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SHANG ZHI WANG
BO YU LI

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A THEOREM ON DIRECTED QUASI-ORDERED SETS AND SOME REMARKS

WANG Shang Zhi and LI Bo Yu

Abstract : We present our result on the minimal cofinal subsets of a directed quasi-ordered set [7] , see also [3], and we add some remarks and problems.

1. Introduction.

Two basic concepts of the theory of posets are those of cofinal set and unbounded set. A subset A of a poset P is cofinal if every $x \in P$ is majorized by some $y \in A$, the subset A is unbounded if no $x \in P$ majorizes all of the elements of A. The cofinality of P, cf(P), is the smallest cardinality of the cofinal subsets of P, similarly $\mu(P)$ is the smallest cardinality of the unbounded subsets of P if P is unbounded and $\mu(P) = 1$ otherwise. These numbers have been considered previously [1], [6], and also called the upper character and the lower character (e.g. [1]). For instance it is known, and easy to see, that :

If P is up-directed (in this context $\mu(P) \neq 2$) then every subset of size at most $\mu(P)$ is bounded (in a natural way, see below) by some well ordered chain of order type $\mu(P)$. Consequently.

1. $\mu(P)$ is a regular cardinal ; and in fact $\mu(P) \leqslant cf(cf(P))$ [1] .
2. $\mu(P) = cf(P)$ iff P has a totally ordered cofinal subset.

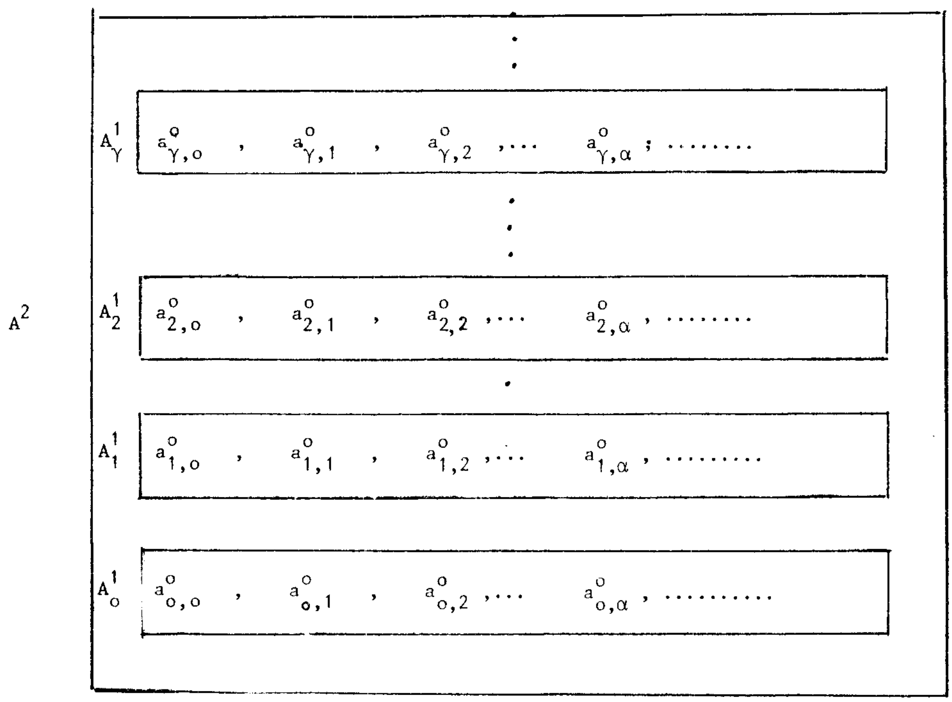
Our result is an attempt to describe the structure of the cofinal subsets of P when these numbers are different. Very roughly it says that there is a least ordinal number λ such that every cofinal subset P' of contains a cofinal subset which can be decomposed into a chain of chain iterated λ times.

2. The theorem.

For convenience we consider quasi-ordered sets instead of posets. Let P be such a quasi-ordered set ; for A,B subsets of P we write $A \leqslant B$ if for every $a \in A$ there is some $b \in B$ such that $a \leqslant b$, and we write $A < B$ if $A \leqslant B$ and $B \not\leqslant A$. A subset A of P is non extendable if it has no strict upper bound (i.e. there is no q such that $A < \{q\}$). To P we associate the quasi-ordered set $\mathcal{D}(P)$ of the non extendable well ordered subsets of P with minimal type. According to our definition of $\mu(P)$ these chains have type $\mu(P)$ whenever P is up-directed. So as follows, we can rewrite the basic facts mentionned above as follows:

LEMMA : *If* **P** *is up-directed then* $\mathcal{D}(P)$ *is up-directed and either* $\mu(\mathcal{D}(P)) = 1$, *that is* P *has a cofinal chain, or* $\mu(P) < \mu(\mathcal{D}(P)) \leqslant cf(P)$.

One can see easily that if $\mu(\mathcal{D}(P)) = cf(P)$ then $\mathcal{D}(P)$ has a cofinal chain, and in this case P has a cofinal subset consisting of a chain in $(\mathcal{D}(P))$ of disjoint unbounded chains of P, as indicated in fig. 1 :



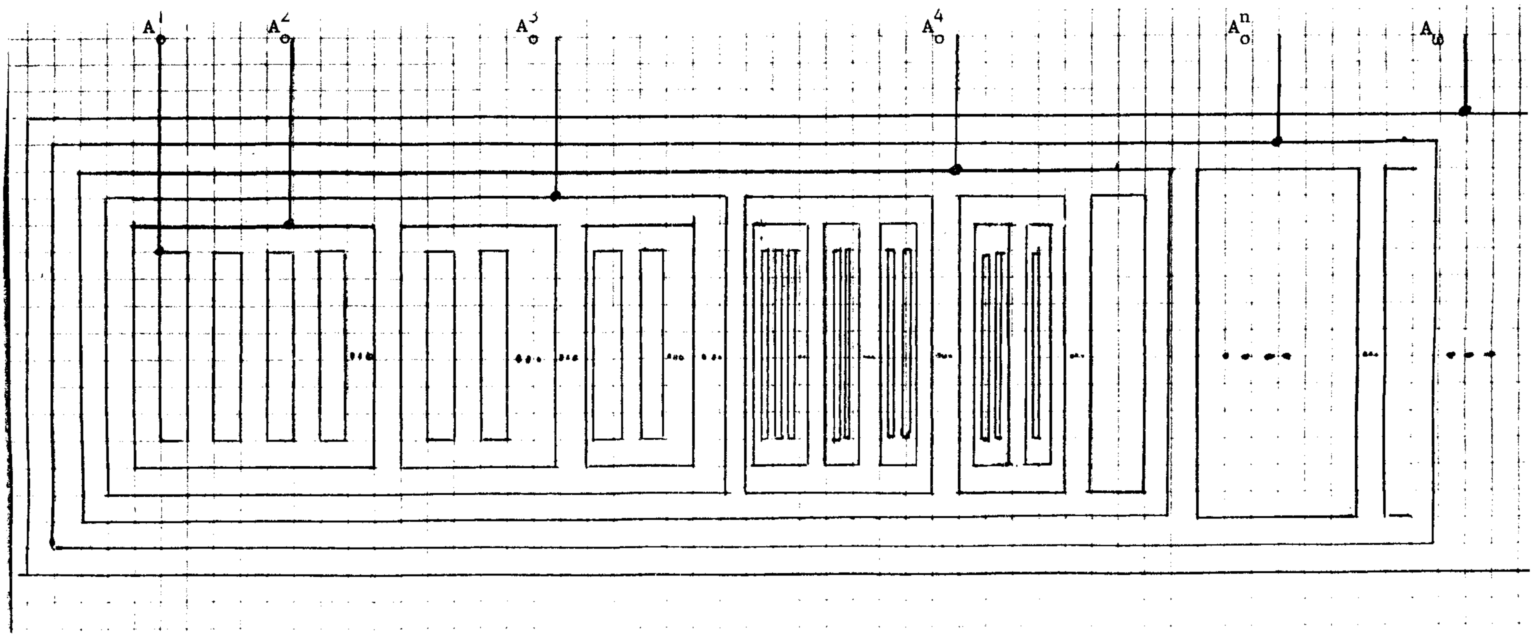
If $\mathfrak{D}(P)$ has no cofinal chain then considering $\mathfrak{D}(\mathfrak{D}(P))$ we get $\mu(P) < \mu(\mathfrak{D}(P)) < \mu(\mathfrak{D}(\mathfrak{D}(P))) \leqslant cf(P)$. If $\mu(\mathfrak{D}(\mathfrak{D}(P))) = cf(P)$ then $\mathfrak{D}(\mathfrak{D}(P))$ has a cofinal chain and P has a cofinal subset coming from a chain (in $\mathfrak{D}(\mathfrak{D}(P))$) of chains in $\mathfrak{D}(P)$ of chains of P.

Obviously we can repeat that indefinitely. In order to do it transfinitely let us define for every ordinal α a set $\mathfrak{D}^{\alpha}(P)$ whose member are chains, that we call the α^{th}-class chains, and a cardinal number μ_{α} as follows :

We set $\mu_o = 0$, $\mathfrak{D}^{o}(P) = P$, $\mu_{\alpha+1} = \mu(\mathfrak{D}^{\alpha}(P))$ and $\mathfrak{D}^{\alpha+1}(P) = \mathfrak{D}(\mathfrak{D}^{\alpha}(P))$.
For α limit let $\mu_{\alpha} = \text{Sup}\,\{\mu_{\beta} / \beta < \alpha\}$, $\mathfrak{D}^{\alpha}(P) = \{\{A^{1},\ldots,A^{\beta},\ldots,/\beta < \alpha\}$ where $A^{\beta} \in \mathfrak{D}^{\beta}(P)$ and A^{β} is the first element of $A^{\beta+1}$ }. In order to define the quasi-ordering on $\mathfrak{D}^{\alpha}(P)$ we associate to every member $A^{\alpha} \in \mathfrak{D}^{\alpha}(P)$ a subset $U_o(A^{\alpha})$ of P, that we call the 0^{th} class original chain defined inductively be $U_o(A^{\alpha}) = \bigcup_{X \in A^{\alpha}} U_o(X)$, with the convention that $U_o(x) = \{x\}$ for every $x \in P$ (roughly $U_o(A^{\alpha})$ is the collection of elements of P which occurs in the construction of A^{β}). Then for two members $A^{\alpha} = \{A^{1},\ldots A^{\beta},\ldots,\beta<\alpha\}$, $A'^{\alpha} = \{A'^{1},\ldots A'^{\beta},\ldots\beta < \alpha\}$ we set $A^{\alpha} < A'^{\alpha}$ iff $U_o(A^{\alpha}) < U_o(A'^{\alpha})$.

With this material we are ready now to show that there is an ordinal λ such that $\mu_{\lambda} = cf(P)$ and a λ^{th}-class chain A^{λ} such that the 0^{th} class original chain $U_o(A^{\lambda})$ is cofinal in P. Indeed let $(a_{\alpha})_{\alpha < cf(P)}$ be a list of the elements of a cofinal subset of P :

From the facts collected in our introduction we can choose a 1^{st} class chain A^{1} above $\{a_{\alpha} / \alpha < \mu_1\}$ (for the quasi-ordering on subsets of P). If $\mu_1 = cf(P)$ then we are done. Otherwise the same facts allows us to choose a 2^{nd} class chain $A^{2} = \{A^{1}_{o}, A^{1}_{1},\ldots,A^{1}_{\beta}\ \ldots, / \beta < \mu_2\}$ with $A^{1}_{o} = A^{1}$ and each A^{1}_{β} above $\{a_{\alpha} / \mu_1 . \beta \leqslant \alpha < \mu_1.(\beta+1)\}$ (for the quasi ordering on subsets of $\mathfrak{D}(P)$). Since $\mu_1 < \mu_2$ this condition simply means that $U_o(A^{2})$ is above $\{a_{\alpha} / \alpha < \mu_2\}$. Consequently if $\mu_2 = cf(P)$ we are done. Otherwise we can again choose a 3^{rd}-class chain $A^{3} = \{A^{2}_{o},\ldots A^{2}_{\beta}\ \ldots / \beta < \mu_3\}$ with $A^{2}_{o} = A^{2}$ and $U_o(A^{3})$ above $\{a_{\alpha} / \alpha < \mu_3\}$.
If this does not stop at a finite step then at the ω^{th} step we can choose for the ω^{th}-class chain the set $A^{\omega} = \{A^{1},A^{2},A^{3},\ldots,A^{n}\ \ldots/n < \omega\}$.
See fig. 2 :



Since $U_o(A^\omega)$ is above $\{ a_\alpha / \alpha < \mu_\omega \}$, if $\mu_\omega = cf(P)$ we are done. Otherwise we can again select an $\omega + 1^{th}$ class chain $A^{\omega+1} = A_o^\omega, A_1^\omega, \dots A_\beta^\omega, \dots \beta < \mu_{\omega+1}$ with $A_o^\omega = A^\omega$ and every $U_o(A_\beta^\omega)$ above $\{ a_\alpha / \mu_\omega \cdot \beta \leqslant \alpha < \mu_\omega \cdot (\beta+1) \}$. A straight forward diagonalization shows that $\mu_{\omega+1} > \mu_\omega$. It follows that $U_o(A^{\omega+1})$ is above $\{a_\alpha / \alpha < \mu_{\omega+1} \}$. So if $\mu_{\omega+1} = cf(P)$ we are done. Since the sequence $\mu_1, \mu_2, \mu_3, \dots, \mu_\omega, \mu_{\omega+1}, \dots$ is strictly increasing then at some step λ we reach $cf(P)$ and we get A^λ . If we do this construction by choosing each $A^{\alpha+1}$ in $P \setminus (\leftarrow U_o(A^\alpha)]$ (the notation $(\leftarrow X]$ stands for $\{y / y \leqslant x$ for some $x \in X\}$) then the resulting chain A^λ is that we call a pure λ^{th}-class chain . Also we can do it in a cofinal subset Q. In résumé we get :

THEOREM [7] : *For every directed quasi-ordered set* P *there is an ordinal* λ *such that* $\mu_\lambda = cf(P)$ *and every cofinal subset* Q *of* P *contains a cofinal subset which is the* 0^{th}*-class original set* $U_o(A^\lambda)$ *of a pure* λ^{th}*-class chain of* P .

3. Examples of application of the theorem and problems.

This result allows us to use transfinite induction in order to discuss some properties of posets. Let us illustrate this with the following well known result.

3.1. THEOREM [2] . *If every chain of a poset has a supremum then every up-directed subset has a supremum too.*

Indeed let us suppose that our up-directed subset P has, for example, a cofinal second class chain (see fig. 1).

Then every chain A^1_α has a supremum $\hat{a}_\alpha$. From $A^1_{\alpha'} \leqslant A^1_{\alpha''}$ follows $\hat{a}_{\alpha'} \leqslant \hat{a}_{\alpha''}$. Thus $\{\hat{a}_\alpha \;/\; \alpha < \mu_2\}$ is a chain. The supremum $\hat{a}$ of this chain is the supremum of P. Here is an other consequence :

Here is an other consequence :

3.2. THEOREM : *Let λ be a cardinal and P be a λ-directed poset (that is $\mu(P) \geqslant \lambda$) If P has no largest element then, under the assumption $cf(P) \leqslant \omega_\omega$, P is the union of an increasing sequence of λ^+ directed subsets P_i such that $cf(P_i) < cf(P)$.*

We don't know if the assumtion on cf(P) can be removed.

3.3. Concerning the cofinality of $\mathfrak{D}(P)$, one can observe that for a directed poset P with no largest element, $cf(P) = cf(\mathfrak{D}(P)$ implies $\mu(P) < cf(cf(P))$. The converse trivially holds under GCH (or the weaker assumption $cf(P)^{<cf\,cf(P)} = cf(P)$) From our theorem it also holds if the ordinal λ such that $\mu_\lambda = cf(P)$ satisfies $\lambda < \mu(P)$. We don't knows if this holds in full generality.

3.4. E.C.Milner and M.Pouzet considered some extensions of the cofinality concept (see [3], [4]). For example they considered the following statement, denoted by CF(P,A), "Every cofinal subset of P contains a cofinal subset isomorphic to A".

For convenience we will call such an A a <u>cofinal invariant model</u>. Only few posets have a cofinal invariant model. On an other hand, in the above theorem the sequence λ, $\mu_0, \mu_1, \ldots, \mu_\lambda$ attached to an up directed set P is the same for every cofinal subset of P. This leeds to generalizations of the above concept. We say that a property $\mathcal{P}$ of a poset P is a <u>cofinal invariant property</u> if every cofinal subset P' of P has the property $\mathcal{P}$.

<u>Problem</u> . What kind of properties are cofinal invariant ?

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LI Bo Yu and WANG Shang-Zhi
Depart. of Mathematics
NORTHWESTERN UNIVERSITY,
XIAN, Shanxi
People's Republic of China