_0} = \left(\lambda + \nu, \dots, \lambda + 1, \lambda_0 - \frac{\nu(\nu+1)}{4}\right)$$

and calculate

$$\hat{\tau}(s_{\lambda, \lambda_0}, Y) = \sum \beta_m q^{-\sum_i (\lambda + \nu + 1 - i) m_i} Y_0^{m_0},$$

where $Y_0 = q^{\frac{\nu(\nu+1)}{2} - \lambda_0} Y$.

For that purpose, we introduce some notations. We denote by $\rho = (\rho_0, \rho_1, \dots, \rho_\varkappa)$ an ordered set of integers such that

$$\rho_0 = 0 < \rho_1 < \dots < \rho_\varkappa = \nu.$$

An element $\mathbf{r} = (r_i)$ satisfying (14) is called of type ρ if

$$r_1 = \dots = r_{\rho_1} > r_{\rho_1 + 1} = \dots = r_{\rho_2} > \dots = r_\nu$$

and, when that is so, we write $\mathbf{r} \in \rho$. Then we obtain the following formula

(23) $$\sum_{(m_i)} \beta_m q^{-\sum(\lambda + \nu + 1 - i) m_i} = \sum_\rho Q_\rho \sum_{\substack{m_0 \geqslant r_1 \geqslant \dots \geqslant r_\nu \geqslant 0 \\ (r_i) \in \rho}} q^{-\sum(\lambda + i) r_i},$$

where

$$Q_\rho = \frac{(1 - q^{-1}) \dots (1 - q^{-\nu})}{\prod_i \{(1 - q^{-1}) \dots (1 - q^{-(\rho_i - \rho_{i-1})})\}}.$$

In fact, we have

$$\sum_{(m_i)} \beta_m q^{-\sum_i (\lambda + \nu + 1 - i) m_i} = \sum_{(m_i), (r_i)} \gamma_{(r_i)(m_i)} q^{-\sum_i (\lambda + \nu + 1 - i) r_i}$$
$$= \sum_{(r_i)} \# (U_1 \backslash U_1 \pi^{(r_i)} U_1) q^{-\sum_i (\lambda + \nu + 1 - i) r_i},$$

where the summations on (r_i) are taken over all (r_i) satisfying (14). Here we have easily

$$\#(U_1\backslash U_1\pi^{(r_i)}U_1) = [U_1 : U_1\cap(\pi^{(r_i)}U\pi^{-(r_i)})]$$
$$= q^{\sum_i(\nu+1-2i)r_i} Q_\rho,$$

if (r_i) is of type ρ. From these follows (23).

From (23) we get

$$(24) \qquad \hat{\tau}(s_{\lambda,\lambda_0}, Y) = \sum_\rho Q_\rho \left(\sum_{\substack{m_0 \geqslant r_1 \geqslant \ldots \geqslant r_\nu \geqslant 0 \\ (r_i)\in\rho}} q^{-\sum_i(\lambda+i)r_i} Y_0^{m_0} \right)$$

$$= \sum_\rho Q_\rho \cdot \frac{1}{1-Y_0} \cdot \left(\prod_{i=1}^{\varkappa-1} \frac{q^{-\rho_i\lambda-\frac{\rho_i(\rho_i+1)}{2}} Y_0}{1-q^{-\rho_i\lambda-\frac{\rho_i(\rho_i+1)}{2}} Y_0} \right) \cdot \frac{1}{1-q^{-\nu\lambda-\frac{\nu(\nu+1)}{2}} Y_0}.$$

Thus we see that $\hat{\tau}(s_{\lambda,\lambda_0}, Y)$ is actually a rational function of the following form:

$$\hat{\tau}(s_{\lambda,\lambda_0}, Y) = \frac{\text{polynomial of degree } \nu-1 \text{ in } Y_0}{(1-Y_0)\prod_{i=1}^{\nu}\left(1-q^{-i\lambda-\frac{i(i+1)}{2}}Y_0\right)}.$$

In particular, we have

$$(25) \qquad \zeta(\lambda_0) = \frac{\text{polynomial of degree } \nu-1 \text{ in } q^{-\lambda_0}}{\prod_{i=0}^{\nu}\left(1-q^{\frac{\nu(\nu+1)}{2}-\frac{i(i+1)}{2}-\lambda_0}\right)}.$$

Appendix II

DETERMINATION OF ZONAL SPHERICAL FUNCTIONS OF POSITIVE TYPE ON PL(2, $\mathfrak{K}$)

1. *Z.s.f. of positive type.* Let G be a unimodular locally compact group and U a compact subgroup. A z.s.f. ω on G relative to U is called " of positive type ", if it satisfies the condition

$$(1) \qquad \int_G\int_G \omega(gg'^{-1})\varphi(g)\overline{\varphi(g')}dgdg' \geqslant 0$$

for all $\varphi\in\mathscr{L}(G, U)$. The totality of such ω's is denoted by $\Omega^+ = \Omega^+(G, U)$. It is well-known that we have, for $\omega\in\Omega^+$,

$$(2) \qquad \omega(g^{-1}) = \overline{\omega(g)},$$

$$(3) \qquad |\omega(g)| \leqslant 1.$$

A character of G (i.e. a continuous homomorphism of G into the multiplicative group of complex numbers of absolute value 1) which is trivial on U is contained in Ω^+; in particular, the constant **1** belongs to Ω^+. For $\varphi\in\mathscr{L}(G, U)$, we have clearly

$$(4) \qquad \hat{\mathbf{1}}(\varphi) = \int_G \varphi(g)dg.$$

Lemma 1. *Let* $\omega\in\Omega^+$.

a) *If* $\varphi\in\mathscr{L}(G, U)$ *is " self-adjoint "* (i.e. $\varphi(g^{-1}) = \overline{\varphi(g)}$), *then* $\hat{\omega}(\varphi)$ *is real.*

b) *If* $\varphi\in\mathscr{L}(G, U)$ *is " non-negative "* (i.e. $\varphi(g)\geqslant 0$), *then we have*

$$(5) \qquad |\hat{\omega}(\varphi)| \leqslant \hat{\mathbf{1}}(\varphi).$$

Proof. *a)* From the assumption and (2), we have

$$\overline{\hat{\omega}(\varphi)} = \int \overline{\varphi(g)}\,\overline{\omega(g^{-1})}dg = \int \varphi(g^{-1})\omega(g)dg = \hat{\omega}(\varphi).$$

b) From the assumption and (3), we have

$$|\hat{\omega}(\varphi)| \leqslant \int \varphi(g)\,|\omega(g^{-1})|\,dg \leqslant \int \varphi(g)dg = \hat{\mathbf{1}}(\varphi), \qquad \text{q.e.d.}$$

In case $\mathscr{L}(G, U)$ is commutative, it is known **[11]** that, for every irreducible unitary representation T of G in a Hilbert space $\mathscr{H}$, the dimension of the subspace ${}^0\mathscr{H}$ of $\mathscr{H}$ formed of all the elements invariant under T_u $(u \in U)$ is at most 1. When we have actually $\dim {}^0\mathscr{H} = 1$ (where $\mathscr{H}$ need not be supposed irreducible), call v_0 an element in ${}^0\mathscr{H}$ with $||v_0|| = 1$; then the function defined by

$$\omega(g) = \langle v_0, T_g v_0 \rangle \tag{6}$$

is a z.s.f. of positive type. Conversely, every $\omega \in \Omega^+$ is obtained in this way. Thus Ω^+ is in one-to-one correspondence with the set of all unitary equivalence classes of the irreducible unitary representations of G such that ${}^0\mathscr{H} \neq \{0\}$ (i.e. " of the first kind ").

2. *Case of* PL(2, $\mathfrak{K}$). In the following, we propose to determine all z.s.f. of positive type on $\overline{G} = PL(2, \mathfrak{K})$ relative to $\overline{U}$ (in the notations of N° 8.4), $\mathfrak{K}$ being a central division algebra over k. By the results in § 8, we know that $\overline{\Omega} = \Omega(\overline{G}, \overline{U})$ is parametrized by $\mathbf{s} = (s, -s)$ with $s \in \mathbf{C}$; hence we write ω_s instead of $\omega_{\mathbf{s}}$. We have $\omega_s = \omega_{s'}$ if and only if

$$s' \equiv \pm s \left(\text{mod.}\ \frac{2\pi i}{\log q}\mathbf{Z}\right). \tag{7}$$

Now let $\mathfrak{X}_m$, τ_m be as defined in Appendix I, N° 1 and denote by $\overline{\tau}_m$ the characteristic function of $\overline{\mathfrak{X}}_m = (\mathfrak{X}_m H_0)/Z$. Then we have

$$\hat{\omega}_s(\overline{\tau}_m) = q^{\frac{dm}{2}} \cdot \frac{q^{(m+1)s} - q^{-(m+1)s}}{q^s - q^{-s}}, \tag{8}$$

d denoting the degree of $\mathfrak{K}/k$. In fact, as is easily seen (or by (7.5), (7.26)), we have $\hat{\omega}_{\mathbf{s}}(\overline{\tau}_m) = \hat{\omega}_{\mathbf{s}}(\tau_m) = \hat{\tau}_m(s, -s)$. By Appendix I, N° 1, we have

$$\begin{aligned}\hat{\tau}_m(s, -s) &= \sum_{\mathbf{m} = (m_1, m_2)} \tilde{\tau}_m(\pi^{\mathbf{m}}) q^{(m_2 - m_1)s} \\ &= \sum_{\substack{m_i \geqslant 0 \\ m_1 + m_2 = m}} q^{\frac{dm}{2} + (m_2 - m_1)s} \\ &= q^{\frac{dm}{2}} \cdot \frac{q^{(m+1)s} - q^{-(m+1)s}}{q^s - q^{-s}}.\end{aligned}$$

In particular, we get

$$\hat{\omega}_s(\overline{\tau}_1) = q^{\frac{d}{2}}(q^s + q^{-s}) \tag{9}$$

and, as $\mathbf{1} = \omega_{\frac{d}{2}}$,

$$\hat{\mathbf{1}}(\overline{\tau}_1) = q^d + 1. \tag{10}$$

Now, let $\omega_s \in \overline{\Omega}^+$. Then, since τ_1 is clearly self-adjoint and non-negative, it follows from Lemma 1 that $\hat{\omega}_s(\overline{\tau}_1)$ is real and that we have

$$|\hat{\omega}_s(\overline{\tau}_1)| \leqslant q^d + 1.$$

From (9), this implies that, for $s \in \mathbf{C}$ with $\omega_s \in \overline{\Omega}^+$, we have only the following three possibilities:

1° Re $s = 0$,

2° $-\frac{d}{2} < \text{Re}\ s < \frac{d}{2}$, Re $s \neq 0$, Im $s \equiv 0 \left(\text{mod.}\ \frac{\pi}{\log q}\mathbf{Z}\right)$,

3° $s \equiv \pm\frac{d}{2}\left(\text{mod.}\ \frac{\pi i}{\log q}\mathbf{Z}\right)$.

We know already, by Corollary 1 to Proposition 5.1, that the case 1° comes from the unitary representations of G of the "principal series". (Note that in general the representation T^α constructed in N° 5.3 is reducible and containing the irreducible representation corresponding to ω_s ($\delta^{\frac{1}{2}}\alpha \longleftrightarrow s$) just once. For the case where T^α itself is irreducible, see Bruhat [3].) On the other hand, it is clear that the case 3° is realized by the usual characters of G, i.e.

$$(11)\qquad \begin{aligned} &\omega_s = \mathbf{1} && \text{for } s \equiv \pm\frac{d}{2} && \left(\text{mod. } \frac{2\pi i}{\log q}\mathbf{Z}\right), \\ &\omega_s(\bar{g}) = (-1)^{\operatorname{ord}_{\mathfrak{p}}\widetilde{N}(g)} && \text{for } s \equiv \pm\frac{d}{2} + \frac{\pi i}{\log q} && \left(\text{mod. } \frac{2\pi i}{\log. q}\mathbf{Z}\right), \end{aligned}$$

$\bar{g}$ denoting the class of $g \in GL(2, \mathfrak{K})$ modulo Z and $\widetilde{N}$ denoting the reduced norm of $M_2(\mathfrak{K})/k$.

In the rest of this Appendix, we shall show that the case 2° with $s \in \mathbf{R}$ comes also from certain unitary representations of $\overline{G}$ (called "supplementary series"). In view of Lemma 7.2, applied to the second character in (11), this implies actually that the whole case 2° takes place. Thus we shall have the following theorem:

THEOREM. *Let $\mathfrak{K}$ be a central division algebra of degree d over k. Then the totality of zonal spherical functions of positive type on $\overline{G} = PL(2, \mathfrak{K})$ relative to $\overline{U}$ is given by the $\omega_s = \omega_{(s,-s)}$ with s in the ranges 1°, 2°, 3° listed above.*

This result, as well as its proof, is quite analogous to that in the real or complex case (see, for instance [**10**]).

3. *Another formulation of the representations of principal series.* In order to construct representations of supplementary series, it will be convenient to modify the representation $(\mathscr{H}^\alpha, T^\alpha)$ given in N° 5.3 in the following way.

Let $G = GL(2, \mathfrak{K})$ and we use the same notations as in N° 8.2. Put further

$$(12)\qquad N' = \left\{ n'_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in \mathfrak{K} \right\}.$$

Then, as is well-known (and easily verified), we have the following "cellular decomposition":

$$G = N'HN \cup \iota HN, \qquad \iota = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

in other words, every $g \in G$ can be written uniquely either in the form $g = n'hn$ or in the form $g = \iota hn$ with $n' \in N'$, $h \in H$, $n \in N$. Hence, N' can be identified with a subset of $\overline{U} = U/(U \cap HN) \underset{=}{\sim} G/HN$, and in fact one has then $\overline{U} = N' \cup \{\text{class of } \iota\}$. If $\overline{U} \ni \bar{u} \longleftrightarrow n'_x$ $(x \in \mathfrak{K})$, and if dx denotes an (additive) Haar measure of $\mathfrak{K}$, the following transformation formula of the measures is easily verified:

$$(13)\qquad d\bar{u} = c\psi_\delta^{-1}(n'_x)\,dx,$$

where c is a constant and ψ_δ is as defined by (5.10) (i.e. if $n'_x = uhn$, we have $\psi_\delta(n'_x) = \delta(h)$). If the Haar measures are normalized in such a way that $\int_{\overline{U}} d\bar{u} = \int_{\mathfrak{D}} dx = 1$, we have $c = (1 + q^{-d})^{-1}$.

Before proceeding further, we prepare some routine properties of (non-commutative) linear fractional transformations.

LEMMA 2. *Let $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in GL(2, \mathfrak{K})$, $x \in \mathfrak{K}$ and suppose that $-xg_{12} + g_{22} \neq 0$. Then we have*

$$(14)\qquad \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ g^{-1}.x & 1 \end{pmatrix} \begin{pmatrix} * & 0 \\ 0 & J(g^{-1}, x) \end{pmatrix} \begin{pmatrix} 1 & -g_{12}J(g^{-1}, x) \\ 0 & 1 \end{pmatrix}$$

where we put

$$(15)\qquad \begin{aligned} g^{-1}.x &= (-xg_{12} + g_{22})^{-1}(xg_{11} - g_{21}), \\ J(g^{-1}, x) &= (-xg_{12} + g_{22})^{-1}. \end{aligned}$$

It follows immediately that we have

$$(16)\qquad \begin{aligned} (gg_1)^{-1}.x &= g_1^{-1}.(g^{-1}.x), \\ J((gg_1)^{-1}, x) &= J(g_1^{-1}, g^{-1}.x)J(g^{-1}, x) \end{aligned}$$

for all $g, g_1 \in GL(2, \mathfrak{K}), x \in \mathfrak{K}$, whenever the right hand side has meaning. Moreover, we have

$$(17) \qquad |N(g^{-1}.x_2 - g^{-1}.x_1)|_{\mathfrak{p}} = |N(x_2 - x_1)|_{\mathfrak{p}} |N(J(g^{-1}, x_1)J(g^{-1}, x_2))\widetilde{N}(g)|_{\mathfrak{p}},$$

$$(18) \qquad d(g^{-1}.x) = |N(J(g^{-1}, x))|_{\mathfrak{p}}^{2d} |\widetilde{N}(g)|_{\mathfrak{p}}^{d} dx,$$

$N, \widetilde{N}$ denoting the reduced norms of $\mathfrak{K}/k$, $M_2(\mathfrak{K})/k$, respectively. By virtue of (16), the proofs of (17), (18) are reduced to the cases of $g = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$.

Now let α be a quasi-character of H such that $\delta^{\frac{1}{2}}\alpha \longleftrightarrow (s_1, s_2)$ and let $\mathscr{H}^{\alpha}$, T^{α} be as defined in N° 5.3. By what we mentioned above, $f \in \mathscr{H}^{\alpha}$ is uniquely determined by its restriction on N'; hence we put $f(x) = f(n'_x)(x \in \mathfrak{K})$ and consider $f \in \mathscr{H}^{\alpha}$ as a function on $\mathfrak{K}$. Then, it is clear that, if $\bar{u} \longleftrightarrow n'_x$, we have

$$f(u) = \psi_{\alpha}^{-1}(n'_x) f(x).$$

Thus, in view of (13) and (5.6), $\mathscr{H}^{\alpha}$ may be regarded as the Hilbert space formed of all (classes of) measurable functions f on $\mathfrak{K}$ such that

$$||f||^2 = c\int_{\mathfrak{K}} |f(x)|^2 \psi_{\delta|\alpha|^2}^{-1}(n'_x) dx < \infty,$$

the inner product being defined by

$$(19) \qquad <f_1, f_2> = c\int_{\mathfrak{K}} \overline{f_1(x)} f_2(x) \psi_{\delta|\alpha|^2}^{-1}(n'_x) dx.$$

By Lemma 2, the operation of $T_g^{\alpha}(g \in G)$ is given by

$$(20) \qquad T_g^{\alpha} f(x) = f(g^{-1}.x) |N(J(g^{-1}, x))|_{\mathfrak{p}}^{-s_1+s_2+d} |\widetilde{N}(g)|_{\mathfrak{p}}^{-s_1+\frac{d}{2}}.$$

Note that, in view of the relation

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix},$$

we have

$$(21) \qquad \psi_{\alpha}(n'_x) = \begin{cases} 1 & \text{if } x \in \mathfrak{O} \\ \alpha\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = |N(x)|_{\mathfrak{p}}^{s_1 - s_2 - d} & \text{if } x \notin \mathfrak{O}. \end{cases}$$

4. *Changing the inner product.* The notations being as before, let hereafter $s_1 = s, s_2 = -s$ (which implies that α is real) and put

$$(22) \qquad \rho = -2 \operatorname{Re} s - d.$$

Let $\mathscr{H}_0^{\alpha}$ be the subspace of $\mathscr{H}^{\alpha}$ formed of all continuous functions on $\mathfrak{K}$ with compact carrier. We now consider an integral operator defined as follows:

$$(23) \qquad Af(x) = \psi_{\delta\alpha^2}(x) \int_{\mathfrak{K}} |N(x - x_1)|_{\mathfrak{p}}^{\rho} f(x_1) dx_1.$$

PROPOSITION. *The notations being as above, if* $\rho > -d$, A *is a hermitian operator on* $\mathscr{H}^{\alpha}$, *whose domain of definition contains* $\mathscr{H}_0^{\alpha}$. *If moreover* $\rho < 0$, A *is a positive semi-definite.*

Proof. Let l be an integer and let $\{x_i\}$ be a system of representatives of $\mathfrak{K}/\mathfrak{P}^l$; denote by c_{l,x_i} the characteristic function of $x_i + \mathfrak{P}^l$. Then, if $\rho > -d$, an easy computation gives

$$(24) \qquad Ac_{l,x_i} = \psi_{\delta\alpha^2}\left\{ q^{-l(\rho+d)} \frac{1 - q^{-d}}{1 - q^{-\rho-d}} c_{l,x_i} + q^{-ld} \sum_{j \neq i} |N(x_j - x_i)|^{\rho} c_{l,j} \right\}.$$

It follows, in particular, that, for $x \in \mathfrak{K}$ with $\operatorname{ord}_{\mathfrak{P}} x$ sufficiently small, we have $(Ac_{l,x_i})(x) = q^{-ld}\psi_{\alpha}(x)$. This shows that $Ac_{l,x_i} \in \mathscr{H}^{\alpha}$, and more generally that $Af \in \mathscr{H}^{\alpha}$ for all $f \in \mathscr{H}_0^{\alpha}$. From the definition, A is clearly hermitian.

Now to prove that $A\,(\rho<0)$ is positive semi-definite, it is enough to show that, for every $\varkappa\geqslant 0$, l, the matrix $(<Ac_{l,x_i}, c_{l,x_j}>)$ of degree $q^{\varkappa d}$, with x_i, x_j ranging over a system of representatives of $\mathfrak{P}^{l-\varkappa}/\mathfrak{P}^l$ is positive definite. From (24), we have

$$<Ac_{l,x_i}, c_{l,x_j}> = \begin{cases} cq^{-l(\rho+2d)}\dfrac{1-q^{-d}}{1-q^{-\rho-d}} & \text{if } i=j, \\ cq^{-2ld}\,|N(x_j-x_i)|^{\rho} & \text{if } i\neq j. \end{cases}$$

Hence, writing the matrix in question, with the indices ordered suitably, in the form $cq^{-l(\rho+2d)}A^{(\varkappa)}$, one has

$$A^{(0)}=\frac{1-q^{-d}}{1-q^{-\rho-d}}\quad(>q^{\rho}),\qquad A^{(\varkappa)}=\begin{pmatrix} A^{(\varkappa-1)} & & q^{\varkappa\rho} \\ & \ddots & \\ q^{\varkappa\rho} & & A^{(\varkappa-1)} \end{pmatrix}\qquad(\varkappa\geqslant 1),$$

whence, putting $E^{(\varkappa)}=\overbrace{\begin{pmatrix} 1 & \dots & 1 \\ \dots & \dots & \dots \\ 1 & \dots & 1\end{pmatrix}}^{q^{\varkappa d}}$ and making use of the assumption $\rho<0$, one can prove by an easy induction on $\varkappa$ that

$$A^{(\varkappa)}>q^{\varkappa\rho}E^{(\varkappa)}\geqslant 0\qquad(\varkappa\geqslant 1).$$

This completes the proof.

It follows that, defining a new inner product in $\mathscr{H}_0^{\alpha}$ by

$$(25)\qquad <f_1,f_2>'=c<Af_1,f_2>=c<f_1,Af_2>$$
$$=\int_{\mathfrak{K}}\int_{\mathfrak{K}}|N(x_2-x_1)|^{\rho}\overline{f_1(x_1)}f_2(x_2)\,dx_1\,dx_2\qquad(-d<\rho<0)$$

and completing $\mathscr{H}_0^{\alpha}$ with respect to this inner product, one obtains a Hilbert space, which we denote by $\mathscr{H}'^{\alpha}$. From (17), (18) one sees at once that the operator $T_g^{\alpha}(g\in G)$ defined by (20) is a unitary operator of $\mathscr{H}'^{\alpha}$. Finally, from (21) and (24), it can readily be seen that $\psi_\alpha\in{}^0\mathscr{H}'^{\alpha}$ and that

$$A\psi_\alpha=\lambda\psi_\alpha,\qquad \lambda=\frac{1-q^{-\rho-2d}}{1-q^{-\rho-d}},$$

whence one gets

$$<\psi_\alpha, T_g^{\alpha}\psi_\alpha>'/<\psi_\alpha,\psi_\alpha>'=<\psi_\alpha, T_g^{\alpha}\psi_\alpha>=\omega^{\alpha}(g).$$

Thus one concludes that the z.s.f. of positive type associated with $(\mathscr{H}'^{\alpha}, T^{\alpha})$ (or, more precisely, with the irreducible component of it containing ψ_α) is precisely ω^{α}.

In particular, if s is real, we have $-d/2<s<0$, and the representation T^{α} gives actually a representation of $\bar{G}=G/Z$. This completes the proof of our Theorem.

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