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ARMAND BOREL

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SOME FINITENESS PROPERTIES OF ADELE GROUPS OVER NUMBER FIELDS

by ARMAND BOREL

The formalism of adeles and ideles, introduced in algebraic number theory by Chevalley, has been recently applied to more general situations. In particular, it allows one to associate to an algebraic matric group G defined over a finite number field k a locally compact group G_A, the adele group of G, in which the group G_k of rational points over k of G is naturally imbedded as a discrete subgroup, in the same way as the multiplicative group k^* of non-zero elements of k is imbedded in the idele group I_k of k.

When $k=\mathbf{Q}$, the pair G_A, G_k may be viewed roughly as the global counterpart of the pair $G_{\mathbf{R}}$, $G_{\mathbf{Z}}$, where, as usual, $G_{\mathbf{R}}$ is the group of real matrices of G, and $G_{\mathbf{Z}}$ the group of integral matrices with determinant ± 1 contained in G (the units of G); and, in fact, the chief purpose of this work is to prove for G_A, G_k the analogues of the main results of [4] on $G_{\mathbf{R}}$ and $G_{\mathbf{Z}}$.

This paper is divided into eight paragraphs. The first one fixes the notation and collects some definitions and elementary facts pertaining to the adele groups. The second one contains some remarks on the double cosets modulo G_A^∞ and G_k, where G_A^∞ is the stability group of the lattice of integral points in the underlying space k^n, chiefly: relation with the ideal classes of k if $G=\mathbf{GL}_n$, with the classes in the genus of a rational quadratic form F, if G is the orthogonal group of F, with strong approximation, and behaviour in semi-direct products. § 3 reviews, reformulates in part, and strengthens slightly some results of [4].

In § 4 we construct, in much the same way as in [4], *fundamental sets* for G_k in G_A (i.e. subsets of G_A which intersect only finitely many of their right translates under G_k and which meet every left coset $x.G_k$), when G is connected. The case of non-connected groups is dealt with in 5.2.

§ 5 proves first that the number of distinct double cosets $G_A^\infty . x . G_A$ is finite. This makes it clear that G_A/G_k is of finite invariant measure, or is compact, if and only if G_∞/G_0 is so; here G_∞ is the product of the groups G_{k_v}, where k_v runs through the completions of k with respect to the archimedean absolute values, and G_0 is the group of units of G, (if $k=\mathbf{Q}$, then $G_\infty=G_{\mathbf{R}}$, $G_0=G_{\mathbf{Z}}$). Theorem 12.3 of [4] then shows that G_A/G_k is of finite invariant volume if and only if the identity component G^0 of G has no non-trivial rational character over k, and is compact if and only if, moreover, every unipotent element of G_k belongs to the radical of G_k. In analogy with Theorem 6.9

of [4], it is also proved in § 5 that if G is reductive, H is a reductive subgroup of G, and $\sigma : G \to G/H$ the natural projection, then $\sigma(G_A) \cap (G/H)_k$ consists of finitely many orbits of G_k.

As a consequence of this last result and of some elementary facts about Galois cohomology, we show in § 6 that if G is reductive, connected, the principal homogeneous spaces over k of G which have rational points over all completions of k form finitely many isomorphism classes.

§ 7 is concerned with the double cosets of G_k modulo G_0 and H_k, where H is a parabolic subgroup (i.e. is such that G/H is a projective variety) over k. Their number is finite and, under suitable assumptions, equal to the number of double cosets $H_A^\infty . x . H_k$ in H_A.

Finally, § 8 extends to S-units and certain subgroups of G_A the results of [4] and of §§ 4, 5 of the present paper pertaining to fundamental sets and closed orbits in rational representations.

The main results of this paper have been announced in [1, 3]. Some of them have already been established for solvable groups in [11] and for certain classical groups in [14]. The compactness criterion for G_A/G_k mentioned above is the adele version of Godement's conjecture and has also been given another proof by G. D. Mostow and T. Tamagawa (*Annals of Math.*, 76 (1962), p. 446-463).

§ 1. Preliminaries.

In this paragraph, we fix the notation and recall some notions and facts about adele groups attached to linear algebraic groups over number fields. This is not meant as a self-contained introduction to the subject; for more details, see [14, Chap. I]. The notation of the sections 1.1 to 1.4 will be used throughout.

1.1. k is an algebraic number field of finite degree, $\mathfrak{o}$ the ring of integers of k, V the set of primes (or of equivalence classes of absolute values) of k, $P \subset V$ the set of finite primes of k, $x \to ||x||_v$ the normalized absolute value associated to $v \in V$, k_v the completion of k with respect to $||x||_v$, $\mathfrak{o}_\mathfrak{p}$ $(\mathfrak{p} \in P)$ the ring of $\mathfrak{p}$-adic integers of $k_\mathfrak{p}$, and A_k or A the ring of adeles of k. If $v \in V - P$ is an infinite (or archimedean) prime, we put $\mathfrak{o}_v = k_v$. If S is a subset of V, then $\mathfrak{o}(S) = \{x \in k \mid x \in \mathfrak{o}_v \ (v \in V - S)\}$. In particular, $\mathfrak{o}(S) = \mathfrak{o}$ if S is contained in $V - P$.

1.2. The letter G will always stand for an algebraic matric group over k, n for its degree, and G^0 for its identity component (see 1.11). Given a subring B of an overfield of k, G_B denotes the group of elements of G with coefficients in B and determinant invertible in B. We shall write G_v for G_{k_v}.

G_{A_k} or G_A is the adele group of G, and

$$G_U = \prod_{\mathfrak{p} \in P} G_{\mathfrak{o}_\mathfrak{p}},$$

$$G_{A(S)} = \prod_{v \in S} G_v \times \prod_{v \in V - S} G_{\mathfrak{o}_v}, \qquad (S \subset V;\ S \text{ finite}).$$

We recall that G_A is by definition a subgroup of the direct product of the G_v's. An element $x=(x_v)_{v\in V}$ of that product is an adele of G if and only if $x_v\in G_{o_v}$ for almost all (i.e. for all but a finite number of) v's. The group G_A contains therefore $G_{A(S)}$. The group G_v $(v\in V)$ is locally compact with respect to the v-adic topology, and G_{o_p} $(p\in P)$ is open and compact in G_p; $G_{A(S)}$ is locally compact and G_U is compact for the product topology. G_A itself is a locally compact group, once endowed with the inductive limit topology of the $G_{A(S)}$, where S runs through the finite subsets of V ordered by inclusion. $G_{A(S)}$ is open in G_A; every compact subset of G_A is contained in the union of finitely many translates of G_A^∞, hence belongs to some group $G_{A(S)}$.

For any $S\subset V$ we put $G_{A(S)}=\{g=(g_v)\in G_A \mid g_v\in G_{o_v}\ (v\in V-S)\}$. If S is finite, this definition of $G_{A(S)}$ coincides with the previous one. π_S will denote the projection of G_A in $\prod_{v\in S} G_v$, G_S the image of G_A, and j_S the natural inclusion of G_S in G_A as the subgroup of adeles whose components outside S are equal to e; often, we identify G_S with $j_S(G_S)$ and G_U with $j_P(G_U)$. We have $G_A\cong G_S\times G_{V-S}$. If S is finite, then $G_S=\prod_{v\in S} G_v$.

If $S=V-P$, we write $G_o, G_\infty, G_A^\infty, \pi_\infty, j_\infty$ for $G_{o(S)}, G_S, G_{A(S)}, \pi_S, j_S$.

To any element $a\in G_k$ corresponds an adele of G, all of whose components are equal to a, called a *principal adele* of G. When viewed as subgroups of G_A, the groups G_k, $G_{o(S)}$, G_o will, unless otherwise said, be identified with groups of principal adeles. They are discrete and $G_{o(S)}=G_{A(S)}\cap G_k$, in particular $G_o=G_A^\infty\cap G_k$. If $S=V$, $G_{A(S)}=G_A$, $G_{o(S)}=G_k$. If S is empty or contained in $V-P$, then $G_{A(S)}=G_A^\infty$, $G_{o(S)}=G_o$. The group G_o is the group of units, and $G_{o(S)}$ the group of S-units of G.

The (possibly infinite) number of double cosets $G_A^\infty . x . G_k$ $(x\in G_A)$ in G_A will be denoted by $c(G)$.

1.3. A rational homomorphism $f: G\to G'$ over k of G into another algebraic matric group over k induces a continuous homomorphism $f_v: G_v\to G'_v$ mapping G_{o_p} into G'_{o_p} for almost all p's, and consequently also induces continuous homomorphisms $f_A: G_A\to G'_A$, $f_S: G_S\to G'_S$, which are isomorphisms if f is.

The rational homomorphisms over k of G into $\mathbf{GL}_1$, usually called the rational characters defined over k of G, form a commutative group, to be denoted $X_k(G)$. Each element $\chi\in X_k(G)$ defines a homomorphism of G_A into $\mathbf{GL}_{1,A}$. The latter group is just the idele group I_k of k.

1.4. When $k=\mathbf{Q}$, the primes of k are of course the rational prime numbers and an infinite prime ∞. As is well-known, and as will be seen repeatedly in this paper, the general case of a number field can in many instances be reduced to the case where $k=\mathbf{Q}$ by the use of the so-called restriction of the groundfield [14, Chap. I]. We recall that, once a vector-space basis (α_i) $(1\leqslant i\leqslant d=[k:\mathbf{Q}])$ of K over $\mathbf{Q}$ is chosen, there is associated to G an algebraic matric group $R_{k/\mathbf{Q}}G=G'$ over $\mathbf{Q}$, of degree $n.d$, and a rational homomorphism $\mu: G'\to G$ over k such that

$$(1)\qquad \mu^0=(\mu^{\sigma_1}, \ldots, \mu^{\sigma_d}): G'\to G^{\sigma_1}\times \ldots \times G^{\sigma_d}$$

is an isomorphism over $\bar{k}$, where $\sigma_1, \ldots, \sigma_d$ are the distinct isomorphisms of k into $\mathbf{C}$. The pair (G', μ) is, up to isomorphism, independent of the choice of the basis (α_i), and can also, up to isomorphism, be characterized by a universal property *(loc. cit.)*. The map μ also induces isomorphisms

$$G'_{\mathbf{Q}} \cong G_k; \quad G'_{\mathbf{R}} \cong G_\infty; \quad G'_{A_{\mathbf{Q}}} \cong G_{A_k}, \tag{2}$$

and, if (α_i) is a module basis of $\mathfrak{o}$ over $\mathbf{Z}$, as we shall always assume, then

$$G'_{\mathbf{Z}} \cong G_{\mathfrak{o}}, \quad G'^{\infty}_{A_{\mathbf{Q}}} \cong G^{\infty}_{A_k}, \quad G'_{U} \cong G_{U}. \tag{3}$$

1.5. There exists a rational homomorphism $\nu : G \to G'$ over k such that

$$\nu^0 = (\nu^{\sigma_1}, \ldots, \nu^{\sigma_d}) : G^{\sigma_1} \times \ldots \times G^{\sigma_d} \to G',$$

is the inverse of μ^0. In fact, if say σ_1 is the identity, the intersection G_1 of the kernels of the homomorphisms μ^{σ_i} $(i \neq 1)$ is stable under every isomorphism of $\mathbf{C}$ over k, hence is defined over k, and it is mapped isomorphically onto G by μ. The map ν is then just the inverse of the restriction of μ to G_1. It clearly has the following properties

$$\mu \circ \nu = id. \; \mu^{\sigma_i} \circ \nu(G) = (e) \qquad (i \neq 1), \tag{4}$$

$$\prod_{1 \leq i \leq d} (\nu^{\sigma_i}(\mu^{\sigma_i}(x'))) = x' \qquad (x' \in G'). \tag{5}$$

Proposition. — The map $\alpha : X_{\mathbf{Q}}(G') \to X_k(G)$ *defined by* $\chi \to \nu \circ \chi$ $(\chi \in X_{\mathbf{Q}}(G'))$ *is an isomorphism.*

Let $\chi \in X_k(G)$. Then $\mu \circ \chi \in X_k(G')$, therefore the product in $X_k(G')$ of the characters $(\mu \circ \chi)^{\sigma_i}$ is defined over $\mathbf{Q}$, whence a map $\beta : X_k(G) \to X_{\mathbf{Q}}(G')$. A routine verification based on (4), (5), which we omit, shows then that $\alpha \circ \beta$ and $\beta \circ \alpha$ are both the identity map.

1.6. *Proposition. — Let* $G = H.N$ *be the semi-direct product of a subgroup* H *and a normal subgroup* N, *both algebraic, over* k. *Then* N_A *is normal in* G_A, *and* G_A *is the semi-direct product of* H_A *and* N_A.

Let $g = (g_v)_{v \in V}$ be an element of G_A. Since G_v is the semi-direct product of H_v and N_v, we may write uniquely $g_v = h_v . n_v$ $(h_v \in H_v, n_v \in N_v)$. We have $h_v = f_v(g_v)$ where $f : G \to G/N = H$ is the natural projection, hence $(h_v)_{v \in V} \in H_A$, which implies $(n_v)_{v \in V} \in N_A$ and $G_A \subset H_A . N_A$. The proposition then follows readily.

1.7. *Proposition. — Let* G' *be an algebraic matric group over* k, *and* $\Phi : G_A \to G'_A$ *an isomorphism which maps* G_v *onto* G'_v *for every* $v \in V$. *Then* $\Phi(G_{A(S)})$ *is commensurable with* $G'_{A(S}$ *for every subset* S *of* V.

(We recall that two subgroups of a group are commensurable with each other if their intersection has finite index in both of them.)

The group $\Phi(G_{\mathfrak{o}_\mathfrak{p}})$ $(\mathfrak{p} \in P)$ is open and compact in $G'_\mathfrak{p}$, hence is commensurable with $G'_{\mathfrak{o}_\mathfrak{p}}$. The groups $\Phi(G_U)$ and $\Phi^{-1}(G'_U)$, being compact, are contained respectively in $G'_{A(T)}$ and $G_{A(T)}$ for a suitable finite subset T of V; hence $\Phi(G_{\mathfrak{o}_\mathfrak{p}}) = G'_{\mathfrak{o}_\mathfrak{p}}$ for $\mathfrak{p} \in V - T$, and the proposition.

Remark. — This proposition applies notably when $\Phi = f_A$, where f is an isomorphism over k of G onto G′, or when $G = G'$ is normal in an algebraic matric group H over k, and Φ is the restriction to G_A of an inner automorphism of H_A. In the former case, it shows that $f(G_{\mathfrak{o}(S)})$ is commensurable with $G'_{\mathfrak{o}(S)}$ for any $S \subset V$.

1.8. *Proposition. — Let* S *be a subset of* V *containing the infinite primes. Then* $\pi_S(G_{\mathfrak{o}(S)})$ *is a discrete subgroup of* G_S *and* $G_{A(S)}/G_{\mathfrak{o}(S)}$ *is a principal fibration over* $G_S/\pi_S(G_{\mathfrak{o}(S)})$, *with compact structural group* $\prod_{\mathfrak{p} \in V-S} G_{\mathfrak{o}_\mathfrak{p}}$.

Let $M = \prod_{\mathfrak{p} \in V-S} G_{\mathfrak{o}_\mathfrak{p}}$. The projection $\pi_S : G_A \to G_S$ induces a homomorphism of $G_{A(S)}$ onto G_S, with kernel M. Since M is compact, $\pi_S(G_{\mathfrak{o}(S)})$ is discrete. Moreover, we have obviously

$$j_S \cdot \pi_S(G_{\mathfrak{o}(S)}) \cdot M = \pi_S^{-1}(\pi_S(G_{\mathfrak{o}(S)})) \cap G_{A(S)} = G_{\mathfrak{o}(S)} \cdot M. \tag{6}$$

The group $N = G_{\mathfrak{o}(S)} \cdot M$ is closed and, since M is normal in $G_{A(S)}$, the space $G_{A(S)}/G_{\mathfrak{o}(S)}$ is a principal fibration over

$$G_{A(S)}/N = (G_S \times M)/N \cong G_S/\pi_S(G_{\mathfrak{o}(S)}),$$

with structural group

$$N/G_{\mathfrak{o}(S)} \cong M/(M \cap G_{\mathfrak{o}(S)}) \cong M.$$

1.9. *Proposition. — The group* $G_A/(G^0)_A$ *is compact.*

We view G/G^0 as a 0-dimensional cycle over k in $\mathbf{GL}_n/G^0$, and denote by σ the restriction to G of the natural projection of $\mathbf{GL}_n$ onto $\mathbf{GL}_n/G^0$. Let $x \in G/G^0$ and $X = \sigma^{-1}(x)$. We show first:

(*) There exists a finite subset S_x of V, such that if $v \notin S_x$, and X_v is not empty, then $X_{\mathfrak{o}_v} = G_{\mathfrak{o}_v} \cap X_v$ is not empty.

The field $k' = k(x)$ is algebraic, of finite degree over k. Let V′ the set of primes of k', S′ the set of finite primes of k', and $\mathfrak{o}'$ the ring of integers of k.

Since X is defined over k', and is irreducible, non-singular, there exists a finite subset S'_x of V′ such that X contains an integral $\mathfrak{P}$-adic point for every $\mathfrak{P} \notin S'_x$ (1). We take then for S_x the set of primes of k which are divided by some element of S'_x. It is finite. Let now $\mathfrak{p} \in P$, $\mathfrak{p} \notin S_x$. If $X_\mathfrak{p}$ is not empty, and $y \in X_\mathfrak{p}$, then $k(x) = k(\sigma(y)) \subset k_\mathfrak{p}$, hence $k' \subset k_\mathfrak{p}$. This implies the existence of $\mathfrak{P} \in P' \cap V'$ such that $\mathfrak{P}|\mathfrak{p}$ and $k'_\mathfrak{P} = k_\mathfrak{p}$, $\mathfrak{o}'_\mathfrak{P} = \mathfrak{o}_\mathfrak{p}$. By definition of S_x, we have $\mathfrak{P} \notin S'_x$, therefore X contains a point with coordinates in $\mathfrak{o}'_\mathfrak{P}$, hence in $\mathfrak{o}_\mathfrak{p}$.

For $\mathfrak{p} \in P$, let us choose for each $x \in G/G^0$ an element of $(\sigma^{-1}(x))_v$ if $(\sigma^{-1}(x))_v \neq \emptyset$, belonging to $(\sigma^{-1}(x))_{\mathfrak{o}_v}$ if the latter is not empty, and let $C_{(v)}$ the set of these elements. It is finite, and belongs to $G_{\mathfrak{o}_v}$ if v is outside the union of the sets S_x $(x \in G/G^0)$, therefore

$$C = \{g = (g_v) \in G_A \mid g_v \in C_{(v)} \ (v \in V)\}$$

is compact. Since clearly $G_A = C \cdot (G^0)_A$, the proposition is proved.

(1) See S. Lang, *Bull. Amer. Math. Soc.*, 66 (1960), p. 240-249, end of introduction.

1.10. Let W be a finite dimensional vector space over k. Then $W_A = W \otimes A$ is the adele space attached to W. More generally, an adele space X_A may be attached to any algebraic variety X defined over k [14, Chap. I]. However, this will occur only incidentally in this paper, and we refer directly to [14] for the relevant facts.

1.11. *Remark.* — The notion of algebraic matric group over k, of degree n, which is meant here, differs slightly in presentation, but of course not in substance, from that of "algebraic matric group defined over k", as used for instance in [4], or in the author's paper on linear algebraic groups (*Annals of Math.*, 64 (1956), p. 20-80), where the matrix coefficients and the fields of definition are assumed to belong to some universal field Ω, given once and for all. This convention would not be consistent with the consideration of all the G_v's. It will be more convenient to say that an algebraic matric group H over k, of degree n, is given by an ideal $\mathfrak{J} \subset k[X_{11}, X_{12}, \ldots, X_{nn}]$ such that for some (and hence for every) algebraic closure $\bar{k}$ of k, the set $H_{\bar{k}}$ of elements of $\mathbf{GL}(n, \bar{k})$ whose coefficients annihilate $\mathfrak{J}$ is a group. H is said to be connected if $H_{\bar{k}}$ is connected in the Zariski topology. If B is as in 1.2, then H_B is the group of elements of $\mathbf{GL}(n, B)$ whose coefficients annihilate $\mathfrak{J}$. In agreement with this point of view, we shall write $\mathbf{GL}_n$, $\mathbf{SL}_n$ rather than $\mathbf{GL}(n, \Omega)$, $\mathbf{SL}(n, \Omega)$ and then use indifferently $\mathbf{GL}_{n,B}$ and $\mathbf{GL}(n, B)$ or $\mathbf{SL}_{n,B}$ and $\mathbf{SL}(n, B)$ (1).

An isomorphism class over k of such groups corresponds to an affine algebraic group over k, of which these groups are matrix realizations. Properties of, or notions relative to, algebraic matric groups over k which are invariant under isomorphisms over k belong in fact to the underlying affine algebraic group. This is the case for G_A, $R_{k/\mathbf{Q}}G$, or also, by 1.7, for the finiteness of $c(G)$, but *not* the case for G_0, G_A^∞ or the actual value of $c(G)$.

We leave it to the reader who feels the need for it to make similar adjustments in the few occasions where we shall have to consider affine or projective varieties.

1.12. It will be convenient to be allowed to consider sometimes rational representations over k and homogeneous spaces of a non-necessarily connected matric algebraic group. In the cases of interest in this paper, this does not present any difficulty but, there being seemingly no handy reference for it, we feel compelled to devote a few sentences to that question.

Let L be an algebraic matric group, of degree m, over a field F. The ring F[L] of regular functions, defined over k, over L, the "coordinate ring" over F of L, is generated by 1, the matrix coefficients and $(\det x)^{-1}$ $(x \in L)$. A rational representation $\rho : L \to \mathbf{GL}_q$ is over F is the coefficients of $\rho(x)$ $(x \in L)$ belong to F[L].

Let M be an algebraic subgroup over F of L. Then $M \backslash L$ will be identified with its image in $M \backslash \mathbf{GL}_m$. The regular functions over F on $M \backslash L$ may be identified with the ring $F[L]^M$ of elements of F[L] which are invariant under left

(1) This way of introducing algebraic matric groups, which in a way brings us back to the pre-universal field days, was suggested to me by P. Cartier; see his paper in the Proceedings of the *Colloque sur la Théorie des Groupes Algébriques*, Bruxelles, 1962 (to appear) for a more complete discussion, and a comparison with the point of view of schemes.

translations by elements of M. If $M\backslash L$ is an affine algebraic set, this determines completely its structure over F. In particular, if $\rho : L \to \mathbf{GL}_q$ is a right rational representation over F, and $w \in F^q$ a point whose orbit X is closed and whose isotropy group is M, then $g \to x.\rho(g)$ induces an isomorphism over F of $M\backslash L$ onto X.

1.13. Let F be a field of characteristic zero, L an algebraic matric group over F. For the sake of reference, we recall that if L_F is Zariski dense in L, in particular [12, Theorem 3] if L is connected, then $L = H.N$ is the semi-direct product of its unipotent radical N by a reductive algebraic subgroup H over F. This globalization of a known fact on algebraic Lie algebras is due to Mostow (*Amer. J. M.*, *78* (1956), 200-221, Theorem 6), see also G. Hochschild (*Illinois J. M.*, *5* (1961), 492-519, Section 3).

Remark (added in proof). — In fact, the preceding result is also valid if L is not connected (see [5]). This allows some simplifications in the sequel. In particular, in 3.6, the second part of the proof, from « In the general case » on, is superfluous ; in 4.6, the proof given is also valid in the non-connected case.

§ 2. The double cosets modulo G_A^∞ and G_k.

This paragraph is devoted to some simple examples and remarks concerning the double cosets $G_A^\infty.x.G_k$. In the sequel, 2.3 will not be used and 2.2 will be needed only when $k = \mathbf{Q}$.

2.1. In this section, we recall some facts on lattices, of which Proposition 2.2 will be an easy consequence.

Let X be a lattice in k^n. Then $\mathfrak{o}_\mathfrak{p}.X = X_\mathfrak{p}$ $(\mathfrak{p} \in P)$ is a lattice in $k_\mathfrak{p}^n$, and $X_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}^n$ for almost all $\mathfrak{p}$'s. Conversely, given lattices $X'_\mathfrak{p}$ $(\mathfrak{p} \in P)$ such that $X'_\mathfrak{p} = \mathfrak{o}_\mathfrak{p}^n$ for almost all $\mathfrak{p}$, then $X' = \bigcap_{\mathfrak{p} \in P} X'_\mathfrak{p}$ is a lattice in k^n and we have $(X')_\mathfrak{p} = X'_\mathfrak{p}$ for all $\mathfrak{p} \in P$.

A lattice X in k^n is isomorphic to the direct sum of $\mathfrak{o}^{n-1}$ and a fractionary ideal of k, whose ideal class $\gamma(X)$ depends only on X. Two lattices X, X′ are isomorphic if and only if $\gamma(X) = \gamma(X')$. (For all this, see e.g. [6, §§ 12, 13]).

To an ordered pair X, X′ of lattices in k^n (resp. in $k_\mathfrak{p}^n$ $(\mathfrak{p} \in P)$), there is associated a fractionary ideal $\chi(X, X')$ of k (resp. $k_\mathfrak{p}$) [13, Chap. III, § 1]. If for example $k = \mathbf{Q}$ and $X' \subset X$, then $\chi(X, X')$ is the ideal generated by the index of X′ in X. If $u \in \mathbf{GL}(n, k)$, then $\chi(X, u(X))$ is the ideal generated by $\det u$ (*loc. cit.*, Prop. 2, p. 58). Since $\chi(\mathfrak{o}, \mathfrak{a}) = \mathfrak{a}$ for any ideal $\mathfrak{a}$ of $\mathfrak{o}$ [13, p. 27], the above and the formal properties of $\chi(X, X')$ [13, Prop. 1, p. 58] imply that the ideal class of $\chi(\mathfrak{o}^n, X)$ is equal to $\gamma(X)$. We note finally that

$$\chi(X, X')_\mathfrak{p} = \chi(X_\mathfrak{p}, X'_\mathfrak{p}) \qquad (\mathfrak{p} \in P),$$

as follows directly from the definition of $\chi(X, X')$.

2.2. *Proposition.* — *The number* $c(G)$ *is equal to the class number of* k *if* $G = \mathbf{GL}_n$, *to one if* $G = \mathbf{SL}_n$.

Let $G = \mathbf{GL}_n$, and $g = (g_v) \in G_A$. By 2.1, given a lattice X in k^n, then $g(X) = \bigcap_{\mathfrak{p} \in P} g_\mathfrak{p}(X_\mathfrak{p})$ is a lattice, and G_A operates in this way transitively on the set of all lattices in k^n. The isotropy group of $\mathfrak{o}^n$ is just G_A^∞, hence the orbits of G_k in G_A/G_A^∞ are in 1—1 correspondence with the isomorphism classes of lattices in k^n and therefore, by 2.1, with the ideal classes of k.

Let $G=\mathbf{SL}_n$, $g=(g_v)\in G_A$, and $X=g(\mathfrak{o}^n)$. Then

$$\chi(\mathfrak{o}_\mathfrak{p}^n, X_\mathfrak{p})=(\det g_\mathfrak{p})=\mathfrak{o}_\mathfrak{p}$$

for every $\mathfrak{p}\in P$, hence $\chi(\mathfrak{o}^n, X)=\mathfrak{o}$. Since $(\chi(\mathfrak{o}^n, X))=\gamma(X)$, there exists then $u\in\mathbf{GL}(n,k)$ such that $X=u(\mathfrak{o}^n)$, and since $(\det u)=\chi(\mathfrak{o}^n, X)$, the determinant of u is a unit of $\mathfrak{o}$. Multiplying u by an element of $\mathbf{GL}(n,\mathfrak{o})$ whose determinant is the inverse of det u, we get an element $u'\in\mathbf{SL}(n,k)$ which brings $\mathfrak{o}^n$ onto X. Thus $\mathbf{SL}(n,k)(\mathfrak{o}^n)=\mathbf{SL}_{n,A}(\mathfrak{o}^n)$, which proves our assertion.

2.3. *Proposition. — Let* $k=\mathbf{Q}$, F *be a non-degenerate quadratic form on* $\mathbf{Q}^n$, *and* $G=O(F)$ *the orthogonal group of* F. *Then the elements of* $G_A^\infty\backslash G_A/G_\mathbf{Q}$ *are in* 1—1 *correspondence with the classes in the genus of* F.

As is usual, we shall write L[M] for ${}^t M.L.M$, where L, M are two $n\times n$ matrices.

Let S, T be two rational quadratic forms on $\mathbf{Q}^n$. We recall that S, T belong to the same class if there exist matrices $B, C\in\mathbf{GL}(n,\mathbf{Z})$ such that

$$T=S[B],\quad S=T[C], \tag{1}$$

and that S, T belong to the same genus if for every prime v of $\mathbf{Q}$ there exist matrices $B_v, C_v\in\mathbf{GL}(n,\mathfrak{o}_v)$ such that

$$T=S[B_v],\quad S=T[C_v]. \tag{2}$$

In the latter case, the quotient det S/det T is >0 and is a p-adic unit for every finite prime p, hence $\det S=\det T$, and $\det B_v=\pm\det C_v=\pm 1$ for every v. Using the fact that a p-adic quadratic form always has a unit of determinant -1, we see that if S, T are in the same genus, then (2) has solutions $B_v, C_v\in\mathbf{SL}(n,\mathbf{Q}_v)$.

Let now $U\in G_A$. By 2.2, we may write

$$U=M.N^{-1}\ (M=(M_v)\in\mathbf{SL}_{n,A}^\infty),\quad N\in\mathbf{GL}(n,\mathbf{Q}),\qquad \det N=\pm 1. \tag{3}$$

We have then $F[N]=F[M_v]$, therefore $T=F[N]$ is a rational form in the genus of F. A straightforward verification shows that the class of T does not change if we use another decomposition of U similar to (3) or if we replace U by an element of $G_A^\infty.U.G_\mathbf{Q}$. Therefore $U\to T$ defines a map Φ of $G_A^\infty\backslash G_A/G_\mathbf{Q}$ into the set of classes in the genus of F. There remains to show that Φ is bijective.

Let T be as above, and $T'=F[M_v']=F[N']$ be obtained similarly from F. It T' belongs to the class of T, then there exists $L\in\mathbf{GL}(n,\mathbf{Z})$ such that $T=T'[L]$; we have

$$F[N']=F[N.L],\qquad F[M_v']=F[M_v.L],$$

hence

$$M'\in G_A^\infty.M.L,\quad N'\in G_\mathbf{Q}.N.L,\quad M'.N'^{-1}\in G_A^\infty.M.N^{-1}.G_\mathbf{Q},$$

which shows that Φ is injective. Let now T be in the genus of F, and $M=(M_v)\in\mathbf{SL}(n,A)^\infty$ be such that $T=F[M_v]$ for every v. By Hasse's theorem, there exists $N\in\mathbf{GL}(n,\mathbf{Q})$

such that $T = F[N]$. We have then $M.N^{-1} \in G_A$, $\det N = \pm 1$, and the class of T belongs to the image of Φ.

2.4. *Proposition.* — *Assume that* $G = H.N$ *is the semi-direct product of a subgroup* H *and a normal subgroup* N, *both algebraic, over* k, *and that* $c(N) = 1$. *Then* $c(G)$ *is finite if* $c(H)$ *so is, and is equal to one if* $c(H)$ *so is.*

Let $(x_i)_{i \in I}$ be a set of representatives of the double cosets $H_A^\infty . x . H_k$ $(x \in H_A)$. We have $G_A = H_A . N_A$ by 1.5, and $N_A = N_A^\infty . N_k$ by assumption, whence

$$G = \bigcup_{i \in I} H_A^\infty . x_i . H_k . N_A = \bigcup_{i \in I} H_A^\infty . x_i . N_A^\infty . x_i^{-1} . x_i . G_k .$$

Since $x_i . N_A^\infty . x_i^{-1}$ is commensurable with N_A^∞ (1.7), there exist finitely many elements $y_{ij} \in N_A$ such that $x_i . N_A^\infty . x_i^{-1} \subset \bigcup_j N_A^\infty . y_{ij}$. The set of products $(y_{ij} . x_i)$ contains then representatives of all double cosets $G_A^\infty . x . G_k$, which proves the first part of 2.4. If moreover $c(H) = 1$, then I has one element, we may assume $x_i = e$, whence $y_{ij} = e$, and $c(G) = 1$.

2.5. *Corollary.* — *If* G *is unipotent, then* $c(G) = 1$.

This follows from the approximation theorem if $\dim G = 1$, and from 2.4 by induction in the general case.

2.6. The group G is said to have the *strong approximation property* if $G_{V-P} . G_k$ is dense in G_A [8, 9]. This property is clearly invariant under isomorphisms over k.

Let G have the strong approximation property. If U is an open subgroup of G_A containing G_{V-P}, then $U . G_k = G_A$, for, given $g \in G_A$, we have $U . g \cap G_{V-P} . G_k \neq \emptyset$, whence $g \in U . G_{V-P} . G_k = U . G_k$. In particular, $G_A = G_A^\infty . G_k$, and $c(G) = 1$. This applies e.g. when $G = \mathbf{SL}_n$ or G is unipotent [8, 9].

2.7. *Proposition.* — *Let* $G = H.N$ *be the semi-direct product of a subgroup* H *and a normal subgroup* N, *both algebraic over* k. *Assume that* N *has the strong approximation property. Then every double coset* $G_A^\infty . x . G_k$ $(x \in G_A)$ *intersects* H_A, *hence* $c(G) \leq c(H)$. *If moreover* $G_A^\infty = H_A^\infty . N_A^\infty$, *then* $c(G) = c(H)$.

For any $x \in G_A$, the group $x^{-1} . N_A^\infty . x$ is open in N_A and contains N_{V-P}, therefore, by 2.6, we have $N_A = N_A^\infty . N_k = x^{-1} . N_A^\infty . x . N_k$, hence also

$$x . N_A^\infty . N_k = N_A^\infty . x . N_k \qquad (x \in G_A). \tag{1}$$

Let now $(x_i)_{i \in I}$ be a set of representatives of $H_A^\infty \backslash H_A / H_k$. We have

$$G_A = \bigcup_i H_A^\infty . x_i . H_k . N_A = \bigcup_i H_A^\infty . x_i . N_A^\infty . N_k . H_k = \bigcup_i H_A^\infty . x_i . N_A^\infty . G_k ,$$

and, using (1)

$$G_A = \bigcup_i H_A^\infty . N_A^\infty . x_i . G_k = \bigcup_i G_A^\infty . x_i . G_k ,$$

which shows that $(x_i)_{i \in I}$ intersects each double coset $G_A^\infty . x . G_k$, and proves the first assertion.

Assume now that $G_A^\infty = H_A^\infty . N_A^\infty$, and let $x, y \in H_A$ be such that $x \in G_A^\infty . y . G_k$. Then we may write

$$x = a.b.y.u.v \qquad (a \in H_A^\infty,\ b \in N_A^\infty,\ u \in N_k,\ v \in H_k).$$

The element $x' = a^{-1}.x.v^{-1}$ of H_A belongs to the same double coset mod H_A^∞, H_k as x, and is also equal to $b.y.u$, hence, by (1)

$$x' = y.b'.u' \qquad (b' \in N_A^\infty,\ u' \in N_k).$$

Since G_A is the semi-direct product of H_A and N_A, this yields $x' = y$, and our second assertion.

§ 3. Siegel domains in $\mathbf{GL}_n$ and fundamental sets in G_∞.

In this paragraph G_0 is identified with $\pi_\infty(G_0)$.

3.1. A subset B of G_∞ is a fundamental set for G_0 if it satisfies the following conditions

$(F\ 0)_\infty$: $K.B = B$ for some maximal compact subgroup K of G.

$(F\ 1)_\infty$: $B.G_0 = G_\infty$.

$(F\ 2)_\infty$: For any $a, b \in G_k$, the set of $x \in G_0$ such that $B.a.x.b \cap B \neq \emptyset$ is finite.

The condition $(F\ 0)_\infty$ has been included chiefly to remain in agreement with [2, 4], but will play no role here. Since G_0 is discrete in G_∞, $(F\ 2)_\infty$ is certainly true if B is relatively compact. Therefore compact, or open and relatively compact, fundamental sets always exist when G_∞/G_0 is compact.

It is clear that the condition $(F\ 2)_\infty$ is not changed if x is allowed to run through any finite union of right or left cosets of G_0 in G_k. For future use, we mention one such apparent strengthening of $(F\ 2)_\infty$:

Lemma. — The condition $(F\ 2)_\infty$ *for a subset* B *of* G_∞ *is equivalent to*:

$(F\ 2)'_\infty$: *For any* $a, b \in G_k$, *and any non-zero algebraic integer* $r \in \mathfrak{o}$, *the set of* $x \in G_r = \{g \in G_k \mid r.g, r.g^{-1} \in \mathbf{M}(n, \mathfrak{o})\}$ *such that* $B.a.x.b \cap B \neq \emptyset$ *is finite.*

Let $x \in G_r$. Then

$$(r).\mathfrak{o}^n \subset x(\mathfrak{o}^n) \subset (r^{-1}).\mathfrak{o}^n.$$

But there are only finitely many lattices between two lattices $\Gamma \subset \Gamma'$ of k^n (since Γ'/Γ is a finite group); therefore G_r is the union of finitely many left cosets modulo the isotropy group of $\mathfrak{o}^n$ in G_k, that is modulo G_0, and G_r is of course equal to G_0 if $r = 1$, whence the lemma.

3.2. The following property of a subset $B \subset G_\infty$, relatively to a given algebraic subgroup H over k of G, plays a considerable role in [4], although it is not stated explicitely there:

$(F\ 3)_\infty$: For any rational integer $m \geqq 1$, any rational (right) representation $\rho : G \to \mathbf{GL}_m$ over k, any point $w \in k^m$ with a closed orbit and isotropy group H, and any lattice $\Gamma \subset k^m$, the intersection $w.\rho(B) \cap \Gamma$ is finite.

In view of the properties of $R_{k/\mathbf{Q}}$, it is clear that if $(F\ 3)_\infty$ is true for $R_{k/\mathbf{Q}}G$, $R_{k/\mathbf{Q}}H$, then it is true for G, H.

3.3. The standard Siegel domain $\mathfrak{S}_{t,u}$ ($t, u\in\mathbf{R}$, $t, u>0$) of $\mathbf{GL}(n, \mathbf{R})$ is by definition the set of products $k.a.n$ where $k\in\mathbf{O}(n)$, $a=\operatorname{diag}(a_1, \ldots, a_n)$ ($a_i\leqslant t.a_{i+1}$; $i=1, \ldots, n-1$) and $n=(n_{ij})$ is upper triangular, with ones in the diagonal, and $|n_{ij}|\leqslant u$ ($i<j$), where the bounds are sufficiently large so that $\mathbf{GL}(n, \mathbf{R})=\mathfrak{S}_{t,u}.\mathbf{SL}(n, \mathbf{Z})$, say $t>4/3$, $u>\frac{1}{2}$ (see [4, § 4.5] for references). It has the property $(F\ 2)_\infty$ by a well-known theorem of Siegel recalled in [4, § 4.5] and is therefore a fundamental set for $\mathbf{GL}(n, \mathbf{Z})$ in $\mathbf{GL}(n, \mathbf{R})$.

3.4. *Lemma. — Let* $k=\mathbf{Q}$, $\mathfrak{S}$ *a standard Siegel domain of* $\mathbf{GL}(n, \mathbf{R})$, I *a finite subset of* $\mathbf{GL}(n, \mathbf{Q})$. *Let* G *be a reductive group,* $a\in\mathbf{SL}(n, \mathbf{R})$ *such that* $a.G_\mathbf{R}.a^{-1}$ *is self-adjoint* (1), *and* $B=\bigcup_{c\in I}(a^{-1}.\mathfrak{S}.c\cap G_\mathbf{R})$. *Then* B *has property* $(F\ 2)_\infty$. *If* H *is an algebraic subgroup over* k *and* $a.H_\mathbf{R}.a^{-1}$ *is also self-adjoint, then* B *has property* $(F\ 3)_\infty$ *with respect to* H.

Property $(F\ 2)_\infty$ follows from the facts recalled in 3.3. As to $(F\ 3)_\infty$, 3.4 is in this case chiefly a restatement of results proved in [4]. In fact, an easy commensurability argument, as given at the end of 6.9 in [4, p. 511], shows that it is enough to prove the existence of one rational representation $\rho' : G\to\mathbf{GL}_m$ over $\mathbf{Q}$ and of one point $w'\in\mathbf{Q}^m$ with closed orbit X′ and isotropy group H such that $w'.\rho'(B)\cap\Gamma'$ is finite for any lattice $\Gamma'\subset\mathbf{Q}^m$. We start then with a representation $\rho' : \mathbf{GL}_n\to\mathbf{GL}_m$ over $\mathbf{Q}$ for which there exists $w'\in k^m$ whose orbit under $\mathbf{GL}_n$ is closed and whose isotropy group in $\mathbf{GL}_n$ is H. This exists by [4, 3.8], taking into account the fact that, $a.H_\mathbf{R}.a^{-1}$ being self-adjoint, H is reductive [4, § 1]. The orbit of w under G is then also closed, and it is *a fortiori* enough to show that $w'.\rho'(a^{-1}.\mathfrak{S}'.b)\cap\Gamma'$ is finite for any lattice $\Gamma'\subset\mathbf{Q}^m$ and any $b\in\mathbf{GL}(m, \mathbf{Q})$. This amounts to proving the finiteness of $w''.\rho'(\mathfrak{S})\cap\Gamma'.\rho'(b^{-1})$, where $w''=w'.\rho(a^{-1})$. Since $\Gamma'.\rho'(b^{-1})$ is also a lattice in $\mathbf{Q}^m$, and the isotropy group $a.H_\mathbf{R}.a^{-1}$ of w'' in $\mathbf{GL}(n, \mathbf{R})$ is self-adjoint, the finiteness in question is a consequence of [4, 5.4].

3.5. It follows from 3.4 and [4] that if G is reductive, and H an algebraic subgroup over k of G, then G_∞ has open or closed fundamental sets which also verify $(F\ 3)_\infty$.

In fact, using the restriction of the ground field, we may assume $k=\mathbf{Q}$. If $(F\ 3)_\infty$ is not an empty condition relatively to H, then H\G admits a realization as a closed orbit, hence H is reductive [4, 3.8], and there exists $a\in\mathbf{SL}(n, \mathbf{R})$ such that $a.G_\mathbf{R}.a^{-1}$ and $a.H_\mathbf{R}.a^{-1}$ are self-adjoint (by a result of Mostow, also proved in [4, § 1]). Then, by [4, 6.5], $G_\infty=G_\mathbf{R}=B.G_\mathbf{Z}$, where B is as in 3.4, and satisfies therefore all our conditions.

3.6. Similarly, we may strengthen slightly [4, 12.3] and assert that G_∞ contains closed or open fundamental sets.

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(1) As in [4], a subgroup of $\mathbf{GL}(n, \mathbf{R})$ is said to be self-adjoint if it is invariant under the map $g\to{}^tg$, where ${}^tg$ is the transpose matrix of g.

If G is unipotent, then G_∞/G_0 is compact, and G_∞ has compact, or open and relatively compact, fundamental sets (see 3.1).

If G_k is Zariski-dense in G, we use the decomposition $G=H.N$ of 1.13. In view of the above, it suffices to show that if $B\subset H_\infty$ and $C\subset N_\infty$, with C relatively compact, are fundamental sets for H_0 and N_0 in H_∞ and N_∞ respectively, then B.C is a fundamental set for G_0 in G_∞. For $(F\,0)_\infty$, this is clear because the maximal compact subgroups of H_∞ are also maximal compact in G_∞ ; as to $(F\,1)_\infty$, this follows from

$$G_\infty=B.H_0.N_\infty=B.N_\infty H_0=B.C.N_0.H_0.$$

The condition $(F\,2)_\infty$ is invariant under a rational change of coordinates, since the latter replaces G_0 by a commensurable group (1.7, Remark). We may therefore assume that k^n is the direct sum of subspaces which are stable under H_k and acted upon trivially by N_k, and that N_k is upper triangular. Then $G_0=H_0.N_0$, and the proof given in [4, 6.12] holds here too.

In the general case, let G_1 be the subgroup of G formed by all connected components of G which intersect G_k. By the theorem of Rosenlicht [12, Theorem 3], G_{1k} is Zariski dense in G_1. By the above, there exists an open or closed fundamental set Ω_1 for G_{1_0} in $G_{1\infty}$. We denote by K_1 a maximal compact subgroup of $G_{1\infty}$ such that $K_1.\Omega_1=\Omega_1$.

Let (a_i) $(1\leq i\leq s)$ be a set of representatives in G_∞ for the cosets $G_\infty/G_{1\infty}$. Then $\Omega=\bigcup\limits_{1\leq i\leq m} a_i.\Omega_1$ obviously verifies $(F\,1)_\infty$. If now $\Omega.axb\cap\Omega\neq\emptyset$ $(a,b\in G_k, x\in G_0)$, then there exist i,j $(1\leq i,j\leq m)$ such that

$$a_i\Omega_1.a.x.b.\cap a_j\Omega_1\neq\emptyset.$$

By definition, $G_{1\infty}$ contains Ω_1, a, x, b, therefore a_i and a_j are in the same coset modulo $G_{1\infty}$, hence $i=j$, $\Omega_1 a.x.b\cap\Omega_1\neq\emptyset$, and the possible x's are finite in number. Thus Ω verifies $(F\,2)_\infty$.

In order to have Ω verify $(F\,0)_\infty$ also, we take a maximal compact subgroup K of G_∞ containing K_1. Then K intersects all connected components (usual topology) of G_∞, hence we may take the a_i's in K. We have then $K=\bigcup\limits_i a_iK_1$,

$$\Omega=\bigcup_i a_i.\Omega_1=\bigcup_i a_i.K_1.\Omega_1=K.\Omega_1,$$

and therefore $K.\Omega=\Omega$.

Remark. — In the last part of the proof we have used the fact that the usual properties of maximal compact subgroups in a connected Lie group are also true in a Lie group L with finitely many connected components, in particular: every compact subgroup is contained in a maximal compact one, the maximal compact subgroups are conjugate by inner automorphisms, and the quotient of L by one of them is homeomorphic to a euclidean space (see G. D. Mostow, *Annals of Math.*, 62 (1955), p. 44-55).

§ 4. Fundamental sets for G_k in G_A.

4.1. *Definition.* — A subset $\Omega \subset G_A$ is a *fundamental set* for G_k if it verifies the two conditions:

$$(\mathrm{F}\ 1)_A : G_A = \Omega . G_k,$$

$$(\mathrm{F}\ 2)_A : \Omega^{-1}.\Omega \cap G_k \quad \text{is finite.}$$

Since G_k is discrete, $(\mathrm{F}\ 2)_A$ is true for any relatively compact set Ω; therefore, if G_A/G_k is compact, there always exist compact, or open and relatively compact, fundamental sets.

4.2. In analogy with 3.2, we also introduce the following condition for a subset Ω of G_A, relatively to a given algebraic subgroup H over k of G.

$(\mathrm{F}\ 3)_A$: For any rational integer $m \geq 1$, any rational representation $\rho : G \to \mathbf{GL}_m$ over k, and any $w \in k^m$ with closed orbit and isotropy group H, the set $w.\rho_A(\Omega) \cap k^m$ is finite.

4.3. *Lemma.* — *Let* B *be a subset of* G_∞ *and* C *a relatively compact subset of* G_P.

a) *If* B *verifies* $(\mathrm{F}\ 2)_\infty$, *then* B.C *verifies* $(\mathrm{F}\ 2)_A$.

b) *If* B *verifies* $(\mathrm{F}\ 3)_\infty$ *relatively to a subgroup* H, *then* B.C *verifies* $(\mathrm{F}\ 3)_A$ *relatively to* H.

Let $x \in G_k \cap (BC)^{-1}.BC$. Then, for any $\mathfrak{p} \in P$, the elements x and x^{-1} belong to the relatively compact sets $\pi_\mathfrak{p}(C^{-1}.C)$ and $\pi_\mathfrak{p}(C.C^{-1})$ respectively. The basic properties of the adele topology (see § 1) show then the existence of $r \in \mathfrak{o}, r \neq 0$, such that $rx, rx^{-1} \in \mathbf{M}(n, \mathfrak{o}_\mathfrak{p})$ for every $\mathfrak{p} \in P$, hence such that $rx, rx^{-1} \in \mathbf{M}(n, \mathfrak{o})$. For the components of infinity, we have $B.x \cap B \neq \emptyset$, whence our assertion (see 3.1).

The proof of *b)* is quite similar. Let ρ, w be as in $(\mathrm{F}\ 3)_A$ and $x \in w.\rho_A(B.C) \cap k^m$. For every $\mathfrak{p} \in P$, the element x belongs to the relatively compact set $\pi_\mathfrak{p}(w.\rho(C))$; there exists then $r \in \mathfrak{o}, r \neq 0$, such that $r.x \in \mathfrak{o}_\mathfrak{p}^m$ for every $\mathfrak{p} \in P$, therefore such that $r.x \in \mathfrak{o}^m$. Thus

$$x \in w.\rho(B.C) \cap k^m \Rightarrow x \in \Gamma,$$

where $\Gamma = (r^{-1}).\mathfrak{o}^m$ is a lattice in k^m. Projecting at infinity, we get $x \in w.\rho_\infty(B) \cap \Gamma$, which is finite by assumption.

Remark. — The proof of $(\mathrm{F}\ 3)_\infty \to (\mathrm{F}\ 3)_A$ shows in fact that if B verifies $(\mathrm{F}\ 3)_\infty$ for one representation ρ, one point $w \in k^m$, and any lattice $\Gamma \subset k^m$, then B.C verifies $(\mathrm{F}\ 3)_A$ for ρ and w.

4.4. *Lemma.* — *Let* $k = \mathbf{Q}$, $G = \mathbf{GL}_n$, *and* $\mathfrak{S}$ *a standard Siegel domain of* $\mathbf{GL}(n, \mathbf{R})$. *Then* $G_A = \mathfrak{S}.G_U.G_\mathbf{Q}$.

This follows from $G_A = G_A^\infty.G_\mathbf{Q}$ (see 2.2) and 1.8 (6). In fact what is proved here is that if $c(G) = 1$, and $B.\pi_\infty(G_0) = G_\infty$ $(B \subset G_\infty)$, then $G_A = B.G_U.G_k$.

4.5. *Theorem.* — *Let* $k = \mathbf{Q}$, G *be reductive. Let* $\mathfrak{S}$ *be a standard Siegel domain of* $\mathbf{GL}(n, \mathbf{R})$ *and* $a \in \mathbf{SL}(n, \mathbf{R})$ *be such that* $a.G_\mathbf{R}.a^{-1}$ *is self-adjoint* [4, § 1]. *Then there* *exist finitely many elements* $b_1, \ldots, b_s \in \mathbf{GL}(n, \mathbf{Q})$ *such that* $\Omega = \bigcup_{i=1}^{s} (a^{-1}.\mathfrak{S}.\mathbf{GL}_{n,\mathrm{U}}.b_i \cap \mathrm{G}_\mathrm{A})$ *is a fundamental set for* $\mathrm{G}_\mathbf{Q}$ *in* G_A. *If* H *is an algebraic subgroup over k of* G *and* $a.\mathrm{H}_\mathbf{R}.a^{-1}$ *is self-adjoint, then* Ω *verifies* $(\mathrm{F}\ 3)_\mathrm{A}$ *relatively to* H.

Here, $a^{-1}.\mathfrak{S}.\mathbf{GL}_{n,\mathrm{U}}$ stands for

$$\{g = (g_v) \in \mathbf{GL}_{n,\mathrm{A}} \mid g_\infty \in a^{-1}.\mathfrak{S},\ g_p \in \mathbf{GL}(n, \mathbf{Z}_p),\ (p \text{ prime})\}.$$

By [4, 3.8] there exists a rational representation $\rho : \mathbf{GL}_n \to \mathbf{GL}_m$ defined over $\mathbf{Q}$ and a point $w \in \mathbf{Q}^m$ whose orbit under $\mathbf{GL}(n, \mathbf{C})$ is closed and whose isotropy group is G.

Let w' be the point of A^m defined by $w'_\infty = w.\rho(a^{-1})$, $w'_p = w_p$ (p prime). The orbit of w'_∞ under $\mathbf{GL}(n, \mathbf{C})$ is the same as that of w, hence is closed, and the isotropy group of w'_∞ in $\mathbf{GL}(n, \mathbf{R})$ is $a.\mathrm{G}_\mathbf{R}.a^{-1}$, hence is self-adjoint. Lemma 3.4 (with G and H replaced by $\mathbf{GL}_n$ and G respectively) and 4.3 show the existence of finitely many elements $b_1, \ldots, b_s \in \mathbf{GL}(n, \mathbf{Q})$ such that

$$w'.\rho_\mathrm{A}(\mathfrak{S}.\mathbf{GL}_{n,\mathrm{U}}) \cap w.\rho_\mathrm{A}(\mathbf{GL}(n, \mathbf{Q}) \subset \{w.\rho_\mathrm{A}(b_1^{-1}), \ldots, w.\rho_\mathrm{A}(b_s^{-1})\}. \tag{1}$$

Let us put

$$\mathrm{H} = \{x \in \mathbf{GL}_{n,\mathrm{A}} \mid w'.\rho_\mathrm{A}(x) = w\}.$$

Then, clearly,

$$\mathrm{H} = j_\infty(a).\mathrm{G}_\mathrm{A}. \tag{2}$$

Let now $x \in \mathrm{H}$. Using 4.4, we may write

$$x = s.b^{-1} \qquad (s \in \mathfrak{S}.\mathbf{GL}_{n,\mathrm{U}},\ b \in \mathbf{GL}(n, \mathbf{Q}))$$

and the relation $w'.\rho_\mathrm{A}(x) = w$ gives

$$w'.\rho_\mathrm{A}(s) = w.\rho_\mathrm{A}(b);$$

which, by (1), shows that $w.\rho_\mathrm{A}(b) = w.\rho_\mathrm{A}(b_i^{-1})$ for some $i (1 \leq i \leq s)$. This yields $b.b_i \in \mathrm{G}_\mathbf{Q}$,

$$\mathrm{H} \subset \bigcup_{i=1}^{s} (\mathfrak{S}.\mathbf{GL}_{n,\mathrm{U}}.b_i.\mathrm{G}_\mathbf{Q})$$

which, together with (2), proves that $\mathrm{G}_\mathrm{A} = \Omega.\mathrm{G}_\mathbf{Q}$.

We may write $\Omega \subset \mathrm{B}.\mathrm{C}$, where

$$\mathrm{B} = \bigcup_{i=1}^{s} (a^{-1}.\mathfrak{S}.\pi_\infty(b_i) \cap \mathrm{G}_\infty) \tag{3}$$

verifies $(\mathrm{F}\ 2)_\infty$ by 3.4, and where

$$\mathrm{C} = \bigcup_{i=1}^{s} (\pi_\mathrm{P}(b_i).\mathbf{GL}_{n,\mathrm{U}} \cap \mathrm{G}_\mathrm{U}) \tag{4}$$

is compact. The validity of $(\mathrm{F}\ 2)_\mathrm{A}$, $(\mathrm{F}\ 3)_\mathrm{A}$ then follows from 3.4 and 4.3.

4.6. *Theorem.* — G_A *contains closed or open fundamental sets for* G_k *if* G *is connected. If* G *is reductive, and* H *is an algebraic subgroup over k of* G, *then* G_A *contains closed or open fundamental sets verifying* $(\mathrm{F}\ 3)_\mathrm{A}$ *relatively to* H.

By use of the restriction of the scalars, the proof is readily reduced to the case where $k=\mathbf{Q}$.

Let first G be reductive. Then 4.4 gives closed fundamental sets. But we can of course replace $\mathfrak{S}$ by the interior of a standard Siegel domain, and $\mathbf{GL}_{n,\mathrm{U}}$ by a bigger open, relatively compact subset of $\mathbf{GL}_{n,\mathrm{P}}$, and then we get open fundamental sets. If now $\mathrm{H}\backslash\mathrm{G}$ is realized as a closed orbit in a vector space, it is an affine variety, hence H is reductive [4, 3.8], and there exists $a\in\mathbf{SL}(n,\mathbf{R})$ such that both $a.\mathrm{G}_{\mathbf{R}}.a^{-1}$ and $a.\mathrm{H}_{\mathbf{R}}.a^{-1}$ are self-adjoint [4, § 1]. With this choice of a, the set Ω of 4.5 verifies $(\mathrm{F}\ 3)_{\mathrm{A}}$ in virtue of 4.3 *b)*.

If now G is connected, we have $\mathrm{G}=\mathrm{H}.\mathrm{N}$, with H reductive, N unipotent and normal in G (1.13). By [11, Prop. 15] $\mathrm{N}_{\mathrm{A}}/\mathrm{N}_{\mathbf{Q}}$ is compact, hence 4.6 is true for $\mathrm{G}=\mathrm{N}$, and it is enough to show that if $\mathrm{B}\subset\mathrm{H}_{\mathrm{A}}$ and $\mathrm{C}\subset\mathrm{N}_{\mathrm{A}}$ are fundamental set for $\mathrm{H}_{\mathbf{Q}}$ and $\mathrm{N}_{\mathbf{Q}}$, with C relatively compact, then B.C is a fundamental set for $\mathrm{G}_{\mathbf{Q}}$ in G_{A}.

We have

$$\mathrm{G}_{\mathrm{A}}=\mathrm{H}_{\mathrm{A}}.\mathrm{N}_{\mathrm{A}}=\mathrm{B}.\mathrm{H}_{\mathbf{Q}}.\mathrm{N}_{\mathrm{A}}=\mathrm{B}.\mathrm{N}_{\mathrm{A}}.\mathrm{H}_{\mathbf{Q}}=\mathrm{B}.\mathrm{C}.\mathrm{G}_{\mathbf{Q}}.$$

Let $g=h.u\ (h\in\mathrm{H}_{\mathbf{Q}},\ u\in\mathrm{N}_{\mathbf{Q}})$ belong to $(\mathrm{BC})^{-1}.\mathrm{BC}$. This is the case if and only if

$$\mathrm{B}.h\cap\mathrm{B}\neq\emptyset,\qquad h^{-1}.\mathrm{C}.h.u\cap\mathrm{C}\neq\emptyset.$$

Thus there are only finitely many possibilities for h, and, C being relatively compact, only finitely many possible u's for a given h, which ends the proof.

Remark. — The first part of 4.6 will be extended to non-connected groups in 5.2.

§ 5. Finiteness theorems for G_{A}, G_k.

5.1. *Theorem.* — *The number $c(\mathrm{G})$ of distinct double cosets $\mathrm{G}_{\mathrm{A}}^{\infty}.x.\mathrm{G}_k\ (x\in\mathrm{G}_{\mathrm{A}})$ is finite.*

Since a compact subset C of G_{A} is contained in the union of finitely many right translates of $\mathrm{G}_{\mathrm{A}}^{\infty}$ (see 1.2), 5.1 is equivalent to the existence of a compact subset C of G_{A} such that

$$\mathrm{G}_{\mathrm{A}}=\mathrm{G}_{\mathrm{A}}^{\infty}.\mathrm{C}.\mathrm{G}_k. \tag{1}$$

Let first G be connected. Using 1.13, 2.4, 2.5, we may assume G to be reductive. By restriction of the ground field (1.4), it is enough to consider the case where $k=\mathbf{Q}$. But then (1) is a consequence of 4.5 (3), (4).

In the general case, there exists a compact set D of G_{A} such that $\mathrm{G}_{\mathrm{A}}=\mathrm{D}.\mathrm{G}_{\mathrm{A}}^{0}$ (1.9). By 1.2 and the above, we may find finite subsets I, J of G_{A} and $\mathrm{G}_{\mathrm{A}}^{0}$ such that

$$\mathrm{G}_{\mathrm{A}}=\bigcup_{x\in\mathrm{I},\,y\in\mathrm{J}}\mathrm{G}_{\mathrm{A}}^{\infty}.x.\mathrm{G}_{\mathrm{A}}^{0\,\infty}.y.\mathrm{G}_k.$$

The theorem follows now from the fact that $x.\mathrm{G}_{\mathrm{A}}^{\infty}.x^{-1}$ is commensurable with $\mathrm{G}_{\mathrm{A}}^{\infty}$ (1.7).

5.2. *Corollary.* — *Let* B *a fundamental set for* G_0 *in* G_{∞}. *Then there exists a compact subset* $\mathrm{C}\subset\mathrm{G}_{\mathrm{P}}$ *such that* $\mathrm{B}.\mathrm{C}'$ *is a fundamental set for* G_k *in* G_{A} *for any relatively compact subset* C' *of* G_{P} *containing* C.

By 5.1, there exists a compact set C_0 in G_P such that $G_A = G_A^\infty . C_0 . G_k$. Let

(2) $$C = G_U . C_0 . G_U .$$

We have

$$G_A^\infty . C = G_\infty . C = B . (j_\infty . \pi_\infty(G_0)) . C;$$

using (2) and 1.8, we get

$$G_A^\infty . C = B . C . (j_\infty . \pi_\infty(G_0)) = B . C . G_0,$$

hence

$$G_A = G_A^\infty . C . G_k = B . C . G_k .$$

The corollary then follows from 4.3. Since B always exists (3.5), this produces fundamental sets also when G is not connected.

5.3. Let G be reductive connected and H an algebraic subgroup over k. When $k = \mathbf{Q}$, it is shown in [4, 3.8] that H is reductive if and only if there exists a rational representation $\rho : G \to GL(W)$ over k and a point $w \in W_k$ whose orbit X is closed and whose isotropy group is H.

This is also valid over a number field, with G not necessarily connected. In fact let $G' = R_{k/\mathbf{Q}} G$, $H' = R_{k/\mathbf{Q}} H$, $X' = R_{k/\mathbf{Q}} X$. We have then $X' = H' \backslash G'$. If H is reductive, then so is H′, hence X′ is affine ([4, 3.8] and 1.12), X is affine, and the existence of ρ follows from [4, 2.4, footnote 2] (1). If ρ exists, then X is affine, hence so is X′, H′ is reductive [4, 3.8], and therefore H is reductive.

5.4. *Theorem. — Let* G *be reductive,* H *an algebraic subgroup over k of* G, *and* σ *the natural projection of* G *onto* $X = H \backslash G$. *Assume that* H *is reductive or, equivalently* (5.3) *that* X *is an affine algebraic set. Then* $\sigma_A(G_A) \cap X_k$ *is the union of finitely many orbits of* G_k (2).

Let $\rho : G \to \mathbf{GL}_m$ be a right rational representation over k and $w \in k^m$ a point whose orbit X′ is closed and whose isotropy group is H (see 5.3). There is an equivariant isomorphism $\Phi : X \to X'$ over k which maps $\sigma(H)$ onto w, whence an equivariant homeomorphism $\Phi_A : X_A \to X'_A$ which sends X_k onto X'_k and $\sigma_A(G_A)$ onto $w . \rho(G_A)$. Thus we are reduced to proving that $w . \rho(G_A) \cap k^m$ consists of finitely many orbits of G_k, or that $w . \rho(\Omega) \cap k^m$ is finite for a suitable fundamental set Ω for G_k in G_A; but this follows from 4.6.

5.5. *Lemma. — The following conditions are equivalent:* a) G_A *is unimodular,* b) G_A^0 *is unimodular,* c) G_A^∞ *is unimodular,* d) G_∞ *is unimodular,* e) *every left invariant rational exterior differential form on* G^0, *of degree* $s = \dim G$, *defined over k, is right invariant. If* $X_k(G^0) = 1$, *then* G_A *is unimodular.*

By 1.9, the quotient group G_A / G_A^0 is compact, therefore $(a) \Leftrightarrow (b)$. The quotient

(1) We do not need to know whether the result of Weil used in the proof of [4, 2.4] holds good for non-irreducible varieties, because $H' \backslash G'$ is embedded in $H' \backslash \mathbf{GL}_n$, which is irreducible. An affine embedding over k of the latter then yields one for the former.

(2) As in [4], we have been led by our conventions on Siegel domains to consider right representations in § 3. This is why 5.8 is formulated for right coset spaces; but it is of course equivalent to the corresponding statement for G/H.

G_∞/G^0_∞ is finite, and 1.9 implies that $G^\infty_A/G^{0\infty}_A$ is compact. Therefore G^∞_A (resp. G_∞) is unimodular if and only if $G^{0\infty}_A$ (resp. G^0_∞) is so. It suffices therefore to prove 5.5 when G is connected.

Let ω be a rational left-invariant differential form of degree s on G, defined over k. Then $\omega.g=\chi(g).\omega$ where $\chi\in X_k(G)$, hence $X_k(G)=1$ implies *e)*.

The form ω induces a left-invariant form ω_v on G_v for every $v\in V$. There exist convergence factors $(\lambda_v)_{v\in V}$ such that the product of the measures $\lambda_v\omega_v$ is defined on $G_{A(S)}$ ($S\subset V$, S finite) and is a Haar measure, and the inductive limit of these Haar measures is a Haar measure on G_A [14, Chap. II]. From this, the equivalence of *e)* with any of *b)*, *c)*, *d)* is clear.

5.6. *Theorem.* — (i) G_A/G_k *carries an invariant measure and has a finite volume for that measure if and only if* $X_k(G^0)=1$.

(ii) G_A/G_k *is compact if and only if* $X_k(G^0)=1$ *and every unipotent element of* G_k *belongs to the radical of* G_k.

By 5.1, there exists a finite subset I of G_A such that

$$G_A=\bigcup_{x\in I} G^\infty_A.x.G_k.$$

We have then

$$G_A/G_k=\bigcup_{x\in I} x.\{(x^{-1}.G^\infty_A.x)/(x^{-1}.G^\infty_A.x\cap G_k)\}. \tag{3}$$

Since $x^{-1}.G^\infty_A.x$ is commensurable with G^∞_A (1.7), the group $x^{-1}.G^\infty_A.x\cap G_k$ is commensurable with $G^\infty_A\cap G_k=G_0$. Taking 5.5 into account, we see that (i) and (ii) are respectively equivalent to the same assertions for G^∞_A/G_0. Since the latter space is fibered over G_∞/G_0 with compact fibers (1.8), (i) and (ii) are also equivalent to the statements obtained from them by writing G_∞/G_0 instead of G_A/G_k, but these follow from [4, 12.3] and from the fact that $(G^0)_\infty$ has a finite index in G_∞.

(Of course, we could also reduce the proof to the case where $k=\mathbf{Q}$ by use of 1.4, 1.5 and then refer to [4, 9.4, 11.8] rather than to [4, 12.3].)

5.7. Let $m_k: I_k\to\mathbf{R}^+$ be the continuous homomorphism of the idele group of k into $\mathbf{R}^+$ which associates to an idele $x=(x_v)_{v\in V}$ its idele module $\prod_{v\in V}\|x_v\|_v$. To each character $\chi\in X_k(G)$ corresponds a continuous homomorphism $m_k\circ\chi_A: G_A\to\mathbf{R}^+$; we put

$$_mG_A=\bigcap_{\chi\in X_k(G)}\ker(m_k\circ\chi_A). \tag{4}$$

If s is a rational integer, then, clearly, $\ker m_k\circ\chi^s_A=\ker m_k\circ\chi_A$. Therefore

$$_mG_A=\bigcap_{\chi\in M}\ker(m_k\circ\chi_A), \tag{5}$$

whenever M is a subgroup of finite index of $X_k(G)$.

Let

$$L=\{g\in G\mid\chi(x)=\pm1,\ (\chi\in X_k(G))\}. \tag{6}$$

This is an algebraic normal subgroup over k of G, whose unipotent radical coincides with that of G. Since G_U is compact, it belongs to ${}_mG_A$, whence

$$(7) \qquad {}_mG_A \cap G_A^\infty = L_\infty \times G_U,$$

which implies

$$(8) \qquad \pi_\infty(G_0) \subset L_\infty, \quad \pi_\infty(G_0) = \pi_\infty(L_0).$$

For connected groups, 5.6 admits the following generalization:

5.8. *Theorem. — Let* G *be connected. Then* ${}_mG_A = \bigcap_{\chi \in X_k(G)} \ker(m_k \circ \chi)$ *is unimodular, contains* G_k, *the space* ${}_mG_A/G_k$ *has a finite invariant measure, and* ${}_mG_A/G_k$ *is compact if and only if every unipotent element of* G_k *belongs to the radical of* G_k.

A classical special case of 5.8 not included in 5.6 is the compactness of the quotient I_k^0/k^* of the group of ideles of k of idele-module one, by k^*.

By the product formula, ${}_mG_A$ contains G_k. Let J be a set of representatives for the distinct double cosets $G_A^\infty . x . G_k$ which meet ${}_mG_A$. It is finite by 5.1, and

$${}_mG_A = \bigcup_{x \in J} ({}_mG_A \cap G_A^\infty) . x . G_k.$$

The group $x^{-1}({}_mG_A \cap G_A^\infty)x = {}_mG_A \cap x^{-1}.G_A^\infty.x$ is commensurable with ${}_mG_A \cap G_A^\infty$ and therefore, as in 5.6, our theorem is equivalent to the corresponding assertion for ${}_mG_A \cap G_A^\infty$ and G_0. Using 5.7 (6), (7), (8) and 1.8, we see that $({}_mG_A \cap G_A^\infty)/G_0$ is fibered over $L_\infty/\pi_\infty(L_0)$ with compact typical fiber G_U. We are thus reduced to proving the statement 5.8 with ${}_mG_A$ and G_0 replaced by L_∞ and L_0, the latter group being now viewed as a subgroup of L_∞. This modified statement follows from 5.5 and [4, 9.4, 11.8] provided that we show that $X_k(L^0) = 1$. But this last fact is a consequence of the more general lemma:

5.9. *Lemma. — Let* F *be a field of characteristic zero,* M *a connected algebraic matric group over* F, *and* B *a connected normal algebraic subgroup over k of* M. *Then the cokernel of the restriction homomorphism* $X_F(M) \to X_F(B)$ *is finite.*

Let R(M) and R(B) be the radicals of M and B. The group R(B) is normal in R(M) and M (resp. B) is isogeneous to the semi-direct product of R(M) (resp. R(B)) with a semi-simple group. The restriction defines therefore an injective homomorphism $X_F(M) \to X_F(R(M))$ (resp. $X_F(B) \to X_F(R(B))$) with finite cokernel, and we may assume M and B to be solvable. The group M is then the semi-direct product, over k, of its unipotent radical N by a maximal algebraic torus T, whence a natural isomorphism $X_F(M) \cong X_F(T)$. Assuming, as we may, that $S = T \cap B$ is a maximal torus of B, we have similarly $B = (T \cap B).(N \cap B)$, $X_F(B) = X_F(T \cap B)$, so that it is enough to prove 5.9 when M is an algebraic torus; in that case it follows for instance from [4, 8.4 *a)*].

5.10. *Remark.* — In Ono's terminology [11], G is of type (F) if $c(G)$ is finite, of type (C) (resp. (M)) if ${}_mG_A/G_k$ is compact (resp. of finite invariant measure), and G has no defect if the restriction homomorphism $X_k(G) \to X_k(G^0)$ has a finite cokernel. If G has no defect then the intersection L' of the kernels of the elements of $X_k(G^0)$ has finite index in the group L of 5.7 (6), hence L'_∞ has a finite index in L_∞ and 5.7 (5), (7) show that ${}_mG_A/{}_mG_A^0$ is compact. Thus ${}_mG_A$ is unimodular if and only if ${}_mG_A^0$ is so. Together with 5.1, 5.6, this proves: G is of type (F), a group with no defect is of type (M), a group with no defect is of type (C) if and only if every unipotent element of G_k belongs to the radical of G_k. For G solvable and G_k Zariski dense in G, this was already proved in [11].

§ 6. Application to principal homogeneous spaces.

Our aim in this paragraph is to give an application of 5.4 to Galois cohomology. As to the latter, we limit ourselves to a minimum of preliminaries, and refer to [5, 10, 13] for more details.

6.1. Let L be a group, and B a group on which L operates on the left. Being interested here only in Galois cohomology, we assume that all the orbits of L in B are finite and we define a 1-cocycle of L, with values in B, as a map $s \to b_s$ of L in B with a *finite image*, such that $b_{st} = s(b_t)$. Two cocycles (b_s), (b'_s) are cohomologous if there exists $c \in B$ such that $b'_s = c^{-1}.b_s.s(c)$. This is an equivalence relation. The set of such equivalence classes is the first Galois cohomology set of L with coefficients in B, denoted $H^1(L; B)$. If B is commutative, it is a commutative group; in general $H^1(L; B)$ is just a set with a distinguished zero element, the class of the coboundaries $s \to b^{-1}.s(b)$ $(b \in B)$. If L operates on another group C, again with finite orbits, then any equivariant map $f: B \to C$ induces a map $f^1 : H^1(L; B) \to H^1(L; C)$ sending the zero element onto the zero element; by definition, the kernel of f^1 is the inverse image of the zero element of $H^1(L; C)$.

The set B^L of fixed points of L in B is, by definition, the 0-cohomology group $H^0(L; B)$ of L in B, and the following lemma is in fact a part of the exactness of a cohomology sequence ([13, p. 133], [5]).

6.2. *Lemma. — We keep the previous notation, and assume that* B *is a subgroup of* C. *Then there exists a natural map* $\delta : (C/B)^L \to H^1(L; B)$ *which induces a bijection* δ_0 *of the set of orbits of* C^L *in* $(C/B)^L$ *onto the kernel* N *of the map* $i^1 : H^1(L; B) \to H^1(L; C)$ *induced by the injection of* B *in* C.

Let $\pi : C \to C/B$ be the canonical projection, and $0 = \pi(B)$. Let $x \in (C/B)^L$, and choose $c \in \pi^{-1}(x)$. Then $x \in (C/B)^L$ implies $c^{-1}.s(c) \in B$, and $s \to c^{-1}.s(c)$ is a cocycle of L in B, whose class does not change when c varies in $\pi^{-1}(x)$, and is $\delta(x)$ by definition. Clearly $\delta(x) \in N$; it is easily checked that δ is constant on the C^L orbits.

If $\delta(x) = \delta(y)$ $(x, y \in (C/B)^L)$, then there exist $c, d \in C$, $z \in B$ such that

$$c.0 = x, \quad d.0 = y, \quad d^{-1}.s(d) = z^{-1}.c^{-1}.s(c).s(z) \quad (s \in L),$$

whence $c.z.d^{-1} \in C^L$. Since $c.z.d^{-1}.y = x$, this shows that δ_0 is injective. If now $(b_s) \in N$, then there exists $c \in C$ such that $b_s = c^{-1}.s(c)$ $(s \in L)$, whence $\pi(c) \in (C/B)^L$ and $(b_s) = \delta(\pi(c))$. Thus $\operatorname{Im} \delta \supset N$.

6.3. We shall be concerned here only with the case where L is the Galois group

Gal(F/k) over k of the field F of all algebraic numbers, and where $B = G_F$; as is usual, we write $H^1(k, G)$ for $H^1(\mathrm{Gal}\,(F/k); G_F)$. This set may also be viewed as the inductive limit of the cohomology sets $H^1(\mathrm{Gal}\,(k'/k); G_{k'})$, where k' runs through the finite algebraic normal extensions of k. Since $H^1(\mathrm{Gal}\,(k'/k), G_{k'}) = 0$ when $G = \mathbf{GL}_n$ (Theorem of Speiser, see e.g. [13, p. 159]), we also have

$$H^1(k, \mathbf{GL}_n) = 0. \tag{1}$$

Together with 6.2, this implies the

Lemma. — The map $\delta : (\mathbf{GL}_n/G)_k \to H^1(k, G)$ *induces a bijection* δ_0 *of the set of orbits of* $\mathbf{GL}(n, k)$ *in* $(\mathbf{GL}_n/G)_k$ *onto* $H^1(k, G)$.

6.4. A *principal homogeneous* space for G is an affine algebraic set X over which G operates, say on the right, as an algebraic transformation group, so that for each $x \in X$ the map $g \to x.g$ is a biregular map of G onto X. Since we are in characteristic zero, it would be equivalent to require that G is simply transitive on X. The principal homogeneous space is over k if X and the action of G on X are defined over k.

A principal homogeneous space over k is said to split over an extension k' of k if it has a point rational over k'. If this is the case, and if $x \in X_{k'}$ then $g \to x.g$ identifies, over k', X with G operating on itself by right translations.

Two principal homogeneous space X, X' for G, over k, are isomorphic over k if there exists an equivariant birational biregular map over k of X onto X'. It is well known that there is a natural 1 — 1 correspondence Φ between these isomorphism classes and $H^1(k; G)$ (see [10] for instance). We sketch the proof:

A principal homogeneous space X over k for G always contains an algebraic point x. Given $s \in \mathrm{Gal}\,(F/k)$, there is a unique $g_s \in G_F$ such that $x.g_s = s(x)$. It is readily checked that $s \to g_s$ is a 1-cocycle whose class depends only on the isomorphism class over k of X, and that the map Φ thus obtained is injective. That Φ is surjective follows from " field-descent " but, in our case, may be deduced from 6.3: in fact, if $x \in (\mathbf{GL}_n/G)_k$, and if X_x is the inverse image of x in $\mathbf{GL}_n$, viewed in the obvious manner as a principal homogeneous space for G, then it follows immediately from the definitions that $\delta(x) = \Phi(X_x)$.

6.5. A principal homogeneous space X over k is said to split locally everywhere if it splits over all completions of k. Since an irreducible variety over k has integral $\mathfrak{p}$-adic points for almost all $\mathfrak{p} \in P$ (see 1.9 for a reference), so does X, if G is connected.

Let now G be connected, G' be an algebraic matric group over k containing G as an algebraic subgroup, and $\sigma : G' \to G'/G$ the natural projection. σ induces a continuous map $\sigma_A : G'_A \to (G'/G)_A$. Let $x \in (G'/G)_k$ and $X = \sigma^{-1}(x)$ be the corresponding principal homogeneous space. If $x \in \sigma_A(G'_A)$, then X obviously splits everywhere; the remark made earlier in this section shows that the converse is true if G is connected. Together with 6.3, 6.4, this proves the following.

6.7. *Lemma. — We keep the previous notation, and assume* G *to be connected. Then the elements of* $\ker\,(H^1(k, G) \to H^1(k, G'))$ *which, viewed as principal homogeneous spaces, split* *locally everywhere, are in* $1-1$ *correspondance with the orbits of* G'_k *in* $\sigma_A(G'_A) \cap (G'/G)_k$.

If now G is reductive and $G' = \mathbf{GL}_n$, then $\sigma_A(G'_A) \cap (G'/G)_k$ is the union of finitely many orbits of G'_k by 5.4. Combined with 6.3, this proves the following:

6.8. *Theorem. — Let* G *be reductive, connected. Then the number of isomorphism classes over k of principal homogeneous spaces over k for* G *which split locally everywhere is finite.*

This theorem will be generalized in [5], where it will be shown that given a principal homogeneous space X over k for G (where G is subject only to our standing assumptions) and a finite subset $S \subset V$, the principal homogeneous spaces over k for G which are isomorphic to X over k_v for all $v \in V - S$ form a finite number of isomorphism classes over k.

§ 7. Application to parabolic subgroups.

7.1. Let G be connected. An algebraic subgroup H of G is *parabolic* if G/H is a complete variety. H is then connected, equal to its normalizer. If, moreover, H is defined over k, the fibering of G by H admits local rational cross sections defined over k (for all this, see [7]); consequently [14, p. 27]:

$$(G/H)_A = G_A/H_A, \quad (G/H)_k = G_k/H_k, \tag{1}$$

and, of course, $(G/H)_A$ is compact.

7.2. *Lemma.* — (Godement). *Let* G *be connected, and* H *a parabolic subgroup. Then there exists a finite subset* $I \subset G_A$ *such that*

$$G_A = \bigcup_{x \in I} G_A^\infty . x . H_k.$$

By (1) and the compactness of $(G/H)_A$, there exists a compact subset C of G_A such that $G_A = C.H_A$. The set C is covered by finitely many translates of G_A^∞ (1.2) and H_A by finitely many double cosets modulo H_A^∞ and H_k (5.1). Therefore G_A is a finite union of subsets $y.G_A^\infty.z.H_k$ ($y \in G_A$, $z \in H_A$). Since $y.G_A^\infty.y^{-1}$ is commensurable with G_A^∞ (1.7), it is contained in the union of finitely many right translates of G_A^∞, whence the lemma.

7.3. *Theorem. — Let* G *be connected and* H *be a parabolic subgroup, defined over k. Then* $(G/H)_k$ *is the union of finitely many orbits of* G_0.

Lemma 7.2 implies the existence of a finite subset I of G_k such that

$$G_k \subset \bigcup_{x \in I} G_A^\infty . x . H_k,$$

whence

$$G_k = \bigcup_{x \in I} (G_A^\infty \cap G_k) . x . H_k,$$

$$G_k = \bigcup_{x \in I} G_0 . x . H_k, \tag{2}$$

which is equivalent to our assertion.

7.4. *Remark.* — The above proof is due to Godement. Another one is given in [2]. In fact, the theorem is proved there only when $k = \mathbf{Q}$, and G is semi-simple. However the general case can easily be reduced to that one by use of the restriction of the scalars, of the surjectivity of $G_k \to (G/H)_k$, and of the fact that H contains the radical of G. We refer to [2, 4.6] for another formulation of this theorem.

We conclude this section with a proposition which is used in proving the assertions made in [2, 4.7], as will be shown elsewhere.

7.5. *Proposition.* — *Let* G *be connected, and* H *a parabolic subgroup. Assume that* $G_A = G_A^\infty . G_k$ *and that* $G_\mathfrak{p} = G_{\mathfrak{o}_\mathfrak{p}} . H_\mathfrak{p}$ *for every* $\mathfrak{p} \in P$. *Then the number* $\nu(G, H)$ *of double cosets of* G_k *modulo* $G_\mathfrak{o}$ *and* H_k *is equal to* $c(H)$.

The assumption on the $G_\mathfrak{p}$ $(\mathfrak{p} \in P)$ implies

$$G_A = G_A^\infty . H_A,$$

whence

$$G_A = \bigcup_{1 \leq i \leq m} G_A^\infty . h_i . H_k, \tag{3}$$

where (h_i) $(1 \leq i \leq m)$ is a set of representatives in H_A for the distinct double cosets $H_A^\infty . h . H_k$. Since $G_A = G_A^\infty . G_k$ by assumption, there exists $x_i \in G_k$ such that $h_i \in G_A^\infty . x_i$, whence

$$G_k = \bigcup_{1 \leq i \leq m} (G_A^\infty \cap G_k) . x_i . H_k = \bigcup_{1 \leq i \leq m} G_\mathfrak{o} . x_i . H_k,$$

and $\nu(G, H) \leq c(H)$.

If now $x_j \in G_\mathfrak{o} . x_i . G_k$, then $h_j \in G_A^\infty . h_i . H_k$, and therefore

$$h_j \in (G_A^\infty \cap H_A) . h_i . H_k = H_A^\infty . h_i . H_k,$$

which shows that $\nu(G, H) \geq c(H)$.

§ 8. Generalization to the groups of S-units.

The results of the previous paragraphs pertain to G_k, G_A, those of [4] to $G_\mathfrak{o}$, G_∞. Here we discuss some extensions to $G_{\mathfrak{o}(S)}$, $G_{A(S)}$, G_S $(S \subset V)$.

As usual, S denotes a subset of V. *Unless otherwise stated,* S *is assumed to contain the infinite primes*, but it is not necessarily finite.

We shall often make no notational distinction between G_k and $\pi_S(G_k)$, or between $G_{\mathfrak{o}(S)}$ and $\pi_S(G_{\mathfrak{o}(S)})$. Otherwise said, when we view $G_k, G_{\mathfrak{o}(S)}$ as subgroups of G_S, then we mean $\pi_S(G_k)$, $\pi_S(G_{\mathfrak{o}(S)})$.

8.1. A subset $\Omega \subset G_{A(S)}$ is a *fundamental set* for $G_{\mathfrak{o}(S)}$ if it satisfies the two conditions:

$(F\ 1)_{A(S)} : \Omega . G_{\mathfrak{o}(S)} = G_{A(S)}$,

$(F\ 2)_{A(S)} : \Omega^{-1} . \Omega \cap G_{\mathfrak{o}(S)}$ is finite.

We could of course define a fundamental set Ω for $G_{\mathfrak{o}(S)}$ in G_S in the same way, but the case where $S = V - P$ suggests rather to require

$(F\ 1)_S : \Omega . G_{\mathfrak{o}(S)} = G_S$.

$(F\ 2)_S$: For any $a, b \in G_k$, the set of elements $x \in G_{\mathfrak{o}(S)}$ such that $\Omega . a . x . b \cap \Omega \neq \emptyset$ is finite.

8.2. It will be convenient to have at one's disposal the analogue of $(F\ 2)'_\infty$ in 3.1. To this effect, we remark first that if Γ is a lattice in k^n, then

$$\mathfrak{o}(S) . \Gamma = \bigcap_{\mathfrak{p} \notin S} (\Gamma_\mathfrak{p} \cap k^n), \qquad (\Gamma_\mathfrak{p} = \mathfrak{o}_\mathfrak{p} . \Gamma). \tag{1}$$

In fact, Γ being the direct sum of $\mathfrak{o}^{n-1}$ and of a fractionary ideal of k [6, Satz 12.5], the proof of (1) reduces to the one-dimensional case, where it follows from ideal theory. Since $\Gamma_{\mathfrak{p}}=(\mathfrak{o}_{\mathfrak{p}})^n$ for almost all $\mathfrak{p}$'s, this implies that if Γ and Γ' are two lattices in k^n, then $\mathfrak{o}(S).\Gamma$ and $\mathfrak{o}(S).\Gamma'$ are commensurable, and therefore the set of groups $\mathfrak{o}(S)\Gamma''$, where Γ'' runs through the lattices in k^n such that

(2) $$\mathfrak{o}(S).\Gamma\subset\mathfrak{o}(S).\Gamma''\subset\mathfrak{o}(S).\Gamma',$$

is finite.

Lemma. — The condition $(F\,2)_S$ *for a subset* Ω *of* G_S *is equivalent to* $(F\,2)'_S$: *For any* $a, b\in G_k$ *and any non-zero algebraic integer* $r\in\mathfrak{o}$, *the set of elements* $x\in G_k$ *such that* $r.x, r.x^{-1}\in\mathbf{M}(n, \mathfrak{o}(S))$ *and that* $\Omega.a.x.b\cap\Omega\neq\emptyset$ *is finite.*

Let $G_{r,S}=\{g\in G_k \mid r.g, r.g^{-1}\in\mathbf{M}(n, \mathfrak{o}(S))\}$ and $x\in G_{r,S}$. Then
$$\mathfrak{o}(S).(r).\mathfrak{o}^n\subset\mathfrak{o}(S).x(\mathfrak{o}^n)\subset\mathfrak{o}(S).(r^{-1}).\mathfrak{o}^n.$$

By the above, $G_{r,S}$ is then a finite union of left cosets modulo the isotropy group of $\mathfrak{o}(S).\mathfrak{o}^n$ in $G_{r,S}$, which is nothing but $G_{\mathfrak{o}(S)}$, whence the lemma.

8.3. The following condition for a subset $\Omega\subset G_S$, relatively to an algebraic subgroup H over k, admits $(F\,3)_\infty$ and $(F\,3)_A$ as special cases.

$(F\,3)_S$: For any integer $m\geqq 1$, any rational representation $\rho : G\rightarrow\mathbf{GL}_m$, over k, any $w\in k^m$ with isotropy group H and closed orbit, and any lattice $\Gamma\subset k^m$, the set $w.\rho_A(\Omega)\cap\mathfrak{o}(S).\Gamma$ is finite.

8.4. *Lemma. — Let* $B\subset G_\infty$, C *a relatively compact subset of* $G_{P\cap S}$. *If* B *verifies* $(F\,2)_\infty$ (resp. $(F\,3)_\infty$ *relatively to a subgroup* H), *then* B.C *verifies* $(F\,2)_S$ (resp. $(F\,3)_S$ *relatively to* H).

Let $a, b\in G_k$ and x be as in $(F\,2)_S$. For $\mathfrak{p}\in P\cap S$, the element x then belong to the relatively compact set $\pi_{\mathfrak{p}}(C^{-1}.C)$, whence the existence of $s\in\mathfrak{o}$ such that $s.x, s.x^{-1}\in\mathbf{M}(n, \mathfrak{o}_{\mathfrak{p}})$ for every $\mathfrak{p}\in S$. Since $x, x^{-1}\in\mathbf{GL}(n, \mathfrak{o}_{\mathfrak{p}})$ for $\mathfrak{p}\notin S$ by assumption, we get $sx, sx^{-1}\in\mathbf{M}(n, \mathfrak{o})$. Since $Bx\cap B\neq\emptyset$, the part of the lemma concerning F 2 follows from 3.1.

Let now $\rho : G\rightarrow\mathbf{GL}_m$, $w\in k^m$ be as in $(F\,3)_S$, and $x\in w.\rho_S(B.C)\cap\mathfrak{o}(S).\Gamma$. In particular, $x\in\pi_{\mathfrak{p}}(w.\rho_S(C))$ for $\mathfrak{p}\in P\cap S$, whence again the existence of $r\in\mathfrak{o}$ such that $rx\in\mathfrak{o}_{\mathfrak{p}}^m$ for all $\mathfrak{p}\in P\cap S$. Two lattices in k^n being commensurable, there exists $s\in\mathfrak{o}$ such that $(s)\Gamma\in\mathfrak{o}^m$; since $x\in\mathfrak{o}(S).\Gamma$, we have then $x\in\Gamma_{\mathfrak{p}}$ for $\mathfrak{p}\notin S$, hence $s.x\in\mathfrak{o}_{\mathfrak{p}}^m$ for $\mathfrak{p}\notin S$. Altogether, we get $r.s.x\in\mathfrak{o}^m$, which shows that x belongs to the lattice $\Gamma=(s^{-1}.r^{-1})\mathfrak{o}^m$, and, combined with $x=w.\rho_\infty(B)$, ends the proof.

Remark. — The remark to 4.3 also applies to the proof of $(F\,3)_\infty\Rightarrow(F\,3)_S$.

8.5. *Theorem. — The groups* $G_{A(S)}$ *and* G_S *contain open or closed fundamental sets for* $G_{\mathfrak{o}(S)}$, *which, in the case of* G_S, *verify* $(F\,3)_S$ *relatively to* H, *if* G *is reductive, and* H *an algebraic subgroup over* k *of* G.

In view of 8.4, 3.4, 3.5, it is enough to show that if $B\subset G_\infty$ satisfies $(F\,1)_\infty$, then there exists a compact set $C\subset G_{P\cap S}$ such that B.C and B.C.M, where $M=\prod_{\mathfrak{p}\notin S} G_{\mathfrak{o}_{\mathfrak{p}}}$, verify $(F\,1)_S$ and $(F\,1)_{A(S)}$ respectively.

By 5.1, there exists a compact (in fact finite) subset $I \subset G_A$ such that

$$G_{A(S)} \subset \bigcup_{x \in I} G_A^\infty . x . G_k .$$

We may of course limit ourselves to the double cosets which intersect $G_{A(S)}$, and therefore may assume that $I \subset G_{A(S)}$. Then

$$(1) \qquad G_{A(S)} = \bigcup_{x \in I} G_A^\infty . x . (G_{A(S)} \cap G_k) = \bigcup_{x \in I} G_A^\infty . x . G_{\mathfrak{o}(S)},$$

which shows the existence of a compact set $C \subset G_{P \cap S}$ such that

$$(2) \qquad G_{A(S)} = G_A^\infty . C . M . G_{\mathfrak{o}(S)} \quad (M = \prod_{\mathfrak{p} \notin S} G_{\mathfrak{o}_\mathfrak{p}})\text{[1]}.$$

Since $L = \prod_{\mathfrak{p} \in S \cap P} G_{\mathfrak{o}_\mathfrak{p}}$ is compact, we may, and shall, assume

$$(3) \qquad L . C . L = C.$$

By assumption, $G_\infty = B . \pi_\infty(G_\mathfrak{o})$, whence

$$G_{A(S)} = B . C . M . \pi_\infty(G_\mathfrak{o}) . G_{\mathfrak{o}(S)},$$

and by (3),

$$G_{A(S)} = B . C . G_U . \pi_\infty(G_\mathfrak{o}) G_{\mathfrak{o}(S)}.$$

Using 1.8 (6), we get

$$G_{A(S)} = B . C . G_U . G_\mathfrak{o} . G_{\mathfrak{o}(S)},$$
$$G_{A(S)} = B . C . G_U . G_{\mathfrak{o}(S)},$$

and, again by (3),

$$G_{A(S)} = B . C . M . G_{\mathfrak{o}(S)},$$

hence also

$$G_S = B . C . G_{\mathfrak{o}(S)},$$

which proves our assertion.

8.6. *Remarks.* — (1) In fact, this proves the existence of open or closed fundamental sets of the form $B . C . M$ or $B . C$, where B is a fundamental set for $G_\mathfrak{o}$ in G_∞, C is relatively compact in $G_{P \cap S}$ and $M = \prod_{\mathfrak{p} \notin S} G_{\mathfrak{o}_\mathfrak{p}}$, which satisfy $(F\ 3)_S$ relatively to H if G is reductive, and H a given algebraic subgroup over k of G.

(2) If $k = \mathbf{Q}$, and G is reductive, then, in the notation of 4.5, there are fundamental sets for $G_\mathfrak{o}$ in $G_{A(S)}$ and in G_S of the form

$$\Omega = \bigcup_{i=1}^{m} (a^{-1} \mathfrak{S} \mathbf{GL}_{n,U} . b_i \cap G_{A(S)}) \qquad (b_i \in \mathbf{GL}(n, \mathbf{Z}(S))$$

and $\pi_S(\Omega)$ respectively. The proof is the same as that of 4.5, except for the fact that it uses the equality

$$(4) \qquad \mathbf{GL}_{n, A(S)} = \mathfrak{S} . \mathbf{GL}_{n,U} . G_{\mathbf{Z}(S)},$$

which is an obvious consequence of 4.4, rather than 4.4 itself.

8.7. Let Γ be a lattice in k^n, $G_{\mathfrak{p},\Gamma}$ the stability group of $\Gamma_\mathfrak{p} = \mathfrak{o}_\mathfrak{p} . \Gamma$ in $G_\mathfrak{p}$ ($\mathfrak{p} \in P$),

[1] This proof of (3) is patterned after an argument of M. Kneser's (unpublished).

and $G_{A,\Gamma} = G_A^\infty . \prod_{\mathfrak{p}\in P} G_{\mathfrak{p},\Gamma}$. The group $G_{\mathfrak{p},\Gamma}$ is open and compact in $G_\mathfrak{p}$ and $G_{A,\Gamma}$ is of course the isotropy group of Γ in G_A, where G_A operates on the set of lattices in k^n by $\Gamma \to g(\Gamma) = \bigcap_{\mathfrak{p}\in P} g_\mathfrak{p}(\Gamma_\mathfrak{p})$ $(g = (g_v) \in G_A)$ (see 2.2). The facts on lattices recalled in 2.1 show that $G_{\mathfrak{p},\Gamma} = G_{\mathfrak{o}_\mathfrak{p}}$ for almost all $\mathfrak{p} \in P$, hence that $G_{A,\Gamma}$ is commensurable with G_A^∞, or with $G_{A,\Gamma'}$ for any lattice $\Gamma' \subset k^n$.

Let now $\rho : G \to \mathbf{GL}_m$ be a rational representation over k. Then any lattice Γ in k^m is contained in a lattice $\Gamma' \subset k^m$ stable under $\rho(G_A^\infty)$. In fact, $\rho(G_U)$ is compact, and therefore $\rho(G_U) \cap \mathbf{GL}_{m,A,\Gamma}$ is commensurable with $\rho(G_U)$. Consequently, the set of transforms of Γ under $\rho(G_A^\infty)$ is finite, and the sum Γ' of those transforms is the desired lattice.

8.8. *Theorem. — Let* G *be reductive* $\rho : G \to \mathbf{GL}_m$ *a rational representation over* k, $w \in k^m$ *a point whose orbit* X *is closed, and* Γ *a lattice in* k^m. *Then* $w . \rho_A(G_{A(S)}) \cap k^m$ (resp. $w . \rho_S(G_S) \cap \mathfrak{o}(S) . \Gamma$) *is contained in the union of finitely many orbits of* $G_{\mathfrak{o}(S)}$.

In view of 8.7, we may assume Γ to be invariant under G_A^∞, hence $\mathfrak{o}(S) . \Gamma$ to be invariant under $G_{\mathfrak{o}(S)}$. The theorem follows then from the existence of fundamental sets in $G_{A(S)}$ (resp. G_S) which verify $(F\ 3)_{A(S)}$ (resp. $(F\ 3)_S$).

If $S = V$, then both parts of 8.8 reduce to 5.4.

8.9. In the next section, we shall need the following fact, whose proof uses a result of [5].

Lemma. — Let S *be any finite subset of* V. *Let* X *be a homogeneous space over* k *for* G. *Then* $X_S = \prod_{v\in S} X_v$ *is the union of finitely many orbits of* G_S.

It is enough to prove this when S consists of one element v. If $k_v = \mathbf{C}$, then G_v is transitive on X_v; if $k_v = \mathbf{R}$, this assertion is elementary (see [4, 2.3]). If v is finite, the only proof known to the author is a cohomological one, to be given in [5].

8.10. *Theorem. — Let* G *be reductive,* $\rho : G \to \mathbf{GL}_m$ *a rational representation over* k *of* G, X *a closed orbit and* Γ *a lattice in* k^m. *If* S *is finite, then* $X_S \cap \mathfrak{o}(S) . \Gamma$ *is contained in finitely many orbits of* $G_{\mathfrak{o}(S)}$.

By 8.9, $X_S \cap \mathfrak{o}(S) . \Gamma$ is contained in finitely many sets of the form $w . \rho_S(G_S)$, with $w \in k^m$, to each of which we can apply 8.8.

Here, it would be in fact enough to prove this for G connected, since $(G_o)_S$ has finite index in G_S. We give two applications of 8.10.

8.11. *Corollary. — Let* S *be finite. Then the number of orbits of* $\mathbf{SL}(n, \mathfrak{o}(S))$ *in the set of all symmetric* $n \times n$ *matrices, with coefficients in* $\mathfrak{o}(S)$, *and a given non-zero determinant, is finite.*

This follows from 8.10 applied to the case where ρ is the natural representation of $\mathbf{SL}_n$ in the space of symmetric $n \times n$ matrices and X the set of symmetric matrices with the given determinant. If $S = V - P$, $k = \mathbf{Q}$, then $\mathfrak{o}(S) = \mathbf{Z}$, and we get of course the well-known finiteness of the number of classes of integral quadratic forms with a given non-zero determinant.

8.12. *Proposition. — Let* $\mu : G \to G'$ *be an isogeny over* k *of* G *onto an algebraic matric group* G' *over* k. *Let* S *be finite. Then* $\mu(G_{\mathfrak{o}(S)})$ *is commensurable with* $G'_{\mathfrak{o}(S)}$.

It is clearly enough to prove this when G is connected. Adding one variable if necessary, we may, by a familiar trick, (see [4, 2.1] for instance), assume that G′ consists of matrices of determinant one, hence is closed. Let n' be the degree of G′. Assume first that G is reductive. We let then operate G on the space of $n' \times n'$ matrices by $x \to x.\mu(g)$ and define in this way a rational representation $\rho : G \to \mathbf{GL}_m$ $(m = n^2)$ over k. Our assertion follows then from 8.10 applied to ρ and to G′, viewed as the orbit of e.

If μ is an isomorphism, then 8.11 is elementary, and is a consequence of 1.7. This is necessarily the case if N is unipotent. In general, we use the semi-direct product decompositions $G = H.N$, $G' = H'.N'$ of 1.13. Since we are allowed to replace G (resp. G′) by a group isomorphic to G (resp. G′) over k, we may assume that $\mathbf{C}^n$ (resp. $\mathbf{C}^{n'}$) is the direct sum of subspaces spanned by canonical basis vectors, stable under H (resp. H′), on which N (resp. N′) acts trivially, and that N (resp. N′) is upper triangular. Then $G_{o(S)} = H_{o(S)}.N_{o(S)}$ (resp. $G'_{o(S)} = H'_{o(S)}.N'_{o(S)}$) and we are reduced to the two already discussed special cases.

Remark. — This extension to S-units of [4, 6.11] is not really new. K. Honda (*Jap. J. Math.*, *30* (1960), 84-101) has proved it when S is big enough (for group varieties, not necessarily linear ones, in fact), and Serre (unpublished) has removed the assumption on S by a suitable modification of Honda's method.

8.12. By the argument used in 5.6, it is easily derived from 8.5 (1) that $G_{A(S)}/G_{o(S)}$, G connected, is the union of finitely many open subsets, isomorphic to quotients G_A^∞/H_i, where H_i is commensurable with G_o. Thus the criteria of 5.6 for the finiteness of the volume or the compactness of G_A/G_k also apply to $G_{A(S)}/G_{o(S)}$ and $G_S/G_{o(S)}$.

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