PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 3 (1959), p. 19-27
<http://www.numdam.org/item?id=PMIHES_1959__3__19_0>

ON THE STRUCTURE OF CERTAIN SEMI-GROUPS OF SPHERICAL KNOT CLASSES

By Barry MAZUR

§ 1. Introduction.

The problem of classification of k-sphere knots in r-spheres is the problem of classifying "knot pairs": $S=(S_1, S_2)$, where S_2 is an oriented combinatorial r-sphere, S_1 a subcomplex of S_2 (isomorphic to a standard k-sphere), and the pair S is considered equivalent to S' $(S \sim S')$ if there is a combinatorial orientation-preserving homeomorphism of S_1 onto S_1' bringing S_2 onto S_2'.

Thus it is the problem of classifying certain relative combinatorial structures. The set of all such, for fixed k and r, will be called Σ_k^r, and can be given, in a natural manner, the structure of a semi-group. There is a certain sub-semi-group of Σ_k^r to be singled out — the semi-group S_k^r of all pairs $S=(S_1, S_2)$ where S_1 is smoothly imbedded in S_2 (locally unknotted).

In this paper I shall define a notion of equivalence (which I call $*$-equivalence) between knot pairs which is (seemingly) weaker than the equivalence defined above.

Two knot pairs S and S' are $*$-equivalent if (again) there is an orientation-preserving homeomorphism

$$\varphi : S_2 \rightarrow S_2'$$

bringing S_1 onto S_1'. However φ is required to be combinatorial (not on all of S_2, as before, but) merely on $S_2^* = S_2 - (p_1, \ldots, p_n)$, where $p_1, \ldots, p_n \in S_2$, where S_2^* is considered as an open infinite complex. Thus $*$-equivalence neglects some of the combinatorial structure of the pair (S_1, S_2). The set of all $*$-equivalence classes of knot pairs forms a semi-group again, called ${}^*\Sigma_k^r$.

Finally the subsemi-group of smoothly imbedded knots in ${}^*\Sigma_k^r$ I call ${}^*S_k^r$. The purpose of this paper is to prove a generalized knot theoretic restatement of lemma 3 in [1].

Inverse Theorem: A knot S_k^r is invertible if and only if it is $*$-trivial.

And in application, derive the following fact concerning the structure of the knot semi-groups:

There are no inverses in ${}^*S_k^r$.

§ 2. Terminology.

My general use of combinatorial topology terms is as in [2]. It is clear what is meant by the "usual" or "standard" imbedding of a k-sphere or a k-cell in E^r. Similarly an unknotted sphere or disc in E^r means one that may be thrown onto the usual by a combinatorial automorphism of E^r.

Definition 1. Let M^k be a subcomplex (a k-manifold) of E^r. Then M^k is *locally unknotted at a point* m $(m \in M)$ if the following condition is met with:

1) There is an r-simplex Δ^r drawn about m so that $\Delta^r \cap M \subset \mathrm{St}(m)$, and $\Delta^r \cap M$ is then a k-cell $B^k \subset \Delta^r$, and $\partial B^k \subset \partial \Delta^r$.

2) There is a combinatorial automorphism of Δ^r, sending B^k onto the "standard k-cell in Δ^r". M is plain *locally unknotted* if it is locally unknotted at all points.

Semi-Groups:

All semi-groups to be discussed will be countable, commutative, and possess zero elements.

Definition 2. A semi-group F is *positive* if:

$$X + Y = 0 \text{ implies } X = 0$$

(i.e.if F has no inverses).

Definition 3. A *minimal base* of a semi-group F is a collection $J = (\chi_1, \ldots)$ of elements of F such that every element of F is a sum of elements in J, and there is no smaller $J' \subset J$ with the same property.

Definition 4. A *prime element* p in the semi-group F is an element for which $p = x + y$ implies either $x = 0$ or $y = 0$.

Clearly, if a positive semi-group F possesses a minimal base, that minimal base has to be precisely the set of primes in F, and F has the property that every element is expressible as a finite sum of primes.

Definition 5. An element $x \in F$ is *invertible* if there is a $y \in F$ such that

$$x + y = 0.$$

§ 3. (*)-homeomorphism.

Definition 6. A $(p_1, \ldots, p_n)$-homeomorphism, $h : E^r \to E^r$ will be an orientation preserving homeomorphism which is combinatorial except at the points $p_i \in E^r$. It is a homeomorphism such that $h \mid E^r - (p_i)$ is a combinatorial map — simplicial with respect to a possibly infinite subdivision of the open complexes involved. When there is no reason to call special attention to the points $p_1, \ldots, p_n$, I shall call such: a (*)-homeomorphism.

DEFINITION 7. Two subcomplexes K, $K' \subset E^r$ will be called *$*$-equivalent* ($K \underset{*}{\sim} K'$) if there is a $*$-homeomorphism h of E^r onto itself bringing K onto K'. (If h is a (p_i)-homeomorphism I shall also say $K \underset{(p_i)}{\sim} K'$.) To keep from using too many subscripts, whenever a $(*)$-equivalence comes up in a subsequent proof, I shall act as if it were a (p)-equivalence for a single point p. This logical gap, used merely as a notation-saving device, can be trivally filled by the reader.

I'll say a sphere knot is *$*$-trivial* if it is $*$-equivalent to the standard sphere.

§ 4. Knot Addition.

There is a standard additive structure that can be put on Σ^r_k, the set of combinatorial k-sphere knots in E^r (two k-sphere knots are equivalent if there is an orientation-preserving combinatorial automorphism of E^r bringing the one knot onto the other). (For details see [**2**]).

I shall outline the procedure of "adding two knots" S_0, S_1. Separate S_0 and S_1 by a hyperplane H (possibly after translating one of them). Take a k-simplex Δ_i from each S_i, $i = 0, 1$. And lead a "tube" from Δ_0 to Δ_1 (by "thickening" a polygonal arc joining a point $p_0 \in \Delta_0$ to $p_1 \in \Delta_1$, which doesn't intersect the S_i except at Δ_i). Then remove the Δ_i and replace them by the tube $T = S^{k-1} \times I$, where one end, $S^{k-1} \times 0$ is attached to $\partial \Delta_0$ by a combinatorial homeomorphism, and the other $S^{k-1} \times 1$ is attached to $\partial \Delta_1$ similarly. The resulting knot is called the sum: $S_0 + S_1$, and its knot-equivalence class is unique.

If one added the point at infinity to E^r, to obtain S^r, the hyperplane H would become an unknotted $S^{r-1} \subset S^r$, separating the knot $S_0 + S_1$ into its components S_0 and S_1. In analytic fashion, then, we can say that a k-sphere knot $S \subset S^r$ is *split* by an $S^{r-1} \subset S^r$ if:

1) $S^{r-1} \cap S$ is an unknotted $(k-1)$-sphere knot in S.
2) S^{r-1} is unknotted in S^r.
3) $S^{r-1} \cap S$ is unknotted in S^{r-1}.

Let A_0 and A_1 be the two complementary components of $S^{r-1} \cap S$ in S, and let B be an unknotted k-disc that $S^{r-1} \cap S$ bounds in S^{r-1}. Then $S_0 = A_0 \cup B$, $S_1 = A_1 \cup B$ are knotted spheres again, and clearly $S \sim S_0 + S_1$.

Thus I'll say: S^{r-1} splits S into $S_0 + S_1$; if E_0 and E_1 are the complementary regions of S^{r-1} in S^r, I'll refer to S_1 as *that « part of S » lying in E_1*, and similarly for S_0. Working in the semi-group $^*\Sigma^r_k$, one can be slightly cruder, and say: *S^{r-1} $*$-splits S* if only 1) and 3) hold. Clearly by [**1**], every S^{r-1} is $*$-trivial in S^r.

LEMMA 1: If S^{r-1} $*$-splits S, and S_0, S_1 are constructed in a manner analogous to the above, then $S \underset{*}{\sim} S_0 + S_1$.

§ 5. The Semi-Groups of Spherical Knots.

This operation of addition, discussed in the previous section, turns Σ^r_k into a commutative semi-group with zero. Our object is to study the algebraic structure of the subsemi-group $S_k^r \subset \Sigma_k^r$ of locally unknotted k-sphere knots. Let ${}^*\Sigma_k^r$ be the semi-group of classes of spherical knots under $*$-equivalence. Let $G_k^r \subset \Sigma_k^r$ be the maximal subgroup of Σ_k^r, that is: the subgroup of invertible knots.

Inverse Theorem: There is an exact sequence

$$0 \to G_k^r \to S_k^r \to {}^*S_k^r \to 0$$

(where ${}^*S_k^r$ is the image of S_k^r in ${}^*\Sigma_k^r$)

or, equivalently, a knot in S_k^r is $*$-trivial if and only if it is invertible.

§ 6. Proof of the Inverse Theorem.

a) If S is invertible, then $S \underset{(*)}{\sim} 0$. The proof is quite as in [1]. Let $S + S' \sim 0$. Then consider the knots:

$$S_\infty = S + S' + S + S' + \ldots \cup p_\infty$$
$$S'_\infty = S' + S + S' + S + \ldots \cup p_\infty$$

(See figure 1)

and notice: (as was done in detail in [1])

$$S_\infty \underset{(p_\infty)}{\sim} 0$$
$$S'_\infty \underset{(p_\infty)}{\sim} 0$$
$$S_\infty = S + S'_\infty$$

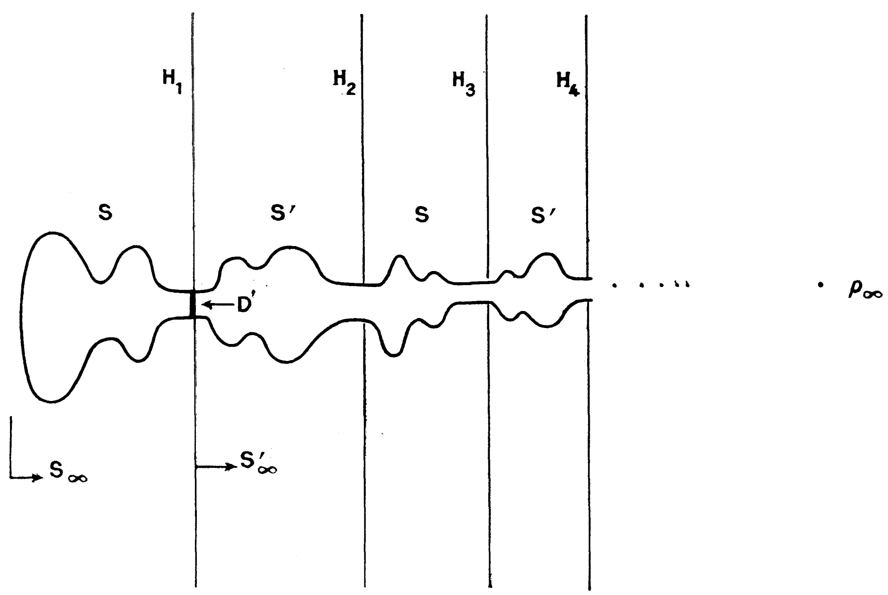

Fig. 1

Lemma 2: There is a $(*)$-homeomorphism

$$f: E^r \to E^r \text{ such that}$$
$$f: S \to S + S'_\infty.$$

PROOF: Let D be the k-cell on which the addition of S to S'_∞ takes place. Since $S'_\infty \underset{(p_\infty)}{\sim} 0$, we may transform figure 1 to figure 2 by a (p_∞)-homeomorphism g which leaves everything to the left of the hyperplane H_1 fixed, and sends S' to the " standard k-sphere " to the right of H_1. (See figure 2.)

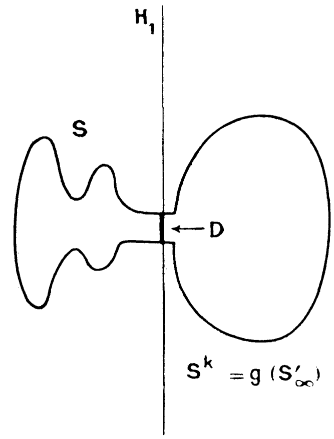

Fig. 2

Then, in figure 2, clearly one can construct an automorphism f' which leaves S fixed and sends D onto $g(S'_\infty)$—int D.

Take $f = g^{-1} f' g$, and f has the properties required, and is a $(*)$-homeomorphism. Therefore, by the above lemma,

$$S \underset{(*)}{\sim} S + S'_\infty = S_\infty \underset{(*)}{\sim} 0$$

and finally:

$$S \underset{(*)}{\sim} 0$$

which proves (a).

b) If $S \in S^r_k$ and $S \underset{(p)}{\sim} 0$, then S is invertible.

PROOF: First observe that if $k = r - 1$, invertibility of knots is generally true (by [1]), and so we needn't prove anything.

LEMMA 3: If $k < r - 1$, and $S \in S^r_k$, $S \underset{(p)}{\sim} 0$ for $p \notin S$, then $S \sim 0$.

PROOF: There is an r-cell Δ containing S but not p. Then $f|\Delta$ is combinatorial, and by a standard lemma:

LEMMA 4: If $g : \Delta \to \Delta'$ is a combinatorial homeomorphism of an r-cell $\Delta \subset E^r$ to an r-cell $\Delta' \subset E^r$, then g can be extended to a combinatorial automorphism of E^r (see [2]). Thus, restrict f to Δ, and extend $f|\Delta$ to a combinatorial automorphism g of E^r. This g yields the equivalence $S \sim 0$. Therefore, assume $S \underset{(p)}{\sim} 0$, and $p \in S$.

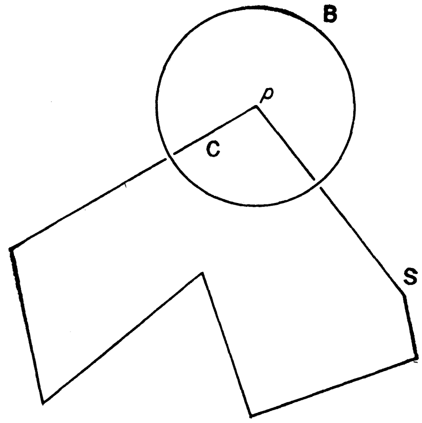

Fig. 3

Let B be a small r-cell about p, so that $C = B \cap S$ is in $St(p)$, and hence an unknotted k-cell, by the local unknottedness of S. $\partial B \cap S = \partial C$ and ∂C is unknotted in ∂B.

Let f be the (p)-homeomorphism taking S onto the standard S^k.

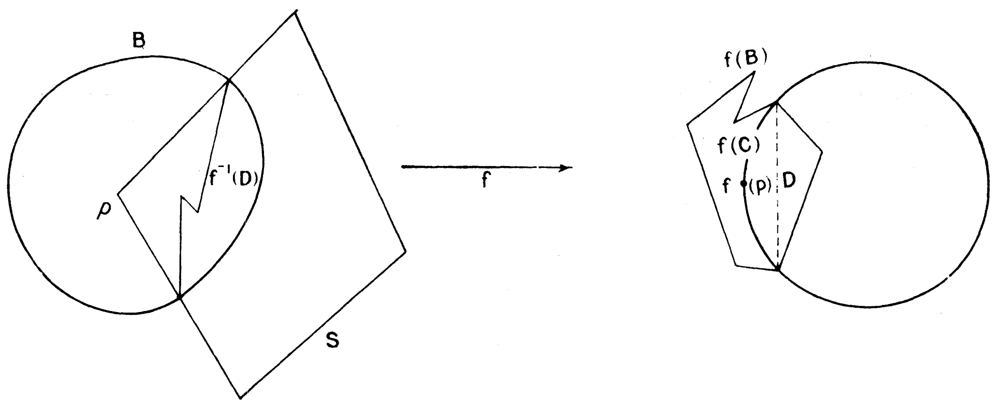

Fig. 4

Now let D be an unknotted disc, the image of a perturbation of $f(C)$ with the properties:

i) $\partial(f(C)) = \partial D$;
ii) int $D \subset$ int B ;
iii) $f(p) \notin D$;
iv) the knot $K = D \cup (S^k - f(C))$ is still trivial.

Then f^{-1} takes K to a knot $K' = f^{-1}(K)$, split by ∂B into the sum:

$$K' = S + S'$$

where S is the knot lying in the exterior component of ∂B, and S' in the interior.

But $K \sim o$, and $K' \underset{f(p)}{\sim} K$ where $f(p) \notin f(K)$, therefore by lemma 3, $f(K) \sim K$. So:

$$S + S' \sim f(K) \sim K \sim o,$$

and S' is invertible.

COROLLARY: $^*S^r_k$ is a positive semi-group.

So we have that $^*S^r_k$ is precisely S^r_k « modulo units ».

§ 7. Infinite Sums in $^*\Sigma^r_k$.

Let X_i, $i = 1, \ldots,$ be knots representing the classes $\chi_i \in \Sigma^r_k$. Define $\sum_{i=1}^{\infty} X_i$ to be the infinite one point compactified sum of the knots X_i, in that order (figure 5).

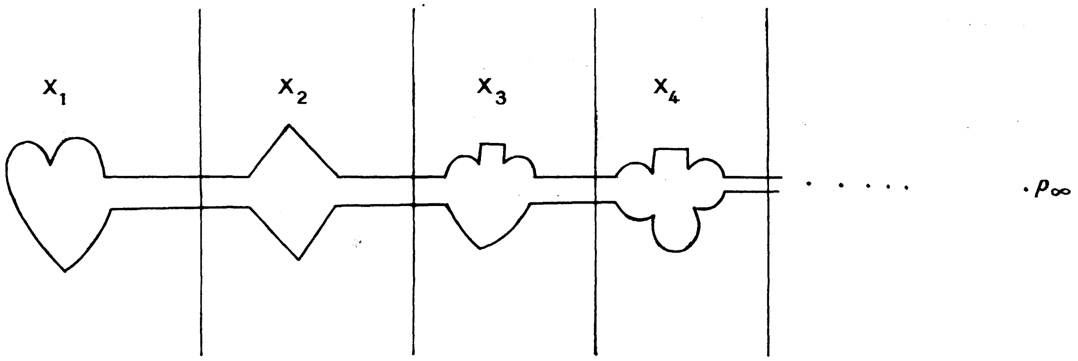

Fig. 5

As it stands, $X = \sum_{i=1}^{\infty} X_i$ will not represent a knot in Σ^r_k, because X is not combinatorially imbedded (at p_∞).

DEFINITION 8. $\sum_{i=1}^{\infty} X_i = X$ *converges* if there is a (p_∞)-homeomorphism $H : X \to Y$, where Y is combinatorially imbedded. In that case, the knot class $y \in {}^*\Sigma^r_k$ is uniquely determined by the $X_i \in \Sigma^r_k$, and I shall say $\sum_{i=1}^{\infty} \chi_i = y$.

If $\sum_{i=1}^{\infty} \chi_i$ is in $^*S^r_k$, I'll say that $\sum_{i=1}^{\infty} \chi_i$ *converges in* $^*S^r_k$.

THEOREM 1. If $\sum_{i=1}^{\infty} \chi_i$ converges in $^*S^r_k$, then it does so finitely. That is, there is an N such that

$$\chi_i \underset{*}{\sim} o, \qquad i > N.$$

PROOF: Notice that by the inverse theorem, there are no inverses in $^*S^r_k$.

Let $X = \sum_{i=1}^{\infty} X_i$, and $H : X \to Y$ where Y is a subcomplex of E^r and H a $(*)$-homeomorphism.

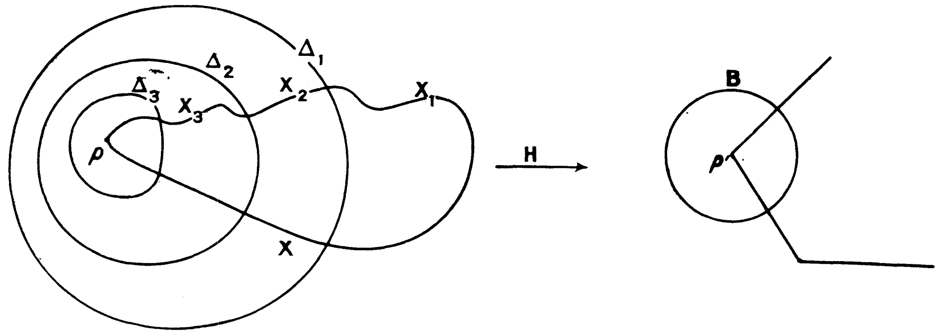

Fig. 6

Let B be a ball about p' such that $B \cap Y$ is a disc in $St(p')$, and by the local unknottedness of Y, ∂B splits Y into two knots,

$$Y = Y^{(1)} + Y^{(2)}$$

where $Y_1 \subset B$ is trivial, and $Y \sim Y_2$.

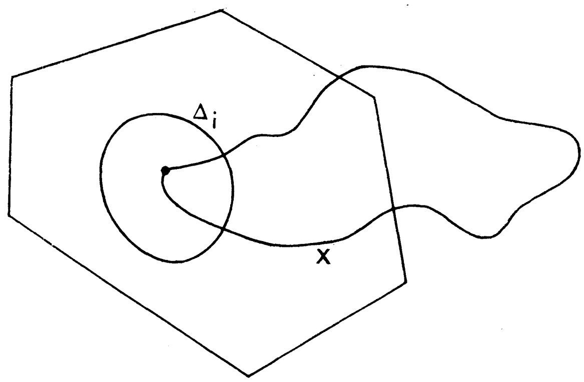

Fig. 7

Now transform the situation by H^{-1}. Let $B' = H^{-1}(B)$, and we have that $\partial B'$ $*$-splits X into:

$$X \underset{*}{\sim} X^{(1)} + X^{(2)}$$

and H yields the $*$-equivalences:

$$X^{(1)} \underset{*}{\sim} Y^{(1)} \sim o$$
$$X^{(2)} \underset{*}{\sim} Y^{(2)}$$

Find an i so large that $\Delta_i \subset \operatorname{int} B'$. Then $\partial \Delta_i$ splits $X^{(1)}$ further:

$$X^{(1)} \sim X^{(3)} + X^{(4)}$$

where $X^{(3)}$ is the part of $X^{(1)}$ lying in Δ_i. But then, by figure 6, $X^{(3)}$ is nothing more than:

$$X^{(3)} \sim \sum_{j=i}^{\infty} X_j.$$

Passing to equivalence classes in ${}^*S_k^r$, one has:

$$\chi^{(3)} + \chi^{(4)} = 0$$

$$\chi^{(3)} = \sum_{j=i}^{\infty} \chi_j$$

(where x the $*$-equivalence class of X). But repeated application of the fact that ${}^*S_k^r$ has no inverses yields $\chi_j = 0$ for $j \geq i$, which proves the theorem.

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[2] V. K. A. M. Guggenheim, Piece wise Linear Isotopy, *Journal of the London Math. Soc.*, vol. 46.

Reçu le 16 *novembre* 1959.