MÉMOIRES DE LA S. M. F.

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*Mémoires de la S. M. F.*, tome 38 (1974), p. 69-72

<http://www.numdam.org/item?id=MSMF_1974__38__69_0>

Journées Géom. analyt. [1972. Poitiers]
Bull. Soc. math. France,
Mémoire 38, 1974, p. 69-72.



# FROBENIUS' THEOREM FOR DIFFERENTIAL FORMS ON ANALYTIC SPACES

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Let be $A = \mathbb{C}\{Z\}_{k/\mathfrak{a}}$, $B = \mathbb{C}\{X\}_{m/\mathfrak{b}}$, $C = \mathbb{C}\{Y\}_{n/\mathfrak{c}}$, where $\mathbb{C}\{.\}$ is the ring of all convergent power series. The images of $Z_\chi, X_\mu, Y_\nu$ in $A$, $B$, $C$ will be denoted by $z_\chi, x_\mu, y_\nu$. $\Omega^1(A)$ is the finite differential module of $A$ over $\mathbb{C}$. The module $\overset{r}{\Lambda}\, \Omega^1(A)$ of all differential forms of degree $r$ is denoted by $\Omega^r(A)$. $d : \Omega^r(A) \to \Omega^{r+1}(A)$ is the natural derivation.

The rings $B$, $C$ are called a decomposition of $A$ if $A$ is the analytic tensor product of $B$ and $C$ :

$$A = B \hat{\otimes} C = \mathbb{C}\{X,Y\}/\mathbb{C}\{X,Y\}.(\mathfrak{b}, \mathfrak{c}).$$

Let $M_o$, $N_o$ be germs of complex analytic varieties and let $B = \mathcal{O}(M_o)$, $C = \mathcal{O}(N_o)$ be the (reduced) structure rings of the germs, then we have

$$B \hat{\otimes} C = \mathcal{O}(M_o \times N_o).$$

If $B$, $C$ are a decomposition of $A$, the module $\Omega^r(A)$ is a direct sum :

$$\Omega^r(A) = \sum_{p+q=r} \Omega^{p,q}(B,C),$$

where $\Omega^{p,q}(B,C)$ is the module generated by

$$\{dx_{\mu_1} \wedge \ldots \wedge dx_{\mu_p} \wedge dy_{\nu_1} \wedge \ldots \wedge dy_{\nu_q} \; : \; 1 \leqslant \mu_1 < \ldots < \mu_p \leqslant m, 1 \leqslant \nu_1 < \ldots < \nu_q \leqslant n\} .$$

We have $d = d_B + d_C$, where $d_B$, $d_C$ are the derivations relatively to $B$ resp. $C$, and we have a differential sequence

$$0 \to \Omega^p(B) \xrightarrow{\varepsilon} \Omega^{p,o}(B,C) \xrightarrow{d_C} \Omega^{p,1}(B,C) \to \ldots ,$$

where $\varepsilon$ is the natural injection. If $C$ is contractible, then this sequence is exact. Il $A$ is reduced, the sequence

$$0 \to B \to A \to \Omega^{o,1}(B,C)$$

is exact, and we have

$$B = d^{-1}(\Omega^{1,o}(B,C)), C = d^{-1}(\Omega^{o,1}(B,C)).$$

In this case the sum $\Omega^1(A) = \Omega^{1,o}(B,C) + \Omega^{o,1}(B,C)$ determines the rings $B$, $C$.

We now will study the following problem : Given a direct sum $\Omega^1(A) = \Omega' + \Omega''$ , can it be obtained from a decomposition of $A$ ?

THEOREM 1. Let $A$ be a domain and let the summands $\Omega'$ , $\Omega''$ of the direct sum $\Omega^1(A) = \Omega' + \Omega''$ be generated by elements $df$ , $f \in A$ . Then there are rings $B$ , $C$ such that $A = B \hat{\otimes} C$ , $\Omega' = \Omega^{1,0}(B,C)$, $\Omega'' = \Omega^{0,1}(B,C)$.-

Proof. We have

$$A = \mathbb{C}\{X,Y\}/\alpha, \Omega' = A.(dx_1,\ldots,dx_m),$$
$$\Omega'' = A.(dy_1,\ldots,dy_n).$$

If $A$ is regular, then there is an isomorphism $\varphi : A \to R$ , where $R$ is a ring of power series. By $\varphi$ we have an isomorphism $\varphi^1 : \Omega^1(A) \to \Omega^1(R)$ . We may suppose, that

$$R = \mathbb{C}\{U,V\}, \varphi^1(\Omega') = R.(dU_1,\ldots,dU_p),$$
$$\varphi^1(\Omega'') = R.(dV_1,\ldots,dV_q).$$

If $\varphi$ is given by the substitution of $\Phi = (\Phi',\Phi'')$ , $\Phi' = (\Phi'_1,\ldots,\Phi'_m)$ , $\Phi'' = (\Phi''_1,\ldots,\Phi''_n)$ , we have $d\Phi'_\mu = d\varphi(x_\mu) \in R.(dU_1,\ldots,dU_p)$, $\Phi'_\mu \in \mathbb{C}\{U\}$ and $\Phi''_\nu \in \mathbb{C}\{V\}$ . Then $\Phi'$ , $\Phi''$ are biholomorphic mappings of the germs $\mathbb{C}^p_o, \mathbb{C}^q_o$ onto germs $M_o \subset \mathbb{C}^m_o$, $N_o \subset \mathbb{C}^n_o$ . We have $A = \mathcal{O}(M_o \times N_o)$ .

In the general case $A$ is the structure ring of an irreducible germ $K_o$ . Let $K$ represent $K_o$ in an open neighbourhood $W = W' \times W''$ , $W' \subset \mathbb{C}^m$ , $W'' \subset \mathbb{C}^n$ , of $0$ . We use the following notations.

$\mathcal{O}$ , $\tilde{\mathcal{O}}$ are the structure sheaves of $W$ resp. $K$ , $\mathfrak{J}$ is the ideal sheaf of $K$ , $\tilde{\Omega}^1$ is the sheaf of differential forms of degree 1 on K. We set $\tilde{\Omega}' := \tilde{\mathcal{O}}.(dx_1,\ldots,dx_m)$ , $\Omega'' := \tilde{\mathcal{O}}.(dy_1,\ldots,dy_n)$ .

We may suppose, that the sum $\tilde{\Omega}^1 = \tilde{\Omega}' + \tilde{\Omega}''$ is direct and that $\mathfrak{J}$ is generated by holomorphic functions $h_1,\ldots,h_t$ on $U$ .

If $w^o \in K$ is a regular point, we have

$$\mathfrak{J}_{w^o} = \mathcal{O}_{w^o}.(h_1,\ldots,h_t) = \mathcal{O}_{w^o}.(f_1,\ldots,f_r,g_1,\ldots,g_s) , f_g \in \mathbb{C}\{X\}, g_\sigma \in \mathbb{C}\{Y\} .$$

Setting

$$M := \{w' \in W' : h_\tau(w',w^o_{m+1},\ldots,w^o_{m+n}) = 0 , \tau = 1,\ldots,t\},$$

$$N := \{w'' \in W'' : h_\tau(w^o_1,\ldots,w^o_m,w'') = 0 , \tau = 1,\ldots,t\}$$

we get $K_{w^o} = (M \times N)_{w^o}$ . Then $K_o$ must be an irreducible component of $(M \times N)_o$.-

STORCH has given an algebraic proof for theorem 1 ([3]). By STORCH's proof theorem 1 is valid in the complete case too.

If $A$ is regular, the theorem of FROBENIUS gives a condition for $\Omega'$ being generated by elements $df$, $f \in A$ :

Let $A$ be regular and let $\Omega^1(A) = \Omega' + \Omega''$ be a direct sum. Then we have : $\Omega'$ is generated by elements $df$, $f \in A$ iff $d\Omega' \subset \Omega^1 \wedge \Omega'$.–

In the singular case we have :

THEOREM 2. Let $A$ and the direct sum $\Omega^1(A) = \Omega' + \Omega''$ satisfy the following conditions : $A$ is a domain, $\Omega^1(A)$ is torsionless, there is a contraction vector field $v$ on $A$ such that $v(\Omega') = 0$, $\operatorname{emdim} A/v(\Omega^1(A)) = \dim \Omega'/\mathfrak{m}\Omega'$. Then we have : $\Omega'$ is generated by elements $df$, $f \in A$ iff $d\Omega' \subset \Omega^1 \wedge \Omega'$.–

An $A$-module $M$ is called torsionless if the natural mapping $M \to M^{**}$ ($M^{**}$ bidual module) is injective. For a reduced complete intersection the following are equivalent :

(i) $\Omega^1(A)$ is torsionless.

(ii) The codimension of the singular locus of $A$ is $\geqslant 2$.

(iii) $A$ is normal.

A contraction vector field $v$ on $A$ is a vector field on $A$, which in an appropriate coordinate system $Z_1, \ldots, Z_k$ can be represented by a vector field $\Sigma\, m_\varkappa Z_\varkappa\, \partial/\partial Z_\varkappa$, $m_\varkappa \geqslant 0$ integer. For the embedding dimension of $A_v := A/v(\Omega^1(A))$ we have the formula $\operatorname{emdim} A_v = \operatorname{emdim} A - \operatorname{rank} dv$, where $dv$ is the linear mapping in the tangent space given by the matrix

$$\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & \cdots & 0 & m_k \end{bmatrix} .$$

For the proof of theorem 2, see [2].

We give an application ([1], [2]).

The analytic ring $A = \mathbb{C}\{U\}/\mathfrak{a}$ is called real if we have $\bar{f} := \Sigma \bar{a}_\alpha U^\alpha \in \mathfrak{a}$ for all $f = \Sigma a_\alpha U^\alpha \in \mathfrak{a}$. The morphisms in the category of real analytic rings are given by substitutions of real power series.

If $K_o \subset \mathbb{R}^k_o$ is the germ of a real analytic variety, the ring $A = \mathcal{R}(K_o)$ of all germs of complex-valued real analytic functions on $K_o$ is a real analytic ring. We have $A = \mathcal{O}(\tilde{K}_o)$, where $\tilde{K}_o$ is the complexification of $K_o$.

A direct sum $\Omega^1(A) = \Omega' + \Omega''$ is called an almost holomorphic structure on the real analytic ring $A$ if we have $\overline{\Omega'} = \Omega''$. The quasi-local ring $H(A) := d^{-1}(\Omega')$ is called the ring of almost holomorphic functions. In general $H(A)$ is no analytic ring.

The germ $K_o \subset \mathbb{K}^k_o$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of a $\mathbb{K}$-analytic variety is called a cone if there is a coordinate system such that the ideal of $K_o$ in this coordinate system is homogeneous of a type $(m_1,\ldots,m_\ell)$, $m_\lambda > 0$.

We have ([2]) :

Let $K_o \subset \mathbb{C}^k_o$ be an irreducible germ of a complex analytic variety with an isolated singularity. Then $K_o$ is a complex cone iff $K_o$ is a real cone.-

Hereby and by theorem 2 we have ([2]) :

THEOREM 3. Let $K_o \subset \mathbb{R}^k_o$ be an irreducible real cone with an isolated singularity and let $\Omega^1(A)$, $A := \mathcal{R}(K_o)$, be torsionless. Then for an almost holomorphic structure $\Omega^1(A) = \Omega' + \Omega''$ the following are equivalent :

(i) $K_o$ is complex analytic with holomorphic structure ring $H(A)$.

(ii) We have $d\Omega' \subset \Omega^1 \wedge \Omega'$, and there is a contraction vector field $v$ on $A$ such that $v(\Omega') = 0$, rank $dv = 1/2$ emdim $A$.-

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(Texte reçu le 18/VII/1972)

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