

Intuitionistic Logic Considered As An Extension of Classical Logic: Some Critical Remarks*

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Résumé. Dans cet article, nous analysons la conception de la logique intuitionniste comme une extension de la logique classique. Ce point de vue — surprenant au premier abord — a été explicitement soutenu par Jan Łukasiewicz sur la base d'une projection de la logique propositionnelle classique dans la logique propositionnelle intuitionniste, réalisée par Kurt Gödel en 1933. Au même moment, Gerhard Gentzen proposait une autre projection de l'arithmétique de Peano dans l'arithmétique de Heyting. Nous discutons ces projections en lien avec le problème de la détermination des symboles logiques qui expriment adéquatement les idiosyncrasies de la logique intuitionniste. De nombreux philosophes et logiciens ne semblent pas suffisamment conscients des difficultés soulevées par le fait de considérer la logique classique comme un sous-système de logique intuitionniste. Un résultat de cette discussion sera de faire ressortir ces difficultés. La notion de traduction logique jouera un rôle

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essentiel dans l'argumentation, et nous esquisserons quelques conséquences concernant la signification des constantes logiques.

Abstract. In this paper we analyze the consideration of intuitionistic logic as an extension of classical logic. This — at first sight surprising — point of view has been sustained explicitly by Jan Łukasiewicz on the basis of a mapping of classical propositional logic into intuitionistic propositional logic by Kurt Gödel in 1933. Simultaneously with Gödel, Gerhard Gentzen had proposed another mapping of Peano's arithmetic into Heyting's arithmetic. We shall discuss these mappings in connection with the problem of determining what are the logical symbols that properly express the idiosyncrasy of intuitionistic logic. Many philosophers and logicians do not seem to be sufficiently aware of the difficulties that arise when classical logic is considered as a subsystem of intuitionistic logic. As an outcome of the whole discussion these difficulties will be brought out. The notion of logical translation will play an essential role in the argumentation and some consequences related to the meaning of logical constants will be drawn.

During the development of symbolic logic it has been a debatable question what are the relations between intuitionistic logic (IL) and classical logic (CL). How to conceive these relations depends on our assumptions on the nature of logic. In this paper we shall assume the distinction formulated by Susan Haack between logics which are *deviant* from classical logic and logics which are *extensions* of classical logic, for this distinction will serve us to the purpose of understanding the historical discussions on the philosophical significance of the translations of CL into IL.

In her book *Deviant Logic* [Haack 1973] Susan Haack distinguished between logics which are deviant from classical logic and logics which are extensions of classical logic. Both distinctions are defined in syntactical (or proof-theoretical) terms. A logic L is *deviant* from another logic L^* constructed over the same language whenever the set of theorems and valid inferences of L is different from the set of theorems and valid inferences of L^* . As L and L^* have the same language, the difference between the set of theorems of L and the set of theorems of L^* should be due to differences of meaning between the logical constants of L and the logical constants of L^* . For this reason, those who think of IL as a deviant logic from CL assert that the intuitionistic connectives and quantifiers have different meaning from their classical counterparts. A logic L^* is an *extension* of another logic L if the symbols of L belong to L^* and if the theorems or valid inferences of L are theorems or valid inferences of

L^* . Logics that are deviant from classical logic are — traditionally — intuitionistic logic, quantum logic and many-valued logics; extensions of classical logic are, e.g., the modal logics. If L^* is an extension of L , then L is said to be a *subsystem* of L^* .

Haack's distinction presupposes that the difference between IL and CL lies on the logical constants. This point of view has been recently contested by [Došen 1993] who argued that in the formulation of both logics by means of sequent systems, the difference between them lies not in the rules for logical constants, but in the structural rules. Thus, it seems as if the logical constants played a secondary role; the difference between both logics would depend on their structural parts. Such a view, although well-founded, will not be adopted in this paper. The philosophical discussions on the relations between IL and CL until the middle of the seventies presupposed the view that it was in the logical constants where the difference between both logics should be sought; see v.g. [Dummett 1977].

Haack's distinction is suitable for v.g. modal logics considered in relation to classical logic (as extensions of it), but it makes problem in order to classify other logics (this is obvious in the case of defeasible logics, for example, but it is valid also for well known deductive logics, such as relevant logic). Moreover, Haack's very definition of logic resting on the notion of theorematicity is problematic. In fact, we dispose currently of other theoretical frameworks to analyze a logic. The idea of substructural rules would be a good example. Notwithstanding, Haack's proposal remains valid in order to investigate the historical and philosophical significance of some of the translations of CL into IL (such as Łukasiewicz's), because it was on the basis of the notion of theorematicity that those translations were conceived.

Using Haack's distinction, some logicians considered IL as a deviant logic and consequently incomparable with CL in terms of the relation "to be a part of a totality". But other logicians thought that IL is a part, a subsystem, of CL. However, in this paper we shall analyze a third, and altogether different point of view, namely, the consideration of IL as an extension of CL.

This — at first sight surprising — point of view, has been sustained explicitly by Jan Łukasiewicz (1878-1956), on the basis of the existence of a mapping of the classical propositional calculus (CPC) into the intuitionistic propositional calculus (IPC) preserving theoremhood [Łukasiewicz 1952]. Łukasiewicz was not the first logician to propose a mapping of CL into IL. In 1929 Valerij Ivanovic Glivenko (1897-1940)

had shown that (i) if a sentence p is provable in CPC, then its double negation is provable in IPC, and (ii) if a negation is provable in CPC, then it is also provable in IPC; see [Glivenko 1929]. Kurt Gödel (1906-1978) extended this result and constructed a mapping of the classical formal arithmetic PA (Peano's arithmetic) into the intuitionistic formal arithmetic HA (Heyting's arithmetic, [Gödel 1933a]). By the time Gödel had constructed this embedding of the formulas of PA into the formulas of HA, Gerhard Gentzen (1909-1945) had proposed another mapping of PA into HA; see [Gentzen 1933]. All these mappings preserve theoremhood, i.e, a theorem of PA is translated into a theorem of HA.

In this paper we shall discuss the results achieved by Gödel, Gentzen and Łukasiewicz in their historical context. Firstly, because the consideration of IL as an extension of CL is based on the existence of those mappings. Secondly, because those translations were used on several occasions to solve the problem of determining what are the logical symbols that properly express the idiosyncrasy of IL. For example, in the philosophical investigation on IL there arose questions such as whether its difference with CL is due to a different use of disjunction, conditional or negation.

Another source of the consideration of IL as an extension of CL can be found in the so-called modal interpretation of IL. It is well known that we can map IL into the modal system S4. As the modal calculus S4 is considered as an extension of CL, we can draw the conclusion that IL is an extension of CL. In the last section of this paper, we shall refer briefly to this modal interpretation of IL.

As an outcome of the whole discussion, the difficulties that arise when CL is considered as a subsystem of IL will be brought out. The notion of logical translation will play an essential role in the argumentation, for the existence of a translation from one logic to another does not imply the existence of a part-whole relation between them. Perhaps this is nothing new in logic, but many philosophers and logicians do not seem to be sufficiently aware of these difficulties.

I

The standard and traditional view on the relation between CL and IL is that IL is a subsystem of CL. On such a view intuitionistic logical constants arise from limitations inherent in the use of classical logical constants. How to conceive these restrictions depends on the formalism

used to present IL. If IL and CL are represented using a Hilbert type system¹, i.e. a formalism in which the axioms are considered as implicit definitions of the logical constants, the restriction appears as the fact that the axioms which are necessary for deriving all the theorems of IL are a subset of the axioms necessary for deriving all the theorems of CL. In cases where IL and CL are represented through a natural deduction system like Gentzen's NJ and NK respectively, the restriction appears as the fact that all the rules governing an intuitionistic logical constant are admissible from a classical point of view. However there are classical rules, for example, the rule of cancellation of double negation which are not intuitionistically admissible. If IL and CL are represented through Gentzen's sequent calculi, the restriction is not more over the rules governing logical constants but over the so-called structural rules (independent from logical constants): in an intuitionistic sequent no more than one formula is allowed expressing functions from formulas in the antecedent to the succedent of the sequent. So, the difference between both logics can be reduced to a difference in the form of the sequents. Now, those who assert, IL is a subsystem of CL say IL is *weaker* than CL².

As is well-known, there is not a uniform justification of IL as a deviant logic. From Brouwer's writings we can draw a representation of IL as the inventory of the modi of reasoning which are used in intuitionistic mathematics [Brouwer 1929]. In Brouwer's conception the difference between classical mathematics and intuitionistic mathematics lies in a different view of mathematical existence. Intuitionistic mathematicians solely admit the existence of entities which can be effectively constructed. Brouwer justified the rejection of the principle of the excluded middle by introducing indefinite objects such as choice sequences. In his paper "Consciousness, Philosophy and Mathematics" [Brouwer 1949], he gave as an example of his idea an assertion P about a drift (a type of choice sequence) such that the statement P or no P does not hold.

Michael Dummett offered a justification of the deviance of IL by using arguments from the philosophy of language. These semantical arguments were so general that they did not involve any consideration of the character of the entities referred to. In contraposition to Brouwer, who denies

1. We use here the terminology of Kleene 1952. A Hilbert type formalism contains axioms and rules of inference. A Gentzen type formalism contains (usually) only rules of inference (with suppositions in some cases).

2. We say that a logic L_1 is *weaker* than a logic L_2 in the case where all the theorems of L_1 are theorems of L_2 , however there could be theorems of L_2 which are not theorems of L_1 .

any autonomy to IL relating to intuitionistic mathematics (see [Brouwer 1929]), Dummett analyzes the deviance of IL without committing himself to any view concerning the nature of Mathematics. Following Dummett, the arguments for employing IL instead of CL, stem from the conception that linguistic meaning must be explained in terms of use. In this view the meaning of an assertion ought to be explained by indicating the conditions of its assertion, not by the conditions of its truth [Dummett 1978, 215-247].

From Dummett's point of view we can draw the conclusion that intuitionistic logic is deviant from classical logic because intuitionistic logical constants have a different meaning from classical logical constants. On this view, the intuitionistic logical constants have at most a certain analogy with their homophonical logical constants ([Dummett 1978, chap. 14]. Two logical constants being "homophonical" means here that they are referred to by the same expressions). A similar view although based on other reasons can be found in Quine [Quine 1970, chap. 6].

Let us assume that IL is a deviant logic from CL, and let us also assume that the divergences between both logics are due to a difference of meaning between the logical constants of IL and their classical counterparts. The following question arise: are there logical intuitionistic constants which by differing from their homophonic classical counterparts express the idiosyncrasy of IL or it is precisely the whole system of intuitionistic constants that expresses the difference between IL and CL? The answers to these questions were controversial³. For our purposes, the main question is the following: if a mapping from CL into IL is given to us, with the property that it assigns to some classical logical constants their homophonical intuitionistic counterparts, can we say that those intuitionistic logical constants which are not the assignment of the classical homophonical counterparts, are precisely those which express the idiosyncrasy of IL? We shall discuss this question later.

3. For example, Quine [Quine 1970, 81ff.] suggests that it is the whole system of intuitionistic logical constants which expresses the properties of intuitionistic logic. On the other hand Gabbay seems to think that the difference between IL and CL lies in a different use of the conditional. Thus classical conditional satisfies Peirce's law, while intuitionistic conditional does not; see [Gabbay 1981]. However Gabbay in his presentation of IL does not consider negation as a primitive logical constant but as a constant defined by $\neg_i A =_{df} A \supset_i f$.

II

Gödel, Gentzen and later Łukasiewicz constructed mappings from CL into IL. From the existence of such mappings we could assert that IL is an extension of CL, provided we identify CL with its image through those mappings. Many of those who hold the opposite point of view, namely that CL is an extension of IL, what they really do, is to assign to each intuitionistic logical constant its classical homophone. Thus, implicitly, they assign classical conjunction to intuitionistic conjunction, classical implication to intuitionistic implication and so on.

What we are interested in is the consideration of IL as an extension of CL. We begin by discussing the mapping of CL into IL presented by Gödel in his paper “Zur intuitionistischen Arithmetik und Zahlentheorie”; see [Gödel 1933a]. The aim of Gödel’s paper was to get a proof of the consistency of PA relative to HA. Gödel’s result shows that PA is as secure as HA. For, if it were possible to derive a contradiction in PA, then its image through the mapping — also a contradiction — would be derivable in HA. To refer to this mapping we use the symbol θ . It has the property that if A is a theorem of CL then $\theta(A)$ is a theorem of IL, i.e θ preserves theoremhood. The mapping θ assigns “homophonically” the intuitionistic negation, conjunction, and universal quantifier to the classical negation, conjunction and universal quantifier respectively. We shall distinguish between classical and intuitionistic logical constants by means of the subscripts c and i respectively. Thus obtains:

$$\begin{aligned} \theta(p) &= p \\ \theta(A \&_c B) &= \theta(A) \&_i \theta(B) \\ \theta(\neg_c A) &= \neg_i \theta(A) \\ \theta(\forall_c x A(x)) &= \forall_i x \theta(A(x)) \end{aligned}$$

where p is an atomic formula of the classical language C and A and B are formulae of any type of the classical language C. As any classical logical symbol can be defined in terms of the set $\{\neg_c, \&_c, \forall_c\}$, θ is a mapping of the totality of CL into IL. Thus the images of the other logical constants are:

$$\begin{aligned} \theta(A \vee_c B) &= \neg_i(\neg_i \theta(A) \&_i \neg_i \theta(B)) \\ \theta(A \supset_i B) &= \neg_i(\theta(A) \&_i \neg_i \theta(B)) \\ \theta(\exists_c x A(x)) &= \neg_i(\forall_i x \neg_i \theta(A(x))). \end{aligned}$$

Gödel considered these assignments of classical logical constants to intuitionistic ones as a “translation” in a syntactical sense of the word.

(The notion of logical translation will be discussed later in section V.)

What Gödel did, was to extend the results attained by Glivenko for propositional logic. Glivenko had proved that given a provable formula A in CL whose logical symbols are only $\&_c$ and \neg_c , the formula A^* obtained by substitution of $\&_i$ and \neg_i for both constants respectively is provable in IL. From this result it follows immediately that if A is a formula derivable in CPC then $\theta(A)$ is provable in IPC. Gödel extended Glivenko's result by proving that if a formula A is derivable in classical formal arithmetic (i.e. PA), then $\theta(A)$ is provable in intuitionistic formal arithmetic (Heyting's arithmetic HA). Thus θ can be considered as an embedding of HA into PA.

Can we say that the existence of θ proves IL to be an extension of CL? As was said above, Gödel's aim was to construct a proof of the consistency of PA relative to HA. But in his paper he also argued CL to be a subsystem (*ein Teilsystem*) of IL. However, from the existence of Gödel's mapping of CL into IL we cannot infer that IL is an extension of CL. For it is obvious that from $\Gamma \vdash_{LC} A$ we can not infer $\theta(\Gamma) \vdash_{LI} \theta(A)$, as is shown by the fact that $\neg_c \neg_c p \vdash_{LC} p$ but not $\neg_i \neg_i p \vdash_{LI} p$. This follows from the fact that the image through θ of Modus Ponens, namely

$$\frac{\neg(\theta(A) \& \neg\theta(B)), \theta(A)}{\theta(B)}$$

is not a valid rule of inference in IL. Thus, the mapping does not preserve Modus Ponens.

Now, in relation with our main question, we see that the translations of classical conjunction, negation and universal quantifier are their intuitionistic homophonical counterparts. Classical disjunction, conditional and existential quantifier are translated into expressions where only intuitionistic conjunction, negation and universal quantifier occur. From this, what Arthur Prior, in his book *Formal Logic*, called *unsophisticated logical feeling*, leads to the conclusion that the logical constants which express the idiosyncrasy of IL are disjunction, conditional and existential quantifier; see [Prior 1962, 254]. According to him, we can support this feeling with the following facts:

- (i) The intuitionistic conditional, in contraposition to the classical one, does not satisfy Peirce's law, so that

$$((p \supset_i q) \supset_i p) \supset_i p$$

is not a theorem of IL.

- (ii) The difference between intuitionistic and classical disjunction is shown by the fact that we can infer intuitionistically $A \vee_i B$ from Γ only if we had previously inferred A from Γ or had inferred B from Γ .
- (iii) Finally, we can intuitionistically assert $\exists_i x A(x)$ only if we could first assert $A(a)$, in contraposition to the classical existential quantifier which can be asserted from $\neg_c \forall_c \neg_c A(x)$.

However, the intuitionistic negation, which is the translation through θ of classical negation, does not satisfy the law of cancellation of double negation. This fact is in conflict with the assertion that what is characteristic of IL can be found only in disjunction, conditional and existential quantification. In fact, only classical conjunction and universal quantification have exactly the same behaviour as their homophonical intuitionistic counterparts. Consequently, θ should translate homophonically only $\&_c$ and \forall_c . In this case, the image through θ of LC should only contain $\&_i$ and \forall_i , if we want to assert that CL is a part of IL. But, given the impossibility of expressing all classical logical constants in terms only of $\{ \&_c, \forall_c \}$, the construction of such a translation function θ remains impossible.

We wonder if it is possible to extend the set of logical symbols translated homophonically by Gödel's mapping θ , while keeping for the others symbols the translation given by θ . For example, can we include in this set the classical conditional? No, for in this case the image of Peirce's law would not be a theorem of IL. Now, can we include in this set the classical existential quantification? No, for in this case the image through θ of the classical logical truth

$$\neg_c \forall_c x A[x] \supset_c \exists_c x \neg_c A[x]$$

would be

$$\neg_i (\neg_i \forall_i x \theta(A[x]) \&_i \neg_i \exists_i x \neg_i \theta(A[x]))$$

which is not a theorem of IL because the conjunction $\neg_i \forall_i x P[x] \&_i \neg_i \exists_i x \neg_i P[x]$ is intuitionistically consistent. For a sentence of the form $\neg_i A$ means that the assumption that A is true leads to a contradiction. Then, from an intuitionistic point of view, we could be in a situation where supposing that all the individuals of the domain of discourse satisfy the property P we are led to a contradiction, while at the same

time, the assertion that there exists a determined individuuum that does not satisfy the property P would lead us to a contradiction⁴.

The preceding analysis shows that the set of logical symbols $\{ \&_c, \neg_c, \forall_c \}$ which are translated homophonically by the embedding θ of Gödel is a maximal set, if we wanted to keep the property that the image through θ of a theorem of CL be a theorem of IL.

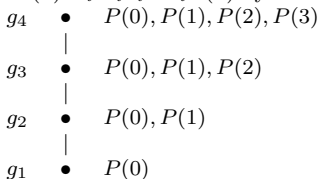
III

In 1952 Łukasiewicz published his paper “On the intuitionistic theory of deduction” (reprinted in [Łukasiewicz 1970, 325-340]). The aim of this paper was to prove that the intuitionistic propositional calculus IPC (called by Łukasiewicz the intuitionistic theory of deduction) contains as a proper part the classical propositional calculus CPC (classical theory of deduction in Łukasiewicz’s terminology). As a conclusion of the arguments expounded in this paper appears the amazing result that “.the principle of excluded middle can be proved in the intuitionistic theory of deduction, because the whole classical theory of deduction is contained in it” [Łukasiewicz 1952, 332].

Although he never explains his results in terms like “mapping” or “translation”, what Łukasiewicz really did in this paper was to construct a mapping τ from CPC to IPC with the property that if A is a theorem (what Łukasiewicz called a thesis) of CPC then $\tau(A)$ is a theorem of IPC. He thought that the mapping τ tacitly defined by himself preserved Modus Ponens. He believed also to have demonstrated that the intuitionistic connectives are stronger than the classical ones.

But the conclusions obtained by Łukasiewicz rest on two mistakes. Firstly, Łukasiewicz mixed up two types of logical rules. Let L be a logic. A rule R of L is said to be a *valid rule* whenever it allows us to

4. Readers can convince themselves of the intuitionistic acceptability of the scheme $\neg_i \forall_i x P(x) \&_i \neg_i \exists_i x \neg_i P(x)$ by the following Kripke tree \mathbf{G} :



The formula $\neg_i \forall_i x P(x) \&_i \neg_i \exists_i x \neg_i P(x)$ is satisfied in the Kripke tree \mathbf{G} . If $\neg_i (\neg_i \forall_i x P(x) \&_i \neg_i \exists_i x \neg_i P(x))$ were a theorem of IL, then by soundness it should be also satisfied in \mathbf{G} , which is absurd.

draw a formula B from premisses A_1, A_2, \dots, A_n . A rule R' is said to be a *proof rule* whenever it allows us to infer a theorem B of L from theorems A_1, A_2, \dots, A_n of L ⁵. From a semantical point of view, a valid rule allows us to draw true sentences from true sentences, while a proof rule satisfies only this weaker condition: If the premisses of the rule are logic true, then the conclusion will be a logical truth. Two examples can help to understand the difference between these two concepts: In many modal logical calculi M the rule of necessitation is a proof rule but not a valid rule. If a formula A is a theorem in M , then the theorem $\mathbf{N}A$ is derivable in M (\mathbf{N} is the logical operator of necessity), but does not obtain if A is merely a supposition — that is, it is not a theorem (a logical truth). In the intuitionistic propositional calculus IPC from $\vdash_{IPC} \neg_i p \supset_i (q \vee_i r)$ we can infer $\vdash_{IPC} (\neg_i p \supset_i q) \vee_i (\neg_i p \supset_i r)$, but it does not obtain when $\neg_i p \supset_i (q \vee_i r)$ is only a supposition⁶. This last example shows that the distinction between proof rules and valid rules makes sense when studying intuitionistic logic. As we shall see later Modus Ponens is preserved by the mapping τ tacitly defined in Łukasiewicz's paper, provided this rule is understood as a proof rule but not as a valid rule.

Secondly, Łukasiewicz identified the expressions which are of the form $\tau(p \supset_c q)$ with $p \supset_i q$ (in Łukasiewicz's terminology $NTpNq$ with Cpq)⁷. For those expressions we use $p \supset^* q$ to emphasize that Łukasiewicz had in fact introduced a new intuitionistic connective. That τ preserves Modus Ponens as a proof rule but not as a valid rule means that if $p \supset^* q$ and p are theorems of IPC then q is a theorem of IPC but from the supposition of $p \supset^* q$ and p we cannot infer q .

5. The same distinction between valid rules and proof rules (under the names 'rule of inference' and 'rule of proof') was made by Göran Sundholm in [Sundholm 1983] and is probably due to Dana Scott.

6. This example is linked with the so-called disjunction property of the intuitionistic propositional calculus.

7. In this paper we use another notation than that used by Łukasiewicz. In Łukasiewicz's prefixed notation Cpq denotes the classical implication, Fpq the intuitionistic implication, Kpq the classical conjunction, Tpq the intuitionistic conjunction, Apq the classical disjunction, Opq the intuitionistic disjunction. Np denotes both the classical and the intuitionistic negation. But he used Cpq also to refer to $NTpNq$ (where we shall use sometimes $\tau(p \supset_c q)$, sometimes $\neg_i(p \&_i \neg_i q)$ and sometimes $p \supset^* q$). In an analogous ambiguous way Łukasiewicz used Kpq to refer also to $NCpNq$ ($\tau(p \&_c q) = \neg_i \neg_i(p \&_i \neg_i \neg_i q) = p \&^* q$ in our terminology) and Apq to refer also to $CNpq$ (in our terminology $\tau(p \vee_c q) = \neg_i(\neg_i p \&_i \neg_i q) = p \vee^* q$). Łukasiewicz's notation is misleading in that it does not allow us to distinguish classical connectives from the new intuitionistic connectives which Łukasiewicz really had introduced.

Let us introduce τ .

$$\begin{aligned}\tau(p) &= p \\ \tau(A \&_c B) &= \neg_i \neg_i (\tau(A) \&_i \neg_i \neg_i \tau(B)) \\ \tau(A \vee_c B) &= \neg_i (\neg_i \tau(A) \&_i \neg_i \tau(B)) \\ \tau(A \supset_c B) &= \neg_i (\tau(A) \&_i \neg_i \tau(B)) \\ \tau(\neg_c A) &= \neg_i \tau(A)\end{aligned}$$

We can reconstruct the procedure followed by Łukasiewicz in order to prove that IPC contains CPC in the following way: Firstly, Łukasiewicz proves that the image of an axiom of CPC is a theorem of IPC. The axiomatization of CPC offered by Łukasiewicz is listed below.

$$\begin{aligned}(1) \quad & (\neg_c p \supset_c p) \supset_c p; \\ (2) \quad & p \supset_c (\neg_c p \supset_c q); \\ (3) \quad & (p \supset_c q) \supset_c ((q \supset_c r) \supset_c (p \supset_c r)).\end{aligned}$$

See his paper of 1951 entitled “On Variable Functors of Propositional Arguments”, reprinted in [Łukasiewicz 1970, 311-324]. Łukasiewicz proves that the image through τ of any of these axioms is a theorem of IPC. That is, he proves that the formulae

$$\begin{aligned}(1') \quad & \neg_i (\neg_i (\neg_i p \&_i \neg_i p) \&_i \neg_i p), \\ (2') \quad & \neg_i (p \&_i \neg_i \neg_i (\neg_i p \&_i \neg_i q)), \\ (3') \quad & \neg_i (\neg_i p \&_i (p \&_i \neg_i q) \&_i \neg_i \neg_i (\neg_i (q \&_i \neg_i r) \&_i \neg_i \neg_i (p \&_i \neg_i r))),\end{aligned}$$

are theorems of IPC. In our notation, where $p \supset^* q$ is defined as $\neg_i (p \&_i \neg_i q)$, these formulae are written as follows:

$$\begin{aligned}* \quad & (\neg_i p \supset^* p) \supset^* p; \\ ** \quad & p \supset^* (\neg_i p \supset^* q); \\ *** \quad & (p \supset^* q) \supset^* ((q \supset^* r) \supset^* (p \supset^* r)).\end{aligned}$$

Our notation allows us to see clearly what Łukasiewicz, due to his notation, did not see, namely, that through $\tau(p \supset_c q) = \neg_i (p \&_i \neg_i q)$ a new intuitionistic connective is defined⁸. The new intuitionistic connective \supset^* satisfies the principles *, **, and *** which are known as the

8. Once more we should say that Łukasiewicz neither used the concept of a translation function τ , nor said he was introducing new intuitionistic connectives. However, the reconstruction of Łukasiewicz’s exposition using the concept of translation allows us to see a lot of distinctions masked by his ambiguous notations. We can define in IPC a lot of connectives which are not definable in terms of the connectives belonging to the set $\{\neg_i, \&_i, \vee_i, \supset_i\}$. This is due to the fact that the intuitionistic connectives are not truth-valued. Readers interested in this problem, see [Gabbay 1981].

principle of Clavius, the principle of Duns Scotus and the principle of syllogism respectively.

We must not identify, erroneously, the new intuitionistic connective with either the intuitionistic conditional or with the classical conditional. The former satisfies the principles ** and *** but not the * listed above. The difference between the intuitionistic conditional and \supset^* lies in the fact that the former satisfies Modus Ponens understood as a valid rule and the latter does not. This is owed to the fact, already noticed in last section, that

$$\frac{\neg_i(A \&_i \neg_i B), A}{B}$$

is not a valid scheme of inference of IPC. In fact, Łukasiewicz adds nothing essentially new to Gödel's translation. In analogy to \supset^* , we can introduce new intuitionistic constants $\&^*$ and \vee^* defined by $A \&^* B =_{df} \neg_i \neg_i A \&_i \neg_i B$ and $A \vee^* B =_{df} \neg_i(\neg_i A \&_i \neg_i B)$.

However, Łukasiewicz succeeded in proving that the image of a theorem of CPC through the mapping τ is a theorem of IPC. Indeed, for Łukasiewicz's purposes it is sufficient to prove that \supset^* satisfies a weak form of Modus Ponens, i.e, when this rule is understood as a proof rule. In other words, it is sufficient to show that if $p \supset^* q$ and p both are theorems of IPC then q is a theorem of IPC. Generally, Łukasiewicz shows that the rule

$$\frac{A \supset^* B, A}{B}$$

is a proof rule of IPC whenever A and B do not contain other connectives than \neg_i and \supset^* . The proof proceeds by cases: firstly considering that B is a propositional variable, secondly considering that B is of the form $\neg_i C$ and thirdly considering that B is of the form $C \supset^* D$. To prove the first case we need the axiom of IPC $p \supset_i (\neg_i p \supset_i q)$ ⁹. If we drop this axiom from the IPC we obtain the minimal calculus of Johansson. (Thus, τ does not define a theorem preserving embedding of CPC in the minimal logic.)

9. Readers interested in the details of this proof of Łukasiewicz, see [Prior 1962, 256].

Łukasiewicz succeeded in demonstrating that given a theorem A of CPC, $\tau(A)$ is a theorem of IPC. But what grounds could Łukasiewicz have for saying that IPC contains the proper CPC? Let A be a formula of CPC. Without loss of generality we can suppose that A contains only the logical symbols \neg_c and $\&_c$, due to the fact that the other propositional connectives are defined in terms of them. Clearly $\tau(A)$ contains only its intuitionistic homophone. If we identify both pairs of symbols, we can consider that $\tau(A)$ is equivalent with A due to the fact that both formulae have the same truth-values. Probably it was in virtue of this equivalence that Łukasiewicz asserted that IPC contains A . But this equivalence holds only provided we identify classical negation and conjunction with their intuitionistic counterparts. But this identification is highly questionable. It presupposes that the classical and the intuitionistic connectives have the same meaning. Moreover, it was well known to Łukasiewicz that to IPC corresponds a matrix with infinite values.

Now, even though we could think that in a sense IPC contains all the theorems of CPC, we cannot say that IPC contains the CPC as a whole. For a logic is not defined only in terms of the theorems which it contains, but also through its schemes of inference. When we translate the expressions $p \supset_c q$, $\neg_c p$, $p \&_c q$ and $p \vee_c q$ into $p \supset^* q$, $\neg_i p$, $p \&^* q$ and $p \vee^* q$ respectively we are in fact defining a new, *pseudo-classical* logic CPC*, which is not equivalent to CPC, although we can say that in a sense any theorem of CPC is a theorem of CPC*. We cannot map any valid derivation Δ in CPC into a valid derivation Δ^* in CPC*. For, as we have said, τ is not Modus Ponens preserving.

The new intuitionistic connectives belonging to CPC* have certain interesting properties in relation to the intuitionistic ones. In fact, the formulae $(p \supset_i q) \supset_i (p \supset^* q)$, $(p \&_i q) \supset_i (p \&^* q)$, $(p \vee_i q) \supset_i (p \vee^* q)$ are theorems of IPC. However, their converse formulae can be disproved. This can be seen through a semantic-type argument¹⁰. From these facts

10. Łukasiewicz considered the following matrixes [Łukasiewicz 1952, 330ff.]

\supset_i	1 2 3	\neg_i	$\&_i$	1 2 3	\vee_i	1 2 3
1*	1 2 3	3	1	1 2 3	1	1 1 1
2	1 1 3	3	2	2 2 3	2	1 2 2
3	1 1 1	1	3	3 3 3	3	1 2 3

These matrixes were formulated by Heyting [Heyting 1932]. Let Ax be an axiom of IPC. For any assignation of one of the three values $\{1, 2, 3\}$ to the propositional variables of Ax , the result of doing the computations indicated by the matrixes above is 1. Besides, if the premises of Modus Ponens receive the value 1 for some assignation θ of truth value to its propositional variables, then the conclusion receives for θ the value 1.

Łukasiewicz infers that the intuitionistic connectives are stronger than the pseudo-classical ones. But he did not realize that the second group of connectives is also intuitionistic. He identified them with the classical connectives. On account of this erroneous identification, he concluded that the intuitionistic propositional logic is stronger than the classical propositional logic.

All the theses in F, T or O [$\supset_i, \&_i, \vee_i$ in our notation] remain true if we replace these stronger functors by the corresponding weaker ones. On the contrary, it is not always the case that a thesis in C, K, or A [$\supset_c, \&_c, \vee_c$ in our notation] remains true, if we replace these weaker functors by the corresponding stronger ones. [Łukasiewicz 1952, 331]

Thus the “strong” principle of Clavius ($\neg_i p \supset_i p$) $\supset_i p$, the “strong” principle of double negation $\neg_i \neg_i p \supset_i p$ and the “strong” principle of excluded middle $p \vee_i \neg_i p$ are not theorems of IPC.

The interpretation of IPC as an extension of CPC given by Łukasiewicz, never gained much popularity among the logicians. Łukasiewicz’s views on intuitionism appeared so strange because they were in conflict with the widely-held conception of intuitionism as a weaker system than classical logic. Nearly all contemporary research on intuitionistic logic disregards Łukasiewicz’s interpretation. For example in the well-known works of Dummett [Dummett 1978], Van Dalen [Van Dalen 1973] and Troelstra [Troelstra 1969] no reference is made to Łukasiewicz’s interpretation. An exception can be found in [Wójciki 1988].

IV

Independently but simultaneously with Gödel, Gerhard Gentzen, at that time a young mathematician who was writing his doctoral dissertation,

Now let us consider the following matrixes below

\supset^*	1 2 3	$\&^*$	1 2 3	\vee^*	1 2 3
1	1 1 3	1	1 1 3	1	1 1 1
2	1 1 3	2	1 1 3	2	1 1 1
3	1 1 1	3	3 3 3	3	1 1 3

These matrixes can be constructed using the former ones by considering the definitons of $\supset^*, \&^*, \vee^*$, in terms of the intuitionistic connectives $\supset_i, \&_i, \vee_i$. By setting $p = 1, q = 2, (p \supset^* q) \supset_i (p \supset_i q)$ receives the value 2, and consequently it is not derivable from the axioms of IPC. By setting $p = 2, q = 2, (p \&^* q) \supset_i (p \&_i q)$ receives the value 2 and $(p \vee^* q) \supset_i (p \vee_i q)$ receives also the value 2. Consequently both formulae are not derivable from the axioms of IPC.

also proved the consistency of classical arithmetic relative to intuitionistic arithmetic. (However, the paper containing the proof was never published during Gentzen's lifetime because Gentzen withdrew the galley proofs when he heard of Gödel's equivalent result. It was first published in an English translation in Szabo's edition of Gentzen's work.)

The proof, quite different from Gödel's, includes a more accurate translation of classical into intuitionistic logic, its methodology and ideas being taken from Hilbert's program with its focus on proof-theoretical aspects, and Gentzen's intentions are determined by his own constructive approach because of the possibility of reducing classical arithmetic to intuitionistic. This will be more evident in his paper on the consistency of number theory [Gentzen 1936], where this embedding is discussed again, this time in connection with the transfinite meanings of disjunction and existential quantifier. (However, the translation is slightly different, see [Gentzen 1936, 169f.]

Now, like Gödel, Gentzen adopted Herbrand's system for classical arithmetic [Herbrand 1932] and Heyting's axiomatization for intuitionistic predicate logic [Heyting 1930] which together constitute Heyting's arithmetic HA. The main result can be stated in the following way: for every derivation in classical arithmetic there is a corresponding derivation in intuitionistic arithmetic. This is achieved by the following steps. Firstly, Gentzen shows a certain restricted validity of the law of double negation in intuitionistic logic for formulae not containing disjunction or existential quantification and provided the double negation is valid for its atomic subformulae (theorem I). Moreover, if A is a formula without those logical constants and if every atomic subformula of A is prefixed with a negation, the law of double negation is intuitionistically valid (i.e. there is a valid derivation in intuitionistic logic for $\neg_i \neg_i A \supset_i A$). Thereafter, the problem becomes finding a translation of CL into IL leading from theorems of CL to theorems in IL for which the preceding holds true. (In the case of a homophonic translation, which Gentzen's is, it would be the same as reducing all classical valid formulae to equivalent formulae to which the rule of double negation could be intuitionistically applicable.)

The desired translation is given through the following function δ :

$$\begin{aligned} \delta(p) &= \neg_i \neg_i p \quad (p \text{ is atomic}); \\ \delta(A \&_c B) &= \delta(A) \&_i \delta(B); \\ \delta(A \vee_c B) &= \neg_i (\neg_i \delta(A) \&_i \neg \delta(B)); \end{aligned}$$

$$\begin{aligned} \delta(A \supset_c B) &= \delta(A) \supset_i \delta(B); \\ \delta(\neg_c A) &= \neg_i \delta(A); \\ \delta(\forall_c x A[x]) &= \forall_i x \delta(A[x]); \\ \delta(\exists_i x A[x]) &= \neg \forall_i x \neg \delta(A[x]). \end{aligned}$$

Unlike Gödel's translation, only atomic formulae, disjunction and existential quantification are not translated homophonically [Gentzen 1933 section 4]. Through this translation, Gentzen arrived at the following result: There is a derivation of a formula C in CL from the premisses A_1, \dots, A_n if, and only if, there is a derivation in IL of $\delta(C)$ from $\delta(A_1), \dots, \delta(A_n)$. This fact could then be extended to arithmetic. (A slightly different reconstruction of the whole proceeding can be found in [Prawitz & Malmnäs 1968].

It must be stressed that this translation provides a way of transforming classical derivations into intuitionistic ones (although the resulting intuitionistic proofs are obviously not the same as the original ones). So, a correlation between classical and intuitionistic derivability can be established. This was clearly acknowledged by Prawitz and Malmnäs in their purely syntactical distinction between "interpretability" and "interpretability with respect to derivability" [Prawitz & Malmnäs 1968]¹¹. The first concept applies to Gödel's and Łukasiewicz's translations and the second to Gentzen's. For example, rules like Modus Ponens are preserved through the translation. Unlike Gödel and Łukasiewicz, Gentzen obtains an inference-validity-preserving translation (and not only a "thesis-preserving" one). The entire "logical machinery" of classical logic finds its image in intuitionistic logic.

Now, it should be stressed that it is only an image. For, strictly speaking, the former cannot be said to be a subsystem of the latter (as also Gödel asserted in 1932-1933). When Gentzen speaks of a (restricted) validity of the law of double negation in intuitionistic logic, he is referring to particular cases in which it is possible to construct this image of classical logic. This restricted validity of the principle of excluded middle is just integrated in the form of the translation through the fact that classical disjunction and existential quantifier are translated by intuitionistic conjunction, universal quantifier and negation. To sum up, Gentzen's translation offers two advantages with respect to Łukasiewicz's: (i) Classical formulae are not confused with their intuitionistic images, and (ii) the translation is derivability-preserving.

11. A similar problem was discussed afterwards in terms of the notion of definability by Riszard Wójciki; see [Wójciki 1988].

V

As was said at the beginning of this paper, our remarks contain little that is new in modern logic, but they are about aspects that seem to have been overlooked by logicians and philosophers. Now, the notion of logical translation, which has been naively used in the preceding sections ought to be more precisely defined and it should be clearly explained what is aimed at with a certain translation.

From the preceding sections, two different notions of logical translation seem to arise. They are conceived in a syntactical way. Provided two logics L_1 and L_2 are formulated respectively in two different languages, a translation of L_1 into L_2 consists in a function from formulas in L_1 into formulas in L_2 , so that, according to the first notion, to every theorem in L_1 the function assigns a theorem in L_2 . Alternatively, according to the second notion, it consists in a function satisfying the following condition: If a formula C is derivable in L_1 from formulas A_1, \dots, A_n , then the function assigns formulas C', A'_1, \dots, A'_2 in L_2 , so that C' is derivable in L_2 from A'_1, \dots, A'_2 .

According to both notions everything that can be asserted in L_1 can be also asserted in L_2 , and that is the basic goal of every translation. Now, the first notion preserves theoremhood; the second preserves derivability throughout the translation and is, therefore, stronger than the first. Gödel's and Łukasiewicz's translations are examples of the first notion; Gentzen's translation illustrates the second. Now, we have shown that neither of them yield a subsystem of IL. For the logic resulting from the translation (that is, the image of CL in IL) does not satisfy the condition for extensions and subsystems formulated in section I. For example, in CPC*, Modus Ponens lacks the properties it has in IPC. Consequently, the existence of a translation from a logic L_1 to a logic L_2 , in the first sense, does not warrant the existence of a subsystem of L_2 .

Furthermore, the accuracy of the first notion depends on what is meant by a logic. A tradition stemming from Frege and the *Principia Mathematica* considers that a logic is defined by the its set of its theorems or logical truths. (See also [Quine 1970, 80] for a similar opinion). Against this characterization, we can also consider relevant for a logic the structure of the derivations set forth within this logic. If the translation doesn't preserve derivability, then some properties of the derivability relation underlying the logic to be translated may be not represented. For example, let us consider CPC*. In it the translation of the classical rule

of Modus Ponens came to be, as was shown in the last section, a proof rule and not a valid rule. If the derivability relation is what counts in characterising a logic, as is often argued, then the second seems to be a more adequate notion of translation¹². In [Gödel 1933a] and, above all, in [Łukasiewicz 1952] the limitations of the first notion are not even mentioned. This is probably due to the axiomatic approach to logic they both had. For this approach diverted attention from the proper object of logical research, namely the process of derivation itself, by focusing it on the derivation of true sentences from true sentences. So, logic came to be a theory rather than the deductive device underlying non-logical theories.

Now, at this point a problem arises concerning the very notion of translation in general. An accurate translation should be expected to preserve meaning. It does not seem to be enough to preserve theoremhood or derivability. So, a translation should presuppose the construction of an adequate semantics for both systems CL and IL, where the meaning-preservation can be checked. In general, this semantic condition has been overlooked (e. g. the above mentioned notion of interpretability in [Prawitz & Malmnäs 1968]).

In order to appreciate these difficulties, let's consider now the case of the well-known *modal* translation of intuitionistic logic, mentioned in the introductory section. This translation, also due to Kurt Gödel, is based on transforming formulas of the intuitionistic language into formulas of the modal language; see [Gödel 1933b]. This can be achieved in several ways by constructing different mappings. One of these is the following: for p atomic and A and B arbitrary formulas of the intuitionistic language:

- (i) $\mu(p) = Np$;
- (ii) $\mu(A \&_i B) = \mu(A) \&_c \mu(B)$;
- (iii) $\mu(A \vee_i B) = \mu(A) \vee_c \mu(B)$;
- (iv) $\mu(A \supset_i B) = N(\mu(A) \supset_c \mu(B))$;
- (v) $\mu(\neg_i A) = N(\neg_c \mu(A))$;
- (vii) $\mu(\forall_i x A[x]) = N(\forall_c x \mu(A[x]))$;
- (vi) $\mu(\exists_i x A[x]) = \exists_c x \mu(A[x])$.

Accordingly, it could be shown that if a formula C is derivable in IL from A_1, \dots, A_n , then $\mu(C)$ is derivable in the quantifier extension of modal

12. In terms of Wójciki they are not definitional translations; see [Wójciki 1988, 69].

calculus **S4** from $\mu(A_1), \dots, \mu(A_n)$ (see [Prawitz & Malmnäs 1968], but also [Schütte 1968, 37], so that this translation preserves derivability, being an example of the second notion of translation.

Moreover, it is also, at least from a classical point of view, meaning preserving, for it induces an adequate possible-worlds semantics for IL, first developed in [Kripke 1965]. The truth conditions that make a sentence A of the intuitionistic language true, make its image in the modal language $\mu(A)$ also true (and conversely). This modal interpretation provided an analysis of the meaning of intuitionistic logical constants from a classical point of view, if the necessity operator is interpreted as meaning “is provable” (like Gödel suggested in [Gödel 1933b]; see also [Legris 1990, chap. 8] for further details). So a classical logician would say, on the basis of this translation, that intuitionistic logic makes no distinction between asserting a formula and asserting the provability of the formula.

Something like this should also be said with respect to Gentzen’s translation of CL into IL. Given an adequate semantics for IL, this translation could be meaning preserving. For example, according to the intuitive meaning conditions for intuitionistic logical constants in terms of constructions (settled down by Arendt Heyting 1956), a sentence $\neg_i \neg_i p$ of the intuitionistic language is true, if there is a proof of an absurdity from the supposition that p is false. Then, according to Gentzen’s translation, a truth condition for classical p can be formulated: a sentence p is true, if an absurdity follows from the supposition that p is false — and this is exactly the condition for *classical* absurdity. (Once more, the meaning of negation seems to be an important point in understanding the differences between CL and IL, not expressed in other translations.)

Thus, Gentzen’s translation would provide an interpretation of classical logic from the intuitionistic point of view. For the classical logician makes no distinction between asserting that a formula A does imply a contradiction and asserting that the negation of that formula $\neg A$ is provable. In other words, from the intuitionistic point of view CL arises from IL when the rule of double negation (or an equivalent principle), which precisely eliminates this distinction, is added to IL. This interpretation matches perfectly with the idea of CL as an extension of IL: if the rule of double negation is added to IL, we obtain CL. So, it cannot be held that the principle of double negation “is contained” in IL.

It must be noticed that in both translations atomic formulas are not preserved, i.e. atomic formulas in one logic are not translated by atomic formulas in the other logic. For this reason, the translated logic cannot

be seen as properly contained in the other. This fact was always clear in the modal translation, but not in Gentzen's translation. They fail to share a common non-logical basis and it can therefore be conjectured that atomic formulas — expressing non-logical facts — should be understood in different ways both in classical and in intuitionistic logic and refer (in some sense) to different objects domains. Hence, it would be impossible to mix them up in one single system, as Łukasiewicz suggested. In the case of the modal translation, it would be erroneous to regard intuitionistic logic as a modal extension of classical logic. Moreover, if one were tempted to look via this translation for a joint system of intuitionistic and classical logic, establishing links between the modal images of intuitionistic logic and the classical basis, the modal system would collapse in classical logic. Again, classical logic cannot be considered a subsystem of intuitionistic logic.

Consider, again, the case of IL as a subsystem of CL as a homophonic translation of IL into CL. For it is already known that if both logics, formulated in different languages, are put together to form a mixed new system, this new system collapses in CL, being only a notational variant of CL (see, e.g., the discussion in [Lenzen 1991]). Any joint formulation of IL and CL is therefore impossible. Notwithstanding, the two can be naturally related in two other different ways. From a classical point of view, IL becomes a proper (even if odd or extravagant) subsystem of CL. In this sense, the former is “contained” in the latter. From the intuitionistic point of view, CL results from restrictions imposed to IL (for example, when the language is restricted only to decidable predicates or something equivalent). Then, CL is regarded as a supersystem of IL. At any rate, both systems remain, under the two perspectives, formally the same, the differences lying in the justification of the meaning of the logical constants (that is, the semantics is different in each case). Finally, as was shown, they constitute the only way of considering IL and CL together in (syntactically) the same language.

As the reader can notice, our analyses are based on Haack's distinction between deviant logics and extensions of classical logic. We recognize the limitations of this distinction. However, it works well in order to understand the results achieved by Gödel, Łukasiewicz and Gentzen in their historical context. Our remarks can be summed up in the following propositions:

- (i) CL is, from the intuitionistic point of view, a special case of IL.
- (ii) IL is, from the classical point of view, a subsystem of CL.

- (iii) There is a derivability-preserving translation from CL to IL.
- (iv) There is a derivability-preserving translation from IL to the quantificational extension of S4.

In (i) and (ii) IL can be seen as the same formal system but with different interpretations of its logical constants. The translation of CL into IL presupposes adopting an *intuitionistic* point of view. So, it does not seem reasonable to call IL an extension of CL on this basis.

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