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COMPACTNESS IN $C_s(T)$ AND APPLICATIONS

Richard HAYDON (*)

1. - INTRODUCTION.

In this paper I look at some properties of compact subsets of $C_s(T)$ which have applications to the "more interesting" space $C_c(T)$. A little light is cast on the difficult problem of when $C_c(T)$ may be a Kelley space, the concept of infra- k_R -space is examined, and lastly I offer two generalizations of a theorem of BUCHWALTER concerning the repletion $\cup T$.

The notations throughout are "standard Lyon". The algebra $C(T)$ of all continuous real-valued functions on the completely regular space T may be endowed with the topology either of simple, compact or bounded convergence on T and is then denoted by $C_s(T)$, $C_c(T)$ or $C_b(T)$, respectively.

2. - ON KELLEY SPACES $C_c(T)$.

The characterization of $M(T)$ as the space $C_c(\theta T)' = M_c(\theta T)$, of all measures of compact support on the c -repletion θT , enables one to deduce ($\{B_1\}$ and $\{H_1\}$) that $C_c(\theta T)$ is always a Kelley space ($\{B_1\}$) and that, when T is a k_R -space, $C_c(T)$ is Kelley if and only if T is c -replete. Put into an attractively symmetric form :

$C_c(T)$ is a complete Kelley space $\iff T$ is a c -replete k_R -space.

One can, however, say more, namely that, when T is a k_R -space, $C_c(\theta T)$ is the *Kelleyfié* $\bar{k} C_c(T)$ ($\{B_1\}$) of $C_c(T)$.

But what can we say if we do not assume T to be a k_R -space ? We can note first that the property used to prove the above results is not the full strength

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of being a k_R -space, but only that the compact discs of $C_c(T)$ should be equicontinuous. H. BUCHWALTER has introduced the definition of a property intermediate between these :

(2.1) DEFINITION. - T is said to be an infra- k_R -space if every precompact subset of $C_c(T)$ is equicontinuous.

If T is the space of (H_2) , θT is infra- k_R and not k_R . Evidently, when T is an infra- k_R -space, θT is also infra- k_R and we have $C_c(\theta T) = \bar{k} C_c(T)$.

But this last equality does not hold for arbitrary T , as has been pointed out in (H_1) . I want to consider here the problem posed at the end of that Note :

If $C_c(T)$ is Kelley, need T be c -replete ?

This question remains open still, but I am able to give some partial results and to show how it is linked to properties of compactness in $C_s(T)$.

(2.2) PROPOSITION. - Let T be non- c -replete and suppose that $C_c(T)$ is a Kelley space. Then there is a compact disc in $C_c(T)$ that is not compact in $C_s(\theta T)$.

Proof. - The continuous characters of the algebra $C_c(T)$ are the evaluations $\delta_t (t \in T)$. If $u \in \theta T \setminus T$, u is not continuous on $C_c(T)$ and, since $C_c(T)$ is Kelley, not continuous on some compact disc of $C_c(T)$. This disc is not compact in $C_s(\theta T)$.

I know of no example of a space T for which some compact subset of $C_c(T)$, even of $C_s(T)$, fails to be compact in $C_s(\theta T)$. Propositions (2.4) and (2.8) suggest that such a space (if one exists !) would be difficult to construct.

Let us denote by $R(T)$ the set of all closures in T of K_G subsets of T and consider the property :

(A) Every function $\psi \in \mathbb{R}^T$ which coincides on each $C \in R$ with a suitable $f \in C(T)$ is itself in $C(T)$.

This property was introduced by J.D. PRYCE who proved :

(2.3) THEOREME ((P), Theorem 2.4). - If T has property (A) then every relatively countably compact (rcc) subset of $C_s(T)$ is relatively compact (rc) in $C_s(T)$.

(2.4) PROPOSITION. - When θT has property (A) the compact subsets of $C_s(T)$ are compact in $C_s(\theta T)$.

Proof. - When (f_n) is a sequence in $C(T)$ and $u \in \theta T$, there exists $t \in T$ such that $f_n(t) = f_n^u(u)$ for every integer n . It follows at once from this that the rcc subsets of $C_s(T)$ and $C_s(\theta T)$ (hence also of $C_s(\theta T)$) are the same. Thus an rc subset of $C_s(T)$ is rcc in $C_s(\theta T)$ and, by the theorem of PRYCE, rc in $C_s(\theta T)$. If a subset is compact in $C_s(T)$ it is rc and closed, hence compact, in $C_s(\theta T)$.

We can note that θT satisfies (A) if θT is k_R or if there is a dense K_σ subset of θT , in particular if T is pseudocompact or has a dense B_σ (σ -bounded) subset.

Write $R'(T)$ for the set of all closures in T of B_σ subsets of T . PRYCE's theorem allows the generalization below.

(2.5) PROPOSITION. - Let T be a completely regular space that satisfies :

(A') Every $\psi \in \mathbb{R}^T$ which coincides on each $C \in R'$ with a suitable $f \in C(T)$ is itself in $C(T)$.

Then every rcc subset of $C_s(T)$ is rc in $C_s(T)$.

Proof. - Suppose first that T satisfies (A'). I shall show that the bidual T'' satisfies (A). Recall that the bidual of T is defined $((B_2))$ as the space T'' of all continuous characters of the algebra $C_b(T)$, embedded as a subspace of θT .

Let ψ be a real-valued function on T'' and suppose that for all $C \in R(T'')$ there is an $f \in C(T'')$ with $f|_C = \psi|_C$. Now if B is a bounded subset of T , $\bar{B}$, taken in T'' , is compact, so that the T'' closure $\bar{D}$ of any $D \in R'(T)$ is in $R(T'')$. Thus, for every such D , there is a $g \in C(T)$ such that $g|_D = \psi|_D$. Applying (A'), we see that $\psi|_T \in C(T)$. Let us denote by ϕ the continuous extension of $\psi|_T$ to T'' . It will be enough to prove that $\phi = \psi$. If B is bounded in T , $\bar{B}$ is compact in T'' ; ϕ and ψ are both continuous on $\bar{B}$ and coincide on B . Hence ϕ and ψ coincide on $\bar{B}$. But by proposition 2 of (B_2) we know that $T'' = \bigcup \{\bar{B} ; B \text{ bounded in } T\}$ and we can deduce that ϕ and ψ coincide on T'' .

If now A is rcc in $C_s(T)$, A is rcc in $C_s(T'')$ by the same reasoning as was used in proposition (2.4). A is therefore rc in $C_s(T'')$ and so certainly rc in

$C_s(T)$.

(2.6) DEFINITION. - A space X is said to be angelic ([P], p. 534) if

- (i) $rcc \implies rc$ for the subsets of X , and
- (ii) every element of the closure of an rc subset A of X is the limit of some sequence in A .

If $T \in R'(T)$, we know already by the first part that (i) is satisfied. PRYCE showed that $C_s(T)$ is angelic if $T \in R(T)$ ([P], theorem 2.5). Therefore $C_s(T'')$ is angelic. If A is rc in $C_s(T)$ (and hence also in $C_s(T'')$) and $f \in \bar{A}$ (the closure being the same in the two topologies), there is a sequence in A that converges to f in $C_s(T'')$, and which converges to f , a fortiori, in $C_s(T)$. Then :

(2.7) PROPOSITION. - If $T \in R'(T)$ (particularly if T is pseudocompact), $C_s(T)$ is angelic.

(2.8) PROPOSITION. - Let T be a (completely regular) space in which all closed and discrete subspaces are C^∞ -embedded (particularly if T is normal or countably compact) and that satisfies :

- (B) For every $u \in \theta T \setminus T$ there is a base U of neighbourhoods of u in θT such that, whenever $V \subset U$ and the cardinality of V is strictly less than that of U , then $T \cap (\bigcap V)$ is nonempty.

Then the compact subsets of $C_s(T)$ are compact in $C_s(\theta T)$.

Proof. - It is enough to show that every character $u \in \theta T$ is continuous on each compact $A \subset C_s(T)$. Suppose then that u is not continuous on such an A ; there is a net (f_α) in A such that $f_\alpha \rightarrow f$ in $C_s(T)$ while $f_\alpha^\theta(u) \rightarrow l \neq f^\theta(u)$. We can assume that the f_α are uniformly bounded by 1, that $f_\alpha \rightarrow 0$ in $C_s(T)$ and that $f_\alpha^\theta(u) = 1$ for all α .

Let U be a base of neighbourhoods of u in θT with the property of (B). Then if $B \subset C(T)$, $V \subset U$ and $\text{card } B, \text{card } V$ are strictly less than $\text{card } U$, there exists $t \in T$ such that $t \in \bigcap V$ and $f(t) = f^\theta(u)$ for every $f \in B$. Let us denote by Ω the first ordinal of cardinality $\text{card } U$ and index U as $(U_\xi)_{\xi < \Omega}$. I shall define, by transfinite induction, families (x_ξ) in T and (g_ξ) in A with the properties :

- (a) $g_\xi(x_\zeta) \leq 1/2$ ($\xi \geq \zeta$),
- (b) $g_\xi(x_\zeta) = 1 = g_\xi^\theta(u)$ ($\xi < \zeta$),
- (c) $x_\xi \rightarrow u$ in θT .

Let x_0 be an arbitrary point of T and choose α_0 such that $f_{\alpha_0}(x_0) \leq 1/2$. Put $g_0 = f_{\alpha_0}$. Suppose that x_ξ and g_ξ have been defined for all ξ less than some $\eta < \Omega$ and that (a) and (b) are satisfied. Since the cardinality of $[0, \eta[$ is less than $\text{card } U$, there exists $x_\eta \in T \cap (\bigcap_{\xi < \eta} U_\xi)$ such that $g_\xi(x_\eta) = g_\xi^\theta(u) = 1$ ($\xi < \eta$).

Let us now choose, for each finite subset S of $[0, \eta]$, an α_S such that $f_{\alpha_S}(x_\xi) \leq 1/2$ ($\xi \in S$). Let g_η be a cluster point of the net (f_{α_S}) , directed by the upward filtering set of finite subsets of $[0, \eta]$. Then we have $g_\eta(x_\xi) \leq 1/2$ ($\xi \leq \eta$) and $g_\eta^\theta(u) = 1$ (because there is $t \in T$ with $g_\eta(t) = g_\eta^\theta(u)$ and $f_{\alpha_S}(t) = f_{\alpha_S}^\theta(u)$ for every finite set $S \subset [0, \eta]$).

Since, by construction, each x_η is in $\bigcap_{\xi < \eta} U_\xi$, we see that $x_\eta \rightarrow u$ in θT . I shall now show that $\{x_\eta ; \eta < \Omega\}$ is a closed discrete subspace of T . If not, there is $\zeta < \Omega$ such that $\{x_\eta ; \eta < \zeta\}$ has an accumulation point x in T . Choose to be the least such ordinal ; then x is in the closure of $\{x_\eta ; \xi < \eta < \zeta\}$ for each $\xi < \zeta$. Hence $g_\xi(x) = 1$ for every $\xi < \zeta$. Let g be a cluster point of the net $(g_\xi)_{\xi < \zeta}$. Then

$g(x) = 1$, but

$g(x_\eta) \leq 1/2$ ($\eta < \zeta$), since

$g_\xi(x_\eta) \leq 1/2$ ($\eta < \xi < \zeta$). This contradicts the continuity of g at x .

Since $\{x_\xi ; \xi < \Omega\}$ is a closed discrete subspace of the space T , there is a continuous function $f \in C(T)$ with $f(x_\xi) = 0$ (ξ an isolated ordinal)
 $f(x_\xi) = 1$ (ξ a limit ordinal).

But such an f can have no extension that is continuous on θT , and this contradiction ends the proof.

Proposition (2.8) applies in particular to the non-c-replete P -space of $((GJ), 9.L)$. In this case there exists, for every $\psi \in \mathbb{R}^{\theta T}$ and every $C \in R(\theta T)$,

a function $f \in C(\theta T)$ with $f|_C = \psi|_C$; a situation very different from that considered in proposition (2.4).

For the last result in this paragraph, we return to the methods of propositions (2.4) and (2.5).

(2.9) PROPOSITION. - A compact subset of $C_s(T)$ remains compact in $C_s(\mu T)$.

Proof. - By the characterization of μT as the space obtained by transfinite iteration of the bidual operation ($\{B_2\}$, théorème 2), it is enough to prove that a compact subset A of $C_s(T)$ is compact in $C_s(T'')$. Such an A is countably compact in $C_s(T'')$ and hence, for each bounded $B \subset T$, $A|_{\vec{B}^\cup}$ is countably compact in $C_s(\vec{B}^\cup)$. But countable compactness and compactness coincide in this space, since $\vec{B}^\cup$ is compact. Thus, for all characters u in $\vec{B}^\cup$, $u|_A$ is continuous for the topology of pointwise convergence on B , and we deduce that $u|_A$ is $C_s(T)$ -continuous for every $u \in T''$.

(2.10) COROLLARY. - If $C_c(T)$ is a Kelley space then T is a μ -space, i.e. $C_c(T)$ cannot be Kelley without being barrelled.

3. - INFRA- k_R -SPACES.

The space T of (H_2) has given us an example of a complete lcs $E = C_c(T)$, the Kelleyfié of which, $F = \bar{k}E = C_c(\theta T)$, is not quasi-complete. F is, however, a *p-semi-reflexive* space ($\{DJ\}$), that is to say, every precompact subset is relatively compact. In this example θT happens to be an infra- k_R -space, but it would seem, a priori, that the property "every precompact set is relatively compact" was a good deal weaker than the infra- k_R -property, "every precompact set is equicontinuous". But it turns out that this is not the case.

(3.1) THEOREM. - T is an infra- k_R -space if and only if every precompact subset of $C_c(T)$ is relatively compact in $C_s(T)$.

(3.2) COROLLARY. - T is an infra- k_R -space if and only if $C_c(T)$ is *p-semi-reflexive*.

We shall need a definition and two preliminary results.

(3.3) DEFINITION. - Let us say that a subset H of $C(T)$ is closed under lattice operations (or, more simply, lattice-closed) if $f \vee g \in H$ and $f \wedge g \in H$ whenever $f, g \in H$. If $H \subset C(T)$, define the lattice-closed hull ΛH of H to be the smallest lattice closed set that contains H .

(3.4) LEMMA. - For a subset H of $C(T)$ the following are equivalent :

- (a) H is precompact in $C_c(T)$;
- (a') for every compact $K \subset T$, $H|K$ is bounded and equicontinuous in $C(K)$ (i.e. $H|K \in H(K)$) ;
- (b) ΛH is precompact in $C_c(T)$;
- (b') for every compact $K \subset T$, $\Lambda H|K \in H(K)$.

Proof. - The equivalences (a) $\Leftrightarrow$ (a') and (b) $\Leftrightarrow$ (b') are consequences of ASCOLI's theorem. (a') is equivalent to (b') since the lattice-closed hull of an equicontinuous set is equicontinuous.

(3.5) PROPOSITION. - A lattice-closed, relatively compact subset of $C_s(T)$ is equicontinuous.

Proof. - Let H be such a set and suppose, if possible, that H is not equicontinuous at some $t \in T$. We can assume that, for some $\varepsilon > 0$, there are, for each neighbourhood U of t , a function $h_U \in H$ and a point $t_U \in U$ such that

$$h_U(t_U) \geq h_U(t) + \varepsilon.$$

Now the set $\{h(t) ; h \in H\}$ is bounded in $\mathbb{R}$ and there exists a subnet of $(h_U(t))$ convergent to some $\alpha \in \mathbb{R}$. That is to say that there is a base $\mathcal{U}$ of neighbourhoods of t such that $h_U(t) \rightarrow \alpha$ as U decreases through $\mathcal{U}$. We can suppose that $|h_U(t) - \alpha| \leq \varepsilon/3$ ($U \in \mathcal{U}$), so that $h_U(t) \leq \alpha + \varepsilon/3$ and $h_U(t_U) \geq \alpha + 2\varepsilon/3$ for all $U \in \mathcal{U}$.

Now let us define, for each finite subset $F = \{U_1, \dots, U_n\}$ of $\mathcal{U}$, $g_F = h_{U_1} \vee \dots \vee h_{U_n}$ and note that $g_F(t) \leq \alpha + \varepsilon/3$ for all F , and $g_F(t_U) \geq \alpha + 2\varepsilon/3$ whenever $U \in F$.

Each g_F is in H and so there is a subnet of (g_F) convergent in $C_s(T)$ to some g (in fact, to $g = \sup_{U \in \mathcal{U}} h_U$) and we see that $g(t) \leq \alpha + \varepsilon/3$ while $g(t_U) \geq \alpha + 2\varepsilon/3$ ($U \in \mathcal{U}$). This contradicts the continuity of g at t .

Proof of theorem (3.1). - The necessity of the condition comes from the fact that a pointwise bounded equicontinuous subset of $C(T)$ is relatively compact in $C_s(T)$.

Suppose now that the condition is satisfied and that H is a precompact subset of $C_c(T)$. By lemma (3.4), ΛH is precompact in $C_c(T)$, and hence relatively compact in $C_s(T)$. But now, by proposition (3.5), we deduce that ΛH is equicontinuous.

4. - TWO GENERALIZATIONS OF A THEOREM OF BUCHWALTER.

H . BUCHWALTER has shown that, if $\cup T$ is a k_R -space, then necessarily $\cup T = \theta T$. There follow two generalizations of this result.

(4.1) LEMMA. - If $H \in \mathcal{H}(T)$ and $\text{card } H$ is non-measurable, then the metrizable space T_H is replete and $H^\cup \in \mathcal{H}(\cup T)$.

Proof. - Recall that T_H is defined to be the Hausdorff quotient of T endowed with the pseudometric $d(s, t) = \sup_{h \in H} |h(s) - h(t)|$. There is an injection $T_H \rightarrow \mathbb{R}^H$ so that $\text{card } T_H \leq c^{\text{card } H}$. Now if m, n are non-measurable cardinals, so is m^n ([I], p. 128) and it follows that T_H is replete.

H factors through the quotient mapping $\pi_H : T \rightarrow T_H$, as $H = H_1 \circ \pi_H$ where $H_1 \in \mathcal{H}(T_H)$. Since T_H is replete, π_H extends to $\pi_H^\cup : \cup T \rightarrow T_H$ and $H^\cup = H_1 \circ \pi_H^\cup \in \mathcal{H}(\cup T)$.

(4.2) THEOREM. - Let T be a completely regular space and suppose either :

- (a) $\cup T$ has property (A), or
- (b) $\cup T$ is an infra- k_R -space.

Then $\cup T = \theta T$.

Proof :

(a) Let $H \in \mathcal{H}(T)$. H is relatively compact in $C_s(T)$ and hence relatively countably compact in $C_s(\cup T)$. By the theorem of PRYCE, H is relatively compact in $C_s(\cup T)$. We can deduce that the topologies of $C_s(T)$ and of $C_s(\cup T)$ coincide on H and hence that, for any $u \in \cup T$, $u|_H$ is continuous for the topology of simple convergence on T . But this is exactly the condition for a character u to be in θT .

(b) Again suppose $H \in \mathcal{H}(T)$. As above, it will be enough to show that $H^\cup$ is relatively compact in $C_s(\cup T)$ and hence enough to show that $H^\cup$ is precompact in $C_c(\cup T)$. This will be true provided that $J^\cup$ is precompact in $C_c(\cup T)$ for each countable $J \subset H$. But, by lemma (4.1), we know that each $J^\cup$ is even in $\mathcal{H}(\cup T)$.

REFERENCES

- (B₁) H. BUCHWALTER, *Topologies et compactologies*, Publ. Dép. Math. Lyon, t. 6-2, 1969, p. 1-74.
- (B₂) H. BUCHWALTER, *Parties bornées d'un espace topologique complètement régulier*, Séminaire Choquet, 9e année, 1969-1970, n° 14, 15 pages.
- (BI) J. BERRUYER - B. IVOL, *Comptes rendus*, 275, série A, 1972, p. 33.
- (DJ) J. DAZORD - M. JOURLIN, *Comptes rendus*, 274, série A, 1972, p. 463 et p. 550.
- (GJ) L. GILLMAN - M. JERISON, *Rings of continuous functions*, Princeton, 1960.
- (H₁) R. HAYDON, *Comptes rendus*, 275, série A, 1972, p. 989.
- (H₂) R. HAYDON, *Comptes rendus*, 275, série A, 1972, p. 1077.
- (I) J. ISBELL, *Uniform spaces*, Providence, 1964.
- (P) J.D. PRYCE, *A device of R.J. Whitley's applied to pointwise compactness in spaces of continuous functions*, Proc. London Math. Soc., 3, 23, 1971, p. 532-546.

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