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ON THE PERMANENCE OF THE HAHN-BANACH PROPERTY IN BORNOLOGY

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Little is known about the permanence of the Hahn-Banach property in bornological convex spaces (b. c. s.). It is known that this property is not inherited by arbitrary subspaces nor preserved by countable direct sums. In fact, the situation is worse, for we give an example below which shows that it is not preserved even under the formation of finite products. However, we prove some positive results describing cases in which the Hahn-Banach property is inherited by certain subspaces and preserved by products.

1 - We follow the usual notation and terminology (e.g. [5]). A b. c. s. E is said to have the Hahn-Banach property if every bounded linear functional on a b-closed subspace of E can be extended to a bounded linear functional on all of E. As remarked by Hogbe-Nlend [4], the hypothesis that the subspace be b-closed is necessary, since a bounded linear functional need not have a bounded extension from a subspace $F \subset E$ to its b-closure $\bar{F}$.

Two bornologies on a linear space E are called consistent if they generate the same Mackey topology (i.e. the same dual) and strictly consistent if they generate the same Mackey-closure topology. In [6, proposition 2] we have shown that not all consistent bornologies have the Hahn-Banach property; in other words, this property does not depend only on the duality. On the other hand, [6, theorem] immediately yields that if a b. c. s. E has the Hahn-Banach property, then it also has this property under any other bornology which is either strictly consistent or coarser and consistent with the bornology of E, whence in particular under the weak bornology. b-closed subspaces and separated quotients inherit the Hahn-Banach property [6, theorem and proposition 3] but as we shall see finite products (direct sums) do not. In particular, by using an idea due to Grothendieck we show that even the product of a Banach space and a b.c.s. with the Hahn-Banach property may fail to have the Hahn-Banach property. In fact, with the notations of [2, p. 90] let $\lambda = \ell^1$ written as a space of double sequences and let $\mu = \bigoplus_{n=1}^{\infty} \ell^2$ (μ has the Hahn-Banach property by [4, corollaire 1]). Then $E = \lambda \cap \mu = \bigoplus_{n=1}^{\infty} \ell^1$ and since $\lambda^\times = \ell^\infty$, $\mu^\times = \prod_{n=1}^{\infty} \ell^2$ and $E^\times = \prod_{n=1}^{\infty} \ell^\infty$, the hypotheses of [2, p. 90] are easily seen to be satisfied. It follows that the image of E in $F = \lambda \times \mu$ under the map $x \longrightarrow (x,x)$ is a (weakly) closed subspace of F on which there are [2, p. 92] bounded linear functionals that have no bounded extension to F. This simple example shows that for complete, non-normable b.c.s. of countable type the conditions of [4, théorème 1] cannot be relaxed too much if we wish to retain the Hahn-Banach property. However, it will be shown below (proposition 4) that for such spaces this property does not imply reflexivity.

2 - For (not necessarily b-closed) subspaces we have:

PROPOSITION 1. <u>Let</u> E <u>be a b.c.s. with the Hahn—Banach property. Then:</u>

(a) <u>If</u> H <u>is a subspace of</u> E <u>with finite codimension, then</u> H <u>has the Hahn-Banach property.</u>

(b) <u>If</u> E <u>is the bornological inductive limit of a sequence of b.c.s. of type</u> (F) [5,p. 42] <u>and if</u> H <u>is a subspace of</u> E <u>with countable codimension, then</u> H <u>has the Hahn-Banach property.</u>

Proof. (a) If F is a b-closed subspace of H, then the b-closure $\bar{F}$ and the b-adherence $F^{(1)}$ of F in E coincide. Thus a bounded linear functional f on F has a bounded extension to $\overline{F}$, whence to E, which restricted to H is a bounded extension of f to H.

(b) Let E be the inductive limit of a sequence (E_n) of b.c.s. of type (F), and let F be a b-closed subspace of H. For every n denote by $\hat{F}_n$ the b-closure of $F \cap E_n$ in the b.c.s. E_n. Each $\hat{F}_n$ is clearly a b.c.s. of type (F) for the bornology induced by E_n. The inductive limit $\hat{F}$ of the sequence $(\hat{F}_n)$ has an algebraic complement G in $\overline{F}$ with at most countable dimension. If we give G the finest convex bornology, then the identity map $\hat{F} \oplus G \longrightarrow \overline{F}$ is a bounded bijection, whence a bornological isomorphism by [5, corollaire 2, p. 43]. But $\hat{F}$ is complete, hence closed in $\overline{F}$, which implies that $\hat{F} = \overline{F}$.

Now let f be a bounded linear functional on F and for each n let f_n be the restriction of f to $F \cap E_n$. Then f_n has a bounded extension $\hat{f}_n$ to $\hat{F}_n$ [3, proposition 8, p. 220 and corollaire 2, p. 230] and the sequence $(\hat{f}_n)$ defines a bounded extension $\hat{f}$ of f to $\hat{F}$. The rest of the proof is as in (a).

We are now going to describe two special cases of products in which the Hahn-Banach property is preserved. For any cardinal d let ϕ_d (ω_d) denote the direct sum (product) of d copies of the scalar field. For countable d we simply write ϕ and ω. Moreover, we denote by E_τ the locally convex space associated with a b.c.s. E.

PROPOSITION 2. Let E be a b.c.s. with the Hahn-Banach property. Then $\phi_d \times E$ has the Hahn-Banach property.

Proof. Let F be a b-closed subspace of $\phi_d \times E$. If $F \subset E$ then F is weakly closed by [6, theorem]; hence suppose that $F \not\subset E$. The projection of F onto ϕ_d is weakly closed in ϕ_d and bornologically isomorphic to ϕ_{d_0} for some cardinal $d_0 \leq d$. Since F is a b-closed subspace of $\phi_{d_0} \times E$ and this space is weakly closed in $\phi_d \times E$, we may assume that the projection of F onto ϕ_d is all of ϕ_d. The subspace $L = F \cap E$ is b-closed in E, hence weakly closed [6, theorem] and, therefore, $(\phi_d \times E)/L$ is a regular b.c.s.

bornologically isomorphic to $\phi_d \times (E/L)$. If now Φ denotes the canonical map of $\phi_d \times E$ onto $\phi_d \times (E/L)$, then $\Phi(F)$ can be taken as the graph of a linear map $u : \phi_d \longrightarrow E/L$. Since u is continuous from $(\phi_d)_\tau$ to $(E/L)_\tau$, $\Phi(F)$ is closed in $(\phi_d)_\tau \times (E/L)_\tau$, whence in $((\phi_d)_\tau \times E_\tau)/L = (\phi_d \times E)_\tau/L$ and finally, $F = \Phi^{-1}(\Phi(F))$ is closed in $(\phi_d \times E)_\tau$, for Φ is continuous from $(\phi_d \times E)_\tau$ to $(\phi_d \times E)_\tau/L$.

We have shown that every b-closed subspace of $\phi_d \times E$ is weakly closed and the assertion now follows from [6, theorem].

Note that for locally convex spaces a proof similar to that of proposition 3 gives the following permanence property of $B_{(r)}$-completeness.

COROLLARY 1. If E is a $B_{(r)}$-complete locally convex space, then so is $E \times (\omega_d)_\tau$.

PROPOSITION 3. Let E be a topological b.c.s. of type (S) [5, p.59] with the Hahn-Banach property. Then $E \times \omega$ has the Hahn-Banach property.

Proof. Under the topology of uniform convergence on the bounded subsets of E the dual $E^\times$ is barreled and B-complete by [7, theorem 5 (i = 3)]. It follows [1, theorem 1] that $E^\times \times \phi_\tau = (E \times \omega)^\times$ is B-complete, whence $E \times \omega$ has the Hahn-Banach property in virtue of [7, theorem 5 (i = 3)].

3 - If in the definition of the Hahn-Banach property we drop the requirement that the subspace from which a bounded linear functional is to be extended be b-closed, we obtain the definition of what we call the strict Hahn-Banach property. The problem of giving necessary and sufficient conditions for a b.c.s. to have this property appears in general to be even more delicate than that concerning the Hahn-Banach property. In the case of

complete b.c.s. of countable type, however, the situation is reversed. In fact, while the problem of characterising the class of complete b.c.s. of countable type which have the Hahn-Banach property is still open, those which have the strict Hahn-Banach property can be described exactly. Precisely, we have

PROPOSITION 4. A complete b.c.s. E of countable type has the strict Hahn-Banach property if and only if it is

(a) a Banach space,

(b) ϕ,

(c) the product of a Banach space and ϕ.

Proof. If E is as in (a), (b) or (c), then the b-adherence $F^{(1)}$ of a subspace $F \subset E$ is identical with the b-closure $\bar{F}$ [5, théorème, p. 54], hence with the completion $\tilde{F}$. A bounded linear functional on F has then a bounded extension to $\bar{F}$ [3, proposition 8, p. 220] and, by proposition 2, to E. Conversely, if E is not as in (a), (b) or (c), then E contains a subspace F whose b-adherence $F^{(1)}$ is not b-closed. Let $x \in \bar{F} \sim F^{(1)}$ and let L be the subspace of E spanned by F and x. Suppose that there is a sequence (z_n) in F which converges to $z \in L$. Then $z \in F^{(1)}$ and, since $z = \alpha x + y$ with $y \in F$, we must have $\alpha x \in F^{(1)}$, which implies $\alpha = 0$ and $z = y \in F$. This shows that F is b-closed in L and, since it has codimension 1, there exists a bounded linear functional f on L such that $f(y) = 0$ for all $y \in F$ and $f(x) = 1$. But then f cannot have a bounded extension to E and, therefore, E does not have the strict Hahn-Banach property.

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