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ON EXTERNAL PROPERTIES OF NONSTANDARD MODELS OF ARITHMETIC

by Denis RICHARD

Résumé : Nous localisons des galaxies sans entiers premiers et donnons deux nouvelles preuves de la réponse négative à la question de KEMENY [7]. Nous décrivons la structure de K idéal dans ${}^*\mathbb{Z}$ des entiers divisibles par tout entier standard. Nous prouvons qu'il y a des premiers non standards dont le corps résiduel contient la clôture abélienne de $\mathbb{Q}$. Par des arguments de théorie élémentaire des nombres, nous montrons que ${}^*\mathbb{Z}/K$ est un sous-anneau du produit des anneaux $\mathbb{Z}_p$ d'entiers p-adiques. Ce sous-anneau est l'anneau quand ${}^*\mathbb{Z}$ est sous-jacent à un modèle intégralement compréhensif, ou quand le groupe additif ${}^*\mathbb{Z}$ est algébriquement compact. Nous donnons quelques propriétés (qui seront démontrées et développées ultérieurement) concernant les sous-groupes G tels que ${}^*\mathbb{Z} = K \oplus G$, où ${}^*\mathbb{Z}$ est sous-jacent à un modèle non standard quelconque. En particulier G n'est, en général, pas isomorphe à un produit direct de sous-groupes des J_p.

To satisfy mathematicians as well as logicians, it must be answered the question «what does from outside a nonstandard model of arithmetic look like ? ». We try here to sum up, find simpler proofs of known results and develop new ones on external properties of such models. Another reason for studying nonstandard integers is the part they play in the so called «nonstandard analysis» framework of infinitesimals and infinitely large numbers. The use of «infinite telescope» or «infinitesimal microscope» of KEISLER in his «ELEMENTARY CALCULUS»

(Prindle, Weber and Schmidt, Boston Massachusetts 1976) show how much it is useful to investigate set of infinitely large integers. The work was begun -among others- by KEMENY [7] ; MENDELSON [9], Mac DOWELL and SPECKER [8], ROBINSON [13], [14], [15], [16], [17], [18] and PHILIPS [10], [11]. The key-set of abnormal numbers was introduced by KEMENY to deal with famous conjectures of elementary number theory : his «new method consists of constructing a pair of models [of the first order arithmetic] in one of which S [a sentence] is true and, in the other one, is false» [7]. In fact KEMENY's structure did not help solving concerned conjectures because this model is not a nonstandard model of arithmetic [we define nonstandard model throughout this paper following CHANG and KEISLER [20] p. 42]. We give (after many others) two new and simple proofs of KEMENY's problem. As a matter of fact and as ROBINSON and Mac DOWELL and SPECKER already noticed, K is recognized as the right tool for building an external description of nonstandard model of «Peano rings» which MENDELSON first made. Noticeably, in MENDELSON, most of the results are negative ($^*\mathbf{Z}$ is neither factorial, nor principal, nor euclidian, nor nœtherian, nor artinian...) and positive results follow from the straight behaviour of F($^*\mathbf{Z}$) the quotient-ring of $^*\mathbf{Z}$ and from his algebraic closure properties (algebraicity play an important part in nonstandard models). We add to ring properties discovered by MENDELSON, examples of galaxies without nonstandard primes and a description of the ideal **K**.

Next we try to sum up the most part of what it seems to be known on the external ring $^*\mathbf{Z}$. We prove, for any non standard model, the known result ensuring there are $^*\mathbf{Z}/_{p}{}^*\mathbf{Z}$ (where p is a nonstandard prime) containing any quadratic extension of $\mathbb{Q}$. Studying quotient rings of $^*\mathbf{Z}$, we have found nonstandard primes p such that $^*\mathbf{Z}/_{p}{}^*\mathbf{Z}$ contains the abelian closure of $\mathbb{Q}$. Then we prove theorems analogous to those introduced by ROBINSON about links between $^*D/_{\mu}$ and the P-adic completions of the Dedekind ring D, when $^*D$ is ω_1- saturated. However these theorems are proved here thru simpler ways : ω_1- saturation is not necessary and we introduce integrally comprehensive models whose definition is given below. and we use additive algebraically compact groups $^*\mathbf{Z}$. We conclude by giving some results on the subgroups G such as $^*\mathbf{Z} = K \oplus G$, showing for instance that G is generally not a direct product of subgroups of the groups of p-adic integers.

NOTATION AND DEFINITIONS :

For a given model, $\mathbb{N}$, $\mathbf{Z}$, $^*\mathbb{N}$, $^*\mathbb{N} \setminus \mathbb{N}$, $^*\mathbf{Z}$, $\mathcal{P}$, μ, μ_p $(p \in \mathcal{P})$ denote the sets of standard positive integers, of standard integers, of nonstandard positive integers, of infinite positives, of nonstandard integers, of standard primes, the intersections $\cap\, n^*\mathbf{Z}$ $(n \in \mathbb{N})$ and $\cap\, p^n\, {}^*\mathbf{Z}$ $(n \in \mathbb{N})$. By $\mathbb{Q}$, J_p, $\mathbf{Z}_p$ $(p \in \mathcal{P})$ are denoted the sets of rationals, the group of p-adic integers and their ring. A nonstandard model of a superstructure $\mathfrak{Z} = \langle U, \in, = \rangle$ (as in (22)) built from $\mathbf{Z}$ will be called $^*\mathfrak{Z}$. We do not mention «$^*\mathbf{Z}$ of $^*\mathfrak{Z}$» when results hold for the $^*\mathbf{Z}$ underlying any nonstandard model. We say that $^*\mathfrak{Z}$ is a comprehensive (resp. denumerably comprehensive) model following definitions of (22). Any ultrapowers is comprehensive and also are the ξ- saturated models for $\xi \geqslant \omega_1$. A denumerably comprehensive model is ω_1- saturated if and only if it is an enlargement (see (22)). We call integrally comprehensive model a model $^*\mathfrak{Z}$ in which every sequence of elements of $^*\mathbf{Z}$ has an extension to a (internal) map from $^*\mathbb{N}$ to $^*\mathbf{Z}$. The class of such models contains the previous (comprehensive and denumerably comprehensive, ξ- saturated for $\xi \geqslant \omega_1$).

1 – ABNORMAL and NONSTANDARD PRIME INTEGERS

An element of $^*\mathbb{Z}$ which is divisible by every standard integer will be called *abnormal*. Let K be the set of those numbers, introduced in [7] by KEMENY. Zero and, for every $\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$, ω ! are abnormal : hence $K \neq \phi$ and $K \neq \{0\}$ - Initial results about K are

Proposition 1 - 1

i/ *There is no first element in* $K \cap {}^*\mathbb{N}\setminus\{0\}$ *and* K *is an external ideal of* $^*\mathbb{Z}$;

ii/ *There is at most one abnormal integer in any galaxy so that* $K + \mathbb{Z} = K \oplus \mathbb{Z}$; [7] ;

iii/ *The ideal* K *is the kernel of* $\varphi : {}^*\mathbb{Z} \longrightarrow \prod_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, *in which* $\varphi(z) = (\ldots, z_n, \ldots)$ *and* z_n *is the residue class modulo* n *of* z [8] ;

iv/ *The ideal* K *is the monad of the topological additive group* $\mathbb{Z}$ *for the* $\mathbb{Z}$*-adic topology ;*

v/ *As a subgroup of* $^*\mathbb{Z}$, K *is divisible and maximal among divisible subgroups* [9]. *Thus there is a subgroup, say* G *of* $^*\mathbb{Z}$ *such as* $^*\mathbb{Z} = K \oplus G$ *and additive groups* $^*\mathbb{Z}/K$ *and* G *are isomorphic ;*

vi/ *As a vector space on* $\mathbb{Q}$, K *is isomorphic to* $\mathbb{Q}^{(\alpha)}$ *where* α *is the external cardinality of* $^*\mathbb{Z}$ [8] ;

vii/ *As a vector space on* $\mathbb{Q}$, K *is isomorphic to the set* $^*\mathbb{Z}/\mathbb{Z}$ *of galaxies of* $\mathbb{Z}$ [10] ;

Proof :

i/ If $k \in K$, it is clear that $k_{/2} \in K$ hence there is no first element in $K - \{0\}$ which is thus external ;

ii/ See [7] ; iii/ See [8] ; iv/ obvious ; v/ See [9] and [21] ; vi/ See [9] ; use classical properties of groups [21] ; vii/ By euclidean division ${}^*\mathbf{Z}_{/\mathbf{Z}}$ is clearly a divisible additive group. There is no galaxy of finite order, so that ${}^*\mathbf{Z}_{/\mathbf{Z}}$ is a $\mathbb{Q}$ - vector space isomorphic to the direct sum of some β copies of $\mathbb{Q}$. Obviously $\beta \leqslant \overline{\overline{{}^*\mathbf{Z}}}$ – let $\{\overline{k^x} \in {}^*\mathbf{Z}_{/\mathbf{Z}} \mid x \in {}^*\mathbf{Z}\}$ be for some $k \in K \setminus \{0\}$; this set, whose cardinal is $\overline{\overline{{}^*\mathbf{Z}}}$, is linearly independent.

Throughout this paper we will return to K, studying K as an ideal, studying ${}^*\mathbf{Z}_{/K}$ and so on. In order to complete proposition 1-1 ii/ we shall ask whether there is exactly one abnormal number in any galaxy, what is equivalent to the KEMENY's conjecture : is $K \oplus \mathbf{Z}$ a nonstandard model of the first order arithmetic ? We can immediately give a first proof of the negative answer of KEMENY's question : if the ring $K \oplus \mathbf{Z}$ is underlying a nonstandard model, the monad of the $\mathbf{Z}$ - adic topology for this model is $\bigcap_{n \in \mathbf{Z}} n(K \oplus \mathbf{Z}) = K$ but for this topology $\mathbf{Z}$ is not compact so that $K \oplus \mathbf{Z} \neq ns(\mathbf{Z})$. A second proof of the solution of the KEMENY's conjecture follows from the proposition 1-2 on nonstandard prime integers, by itself interesting :

Proposition 1 - 2

Let $\overline{a}$ *be a galaxy such that* $\overline{a} \cap K \neq \phi$ *i.e.* $K \cap \overline{a} = \{k\}$ *for some* $k \in K$;

i/ *We have* $K \cap \overline{a} \cap {}^*\mathcal{P} \subset \{k-1, k+1\}$ [7] ;

ii/ *For every* $k \in K \cap {}^*\mathbb{N}$ *such that* $k + 1 \in {}^*\mathcal{T}$, $k! + 1 \notin {}^*\mathcal{T}$;

iii/ *For every* $k \in K \cap {}^*\mathbb{N}$ *and every* $n \notin {}^*\mathbb{N} \setminus \{2^m \mid m \in {}^*\mathbb{N}\}$,

$k^n \cap {}^*\mathcal{T} = \phi$, *i.e. there are galaxies without prime integers.*

Remark :

Last assertion iii/ also follows, by transfer from the fact that, for a given $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ there are intervals in ${}^*\mathbf{Z}$ of length at least equal to ω and without any prime integer.

Proof :

i/ See [7] ii/ If $k + 1 \in {}^*\mathcal{T}$ for some $k \in K$, then by Wilson's theorem $k! + 1 = \lambda(k + 1)$ with $\lambda \in {}^*\mathbf{Z}$. iii/ For every $n \in \mathbb{N} \setminus \{0,1\}$ $k^n - 1$ is divisible by $k - 1$. Moreover we know ([1] p. 15) that $x^n + 1$ is never prime while $x \in \mathbb{N}$ and $n \notin \{2^m \mid m \in \mathbb{N}\}$. Once more we conclude by transfer.

Obviously every interval in ${}^*\mathbf{Z}$ of length at least equal to ω contains galaxies. Hence proposition 1-2 iii/ holds. The point here is that we can give lots of examples of such galaxies.

There is the reason why KEMENY introduced K : in $K \oplus \mathbf{Z}$ famous conjectures are solved, and, if $K \oplus \mathbf{Z}$ is a nonstandard model of arithmetic, then those conjectures are undecidable. For instance the conjecture there are many n - tuples primes is false for $n \geqslant 3$; also, for every integer n, there is an arbitrarily large even integer which cannot be written as the sum of n primes so that, as a particular case, Goldbach's conjecture is false in $K \oplus \mathbf{Z}$. We can add to the previous (mentioned by KEMENY in [7]) this one : «if $n \geqslant 9$, then n is the sum of three odd primes» which is yet false in $K \oplus \mathbf{Z}$. Unfortunately, as already said beyond :

Corollary 1 - 3

$K \oplus \mathbf{Z}$ *is not a nonstandard model of arithmetic* [7].

Second proof :

By transfer of a particular case of Dirichlet's theorem (with a straighfoward proof [see [1] p. 13]) there is, for any integer n, a prime of the form $8m + 5$ greatest than n. If $K + \mathbf{Z}$ is a nonstandard model, and if $8m + 5$ is a prime such that $8m + 5 > \omega \in {}^*\mathbb{N} \setminus \mathbb{N}$, we have $z \in \mathbf{Z}$ satisfying $8z + 5 \in \{-1, +1\}$, what is impossible.

2 – SOME MAIN, EXTERNAL and INTERNAL IDEALS of $^*\mathbf{Z}$

If we denote $\cap\, p^n\, {}^*\mathbf{Z}$ by μ_p , we have $K = \bigcap_{p \in} \mu_p$. Let us give now a description of external properties of K :

Proposition 2 - 1

i/ *The ideal* K *is semi-prime* (rad (K) = K).

ii/ *The ideal* K *is neither maximal, nor prime nor primary.*

iii/ *The ideal* K *is neither intersection, nor product of any finite number (at least equal to two) of prime ideals and the same holds true when «prime» is replaced by «primary» or irreductible.*

Proof :

i/ As usual, $K \subset \mathrm{Rad}\,(K)$. If $x \in \mathrm{Rad}\,(K)$ there is a $n \in \mathbb{N}$ for which x^n is abnormal. For every $(p,m) \in \mathcal{S} \times \mathbb{N}$, p^{mn} divides x^n so that, by transfer, p^m divides x. Thus $x \in K$.

ii/ If K should be primary, K should be prime because rad (K) = K. Let us prove that K is not prime : let $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ be and, by transfer, let $p_1, \ldots, p_\omega$ be the first ω prime integers of $^*\mathbb{N}$. The integers p_1^{ω} and $p_2^{\omega} \ldots p_\omega^{\omega}$

are not abnormal buttheir product is abnormal. A fortiori, K is not maximal.

iii/ We show that K is not intersection of irreductible ideals (then not intersection of prime ideals). Let us assume $K = I_1 \cap \dots \cap I_n$ where I_j is irreductible for $j \in \{1, \dots, n\}$. The sequence $(p_i)_{i \in \mathbb{N}}$ of standard primes can be divide into n + 1 disjoint classes where the j-th class consists of primes p_k where $k \equiv j$ (mod. n + 1). Let $P_j = \bigcap_{k \in j+(n+1)\mathbf{Z}} \mu_{p_k}$; we have $K = \bigcap_{j=1}^{n+1} P_j$, because $K = \bigcap_{p \in \mathcal{T}} \mu_p$. For each $i \in \{1, \dots, n\}$, $I_i \cup K = I_i$ since $K \subseteq I_i$. Thus

$I_i = \bigcap_{j=1}^{n+1} I_i \cup P_j$, but I_i is irreductible so that $I_i \cup P_{j(i)} = I_i$ for some $j(i) \in \{1, \dots, n+1\}$. Hence $P_{j(1)} \cap \dots \cap P_{j(n)} \subset I_1 \cap \dots \cap I_n = K$.
Taking $x \in P_{j(1)} \cap \dots \cap P_{j(n)}$, it is easy to see that $x \notin K$ because x is not divisible by p_{j_0}, where $j_0 \in \{1, \dots, n+1\} \setminus \{j(1), \dots, j(n)\}$. Next we prove that K is not product of irreductible ideals. If $K = I_1 I_2 \dots I_n$, where I_j is irreductible, then $K \subset I_1 \cap \dots \cap I_n$. Just as above, we get $P_{j(1)} \dots P_{j(n)} \subset I_1 \dots I_n = K$.
But $x = p_{j(1)}^{\omega} \dots p_{j(n)}^{\omega} \in P_{j(1)} \dots P_{j(n)}$ when $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $x \notin K$. The primary case (for intersection and product) follows from rad (IJ) = rad $(I \cap J)$ = rad (I) $\cap$ rad (J) and uses the fact that the radical of a primary ideal is prime.

Remarks :

1) The proof of iii/ is analogous of this one of MENDELSON [9] who shows same results on the ideal L of all elements which are divisible by all standard primes.

2) It does not seems that L is a divisible subgroup (and, a fortiori, a maximal divisible subgroup) as stated by MENDELSON [9] : let $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $p_1, \dots, p_\omega$ be the first ω primes of $^*\mathbb{N}$; there is no $x \in L$ satisfying $p_1 x = p_1 p_2 \dots p_\omega$ in spite of the fact $p_1 \dots p_\omega \in L$.

For any nonstandard model we can have «infinitesimal» ideals $k{}^*\mathbf{Z} \subset K$ and $p^{\omega}\,{}^*\mathbf{Z} \subset \mu_p$, for every $k \in K$, $p \in \mathcal{P}$ and $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. They provide us with quotient-rings of ${}^*\mathbf{Z}$ in which the quotient-rings of ${}^*\mathbf{Z}$ by K and the μ_p ' s containing those infinitesimals can be homomorphically mapped.

Next we present some ideals illustrating ring-structures of ${}^*\mathbf{Z}$.

Proposition 2 - 2

i/ *For every* $p \in {}^*\mathcal{P}$, *the external subset* $I(p) = \{z\,p^{\omega} \mid z \in {}^*\mathbf{Z}$ *and* $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}\}$ *is a prime irreductible, but neither maximal nor principal ideal.*

ii/ *For every* $a \in {}^*\mathbb{N}$, $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ *the chain of ideals* $(a^{\omega - n}\,{}^*\mathbf{Z})_{n \in \mathbb{N}}$ *is strictly ascending and the chain* $(a^{n}\,{}^*\mathbf{Z})_{n \in \mathbb{N}}$ *strictly descending.*

Remark *:*

Because of the nœtherian character of $\mathbf{Z}$, an ascending chain must be external. But since $\mathbf{Z}$ is not artinian, we can find an internal strictly ascending chain such as $(z^{n}\,{}^*\mathbf{Z})_{n \in {}^*\mathbb{N}}$.

Proof of proposition *2 - 2 :*

i/ Let $ab \in I(p)$ and -by transfer- denote $v_p(a) = \alpha$, $v_p(b) = \beta$ the valuations of a and b in p. Since $\alpha + \beta \in {}^*\mathbb{N} \setminus \mathbb{N}$, we have $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$ or $\beta \in {}^*\mathbb{N} \setminus \mathbb{N}$: hence I(p) is prime(and thus irreductible). In ${}^*\mathbf{Z}_{/I(p)}$, the equality $\overline{p}\,\overline{q} = \overline{1}$ implies the existence of $a \in I(p)$ such as $p\,(q + a) = 1$ what is not -by transfer- possible in ${}^*\mathbf{Z}$. Thus I(p) is not maximal. If I(p) would be principal, we should have $I(p) = z_0\,p^{\omega_0}\,{}^*\mathbf{Z}$ and $p^{\omega_0 - 1} \notin I(p)$.

ii/ Obvious.

3 – RESULTS ON EXTERNAL STRUCTURE OF THE RING *$\mathbb{Z}$

As straightforward consequences of the proposition 2 - 2 the ring-structure of *$\mathbb{Z}$ is neither factorial, nor principal, nor euclidian, nor nœtherian, nor artinian, nor this one of a Dedekind ring. Another «negative» result (or may-be a «positive» one whose consequences are «negatives») was found by MENDELSON in [9] : any non-zero homomorphism of *$\mathbb{Z}$ into itself is an order preserving homomorphism. Hence there is no subring X of *$\mathbb{Z}$ such that the additive group of *$\mathbb{Z}$ is the direct sum of the additive groups of $\mathbb{Z}$ and X. Also, the ring *$\mathbb{Z}$ has no subring whose intersection with any galaxy is a singleton. As an ordered ring, we first recall that the order type of *$\mathbb{Z}$ is $\omega + (\omega^{-} + \omega)\,\eta$, where ω is the order type of $\mathbb{N}$ and η is an dense order type whithout first and last element ; in particular, if *$\mathbb{Z}$ is denumerable, η is the order type of $\mathbb{Q}$ (see [7], [9] and [16]). The characterization of the order type of *$\mathbb{Z}$ is an open and difficult question : see, for external properties of *$\mathbb{R}$ (and thus of *$\mathbb{N}$) ZAKON [19]. PUKITZ, in [12], uses standard functions to compare infinite integers in galaxies, constellations and skies. Nonstandard models of arithmetic in [12] are ultrapowers and the development depends on the kind of the used ultrafilters. The main result, about comparing nonstandard elements, still in MENDELSON [9], is that *$\mathbb{Z}$ is obviously, non archimedian. Let A be an ordered subring of *$\mathbb{Z}$ and $\ll$ the relation $x \ll y$ iff, for every $n \in \mathbb{N}$, $|nx| \leqslant |y|$. Let $x \approx y$ iff $x \not\ll y$ and $y \not\ll x$. Then this relation $\approx$ is an equivalence. Let h be the canonical map from A onto $H(A) = A/_{\approx}$, whose elements are called heights. This set H(A) is an ordered commutative semi - group for the induced order relation H(A) by $\ll$. The map h is hence a valuation on A. The subring A is archimedian if and only if $H(A) = \{h(1)\}$, so that *$\mathbb{Z}$ is not archimedian once more ; for any infinite cardinal α, there is a non-standard model *$\mathbb{Z}$ whose cardinal is α. Let us close these remarks on orders on *$\mathbb{Z}$ by a result of CONNES [4] : the relation $x\,\mathcal{R}_{\omega}\,y$ given on the interval

$[a,b] \cap {}^*\mathbb{Z}$ whose lenght is $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$, by $|x - y| \ll \omega$ is an equivalence with respect of addition when it is definited : the quotient-set $\frac{[a,b] \cap {}^*\mathbb{Z}}{\mathcal{R}_\omega}$, with the order $<$ when $x < y$ iff there is $z \in I_\omega$ for which $y = x + z$, is isomorphic for this order to $[0,1]$ as an ordered interval of $\mathbb{R}$; furthermore let $f : I_\omega \to [0,1]$ satisfy $f(\omega) = 1$, and for $(x,y) \in ([a,b] \cap {}^*\mathbb{Z})^2$ such that $x + y \ll \omega$, $f(x) + f(y) = f(x + y)$, then $z(z) = (\frac{z}{\omega})^\circ$ for any $z \in [a,b] \cap {}^*\mathbb{Z}$.

Better results on external properties of ${}^*\mathbb{Z}$ come from algebraicity. The quotient field of the entire ring ${}^*\mathbb{Z}$, say $F({}^*\mathbb{Z})$ has a regular behaviour : his additive structure is isomorphic to $\mathbb{Q}^{(\alpha)} \oplus \mathbb{Q}$ where α is the cardinal of ${}^*\mathbb{Z}$ [proof as in [11] by PHILIPS : let us note that we do not know what PHILIPS means by «a nonstandard model of $\mathbb{Q}$ »]. In MENDELSON [9] we find that the ring $\mathbb{Z}$ is integrally closed in ${}^*\mathbb{Z}$ which in turn is integrally closed in $F({}^*\mathbb{Z})$. The extension $[F({}^*\mathbb{Z}) : \mathbb{Q}]$ is transcendental of infinite degree and for any infinite cardinal α, there is a nonstandard model ${}^*\mathbb{Z}$ for which the transcendence degree of $F({}^*\mathbb{Z})$ on $\mathbb{Q}$ has cardinality α. At last, using results of J. ROBINSON, MENDELSON shows that automorphisms of $K({}^*\mathbb{Z})$ preserve order are determined by those of ${}^*\mathbb{Z}$ and determine them.

4 – SOME IMPORTANT QUOTIENT RINGS OF ${}^*\mathbb{Z}$

For what MENDELSON calls «Peano rings», he says in [9] : it can be shown that there is a prime u such that for any t in $\mathbb{Z}^*$ (or any t in $\mathbb{Z}$) $t^{1/2} \in R_u$ (here R_u means, with our notations ${}^*\mathbb{Z}/_{u\,{}^*\mathbb{Z}}$). Let us show this result. Let $p_1 = 2, p_2, \ldots, p_\omega$ be the first prime integers de ${}^*\mathbb{N}$ where $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. It exists $m \in {}^*\mathbb{N}$ such as $q = 8p_2\, p_3 \ldots p_\omega m + 1 \in {}^*\mathcal{P}$. By using symbols of Legendre and reciprocity law, we have :

for every $p \in \mathcal{P} \setminus \{2\}$, $q \equiv 1\,(p)$, $(\frac{p}{q})\,(\frac{q}{p}) = (-1)^{\frac{p-1}{2}\, 4p_2 \ldots p_\omega m} = +1$, thus p is a quadratic residue modulo q. Furthermore

$\left(\frac{2}{q}\right) = (-1)^{\frac{1}{8}(q^2-1)} = +1$ and $\left(\frac{-1}{q}\right) = (-1)^{4m\, p_2 \cdots p_\omega} = +1$ showing (-1)

and 2 are also quadratic residues modulo q. It follows easily that every $r \in \mathbb{Q}$ is a square in ${}^*\mathbb{Z}/_{q\,{}^*\mathbb{Z}}$.

Proposition 4 - 1

There are $p \in {}^*\mathcal{P}\setminus\mathcal{P}$ such that ${}^*\mathbb{Z}/_{p\,{}^*\mathbb{Z}}$ contains the abelian closure of $\mathbb{Q}$.

Proof :

Let a be an abnormal integer. By Dirichlet's theorem transfered, there are $p \in {}^*\mathcal{P}$ and $p > \omega$, where ω is a fixed integer in ${}^*\mathbb{N}\setminus\mathbb{N}$, such that $p = m\,a + 1$ for some $m \in {}^*\mathbb{N}$. Then $p - 1 = m\,a \in K$, and every $d \in \mathbb{N}$ divides $p - 1$. Transfering the number theory theorem (see [1] p. 85) ensuring that : «If p is a prime and d divides $p - 1$, then the polynomial $X^d - 1$ has d roots in $\mathbb{Z}/_{p\mathbb{Z}}$», we know that equations $X^n - 1 = 0$ have every one n roots in ${}^*\mathbb{Z}/_{p\,{}^*\mathbb{Z}}$. Hence, for every $q \in \mathcal{P}$, the field T_q generated by the q^m-th roots of unity when $m \in \mathbb{N}$ is contained in ${}^*\mathbb{Z}/_{p\,{}^*\mathbb{Z}}$. But Ab, the abelian closure of $\mathbb{Q}$, is the subfield generated by $\bigcup_{q\in\mathcal{P}} T_q$, so that Ab is a subfield of ${}^*\mathbb{Z}/_{p\,{}^*\mathbb{Z}}$

For $p \in \mathcal{P}$ and $\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$

we shall now show that ${}^*\mathbb{Z}/_{p^\omega\,{}^*\mathbb{Z}}$ is, in a natural way, the homomorphic image of the ring of the p-adic integers. Let us recall that a p-adic integer is a sequence $(a_n)_{n\in\mathbb{N}\setminus\{0\}}$ such that $0 \leq a_n < p$, $a_{n+1} \equiv a_n\ (p^n)$ for every $n \in \mathbb{N}\setminus\{0\}$. The reason why we introduce p-adics is as follows : let $p \in \mathcal{P}$; there is a representation $\sum_{k=0}^{\omega} a_k\, p^k$, with obvious notations for every integer of ${}^*\mathbb{N}$ and this is nothing else than numeration in the basis p we get by transfer. If we «cut» the sequence of partial sums $\sum_{k=0}^{s} a_k\, p^k$ keeping those for which $s \in \mathbb{N}$, then we have exactly a p-adic integer. First result :

Proposition 4 - 2

If $^*\mathbf{Z}$ *underlies an integrally comprehensive model or is an algebraically compact group, then for every* $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$ *and every* $p \in \mathfrak{T}$, *the ring* $\mathbf{Z}_p$ *of the standard p-adic integers is an homomorphic image of* $^*\mathbf{Z}/_{p^\omega\, ^*\mathbf{Z}}$ *by a map whose kernel is* $\mu_p/_{p^\omega\, ^*\mathbf{Z}}$.

Proof :

Using numeration in the basis $p\,(p \in \mathfrak{T})$ for canonical representants of the cosets of $^*\mathbf{Z}/_{p^\omega\, ^*\mathbf{Z}}$ $(\omega \in {}^*\mathbb{N} \setminus \mathbb{N})$, we can define an external map h_ω over its domain $^*\mathbf{Z}/_{p^\omega\, ^*\mathbf{Z}}$ by $h_\omega\,(\sum_{i=0}^{\omega-1} a_i p^i + p^\omega\, ^*\mathbf{Z}) = (a'_n)_{n \in \mathbb{N}\setminus\{0\}}$ where $a'_i = \sum_{j=1}^{i-1} a_j p^j$. Because $a'_{n+1} \equiv a'_n\ (p^n)$ for every $n \in \mathbb{N} \setminus \{0\}$, $h_\omega(^*\mathbf{Z}/_{p^\omega\, ^*\mathbf{Z}}) \subset \mathbf{Z}_p$. The map h_ω is clearly a ring morphism. Let $\alpha = (b_n)_{n \in \mathbb{N}\setminus\{0\}}$ be a p-adic integer. If $^*\mathbf{Z}$ underlies an integrally comprehensive model, then the sequence $(b_n)_{n \in \mathbb{N}\setminus\{0\}}$ has an extension to $(b_n)_{n \in {}^*\mathbb{N}}$ and we have $h_\omega\,\left(\sum_{i=1}^{\omega-1} b_i p^i\right) = \alpha$. If $^*\mathbf{Z}$ is an algebraically compact group, then the system of equations $(y = b_n + x_n p^n)_{n \in \mathbb{N}\setminus\{0\}}$ is finitely satisfiable and there is consequently at least a general solution $(y_0, \overset{0}{x}_1, \ldots, \overset{0}{x}_n, \ldots)$ such that $h_\omega\,(y_0) = \alpha$. Any coset of the kernel of h_ω contains an integer $\sum_{i=0}^{\omega-1} a_i p^i$ for $a_i = 0$ when $i \in \mathbb{N}$. The set of such representants is the ideal generated by $\bigcup_{\omega' \in {}^*\mathbb{N}\setminus\mathbb{N}} p^{\omega'}\, ^*\mathbf{Z}$. It can be shown that it is μ_p.

Remark :

Since $\mathbb{Z}_p$ is a projective limit of rings $(\mathbb{Z}/_{p^n\mathbb{Z}})_{(n \in \mathbb{N}\setminus\{0\})}$, the last proposition is analogous for rings to ROBINSON's result on groups [16] ensuring that $G = \varprojlim G_\alpha$ is an homomorphic image of $^*G_\omega$, starfinite group of $(G_\alpha)_{\alpha \in {}^*\mathbb{N}}$ for $\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$. However it was not enough to apply this result because we need h_ω to be in a position to investigate the links between the $\mathbb{Z}_p$'s and $^*\mathbb{Z}$, more precisely $^*\mathbb{Z}/_{\mu_p}$ $(p \in \mathcal{P})$ and $\mathbb{Z}/_K$.

5 – LINKS BETWEEN $\mathbb{Z}_p$, $^*\mathbb{Z}/_{\mu_p}$ $(p \in \mathcal{P})$ and $^*\mathbb{Z}/_K$ as RINGS

The facts that $^*\mathbb{Z}/_K$ can be seen as a ring extension of $^*\mathbb{Z}$, and that internal ideals of $^*\mathbb{Z}$ are send by a one-to-one map on the classes of associated principal ideals of $^*\mathbb{Z}/_K$ are obvious (one can see [14]). Also, finite intersections of principal ideals in $^*\mathbb{Z}/_K$ are principal, but $^*\mathbb{Z}/_K$ being not entire, is not a Gauss ring. Let us see what are units and zero divisors in $^*\mathbb{Z}/_K$:

Proposition 5 - 1

i/ *The units of* $^*\mathbb{Z}/_K$ *(whose set we will denote* $v(^*\mathbb{Z}/_K)$*) are cosets* $\bar{a}$, *where* a *has no standard prime divisor.*

ii/ *The zero divisors of* $^*\mathbb{Z}/_K$ *form the set of cosets* $\bar{a}$ *such that is a* $p \in \mathcal{P}$ *for which* $\bar{a} \subset \mu_p$.

Proof :

i/ If $\bar{a} \in v(^*\mathbb{Z}/_K)$, there is $(b,k) \in {}^*\mathbb{Z} \times K$ such that $ba + k = 1$. Hence b is prime to any $p \in \mathcal{P}$. Conversely let p_ω be the infinimum of the set of the primes dividing a. By assumption, $p_\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$ and $(p_\omega - 1)! \in K$ is prime to a. By Bezout's identity, we get $(u,v) \in {}^*\mathbb{Z}$ such that $u(p_\omega - 1)! + a = 1$, hence $\bar{a}\,\bar{b} = \bar{1}$. q. e. d.

ii/ Suppose $\bar{a}\,\bar{b} = \bar{0}$, i.e. $a\,b \in K$ for $a \notin K$ and $b \notin K$. There is $(p,n) \in \mathcal{S} \times \mathbb{N}$ such that p^n does not divide b. For every $m \in \mathbb{N}$, p^m is prime to $b/_{p}v_p(b)$ where $v_p(b)$ is the highest power of p dividing b. Hence p^m divides a for any $m \in \mathbb{N}$, what means $\bar{a} \subset \mu_p$. Conversely if $\bar{a} \subset \mu_p$, $\bar{a} \neq \bar{0}$ and $k \in K$, we consider p^ω the highest power of p dividing k. Then

$$\bar{a}\ \overline{\frac{k}{p^\omega}} = \bar{0}\,,\ \text{but}\ \overline{\frac{k}{p^\omega}} \neq \bar{0}\ \text{and}\ \bar{a} \neq \bar{0}.$$

The main aspect of ${}^*\mathbf{Z}/_K$ follows :

Proposition 5 - 2

For any nonstandard model $\mathbf{Z}^*$.

i/ *the ring ${}^*\mathbf{Z}/_{\mu_p}$ is, for any $p \in \mathcal{S}$, isomorphic to a subring $\mathbf{Z}_p$ of the p-adic integers.*

ii/ *The ring ${}^*\mathbf{Z}/_K$ is isomorphic to a subring of the direct product $\prod_{p \in \mathcal{S}} \mathbf{Z}_p$.*

Corollary 5 - 3

If ${}^*\mathbf{Z}$ underlies an integrally comprehensive model or if ${}^*\mathbf{Z}$ is an algebraically compact group, for any $p \in \mathcal{S}$, $\mathbf{Z}_p$ is isomorphic to ${}^*\mathbf{Z}/_{\mu_p}$ and ${}^*\mathbf{Z}/_K$ to $\prod_{p \in \mathcal{S}} \mathbf{Z}_p$ as rings.

Remarks :

1) Corollary 6 - 3 describes as a particular case, the additive structure of the nonstandard models of arithmetic ${}^*\mathbf{Z}$ when they are ω_1-saturated : for all pratical purpose in nonstandard analysis this is always the case.

2) Proposition 5 - 2 ii/ proves that we shall have to know subgroups of $\prod_{p \in \mathcal{S}} \mathbf{Z}_p$ for a better know ledge of the additive structures of any nonstandard model of arithmetic (see section 6 below).

metic (see section 6 below).

Proofs of proposition 5 - 2 :

i/ For every $(p,\omega) \in \mathcal{T} \times ({}^*\mathbb{N}\setminus\mathbb{N})$, it is clear that $\mu_p \supset p^\omega\, {}^*\mathbf{Z}$. By an isomorphism theorem on rings :

$$ {}^*\mathbf{Z}/_{p^\omega\, {}^*\mathbf{Z}} \Big/ {}^{\mu_p}/_{p^\omega\, {}^*\mathbf{Z}} \simeq {}^*\mathbf{Z}/_{\mu_p} \ ; $$

hence, applying the proposition 4 - 2, ${}^*\mathbf{Z}/_{\mu_p} \simeq \mathbf{Z}_p$ as rings.

ii/ Using i/ above $\prod_{p\in\mathcal{T}} \mathbf{Z}_p \simeq \prod_{p\in\mathcal{T}} {}^*\mathbf{Z}/_{\mu_p}$, so that it is now enough to find an one-to-one morphism of rings from ${}^*\mathbf{Z}/_K$ to $\prod_{p\in\mathcal{T}} {}^*\mathbf{Z}/_{\mu_p}$: for each $z \in {}^*\mathbf{Z}$ let $\varphi(z)$ be $(\bar{z}^{\,p})_{p\in\mathcal{T}}$ where $\bar{x}^{\,p}$ denote the coset of x modulo μ_p. The map φ is obviously a morphism of rings and $\tilde{\varphi} : {}^*\mathbf{Z}/_K \longrightarrow \prod_{p\in\mathcal{T}} {}^*\mathbf{Z}/_{\mu_p}$ given by $\tilde{\varphi}(\bar{z}) = \varphi(z)$ (where x is coset of x modulo K) is one-to-one because $\operatorname{Ker}\varphi = \bigcap_{p\in\mathcal{T}} \mu_p = K$.

Proof of the corollary 5 - 3 :

We must prove that φ is onto when ${}^*\mathbf{Z}$ underlies an integrally comprehensive model let $\omega \in {}^*\mathbb{N}\setminus\mathbb{N}$ be a given nonstandard integer. We have a sequence $s = (\bar{a}^{\,p_i}_{p_i})_{i\in\mathbb{N}}$ where $(p_i)_{i\in\mathbb{N}}$ is the sequence of standard primes and

$\bar{a}^{\,p_i}_{p_i} \in {}^*\mathbf{Z}/_{p_i^\omega\, {}^*\mathbf{Z}}$ let us extend s to $(a_i)_{i\in\mathbb{N}}$ and consider $(a_i)_{i\in[1,\omega]}$ which is a star-finite sequence uniquely determined by s. By chineese remainder theorem there is at least a solution z for the system $x \equiv a_1\ (p_1^\omega)\ ; \ldots , x \equiv a_\omega\ (p_\omega^\omega)$. Since $p^\omega\, {}^*\mathbf{Z} \subset \mu_p$ for any $(p,\omega) \in {}^*\mathcal{T} \times ({}^*\mathbb{N}\setminus\mathbb{N})$, we have $\bar{z} = \bar{a}^{\,p}_p\ (\mu_p)$ for any $p \in \mathcal{T}$ so that $\varphi(z) = s$.

If ${}^*\mathbf{Z}$ is algebraically compact as a group, we denote p_i the i-th standard prime and we note that the system $(y = a_{ij} + x_{ij}\, p_j^{\,i})_{i,j\in\mathbb{N}\setminus\{0\}}$ where $a_{ij} \in {}^*\mathbf{Z}$ is finitely satisfiable so that we can conclude.

6 – ON THE ADDITIVE STRUCTURE OF $^*\mathbb{Z}$

We conclude by giving without proof some results on the subgroups G such that $^*\mathbb{Z} = K \oplus G$. We call φ the one-to-one group homomorphisms mapping G into the $\mathbb{Z}$-adic completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ (i. e. into $\prod_{p \in \mathcal{P}} \mathbb{Z}_p$) and we do not take any other assumption on $^*\mathbb{Z}$ than the one of being a non standard model of arithmetic.

Theorem 6 :

1) The subgroup G *can be choosen such as* $\mathbb{Z} \subset G$ *;*

2) The subgroup $\varphi(G)$ *is pure and dense into* $\widehat{\mathbb{Z}}$ *;*

3) For every $g \in G$, $\varphi(g)$ is an unit of $\widehat{\mathbb{Z}}$ *if and only if his divisors are infinite ;*

4) Canonical projections of G *into the* J_p *'s are not* $\{0\}$ *neither* $\mathbb{Z}$ *;*

5) If $^*\mathbb{Z}$ *is denumerable,* $\varphi(G)$ *is not a direct product of subgroups of the* J_p *'s ;*

6) For any cardinal $\alpha \geqslant \omega$, *there is a model whose cardinal is* α *such that* $\varphi(G)$ *never is a direct product of subgroups of the* J_p *'s.*

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