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Publications mathématiques de l'I.H.É.S., tome 1 (1959), p. 7-23
<http://www.numdam.org/item?id=PMIHES_1959__1__7_0>

THE STRUCTURE OF A UNITARY FACTOR GROUP

By G. E. WALL

Introduction

Let D be a division ring, V a right vector space of finite dimension n over D. A linear transformation, X, on V is called a *transvection* if it has the form $x \to x + a\rho(x)$, where a is a fixed vector and $\rho(x)$ a linear form on V such that $\rho(a) = 0$ (in other words, $X = I + N$, where I is the identity and N a nilpotent linear transformation of rank 1). The group of all non-singular linear transformations on V *(full linear group)* is denoted by $GL(n, D)$, and the invariant subgroup generated by all transvections *(special linear group)* by $SL(n, D)$.

The structure of the factor group GL/SL was elucidated by J. Dieudonné ([1]). Let Δ denote the multiplicative group formed by the non-zero elements of D, Δ_1 the commutator group of Δ. Choose a fixed basis $e_1, \ldots e_n$ of V, and let $X \in GL$. Using the technique of 'elementary transformations' familiar in matrix theory, Dieudonné proved that $X \equiv \Lambda \pmod{SL}$ for some 'diagonal' linear transformation Λ of the form

$$\Lambda e_i = e_i \ (1 \leqslant i \leqslant n-1), \quad \Lambda e_n = e_n \xi \ (\xi \in \Delta);$$

he proved furthermore that ξ is unique modulo Δ_1 and that the mapping $X(SL) \to \xi\Delta_1$ is an isomorphism of GL/SL onto Δ/Δ_1. The coset $\xi\Delta_1$ is the 'noncommutative determinant' of X.

The object of this paper is to prove a similar structure theorem for a class of unitary groups. In order to define unitary groups, we require that D have an involutory anti-automorphism $J: \lambda \to \bar{\lambda}$. As fundamental form we take a function $f = (x, y)$, which is defined for all $x, y \in V$, has values $(x, y) \in D$, and satisfies the conditions:

(1) *f is a sesquilinear form with respect to J, i.e.,*

$$(x, y_1\lambda_1 + y_2\lambda_2) = (x, y_1)\lambda_1 + (x, y_2)\lambda_2,$$

$$(x_1\lambda_1 + x_2\lambda_2, y) = \bar{\lambda}_1(x_1, y) + \bar{\lambda}_2(x_2, y),$$

for all $x, x_i, y, y_i \in V$ and $\lambda_i \in D$;

(2) *f is non-degenerate, i.e., if* $(x,y)=0$ *for all* $y\in V$ *then* $x=0$;

(3) *f is skew-Hermitian, i.e.,* $\overline{(y,x)}=-(x,y)$ *for all* $x,y\in V$.

The *unitary group*, $U(f)$, of f consists of the linear transformations X which leave f invariant: $(Xx, Xy)=(x,y)$ for all $x,y\in V$. (We remark that, unless J is the identity, there is no loss of generality in taking f skew-Hermitian rather than Hermitian ([2], p. 12); thus, our discussion covers the 'properly' unitary, and symplectic, groups but not the orthogonal groups.)

Two subsets M, N of V are *orthogonal* if $(x,y)=0$ for all $x\in M$ and $y\in N$; by (3) the relation of orthogonality is symmetric. If M is a subspace of V, the vectors $x\in V$ which are orthogonal to M form the *orthogonal space*, $M^{\perp}$, of M. By (2) and (3), we have $(M^{\perp})^{\perp}=M$ and $\dim M+\dim M^{\perp}=n$.

It is easy to show that a transvection which belongs to $U(f)$ has the from $x\to x-a\,\omega\,(a,x)$, where ω is a *symmetric* element of Δ (i.e., $\omega=\overline{\omega}$) and a an *isotropic* vector in V (i.e., $(a,a)=0$). Bearing in mind the situation for GL, we make the additional assumption:

(4) *V contains non-zero isotropic vectors.*

The invariant subgroup of $U(f)$ generated by all unitary transvections is denoted by $T(f)$.

By (3), the value (x,x) is *skew* ($\overline{(x,x)}=-(x,x)$) for every $x\in V$. Our final assumption is:

(5) *f is trace-valued, i.e.,* (x,x) *has the form* $\lambda-\overline{\lambda}\ (\lambda\in D)$ *for every* $x\in V$.

Notice that (5) is automatically satisfied when characteristic $D\neq 2$: $(x,x)=\lambda-\overline{\lambda}$, where $\lambda=\frac{1}{2}(x,x)$.

A plane (i.e. 2-dimensional subspace of V) is called *hyperbolic* when it has a basis of two isotropic vectors e_1, e_2 such that $(e_1, e_2)=1$. Condition (5) ensures that *(i)* every isotropic vector can be embedded in a hyperbolic plane and *(ii)* any two hyperbolic planes are equivalent under $U(f)$. From *(i)* and *(ii)* can be deduced an analogue of Witt's theorem on quadratic forms, viz., that the number of members in a maximal set of mutually orthogonal hyperbolic planes is always the same (cf. [2], ch. I, § 11). This number, denoted by ν, is the *Witt index* of f; by *(i)* and (4), $\nu\geqslant 1$.

Let Σ denote the subgroup of Δ generated by the non-zero symmetric elements of D, and Ω the subgroup generated by the $\lambda\in\Delta$ such that $\lambda-\overline{\lambda}=(x,x)$ for some vector $x\in V$ which is orthogonal to a hyperbolic plane. Taking $x=0$, we see that $\Sigma\subseteq\Omega$. It is not difficult to show that Σ, Ω are invariant subgroups of Δ. With these notations, our main result is as follows.

THEOREM 1. *If f satisfies the conditions* (1) - (5), *and if* [1] $U(f) \neq U_3(F_4)$, *then*

$$U(f)/T(f) \cong \Delta/\Sigma[\Delta, \Omega], \tag{6}$$

where $[\Delta, \Omega]$ *is the subgroup of* Δ *generated by the commutators* $\omega^{-1}\delta^{-1}\omega\delta\,(\omega \in \Omega, \delta \in \Delta)$.

It is well known that $SL(n, D)$ is projectively a simple, non-cyclic group, unless $n=2$ and $D=F_2$ or F_3. It follows from the isomorphism $GL/SL \cong \Delta/\Delta_1$, that SL is, except in these two cases, the commutator group of GL. The situation for unitary groups is analogous, but more complicated. Except in some half-dozen cases which we entirely exclude from the discussion, $T(f)$ is projectively a simple, non-cyclic group [2], so that $T(f)$ is the commutator group of $U(f)$ if, and only if, U/T is abelian, i.e., by theorem 1, if, and only if,

$$\Sigma[\Delta, \Omega] \supseteq \Delta_1. \tag{7}$$

Most of the known results on this problem follow fairly easily from (7). We mention only the two results of Dieudonné ([2], ch. II, § 5) that (a) T *is* the commutator group of U whenever $\nu \geqslant 2$, and that (b) T *is not* the commutator group of U when D is the algebra of real quaternions under the usual 'complex conjugate' anti-automorphism and $n=2$. The result which we shall prove is as follows.

THEOREM 2. *Suppose that the conditions of theorem* 1 *hold and that $T(f)$ is projectively a simple group. If $n \geqslant 3$ and D has finite dimension m^2 over its centre Z, then $T(f)$ is the commutator group of $U(f)$.*

It is perhaps unlikely that theorem 2 remains valid whenever D has infinite dimension over its centre, but I have not been able to construct a counterexample.

I am indebted to Professor J. Dieudonné for his helpful comments on this paper.

1. Proof of Theorem 2.

In this section we shall deduce theorem 2 from theorem 1. We assume that J is not the identity, for otherwise $\Delta=\Sigma$ and so, by theorem 1, $U=T$. It follows from this assumption that there exist anisotropic vectors orthogonal to a given hyperbolic plane H; for otherwise the (non-degenerate) restriction of f to $H^{\perp}$ would be a symplectic form, and this would imply that J was the identity. Let a be such an anisotropic vector and λ a fixed element of Δ such that $\lambda - \bar{\lambda} = (a, a)$. Let S denote the set of symmetric elements of D. We consider three cases according to the 'type' of the anti-automorphism J (cf. [2], ch. II, § 5).

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(1) F_q denotes the Galois field with q elements. There is essentially only one properly unitary group over F_q (q a square) for each dimension m, and it is denoted by $U_m(F_q)$.

(2) In order to establish (7) rigorously, we actually need the slightly stronger result that every proper invariant subgroup of T is contained in the centre of T.

TYPE I. ($\mathcal{J}$ leaves every element of $\mathcal{Z}$ invariant and S is a vector space over $\mathcal{Z}$ of dimension $\frac{1}{2}m(m+1)$). This case was considered by Dieudonné ([3], p. 379), whose argument (1) shows that $\Delta = \Sigma$ and therefore $U = T$.

TYPE II. ($\mathcal{J}$ leaves every element of $\mathcal{Z}$ invariant and S is a vector space over $\mathcal{Z}$ of dimension $\frac{1}{2}m(m-1)$; this type occurs only when characteristic $D \neq 2$). We shall prove that U/T is an abelian group by showing that each of its elements has order $\leqslant 2$. Since characteristic $D \neq 2$, the vector space A over $\mathcal{Z}$ formed by the skew elements of D is complementary to S and has dimension $\frac{1}{2}m(m+1)$. Let K denote the vector space over $\mathcal{Z}$ of dimension $1 + \frac{1}{2}m(m-1)$ formed by the elements $\zeta(\lambda+\sigma)$ $(\zeta \in \mathcal{Z}, \sigma \in S)$. Since $(\lambda+\sigma) - \overline{(\lambda+\sigma)} = (a, a)$, every non-zero element of K is in Ω.

Now let $\mu \in \Delta$; it is required to show that $\mu^2 \in \Sigma[\Delta, \Omega]$. Since the sum of the dimensions of the vector spaces μK and A over $\mathcal{Z}$ is $1 + m^2$, these spaces have a non-zero element in common, say μk; as we remarked above, $k \in \Omega$. Then we have $\mu k = -\overline{(\mu k)} = -\bar{k}\bar{\mu}$, and so

$$(k^{-1}\mu k \mu^{-1})\mu^2 = -(k^{-1}\bar{k})(\bar{\mu}\mu). \tag{1.1}$$

Again, the sum of the dimensions of the vector spaces kA and A over $\mathcal{Z}$ is $m^2 + m$, so that there exists a non-zero skew element α such that $k\alpha$ is skew. Thus, $k\alpha = -\bar{\alpha}\bar{k} = \alpha\bar{k}$. Hence

$$k^{-1}\bar{k} = k^{-1}\alpha^{-1}k\alpha. \tag{1. 2}$$

(1. 1) and (1. 2) together show that $\mu^2 \in \Sigma[\Delta, \Omega]$, as we had to prove.

TYPE III. ($\mathcal{J}$ does not leave invariant every element of $\mathcal{Z}$; S, D are vector spaces of respective dimensions m^2, $2m^2$ over the subfield $\mathcal{Z}_0$ formed by symmetric elements of $\mathcal{Z}$). Let K_0 denote the vector space over $\mathcal{Z}_0$ of dimension $1 + m^2$ formed by the elements $\zeta_0(\lambda+\sigma)$ $(\zeta_0 \in \mathcal{Z}_0, \sigma \in S)$. As with type II, every non-zero element of K_0 is in Ω. Let $\mu \in \Delta$. Since the sum of the dimensions of the vector spaces μK_0 and K_0 over $\mathcal{Z}_0$ is $2m^2 + 2$, there exist non-zero elements k_1, k_2 of K_0 such that $\mu k_1 = k_2$. Hence $\mu \in \Omega$ and so $\Delta = \Omega$. It now follows from theorem 1 that U/T is abelian, as required.

2. Two Preliminary Lemmas.

The remainder of the paper is devoted to the proof of theorem 1. We begin with two lemmas on sesquilinear forms (cf. (1)).

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(1) Dieudonné's argument actually applies only when characteristic $D \neq 2$. However, only a slight modification is needed when characteristic $D = 2$.

Lemma 1. *Let W be an m-dimensional right vector space over D, $\Phi(x,y)$ any sesquilinear form on W (with respect to J) and suppose that J is not the identity. Then there exists a basis $e_1,..,e_m$ of W such that $\Phi(e_i,e_j)=0\ (1\leqslant i<j\leqslant m)$.*

Proof. If Φ is not identically zero, then by a familiar argument (using the fact that J is not the identity) there exists an $e_1\in W$ such that $\Phi(e_1,e_1)\neq 0$. The x such that $\Phi(e_1,x)=0$ form an $(m-1)$-dimensional subspace for which (by induction on the dimension) we can choose a basis $e_2,..,e_m$ such that $\Phi(e_i,e_j)=0\ (2\leqslant i<j\leqslant m)$. Then $e_1,..,\ e_m$ satisfy the requirements of the lemma. Q.E.D.

Lemma 2. *Suppose that the conditions of lemma 1 hold and that in addition $\Phi(x,y)$ is non-degenerate. Let $\Psi(x,y)$ be a second sesquilinear* (1) *form on W which is not identically zero. Then, if $m>2$ when $D=F_4$, there exists an $x\in W$ such that both $\Phi(x,x)$ and $\Psi(x,x)$ are non-zero.*

Proof. Choose a basis $e_1,..,e_m$ of W as in lemma 1. As Ψ is not identically zero on W it is not identically zero on every one of the planes $\{e_i,e_j\}$. It therefore suffices to prove the lemma in two cases: (i) $D=F_4, m=3$; (ii) $D\neq F_4, m=2$.

As the first case can be settled by a direct calculation we consider the second only. Suppose that it is not possible to choose x as required by the lemma. Then the matrices of Φ and Ψ with respect to the basis e_1,e_2 must have the forms

$$\begin{pmatrix}\omega_1' & 0\\ \rho' & \omega_2'\end{pmatrix},\quad \begin{pmatrix}0 & \alpha'\\ \beta' & 0\end{pmatrix},$$

where $\omega_1'\omega_2'\neq 0$ and not both α',β' are 0. Moreover, for every $\lambda\in D$, at least one of

$$\Phi(e_1\lambda+e_2,e_1\lambda+e_2)=\bar\lambda\,\omega_1'\lambda+\rho'\lambda+\omega_2'=0$$

and

$$\Psi(e_1\lambda+e_2,e_1\lambda+e_2)=\beta'\lambda+\bar\lambda\,\alpha'=0$$

must hold. By the symmetry of the second equation in α',β', we may assume without loss of generality that $\beta'\neq 0$, and on putting $\mu=\beta'\lambda$ and slightly modifying ω_1', etc., these equations become

(2.1) $\bar\mu\,\omega_1\mu+\rho\mu+\omega_2=0$,

(2.2) $\mu+\bar\mu\,\alpha=0$,

with $\omega_1\omega_2\neq 0$.

If we regard D as a vector space over the field F formed by the symmetric elements in its centre, then it is clear that the solutions of (2.2), and the symmetric elements of D, form subspaces K and S respectively. It is easy to see that either $K=S$ or $K\cap S=\{0\}$. In particular (since it is assumed that $D\neq S$), $K\neq D$.

(1) 'Sesquilinear' means 'sesquilinear with respect to J' unless otherwise stated.

Now every element of D not in K satisfies (2.1). Hence (2.1) has a solution μ such that $\mu\,\sigma$ is also a solution for every $\sigma\,(\neq 0)$ in F. Each such σ satisfies the quadratic equation

$$(\bar{\mu}\,\omega_1\,\mu)\,\sigma^2 + (\rho\,\mu)\,\sigma + \omega_2 = 0$$

and therefore $F = F_2$ or F_3.

(a) $(F = F_3)$. D contains at least 3 additive cosets of K (including K itself), and at least 9 cosets of K if $K = \{0\}$. It is therefore possible to choose μ, ν so that $\mu, \mu + \nu$, $\mu - \nu$ are 3 distinct solutions of (2.1). Putting these in (2.1) we deduce that $\bar{\nu}\,\omega_1\,\nu = 0$ and so $\omega_1 = 0$, a contradiction.

(b) $(F = F_2)$. The centre of D is F or a quadratic extension of F; and by assumption $D \neq F_4$. Hence D is non-commutative, and so, by a result of Dieudonné ([3], *lemma* 1), not every two elements of S commute. In particular, the dimension of $S \geqslant 3$. Let π, σ, τ be any 3 linearly independent elements of S. Since $S = K$ or $S \cap K = \{0\}$, there exists a $\mu \in D$ such that every element of $\mu + S$, with the possible exception of μ itself, satisfies (2.1). Putting $\mu + \pi, \mu + \sigma, \mu + \pi + \sigma$ in (2.1) we deduce that

$$\pi\,\omega_1\,\sigma + \sigma\,\omega_1\,\pi = \bar{\mu}\,\omega_1\,\mu + \omega_2 + \rho\,\mu.$$

Similarly,

$$\tau\,\omega_1\,\sigma + \sigma\,\omega_1\,\tau = \bar{\mu}\,\omega_1\,\mu + \omega_2 + \rho\mu,$$

$$(\pi + \tau)\,\omega_1\,\sigma + \sigma\,\omega_1\,(\pi + \tau) = \bar{\mu}\,\omega_1\,\mu + \omega_2 + \rho\,\mu,$$

whence

(2.3) $\pi\,\omega_1\,\sigma = \sigma\,\omega_1\,\pi.$

(2.3) clearly holds for any two symmetric elements π, σ. For $\sigma = 1$, we get $\pi\,\omega_1 = \omega_1\,\pi$ and so (2.3) becomes $(\pi\,\sigma - \sigma\,\pi)\,\omega_1 = 0$. This is a contradiction because on the one hand not every two symmetric elements commute and on the other $\omega_1 \neq 0$. This proves the lemma.

3. Cayley Parametrization.

In this section we shall obtain a parametrization (without exceptions) for the elements of U. Similar considerations for orthogonal groups lead to a generalization of the ordinary Cayley parametrization.

Let $P \in U$, and write $P = I - Q$, where I is the identity transformation. The space QV will be called the *space of* P, and denoted by V_P. If dim $V_P = r$, P is called an *r-dimensional element* of U.

Since $P \in U$, we have

$$(x, Qy) + (Qx, y) = (Qx, Qy) \tag{3.1}$$

for all $x, y \in V$. This equation obviously shows that the value of (Qx, y) depends only on the values of Qx and Qy. We may therefore write $(Qx, y) = [Qx, Qy]$. This defines for all $u, v \in V_P$, a function $[u, v]$. We denote this function by f_P and call it the *form of P*.

It is easily verified that f_P is a non-degenerate sesquilinear form on V_P (with respect to $\mathcal{J}$), and that (by (3.1))

$$[u,v]-\overline{[v,u]}=(u,v) \tag{3.2}$$

for all $u, v \in V_P$.

Let now $e_1, .., e_r$ be a basis of V_P and set $\omega_{ij}=[e_i, e_j]$, $(\omega_{ij})^{-1}=(\Phi_{ij})$. For each $x \in V$, we have $Qx=\Sigma_1^r e_i \lambda_i$, where the $\lambda_i (\in D)$ depend on x; we shall now determine the λ_i explicitly. Using the equation $(Qx, y) = [Qx, Qy]$, we have

$$(e_i, x)=[e_i, \Sigma_1^r e\ \lambda_j]=\Sigma_1^r \omega_{ij} \lambda_i,$$

and so

$$\lambda_i=\Sigma_1^r \Phi_{ij}(e_j, x)\ (1 \leqslant i \leqslant r).$$

This gives the formula

$$Px=x-\Sigma_{i,j} e_i \Phi_{ij}(e_j, x). \tag{3.3}$$

The following converse of the above holds and justifies the description of (3.3) as a parametrization for U: *if W is any subspace of V and* $[u, v]'$ *any non-degenerate sesquilinear form on W (with respect to* $\mathcal{J}$*) satisfying* (3.2), *then there exists one, and only one,* $P' \in U$ *such that* $V_{P'}=W$ *and* $f_{P'}=[u,v]'$. The straightforward proof will be omitted.

We consider now some properties of the parametrization.

(i) *Conjugate Elements.* If $R \in U$, it is easy to see that

$$RPR^{-1}x=x-\Sigma_{i,j}(Re_i)\ \Phi_{ij}(Re_j, x).$$

Hence $V_{RPR^{-1}}=RV_P$ and the forms f_P and $f_{RPR^{-1}}$ are equivalent ([1]).

(ii) *One-dimensional Elements.* If $V_P=\}e\{$, we have

$$Px=x-e\varphi(e,x),$$

where

$$\varphi^{-1}-\bar{\varphi}^{-1}=(e,e)$$

We denote this element by $(e; \varphi)$.

(iii) *Factorizations of P.* Let W_1 be a subspace of V_P such that the restriction $(f_P)_{W_1}$ of f_P to W_1 is non-degenerate. Let W_2 be the subspace of V_P formed by the u such that $[u, v]=0$ for all $v \in W_1$. Then $V_P=W_1+W_2$ (direct sum) and $(f_P)_{W_2}$ is non-degenerate. Let $P_i (i=1,2)$ be the elements of U such that $V_{P_i}=W_i, f_{P_i}=(f_P)_{W_i}$. *Then we have* $P=P_1P_2$.

To prove this, let $a_1, .., a_s$ and $b_1, .., b_t$ be bases of W_1 and W_2 respectively. The matrix of f_P with respect to the basis $a_1, .., a_s, b_1, .., b_t$ of V_P has the form

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(1) It can be proved that the converse is also true: elements P_1, P_2 of U are conjugate in U if, and only if, their forms f_{P_1} and f_{P_2} are equivalent.

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix},$$

where A refers to the a_i, B to the b_i. By (3.2), the element in the i-th row and j-th column of C is (a_i, b_j). Hence if $A^{-1} = (\alpha_{ij}), B^{-1} = (\beta_{ij})$, we have

$$P_1 x = x - \Sigma\, a_i \alpha_{ij} (a_j, x),$$

$$P_2 x = x - \Sigma\, b_i \beta_{ij} (b_j, x),$$

$$Px = x - \Sigma\, a_i \alpha_{ij} (a_j, x) - \Sigma\, b_i \beta_{ij} (b_j, x) + \Sigma\, a_i \alpha_{ik} (a_k, b_l)\, \beta_{lj} (b_j, x),$$

and direct calculation gives $P = P_1 P_2$, as required.

Essentially the same calculation shows that, conversely, if R_1, R_2 are elements of U such that $P = R_1 R_2$ and $V_P = V_{R_1} + V_{R_2}$ (direct sum), then f_{R_i} is the restriction of f_P to V_{R_i} and $[u, v] = 0$ for all $u \in V_{R_2}$, $v \in V_{R_1}$.

Definition. *We call*

(3.4) $$P = R_1 R_2 \ldots R_s \; (R_i \in U)$$

a direct factorization of P of length s if, firstly, no R_i is the identity I and, secondly, V_P is the direct sum of the V_{R_i}. Any factor occurring in such a direct factorization is called a direct factor of P. A direct factorization is called complete if each factor is a one-dimensional element.

By the above, $R \in U$ is a direct factor of P if, and only if, $\{0\} \neq V_R \subseteq V_P$ and $f_R = (f_P)_{V_R}$. We remark also if $R = R_i$ is the i-th factor in some direct factorization (3.4) of length s, then, for any j such that $1 \leqslant j \leqslant s$, there exists a direct factorization of length s in which R is the j-th factor. If $i < j$, for example, it is easy to prove that

$$P = R_1 \ldots R_{i-1} (R R_{i+1} R^{-1}) \ldots (R R_j R^{-1})\, R R_{j+1} \ldots R_s$$

is such a direct factorization.

Lemma 3. *If $P (\neq I) \in U$ and J is not the identity, then P has a complete direct factorization. Moreover, the space of the first factor can be taken as any line $\{a\} \in V_P$ such that $[a, a] \neq 0$.*

Proof. By lemma 1, V_P has a basis $e_1, \ldots, e_r$ such that $[e_i, e_j] = 0 \; (1 \leqslant j < i \leqslant r)$; and by the proof of lemma 1, $\{e_1\}$ may be taken as any line in V_P satisfying $[e_1, e_1] \neq 0$. Then, if $\varphi_i^{-1} = [e_i, e_i]$,

$$P = (e_1; \varphi_1) \ldots (e_r; \varphi_r)$$

is a complete direct factorization. Q.E.D.

4. A 'Spinor Norm' in U.

Throughout this section, a stands for a fixed element of V chosen (quite arbitrarily) in advance. We shall associate with a a 'spinor norm' which has properties similar to the spinor norm in orthogonal groups. We shall assume that J is not the identity because our construction becomes trivial in the case of symplectic groups. It is not necessary to assume that $\nu \geqslant 1$ or that f be trace-valued.

Let Ω_a denote the subgroup of Δ generated by the $\omega \in \Delta$ such that

$$\omega - \bar{\omega} = (b, b) \tag{4.1}$$

for some $b\,(\in V)$ orthogonal to a. Write $\Gamma_a = \Sigma[\Delta, \Omega_a]$, where $[\Delta, \Omega_a]$ is the subgroup of Δ generated by the commutators $\lambda^{-1}\omega^{-1}\lambda\,\omega\,(\lambda \in \Delta, \omega \in \Omega_a)$.

LEMMA 4. *Σ, Ω_a and Γ_a are invariant subgroups of Δ such that $\Sigma \subseteq \Gamma_a \subseteq \Omega_a$.*

PROOF. If σ is symmetric and $\mu \in \Delta$, then $\mu\,\sigma\,\bar{\mu}$ and $\mu\,\bar{\mu}$ are symmetric so that $\mu\,\sigma\,\mu^{-1} = \mu\,\sigma\,\bar{\mu}\,(\mu\,\bar{\mu})^{-1} \in \Sigma$. Hence Σ is invariant in Δ. Taking $b = 0$ in (4.1) we see that every non-zero symmetric element is in Ω_a and so $\Sigma \subseteq \Omega_a$. Let ω satisfy (4.1) and $\mu \in \Delta$. Then $\mu\,\bar{\mu} \in \Sigma \subseteq \Omega_a$, and $\mu\,\omega\,\bar{\mu} - \overline{(\mu\,\omega\,\bar{\mu})} = (b\,\bar{\mu}, b\,\bar{\mu})$, so that $\mu\,\omega\,\bar{\mu} \in \Omega_a$. Hence $\mu\,\omega\,\mu^{-1} \in \Omega_a$ and therefore Ω_a is an invariant subgroup of Δ. It is now evident that Γ_a is invariant in Δ and $\Sigma \subseteq \Gamma_a \subseteq \Omega_a$. This completes the proof.

LEMMA 5. *Let P be an r -dimensional element of $U\,(r > 0)$, and*

$$P = P_1 \,.\,.\, P_r = P_1' \,.\,.\, P_r'$$

two complete direct factorizations of P. Let

$$P_i = (a_i;\, \omega_i), P_i' = (a_i';\, \omega_i'),$$

the a_i and a_i' being chosen so that each value (a, a_i) and (a, a_i') is either 0 or 1 $(1 \leqslant i \leqslant r)$. Then the cosets $\omega_1\,\omega_2 \,.\,.\, \omega_r\,\Gamma_a$ and $\omega_1'\,\omega_2' \,.\,.\, \omega_r'\,\Gamma_a$ are equal.

PROOF. The lemma is easily proved when D is commutative. For in this case

$$\omega_1 \ldots \omega_r = |A|^{-1}, \omega_1' \,.\,.\, \omega_r' = |B|^{-1},$$

where A, B are the matrices of f_P with respect to the bases $a_1, .\,., a_r$ and $a_1', .\,., a_r'$ of V_P. Since $|A|$ and $|B|$ differ by a factor of the form $\lambda\bar{\lambda}$, we have $\omega_1 \,.\,.\, \omega_r\,\Gamma_a = \omega_1' \,.\,.\, \omega_r'\,\Gamma_a$, as required.

Suppose now that D is non-commutative. It will be convenient to use the following notations: when $\alpha, \beta\,(\in \Delta)$ satisfy $\alpha\,\Gamma_a = \beta\,\Gamma_a$ we write $\alpha \sim \beta$, and when two factorizations $P_1 \ldots . P_r$ and $P_1' \ldots . P_r'$ give rise to the same coset

$$\omega_1 \ldots . \omega_r\,\Gamma_a = \omega_1' \ldots \omega_r'\,\Gamma_a$$

we write $P_1 \ldots .. P_r \sim P_1' \ldots .. P_r'$. Notice that, since $[\Delta, \Omega_a] \subseteq \Gamma_a$, we have $\alpha\,\omega \sim \omega\,\alpha$ whenever $\alpha \in \Delta, \omega \in \Omega_a$. We prove the lemma by induction on r, considering separately the cases $r = 1$, $r = 2$ and $r > 2$.

(i) $(r=1)$. In this case, $V_P=\{a_1\}=\{a_1'\}$, so that $a_1'=a_1\lambda^{-1}$ and $\omega_1'=\lambda\,\omega_1\,\bar\lambda$ for some $\lambda\in\Delta$. Thus,

$$\omega_1^{-1}\,\omega_1'=(\omega_1^{-1}\lambda\,\omega_1\,\lambda^{-1})\,(\lambda\bar\lambda). \tag{4.2}$$

If $\{a\}$ is not orthogonal to V_P, then, by the choice of a_1 and a_1', we have

$$(a,a_1)=(a,a_1')=1.$$

Therefore $\lambda=1$ and so $\omega_1=\omega_1'$. If, on the other hand, $\{a\}$ is orthogonal to V_P, then $\omega_1\in\Omega_a$ and therefore $\omega_1\sim\omega_1'$ by (4.2).

(ii) $(r=2)$. In this case,

$$\left.\begin{aligned} a_1' &= a_1\lambda+a_2\mu \\ a_2' &= a_1\rho+a_2\sigma \end{aligned}\right\}$$

where λ, μ, ρ, $\sigma\in D$, and the matrices of f_P with respect to the bases a_1, a_2 and a_1', a_2' are respectively

$$\begin{pmatrix}\omega_1^{-1} & (a_1,\ a_2)\\ 0 & \omega_2^{-1}\end{pmatrix},\quad \begin{pmatrix}\omega_1'^{-1} & (a_1',\ a_2')\\ 0 & \omega_2'^{-1}\end{pmatrix}.$$

Hence

$$\left.\begin{aligned} \omega_1'^{-1} &= \bar\lambda\,\omega_1^{-1}\lambda+\bar\mu\,\omega_2^{-1}\mu+\bar\lambda\,(a_1,a_2)\,\mu \\ \omega_2'^{-1} &= \bar\rho\,\omega_1^{-1}\rho+\bar\sigma\,\omega_2^{-1}\sigma+\bar\rho\,(a_1,a_2)\,\sigma \end{aligned}\right\}, \tag{4.3}$$

$$0=\bar\rho\,\omega_1^{-1}\lambda+\bar\sigma\,\omega_2^{-1}\mu+\bar\rho\,(a_1,a_2)\,\mu. \tag{4.4}$$

If one of ρ, μ is zero, then by (4.4) so is the other, and therefore $\{a_i\}=\{a_i'\}$ $(i=1,2)$; hence $P_1P_2\sim P_1'P_2'$ by the case $r=1$. We suppose therefore that $\rho\mu\neq0$. Then, by (4.4),

$$\left.\begin{aligned} -\bar\rho\,\omega_1^{-1}\lambda\,\mu^{-1} &= \bar\sigma\,\omega_2^{-1}+\bar\rho\,(a_1,a_2) \\ -\bar\rho^{-1}\bar\sigma\,\omega_2^{-1}\mu &= \omega_1^{-1}\lambda+(a_1,a_2)\,\mu \end{aligned}\right\}$$

and on substituting these values in (4.3) we get

$$\left.\begin{aligned} \omega_1'^{-1} &= (1-\bar\lambda\,\bar\rho^{-1}\bar\sigma\,\bar\mu^{-1})\,\bar\mu\,\omega_2^{-1}\mu \\ \omega_2'^{-1} &= \bar\rho\,\omega_1^{-1}\rho\,(1-\rho^{-1}\lambda\,\mu^{-1}\sigma) \end{aligned}\right\}. \tag{4.5}$$

Suppose firstly that P has a one-dimensional direct factor whose space is orthogonal to $\{a\}$. We may suppose without loss of generality that P_2 is such a factor, so that $(a,a_2)=0$. Then $(a,a_1')=(a,a_1)\,\lambda$ and $(a,a_2')=(a,a_1)\,\rho$. By the definition of the a_i and a_i' (and since $\rho\neq0$), we have $\rho=1$ and $\lambda=0$ or 1 when $(a,a_1)=1$. We may also suppose that $\rho=1$ and $\lambda=0$ or 1 when $(a,a_1)=0$; for, by the case $r=1$, $\omega_1'\,\omega_2'\,\Gamma_a$ is unaltered when a_1',a_2' are replaced by multiples of themselves. With these values of λ, ρ, the element $(1-\bar\lambda\,\bar\rho^{-1}\bar\sigma\,\bar\mu^{-1})\,(1-\rho^{-1}\lambda\,\mu^{-1}\sigma)$ is symmetric and therefore, by (4.5), $\omega_1'\,\omega_2'\sim\mu^{-1}\,\omega_2\,\bar\mu^{-1}\,\omega_1\sim\omega_1\,\omega_2$, since $\omega_2\in\Omega_a$.

Suppose secondly that P has no one-dimensional direct factor whose space is orthogonal to $\{a\}$. Then $(a,a_i)=(a,a_i')=1$ $(i=1,2)$, and so

$$\lambda+\mu=\rho+\sigma=1. \tag{4.6}$$

Further, since $(a, a_1 - a_2) = 0, P$ cannot have a direct factor with space $\{a_1 - a_2\}$. Hence $[a_1 - a_2, a_1 - a_2] = 0$, i.e.,

$$\bar{\omega}_1^{-1} + \bar{\omega}_2^{-1} = (a_1, a_2). \tag{4.7}$$

By (4.4), (4.6) and (4.7), $-\bar{\rho}\, \bar{\omega}_1^{-1} = \bar{\omega}_2^{-1} \mu$, and therefore, by (4.5),

$$\omega_1'^{-1} = (1 - \bar{\rho} - \bar{\mu})\, \bar{\omega}_1^{-1} \text{ and } \omega_2'^{-1} = \bar{\omega}_2^{-1} (1 - \rho - \mu),$$

whence $\omega_1' \omega_2' \sim \omega_1 \omega_2$, as required.

(iii) $(r > 2)$. We assume that the lemma holds for elements of U of dimension $< r$. Consider the subspaces $A = \{a_1, .., a_{r-1}\}$, $B = \{a_1', .., a_{r-1}'\}$. Suppose first that the form $[u, v]$ $(u \in A, v \in B)$ on the pair of spaces A, B is degenerate. Then there exists an $x (\neq 0) \in A$ such that $[x, b] = 0$ for all $b \in B$. Since $[a_r', b] = 0$ for all $b \in B$ we have $\{x\} = \{a_r'\}$. Hence P_r' is a direct factor of $P_1 \ldots P_{r-1}$ and so, by the induction hypothesis, $P_1 \ldots P_{r-1} P_r \sim P_r' R_2 .. R_{r-1} P_r$ for certain R_i. Again, by a double application of the induction hypothesis, $P_1' \ldots P_r' \sim P_1' P_r' S_3 \ldots S_r \sim P_r' T_2 \ldots T_r$, for certain S_i, T_i. But $T_2 \ldots T_r \sim R_2 \ldots R_{r-1} P_r$ by the induction hypothesis, so that $P_1 \ldots P_r \sim P_1' \ldots P_r'$, as required.

Suppose secondly that the form $[u, v]$ on the pair A, B is non-degenerate. Then the equations

$$[u, b] = [u^*, b] \text{ (for all } b \in B)$$

define a one-to-one linear mapping $u \to u^*$ of A onto B. We may define a (non-degenerate) form $[u, v]_1$ on B by the equations

$$[x, y] = [x^*, y^*]_1 \, (x, y \in A).$$

Then, by lemma 2, there is a $b \in B$ such that $[b, b] \neq 0$ and $[b, b]_1 \neq 0$. Since $[u, v]$ is non-degenerate on B and J is not the identity, there exists a vector $d \in B$ such that $[b, d] = 0$ and $[d, d] \neq 0$. Then, if e is the vector in A such that $e^* = b$, we have $[e, e] \neq 0$, $[e, d] = 0, [d, d] \neq 0$.

Now let R, S be respectively the direct factors of P with spaces $\{e\}$, $\{d\}$. By the induction hypothesis, we have, with certain R_i, S_i, T,

$$P_1' \ldots P_{r-1}' \sim S S_2 \ldots S_{r-1},$$

$$P_1 \ldots P_{r-1} \sim R R_2 \ldots R_{r-1},$$

$$SR \sim RT.$$

Also, since $P = S(S_2 \ldots S_{r-1} P_r')$ is a direct factorization and $[e, d] = 0$, e belongs to the space of $S_2 \ldots S_{r-1} P_r'$; therefore, by the induction hypothesis, $S_2 \ldots S_{r-1} P_r' \sim R T_3 \ldots T_r$, for certain T_i.

Hence

$$\begin{aligned} P_1' \dots P_r' &\sim S S_2 \dots S_{r-1} P_r' \\ &\sim S R T_3 \dots T_r \\ &\sim R T T_3 \dots T_r \\ &\sim R R_2 \dots R_{r-1} P_r \text{ (induction hypothèses)} \\ &\sim P_1 \dots P_r. \end{aligned}$$

This completes the proof of the lemma.

DEFINITION. *The coset $\omega_1 \dots \omega_r \Gamma_a$ appearing in lemma 5 is called the spinor norm of P with respect to a and is denoted by $N_a(P)$; $N_a(I)$ is defined to be Γ_a.*

LEMMA 6. *$(N_a(P))^{-1} = N_a(P^{-1})$ $(P \in U)$.*

PROOF. If $P = P_1 \dots P_r$ is a complete direct factorization of P, then $P^{-1} = P_r^{-1} \dots P_1^{-1}$ is evidently a complete direct factorization for P^{-1}. Let $P_i = (a_i; \omega_i)$, where $(a, a_i) = 0$ or 1 $(1 \leqslant i \leqslant r)$. Then $P_i^{-1} = (a_i; -\bar{\omega}_i)$, and so $N_a(P) = \omega_1 \dots \omega_r \Gamma_a$ and $N_a(P^{-1}) = \bar{\omega}_r \dots \bar{\omega}_1 \Gamma_a = (\bar{\omega}_1^{-1} \dots \bar{\omega}_r^{-1} \Gamma_a)^{-1}$.

Since $\bar{\omega}_1^{-1} \dots \bar{\omega}_r^{-1} = (\omega_1 \bar{\omega}_1)^{-1} \omega_1 (\omega_2 \bar{\omega}_2)^{-1} \dots (\omega_r \bar{\omega}_r)^{-1} \omega_r \sim \omega_1 \dots \omega_r$, we have the lemma.

LEMMA 7. *$N_a(P) N_a(Q) = N_a(PQ)$ $(P, Q \in U)$.*

PROOF. We write N for N_a. By lemma 6 and the definition of N, it is sufficient to prove the following statement:
(4.8) *if $P_1, \dots, P_r$ are one-dimensional elements such that $P_1 \dots P_r = I$, then $N(P_1) \dots N(P_r) = \Gamma_a$.*

The proof of (4.8) is by induction on r. Write $Q_s = P_1 \dots P_s$, $V_s = V_{Q_s}$, $\dim V_s = d_s$ $(1 \leqslant s \leqslant r)$. Notice that $s \geqslant d_s$, with equality if, and only if, the factorization

$$Q_s = P_1 \dots P_s \tag{4.9}$$

is direct. Similarly, since V_s is also the space of

$$Q_s^{-1} = P_{s+1} \dots P_r, \tag{4.10}$$

we have $r - s \geqslant d_s$, with equality if, and only if, the factorization (4.10) is direct.

Suppose firstly that for some s (where $1 < s < r$), we have

$$d_s \leqslant d_{s+1},\ d_s \leqslant d_{s-1}. \tag{4.11}$$

Then neither (4.9) nor (4.10) can be direct, so that $s > t, r - s > t$ $(t = d_s)$. Let $R_1 \dots R_t$ be a complete direct factorization for Q_s. Then

$$\left.\begin{aligned} P_1 \dots P_s R_t^{-1} \dots R_1^{-1} &= I \\ R_1 \dots R_t P_{s+1} \dots P_r &= I \end{aligned}\right\}$$

and since $s+t<r$ and $r-s+t<r$, we have by the induction hypothesis and lemma 6,

$$\left.\begin{array}{l} N(P_1)\ldots\ldots N(P_s) = N(R_1)\ldots\ldots N(R_t) \\ N(R_1)\ldots\ldots N(R_t)\, N(P_{s+1})\ldots N(P_r) = \Gamma_a \end{array}\right\}.$$

Hence $N(P_1)\ldots\ldots N(P_r)=\Gamma_a$, as required.

Suppose secondly that (4.11) does not hold for any s. Then it is easy to see that $r=2u$ and $P_1\ldots\ldots P_u, P_{u+1}\ldots\ldots P_r$ are complete direct factorizations of Q_u, Q_u^{-1} respectively. Hence $N(P_1)\ldots N(P_r)=\Gamma_a$ by lemmas 5 and 6. This completes the proof.

5. Proof of Theorem 1.

We shall now assume that the conditions of theorem 1 hold. The theorem being well known for symplectic groups, we shall assume that J is not the identity.

Let e be any fixed non-zero isotropic vector in V, and write $N=N_e$, $\Omega=\Omega_e$, $\Gamma=\Gamma_e$. Then the Ω so defined is the same as the one in the enunciation of theorem 1, and we are required to prove that

$$(5.1) \qquad U/T\cong\Delta/\Gamma.$$

Consider the homomorphism $\theta: P\rightarrow N(P)$, of U into Δ/Γ. We shall prove (5.1) by showing that (i) $\theta(U)=\Delta/\Gamma$, and (ii) $\theta^{-1}(\Gamma)=T$.

Proof of (i). Let $\lambda\in\Delta$. Since f is tracevalued, there exists an isotropic vector e_1 such that $(e,e_1)=1$. Then, if $e_2=e_1-e\lambda^{-1}$, we have $(e_2,e_2)=\lambda^{-1}-\bar{\lambda}^{-1}$ and $(e,e_2)=1$, so that $N((e_2;\lambda))=\lambda\Gamma$. Hence $\theta(U)=\Delta/\Gamma$, as required.

Proof of (ii). It is easy to see that $N(P)=\Gamma$ for every transvection P; hence $T\subseteq\theta^{-1}(\Gamma)$. It remains to prove that if $N(P)=\Gamma$ then $P\in T$. We first consider the case $n=2$, where V itself is a hyperbolic plane. Let e_1 be as in the last paragraph, so that e, e_1 form a basis of V. We note that in the present case $\Omega=\Sigma=\Gamma$ and that therefore $N(P)$ is an element of Δ/Σ.

If $Q\in U$, we have $Qe=e\alpha+e_1\beta$, $Qe_1=e\gamma+e_1\delta$ $(\alpha,\beta,\gamma,\delta\in D)$. We show that

(a) those coefficients out of $\alpha,\beta,\gamma,\delta$ which are not zero all lie in the same coset of Σ; this coset will be denoted by $M(Q)$;

(b) $M(Q_1)M(Q_2)=M(Q_1Q_2)$ $(Q_1,Q_2\in U)$;

(c) $M(Q)=N(Q)$ when Q is a one-dimensional element;

(d) if $M(Q)=\Sigma$, then $Q\in T$.

It is clear that (a) - (c) prove that $M\equiv N$ and that then (d) proves the required result (ii). We shall give only the proofs of (a) and (d), those of (b) and (c) being straightforward verifications.

PROOF OF (a). We remark that $e+e_1\sigma$ is isotropic if, and only if, σ is symmetric. Hence $Qe=e_1\lambda$ or $(e+e_1\sigma)\lambda$, where $\lambda\neq 0, \sigma=\bar{\sigma}$. Since $(e_1, -e)=(e+e_1\sigma, e_1)=1$, we have either

$$(5.2)\qquad \left.\begin{array}{l} Qe=e_1\lambda \\ Qe_1=-(e+e_1\sigma)\bar{\lambda}^{-1} \end{array}\right\},$$

or

$$(5.3)\qquad \left.\begin{array}{l} Qe=(e+e_1\sigma)\lambda \\ Qe_1=(e_1+(e+e_1\sigma)\tau)\bar{\lambda}^{-1} \end{array}\right\},$$

where σ, τ are symmetric. (a) now follows by direct inspection.

PROOF OF (d). We note that

$$(5.4)\qquad Q=\left\{\begin{array}{l} (e_1;\sigma+2)(e-e_1;1)Q_\lambda \ [in\ (5.2)], \\ (e+e_1\sigma;-\tau)(e_1;\sigma)Q_\lambda \ [in\ (5.3)]. \end{array}\right.$$

where

$$\left.\begin{array}{l} Q_\lambda e=e\lambda \\ Q_\lambda e_1=e_1\bar{\lambda}^{-1} \end{array}\right\}.$$

If now $N(Q)=\Sigma$, then $\lambda=\sigma_1 \ldots \sigma_r$, where each σ_i is symmetric, and so

$$(5.5)\qquad Q_\lambda=Q_{\sigma_1}\ldots Q_{\sigma_r}.$$

Finally, when s $(\neq 0)$ is symmetric,

$$(5.6)\qquad Q_s=(e_1;s^{-1})(e;s)(e_1;s^{-1})(e_1;-1)(e;-1)(e_1;-1).$$

We now have $Q\in T$, by (5.4) - (5.6). This proves (ii) when $n=2$.

We suppose finally that $n>2$. Our proof is an adaptation of the argument used by Dieudonné to prove that T is the commutator group of U when the Witt index $\geqslant 2$ ([3], § 16; [4], § 13). The isotropic vector e_1 is chosen as before and the hyperbolic plane $\{e, e_1\}$ is denoted by H. Let f_H and P_H (for $P\in U$) denote the restrictions of f and P to H. It is easy to see that the $P\in U$ such that $V_P\subseteq H$ form a subgroup U^* of U, and that $P\to P_H$ $(P\in U^*)$ is an isomorphism of U^* onto the unitary group $U(f_H)$ of f_H. When $P\in U^*$, we write $N^*(P)=N_e(P_H)$; then $N^*(P)\in\Delta/\Sigma$ and it is clear that $N(P)=N^*(P)\Gamma$.

We shall prove that

(α) if H_1 is any hyperbolic plane in V, there exists a $P\in T$ such that $PH_1=H$ $(U\neq U_3(F_4))$;

(β) if $\lambda\in\Gamma$, there exists an element $Q\in T\cap U^*$ such that $N^*(Q)=\lambda\Sigma$.

Before proving (α) and (β), we show that (ii) follows from them. Suppose then that (α), (β) hold, and let R be an element of U such that $N(R)=\Gamma$; it is required to show that $R\in T$. Let $R_1 \ldots R_r$ be a complete direct factorization of R, and let $V_{R_i}=\{a_i\}$ $(1\leqslant i\leqslant r)$. Since a_i lies in some hyperbolic plane, there exists, by (α), an element $P_i\in T$ such that $P_i a_i\in H$ and therefore $P_i R_i P_i^{-1}\in U^*$ $(1\leqslant i\leqslant r)$. Hence $RT=ST$, where $S=(P_1R_1P_1^{-1})\ldots(P_rR_rP_r^{-1})\in U^*$. Since $\mathcal{N}(R)=\mathcal{N}(S)=\Gamma, \mathcal{N}^*(S)=\lambda\Sigma$, where $\lambda\in\Gamma$. With Q as in (β), we have $\mathcal{N}^*(SQ^{-1})=\mathcal{N}^*(S)\mathcal{N}^*(Q)^{-1}=\Sigma$, and so, by the case $n=2$, $ST=QT=T$. Hence $R\in T$, as required.

Proof of (α). Let $H_1=\rangle a, a_1\langle$, where a, a_1 are isotropic and $(a,a_1)=1$. Since T permutes the isotropic lines of V transitively, there is no loss of generality in supposing that $a=e$. With this assumption, a_1 has the form $e\mu+e_1+b$, where $(b,e)=(b,e_1)=0$ and $\mu-\bar{\mu}=(b,b)$; and we may also suppose that $b\neq 0$, since otherwise (α) is proved. Let P be the element of U such that $V_P=\rangle e,b\langle$ and $f_P=[u,v]$ has matrix

$$\begin{pmatrix}0 & 1\\ 1 & -\bar{\mu}\end{pmatrix}$$

with respect to the basis e,b. It is easily verified that $Pe=e, Pa_1=e_1$, and we shall complete the proof of (α) by showing that $P\in T$ except when $U=U_3(F_4)$.

If b is isotropic, take any complete direct factorization $P=P_1P_2$ of P; since $\rangle e, b\langle$ is totally isotropic, P_1 and P_2 are transvections and so $P\in T$, as required.

Suppose now that $(b,b)\neq 0$. First let $D\neq F_4$. Under this assumption, D contains a symmetric element s distinct from 0 and -1. Write $c=e\bar{\lambda}\mu\lambda+e_1+b\lambda$, where $\lambda=(1-\mu^{-1}\bar{\mu})^{-1}s$; then $(c,c)=0$, $(e,c)=1$, so that $\rangle e,c\langle$ is a hyperbolic plane. Let R be the element of U such that $V_R=\rangle e,c\langle$ and $f_R=\langle u,v\rangle$ has matrix

$$\begin{pmatrix}0 & s+1\\ s & 0\end{pmatrix}$$

with respect to the basis e,c. Then $(e,e)=\langle e,-es^{-1}\rangle$ and $(c,e)=\langle c,-es^{-1}\rangle$, so that $Re=e(1+s^{-1})$. By the case $n=2, R\in T$. Further, $(e,b)=\langle e,-e\bar{\mu}\rangle$ and $(c,b)=\langle c,-e\bar{\mu}\rangle$, so that $Rb=b+e\bar{\mu}$.

Now the matrix of f_P with respect to the basis $b+e\bar{\mu},b$ is

$$\begin{pmatrix}\mu & \mu-\bar{\mu}\\ 0 & -\bar{\mu}\end{pmatrix},$$

so that

$$\begin{aligned}P&=(b+e\bar{\mu};\mu^{-1})(b;-\bar{\mu}^{-1})\\ &=(b+e\bar{\mu};-\bar{\mu}^{-1})^{-1}(b;-\bar{\mu}^{-1})\\ &=R(b;-\bar{\mu}^{-1})^{-1}R^{-1}(b;-\bar{\mu}^{-1});\end{aligned}$$

since $R\in T$, it follows that also $P\in T$, as required.

Finally, let $D=F_4, n\geqslant 4$. As $\nu\geqslant 2$, we can find isotropic vectors d, d_1 orthogonal to both e,e_1 and such that $(d,d_1)=1, d-d_1\bar{\mu}=b$. Then $(d;-1)(e+d;1)b=b+e\bar{\mu}$, and so, by the argument of the last paragraph, $P\in T$. This completes the proof of (α).

PROOF OF (β). By the multiplicative property of $\mathcal{N}^*$, it is sufficient to prove (β) when λ has the form $\mu\rho\mu^{-1}\rho^{-1}$, where $\rho\in\Delta$ and $\mu-\bar{\mu}=(a,a)$ for some non-isotropic vector a orthogonal to both e and e_1.

For any $\alpha\in\Delta$, set $b_\alpha=e\bar{\alpha}^{-1}+e_1\alpha\bar{\mu}$, and let u_α, v_α be the vectors such that $a=u_\alpha+v_\alpha\mu$ and $b_\alpha=u_\alpha+v_\alpha\bar{\mu}$. It is easily verified that $\}a,b_\alpha\{=\}u_\alpha,v_\alpha\{$ is a hyperbolic plane and that u_α, v_α are isotropic vectors such that $(u_\alpha,v_\alpha)=1$.

Let now $R_\alpha=(b_\alpha;\bar{\mu}^{-1})(a;\mu^{-1})$ and $Q=R_\rho R_1^{-1}=(b_\rho;\bar{\mu}^{-1})(b_1;\bar{\mu}^{-1})^{-1}$. Then, in the expression for $R_\alpha v_\alpha$ as a linear combination of u_α and v_α, the coefficient of v_α is -1, so that, by the case $n=2$, $R_\alpha\in T$. Hence $Q\in T\cap U^*$. Finally, in the expression for $(b_\rho;\bar{\mu}^{-1})e$ as a linear combination of e, e_1, the coefficient of e_1 is $\rho\mu\bar{\rho}$, so that $\mathcal{N}^*(Q)=\rho\mu\bar{\rho}\bar{\mu}^{-1}\Sigma=\rho\mu\rho^{-1}\mu^{-1}\Sigma$. This completes the proof of (β) and theorem 1.

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