PUBLICATIONS DU DÉPARTEMENT DE MATHÉMATIQUES DE LYON

C. WATKISS

The Nilpotent Model for a Function Space

Publications du Département de Mathématiques de Lyon, 1976, tome 13, fascicule 4
, p. 73-85

<http://www.numdam.org/item?id=PDML_1976__13_4_73_0>

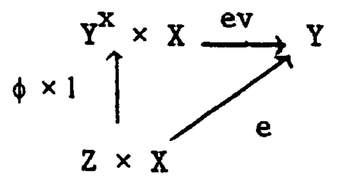
Publications du
Département de
Mathématiques
Lyon 1976 t. 13-4

THE NILPOTENT MODEL FOR A FUNCTION SPACE

by C. WATKISS

INTRODUCTION. - The purpose of these notes is to describe Sullivan's construction of a nilpotent model for a function space Y^X. The motivating idea is to assume the existence of a cocommutative chain theory from topological spaces to coalgebras (which, by abuse of notation, we write as $X \mapsto X$). One then simply translates the universal properties of Y^X into algebraic universal properties, and solves the corresponding algebraic problem.

For example, there is an evaluation map $Y^X \times X \xrightarrow{ev} Y$ such that, given any continuous map $Z \times X \xrightarrow{e} Y$ there exists a unique map $\phi : Z \to Y^X$ such that the diagram

(×) 

commutes. Given a coalgebra X (corresponding to the space X) and a nilpotent algebra $\mathcal{M}$ (corresponding to Y) we construct a nilpotent algebra $\mathcal{A}(\mathcal{M},X)$ which corresponds to the function space Y^X, and satisfies a dual algebraic universal property.

Unfortunately, the function space Y^X is in general not connected. This fact is reflected algebraically by the existence of elements of negative degree in $\mathcal{A}(\mathcal{M},X)$. In effect, since the geometrical significance of these generators is not clear, we calculate the nilpotent model for a connected component of Y^X : given a map $f : X \to Y$; the f-component of Y^X satisfies the universal property but for any pointed space $(Z; *)$ and map $e : Z \times X \to Y$ whose restriction to $* \times X$ is simply f, then there is a unique pointed map $(Z,*) \xrightarrow{\phi} (Y^X,f)$ such that the above diagram (*) commutes. In the algebraic situation the map $f : X \to Y$ corresponds to an algebra homomorphism $\rho : \mathcal{M} \to X^*$; we construct a nilpotent algebra $\mathcal{B}(\rho,X)$ which is a nilpotent model for the f-component of the function space Y^X, and which satisfies an analogous universal property. The essential ingredient in the proof that $\mathcal{B}(\rho,X)$ is indeed a nilpotent model for the f-coponent of Y^X is careful study of the dependence of the algebre $\mathcal{B}(\rho,X)$ on a change of X or $\mathcal{M}$ by a map inducing an isomorphism on(co)homology. It is this algebraic study that we present here.

0. NOTATION.

All algebras, coalgebras, etc. will be over a field k of characteristic 0. We abbreviate commutative differential graded algebra by CDGA and cocommutative differential graded coalgebra by CDGC. (CDGA's are not necessarily positively graded : $A = (A^p)_{p \in Z}$ with $d : A^p \to A^{p+1}$. On the other hand a CDGC is positively graded : $C = (C_p)_{p \geq 0}$ with $\partial : C_p \to C_{p-1}$. For convenience we raise degrees by $C^p = C_{-p}$ so that $C = (C^p)_{p \leq 0}$ is negatively graded and $\partial : C^p \to C^{p+1}$.) If A is CDGA and C a CDGC then Hom(C,A) is a CDGA with the usual notations : $\mathrm{Hom}^p(C,A) = \prod_i \mathrm{Hom}(C^i,A^{i+p})$, $d_{Hom}(\phi) = d_A \circ \phi + (-1)^{|\phi|+1}\phi \circ \partial_C$, $m_{Hom} = \mathrm{Hom}(\Delta_C,m_A)$. Hom(C,k) is denoted C^*.

The free CGA on a graded vector space Q is denoted ΛQ. A CDGA $\mathcal{M}$ is *nilpotent* if $\mathcal{M}$ is free as a CGA ($\mathcal{M} \approx \Lambda Q_n$) and the space of generators Q_M has a well-ordered basis $\{x_\alpha\}$ such that dx_α is a polynomial in the x_p with $\beta \quad \alpha$

A CDGA or CDGC - map $f : A \to B$ is called a *quasi-isomorphism if* $H(f) : H(A) \overset{\approx}{\to} H(B)$ is an isomorphism.

1. THE MODEL FOR A FUNCTION SPACE.

Suppose X is a CDGC and $\mathcal{M} = \Lambda Q_M$ a nilpotent CDGA (with Q_M *strictly* positively graded). Then there is a CDGA $\mathcal{A} = \mathcal{A}(\mathcal{M}, X)$ and a morphism

$$\varepsilon \; : \; \mathcal{M} \to \mathrm{Hom}(X, \quad)$$

with the universal property that for any CDGA Z and map $e : \mathcal{M} \to \mathrm{Hom}(X,Z)$ there is a unique map $\mathcal{A} \overset{\phi}{\to} Z$ such that the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\;\varepsilon\;} & \mathrm{Hom}(X,\mathcal{A}) \\ & {\scriptstyle e}\searrow & \swarrow{\scriptstyle \mathrm{Hom}(X,\phi)} \\ & \mathrm{Hom}(X,Z) & \end{array}$$

commutes. In fact, let $\mathcal{A} = \Lambda(Q_M \otimes X)$ as a CGA. Define the map ε on generators Q_M by

$$\varepsilon(y)\;(x) = y \otimes x \;, \quad y \in Q_M \;, \quad x \in X.$$

The differential in $\mathcal{A}$ is chosen precisely so that ε is a map of CDGA's : $\varepsilon \circ d_{\mathfrak{M}} = d_{Hom} \circ \varepsilon$ forces the definition

$$d_{\mathcal{A}}(y \otimes x) = \varepsilon(d_{\mathfrak{M}}\, y)\,(x) + (-1)^{|y|}\varepsilon(y)(\partial_X x).$$

This differential is extended to a degree λ derivation in $\mathcal{A}$: it is trivial to check that $d_{\mathcal{A}} \circ d_{\mathcal{A}} = o$ and the universal property is satisfied.

1.1. REMARKS. 1. In general $\mathcal{A}$ need not be positively graded (since X is negatively graded).

2. $\mathcal{A}$ is nilpotent.

Now suppose $\rho : \mathcal{M} \to X^{*}$ is a CDGA map. By the universal property for ε, there is a unique augmentation

$ev_{\rho} : \mathcal{A}(\mathcal{M}, X) \to k$

such that the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \varepsilon\ } & Hom(X, \mathcal{A}) \\ {\scriptstyle\rho}\downarrow & & \downarrow {\scriptstyle Hom(X, ev_{\rho})} \\ X^{*} & = & Hom(X,k) \end{array}$$

commutes. Let $I_{\rho} \subset A$ be the ideal generated by $\mathcal{A}^{<o}$ (the elements of negative degree), $\mathcal{A}^{0} \cap Ker\ ev_{\rho}$, and $d_{\mathcal{A}}(\mathcal{A}^{o} \cap Ker\ ev_{\rho})$. Note that I_{ρ} is d is $d_{\mathcal{A}}$-stable : $d_{\mathcal{A}}(\mathcal{A}^{<-1}) \subset \mathcal{A}^{<o}$, and $d_{\mathcal{A}}(\mathcal{A}^{-1}) \subset \mathcal{A}^{o} \cap ker\ ev_{\rho}$ (since ev_{ρ} is a CDGA map).

1.2. DEFINITION . - $\mathcal{B}(\rho, X) = \mathcal{A}(\mathcal{M}, X)/I_{\rho}$.

1.3. LEMMA. - $\mathcal{B}(\rho, X)$ is a nilpotent CDGA.

PROOF. - To see that $\mathscr{B}$ is free, we describe I_S differently. Let Q_A denote the space of generators for $\mathscr{A}$ obtained by replacing each degree 0 generator ξ by $\xi - ev_p \xi$. (Note that we still have $\mathscr{A} \cong \mathscr{A} \Lambda Q_A$ and this doesn't affect the nilpotence of $\mathscr{A}$). Let $K \subset Q_A$ be the graded subspace defined by

$$K^p = \begin{cases} Q_A{}^p & p \leqslant o \\ d_{Q_A}(Q_A) & p = 1 \\ 0 & p > 1 \end{cases}$$

and let $\Pi_Q : Q_A \to Q_A/K = Q_B$ be the projection. Then $\mathscr{B} \cong \Lambda Q_B$ and under this isomorphism $\Pi : \mathscr{A} \to \mathscr{B}$ is just $\Lambda \Pi_Q : \Lambda Q_A \to \Lambda Q_B$. The nilpotence of $\mathscr{B}$ is then an easy argument.

1.4 REMARK. - Let $\bar{\varepsilon} = \mathrm{Hom}(X,\Pi) \circ \varepsilon : \mathscr{M} \to \mathrm{Hom}(X,\mathscr{B})$. Then $\bar{\varepsilon}$ satisfies the following universal property for any augmented CDGA $\overset{\alpha}{\to} k$ and morphism $e : \mathscr{M} \longrightarrow \mathrm{Hom}(X,Z)$ such that the diagram

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\quad e \quad} & \mathrm{Hom}(X.Z) \\ & \searrow & \downarrow \mathrm{Hom}(X,\alpha) \\ & & X^* \end{array}$$

commutes, there exists a unique map $\phi : \mathscr{B} \longrightarrow Z$ such that

$$\begin{array}{ccc} \mathscr{M} & \overset{\bar{\varepsilon}}{\text{———}} & \mathrm{Hom}(X, \mathscr{B}) \\ e \searrow & & \swarrow \mathrm{Hom}(X,\phi) \\ & \mathrm{Hom}(X,Z) & \end{array} .$$

1.5 NATURALITY IN $\mathscr{M}$. Suppose X fixed and $\phi : \mathscr{M}_o \to \mathscr{M}_1$ a map of nilpotent algebras. By the universal property there exists a unique map.

$$\phi_{\mathscr{A}} : \mathscr{A}(\mathscr{M}_o, X) \to \mathscr{A}(\mathscr{M}_1, X)$$

such that the diagram

$$\begin{array}{ccc}
\mathscr{M}_o & \xrightarrow{\ \varepsilon_o\ } & \mathrm{Hom}(X, \mathscr{A}(\mathscr{M}_o, X)) \\
\phi \downarrow & & \downarrow \mathrm{Hom}(X, \phi_{\mathscr{A}}) \\
\mathscr{M}_1 & \xrightarrow[\ \varepsilon_1\ ]{} & \mathrm{Hom}(X, \mathscr{A}(\mathscr{M}_1, X))
\end{array}$$

commutes.

Suppose now that $\mathscr{M}_1 \longrightarrow X^*$. The universal property again implies that $ev_\rho \circ \phi_{\mathscr{A}} = ev_{\rho\circ\phi}$, so that $\phi_{\mathscr{A}}$ is augmentation preservity It follows that $\phi_{\mathscr{A}}(I_{\rho\circ\phi}) \subset I_\rho$, and $\rho_{\mathscr{A}}$ therefore induces a CDGA map-

$$\phi_{\mathscr{B}} : \mathscr{B}(\rho\circ\phi, X) \longrightarrow \mathscr{B}(\rho, X).$$

Clearly in the situation

$$\mathscr{M}_o \xrightarrow{\ \phi\ } \mathscr{M}_1 \xrightarrow{\ \psi\ } \mathscr{M}_2 \longrightarrow X^*$$

we obtain a commutative diagram

$$\begin{array}{ccc}
\mathscr{B}(\rho\psi\phi, X) & \xrightarrow{\ \phi_{\mathscr{B}}\ } & \mathscr{B}(\rho\psi, X) \\
 & {\scriptstyle (\psi\phi)_{\mathscr{B}}} \searrow \quad \swarrow {\scriptstyle \psi_{\mathscr{B}}} & \\
 & \mathscr{B}(\rho, X) &
\end{array}$$

As an example, a pair of maps $\mathcal{M}_o \xrightarrow{\rho_o} X^* \xrightarrow{\rho_1} \mathcal{M}_1$

induces

$$(\rho_o,\rho_1) : \mathcal{M}_o \otimes \mathcal{M}_1 \xrightarrow{\rho_o \otimes \rho_1} X^* \otimes X^* \xrightarrow{m} X^* .$$

Let $j_\nu : \mathcal{M}_\nu \to \mathcal{M}_o \otimes \mathcal{M}_1$, $\nu = o,1$, be the inclusions. The following result is immediate.

1.6. PROPOSITION. - *The maps*

$$(j_\nu)_{\mathcal{B}} : \mathcal{B}(\rho_\nu,X) \to \mathcal{B}((\rho_o,\rho_1),X), \quad \nu = o,1 ,$$

yield a natural isomorphism

$$((j_o)_{\mathcal{B}} ,(j_1)_{\mathcal{B}}) : \mathcal{B}(\rho_o,X) \otimes \mathcal{B}(\rho_1,X) \xrightarrow{\cong} \mathcal{B}((\rho_o,\rho_1),X).$$

1.7 NATURALITY IN X. - Suppose $f : X_o \to X_1$ is a CDGC map and $\mathcal{M} \xrightarrow{\rho} X_1^*$ a CDGA map. Just as above, the universal property implies that f induces

$$f_{\mathcal{B}} : \mathcal{B}(f^* \rho,X_o) \longrightarrow \mathcal{B}(\rho,X_1).$$

Moreover, in the situation $X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2$, $\mathcal{M} \xrightarrow{\rho} X_2^*$,

there is a commutative diagram

$$\begin{array}{ccc} \mathcal{B}(f^* g^* \rho ,X_o) & \xrightarrow{f_{\mathcal{B}}} & \mathcal{B}(g^* \rho ,X_1) \\ & {\scriptstyle (gf)_{\mathcal{B}}} \searrow \quad \swarrow {\scriptstyle g_{\mathcal{B}}} & \\ & \mathcal{B}(\rho ,X_2) . & \end{array}$$

1.8. HOMOTOPY INVARIANCE. The crucial results are the following.

THEOREM A. - *Given* $\mathcal{M}_o \xrightarrow{\phi} \mathcal{M}_1 \xrightarrow{\rho} X^*$ *with* ϕ *a quasi-iso, then*

$$\phi_{\mathcal{B}} : \mathcal{B}(\rho\circ\phi,X) \longrightarrow \mathcal{B}(\rho,X)$$

is a quasi-iso.

THEOREM B. - *Given* $\mathcal{M} \to X_1^*$ *and a quasi-iso* $X_o \xrightarrow{f} X_1$, *then* $f_{\mathcal{B}} : \mathcal{B}(f^*_\circ\rho, X_o) \longrightarrow \mathcal{B}(\rho, X_1)$ *is a quasi-iso.*

These theorems permit the following definition : if $f : X \to Y$ is a morphism of CDGC's, choose a minimal model $\rho : \mathcal{M} \to Y^*$.

1.9 DEFINITION. - $\mathcal{B}(Y^*, f) = \mathcal{B}(\rho, X)$.

It follows that the homotopy type of $\mathcal{B}(Y,f)$ depends only on the homotopy type of f.

2. PROOF OF THEOREM A.

We begin by studying a particularly easy special case of the theorem, which turns out to be the fundamental step in the proof (and at the same time gives a nice illustration of the construction of $\mathcal{B}$) : suppose $\mathcal{M}_1 = \Lambda(x,dx)$, a contractible CDGA with $|x| = n > o$. Then of course $k \overset{\phi}{\to} \mathcal{M}_1$ is a quasi-iso. On the other hand $\mathcal{B}(k \to X^*, X) = k$, since $\mathcal{A}(k,X) = \Lambda X$ is negatively graded. If the theorem is to be true then for any map $\rho : \mathcal{M}_1 = \Lambda(x,dx) \to X^*$ the algebra $\mathcal{B}(\rho,X)$ must be acyclic. In fact the theorem will follow immediately from this.

2.1. LEMMA. - *Let* $\mathcal{M} = \Lambda(x,dx)$ *with* $|x| = n > o$. *Then for any* X *and any* $\rho : \mathcal{M} \to X^*$ *the algebra* $\mathcal{B}(\rho,X)$ *is contractible.*

PROOF. - Choose a basis ξ^m_α for each X^m . Then a basis for the generating space of $\mathcal{A} = \mathcal{A}(\rho,X)$ in dimension p is given by $\{ x \otimes \xi^{p-n}_\alpha , dx \otimes \xi^{p-n-1}_\beta \}$. The differential $d_{\mathcal{A}}$ is entirely linear given by :

$$(2.2)\qquad \begin{aligned} d_{\mathscr{A}}\ (x \otimes \xi_\alpha^{p-n}) &= dx \otimes \xi_\alpha^{p-n} + (-1)^n\, x \otimes \partial\xi_\alpha^{p-n}\ , \\ d_{\mathscr{A}}\ (dx \otimes \xi_\beta^{p-n-1}) &= (-1)^{n+1}\, dx \otimes \partial\xi_\beta^{p-n-1}\ . \end{aligned}$$

Notice that there are no generators in degrees > n+1 because X is negatively graded.

To obtain Q_B we first kill all of the negative generators. We next replace the degree 0 generators by constants.

$$x \otimes \xi_\alpha^{-n} = ev_\rho\ (x \otimes \xi_\alpha^{-n})\ ,$$

$$dx \otimes \xi_\beta^{-n-1} = ev_\rho (dx \otimes \xi_\beta^{-n-1})$$

Finally we kill the differentials of the degree o generators :

$$dx \otimes \xi_\alpha^{-n} = (-1)^{n+1}\ x \otimes \partial\xi_\alpha^{-n}\ ,$$

$$dx \otimes \partial\xi_\beta^{-n-1} = 0.$$

It follows that the generators Q_B of $\mathscr{B}$ are given by

$$Q_B^p = \begin{cases} (x) \otimes X^{-n+1} & , p=1\ , \\ ((x) \otimes X^{p-n}) \oplus ((dx) \otimes X^{p-n-1}) & , 1 < p < n+1, \\ (dx) \otimes X^\circ & , p = n+1, \\ 0 & , \text{otherwise}. \end{cases}$$

Moreover the differential is still given by (2.2).

To show that $\mathscr{B}$ is contractible we decompose the generators $Q_B^p \approx R_B^p \oplus S_B^p$ so that

$$(3.2)\qquad d_B : R_B^p \approx S_B^{p+1},\ S_B^p \to 0.$$

To do this we first decompose $X^r = B^r \oplus H^r \oplus C^r$ in such a way that

$$\partial : C^r \approx B^{r+1}\ ,\ B^r \to o\ ,\ H^r \to 0.$$

(H^r is the (co)-homology and B^r the (co)boundaries in degree r).

We then put

$$R^p_B = (x) \otimes X^{p-n} \ , \quad 1 \leqslant p \leqslant n \ ,$$

$$S^p_B = (dx) \otimes X^{p-n-1} \ , \ 2 \leqslant p \leqslant n+1 .$$

The isomorphism

$$R^p_B \oplus S^p_B \xrightarrow{\cong} Q^p_B$$

is defined to be the inclusion on R^p_B , $(dx) \otimes B^{p-n-1}$ and $(dx) \otimes H^{p-n-1}$, while a generator $dx \otimes \xi^{p-n-1}_\beta$ of $(dx) \otimes C^{p-n-1}$ is sent to

$$dx \otimes \xi^{p-n-1}_\beta + (-1)^n \, x \otimes \partial \xi^{p-n-1}_\beta \ .$$

Property (2.3) is now clear in virtue of the definition of the differential in (2.2).

2.4 COROLLARY. - *For any contratiction CDGA $\mathcal{M}$ and any $\rho : \mathcal{M} \to X^*$, $\mathcal{B}(\rho, X)$ is contractible.*

PROOF. - This follows from lemma 2.1 by taking direct limits.

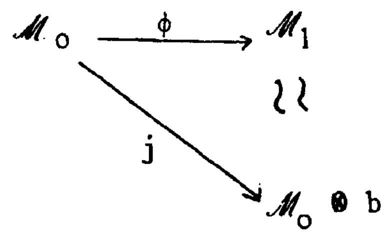
2.5. COROLLARY. - *Theorem A holds in the special case that $\mathcal{M}_1 = \mathcal{M}_o \otimes b$ where b is contractible and $\rho : \mathcal{M}_o \longrightarrow \mathcal{M}_1$ is the inclusion.*

PROOF. - Apply Prop. 1.6 and Cor. 2.4.

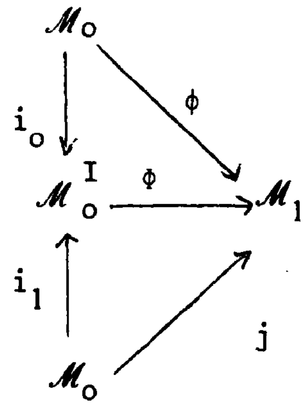
2.6. COROLLARY. - *It is sufficient to prove theorem $\mathcal{A}$ unite case that is minimal.*

PROOF. - By a theorem of Sullivan, any (positively graded) nilpotent algebra decomposes in the form $\mathcal{M}_o \approx \widehat{\mathcal{M}}_o \otimes b$, where $\widehat{\mathcal{M}}_o$ is the minimal model of $\mathcal{M}_o$, b is contractible and the isomorphism is an isomorphism of CDGA's. Then apply Cor. 2.5 and naturality.

2.7. PROOF OF THEOREM A : By Cor. 2.6, we can assume that $\phi : \mathcal{M}_o \longrightarrow \mathcal{M}_1$ is a minimal model of $\mathcal{M}_1$. But $\mathcal{M}_1 \approx \mathcal{M}_o \otimes b$, so uniqueness of minimal models gives a homotopy-commutative diagram



This means that there is a CDGA map $\Phi : \mathcal{M}_o^I \to \mathcal{M}_1$ and a strictly commutative diagram



By Cor. 2.5 each of $j_{\mathcal{B}}$ and $(i_1)_{\mathcal{B}}$ is a quasi-iso, so by naturality $\Phi_{\mathcal{B}}$ is a quasi-iso. Again by Cor. 2.5 ; $(i_o)_{\mathcal{B}}$ is a quasi-iso, so naturality in the upper triangle finally implies that $\phi_{\mathcal{B}}$ is a quasi-iso.

2.8 COROLLARY. - *If* $\mathcal{M} \underset{\rho_1}{\overset{\rho_0}{\rightrightarrows}} X^*$ *are homotopic then* $\mathcal{B}(\rho_0, X)$ *and* $\mathcal{B}(\rho_1, X)$ *are homotopy equivalent.*

PROOF. - A homotopy

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{t_0} & \mathcal{M}^I & \xleftarrow{i_1} & \mathcal{M} \\ & {\scriptstyle \rho_0}\searrow & \downarrow{\scriptstyle \Phi} & \swarrow{\scriptstyle \rho_1} & \\ & & X^X & & \end{array}$$

yields quasi-isos $\mathcal{B}(\rho_0, X) \xrightarrow[(i_0)_{\mathcal{B}}]{} \mathcal{B}(\Phi, X) \xleftarrow[(i_1)_{\mathcal{B}}]{} \mathcal{B}(\rho_1, X)$.

3. PROOF OF THEOREM B.

Recall that $\mathcal{M}$ is a (pos.graded) nilpotent CDGA, $f : X_0 \to X_1$ a quasi-iso of CDGC's and $\rho : \mathcal{M} \to X_1^*$ a CDGA map. We must show that

$$f_{\mathcal{B}} : \mathcal{B}(f^*_\circ \rho, X_0) \longrightarrow \mathcal{B}(\rho, X_1)$$

is a quasi-iso. We give an outline of the proof here.

The idea is to use the nilpotence of $\mathcal{M}$ to give and inductive proof. Choose a basis $\{y_\alpha\}$ for the generators Q_M of $\mathcal{M}$ such that $d_{\mathcal{M}}(y_\alpha) \in \mathcal{M}_{<\alpha}$, the sub-DGA generated by $\{y_\beta \mid \beta<\alpha\}$. Using $\mathcal{M}_{\leqslant\alpha} \hookrightarrow \mathcal{M} \longrightarrow X_1$ we construct CDGA's $(\mathcal{B}_0)_{\leqslant\alpha}$ and $(\mathcal{B}_1)_{\leqslant\alpha}$ with f inducing

$$(f_{\mathcal{B}})_{\leqslant\alpha} : (\mathcal{B}_0)_{\leqslant\alpha} \to (\mathcal{B}_1)_{\leqslant\alpha}.$$

We prove by inducing on α that $(f_{\mathcal{B}})_{\leqslant\alpha}$ is a quasi-iso.

The first step is to decompose the generators $(Q_{\mathcal{M}})_{\leqslant\alpha} \otimes X_i$ as $((Q_{\mathcal{M}})_{<\alpha} \otimes X_i) \oplus ((y_1) \otimes X_i)$. The next step is to decompose X_0 and X_1 in the form $H_i \oplus B_i \oplus C_i$ (as in lemme 2.1) so that the induced map $H_0 \to H_1$ is simply $H(f) : H(X_0) \overset{\approx}{\to} H(X_1)$. Just as in lemme 2.1, this leads to a decomposition

$$(\mathcal{B}_i)_{\leq\alpha} \simeq (\mathcal{B}_i)_{<\alpha} \otimes \Lambda R_i \otimes \Lambda S_i \quad , i = 0,1 ,$$

where, denoting $|y_\alpha| = n$,

$$R_i^p = (y_\alpha) \otimes H_i^{-n+p} \quad , p \geq 1 ,$$

$$S_i^p = \begin{cases} (y_\alpha) \otimes C_i^{-n+1} , & p = 1 , \\ (y_\alpha) \otimes B_i^{-n+p} \oplus (y_\alpha) \otimes C_i^{-n+p} & , p > 1. \end{cases}$$

With these identifications $(f_{\mathcal{B}})_{\leq\alpha}$ is of the form

$$(f_{\mathcal{B}})_{<\alpha} \otimes \Lambda H(f) \otimes \phi : (\mathcal{B}_o)_{<\alpha} \otimes \Lambda R_o \otimes \Lambda S_o \to (\mathcal{B}_1)_{<\alpha} \otimes \Lambda R_1 \otimes \Lambda S_1 .$$

Each ΛS_i is evidently acyclic, while $\Lambda H(f) : \Lambda R_o \xrightarrow{\simeq} \Lambda R_1$ is an isomorphism and, by the inductive hypothesis ,

$$(f_{\mathcal{B}})_{<\alpha} : (\mathcal{B}_o)_{<\alpha} \longrightarrow (\mathcal{B}_1)_{<\alpha}$$

is a quasi-iso. It follows easily that $(f_{\mathcal{B}})_{\leq\alpha}$ is a quasi-iso , and this completes the induction.

REFERENCES.

D. SULLIVAN Infinitesimal computations in topology, preprint, 1975.