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An average form of Artin's conjecture

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AN AVERAGE FORM OF ARTIN'S CONJECTURE

by

P. J. STEPHENS

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A more detailed account of the theorems described in this lecture is published in [1] .

The order, $e_p(a)$, of a modulo p is defined to be the least positive integer n for which $a^r \equiv 1 \pmod p$. It is well-known that there are $\phi(d)$ numbers of order d in any interval of length p . In particular, there are $\phi(p-1)$ primitive roots in any interval of length p . We define

$$M_p(N) = \sum_{\substack{a \leq N \\ e_p(a)=p-1}} 1 , \tag{1}$$

and deduce from the above remark that

$$M_p(N) = \phi(p-1) \{\frac{N}{p} + O(1)\} . \tag{2}$$

Using the notation

$$N_a(x) = \sum_{\substack{p \leq x \\ e_p(a)=p-1}} 1 . \tag{3}$$

Artin conjectured that

$$N_a(x) \sim A(a)\ x / \log x .$$

Forty years elapsed before Hooley proved that

$$N_a(x) = \frac{A(a)\ x}{\log x} + O\left(\frac{x \log\log x}{\log^2 x}\right) ,$$

is a consequence of the Riemann hypothesis holding for certain algebraic number fields.

A year later Goldfeld showed, without any hypothesis, that

$$N_a(x) = A\ \mathrm{li}(x) + O\ (x / (\log x)^D) \tag{4}$$

where $A = \prod_p (1 - \frac{1}{p(p-1)})$ is true for almost all $a \leq N$ where N satisfied

$$x \geq N > \exp \{c_1 (\log x \log\log x)^{\delta}\} .$$

He also found a bound for the number of exceptions.

In this note we shall outline the proof of the following theorem.

THEOREM A . *If* $N > \exp\{4(\log x \log\log x)^{1/2}\}$, *then*

$$\frac{1}{N}\sum_{a\le N} N_a(x) = A\,\mathrm{li}(x) + O\left(\frac{x}{(\log x)^D}\right) \qquad (5)$$

where D *is an arbitrary constant greater than* 1 .

By proving the following theorem it is possible to deduce a bound for the number of exceptions to (4) . We only state it here.

THEOREM B. If $N > \exp\{6(\log x \log\log x)^{1/2}\}$, then

$$\frac{1}{N}\sum_{a\le N}\{N_a(x) - A\,\mathrm{li}(x)\}^2 \ll x^2/(\log x)^E ,$$

where E *is an arbitrary constant greater than* 2 .

The following lemma may be proved using the Siegel-Walfisz theorem.

LEMMA. *For any choice of the constant* D *we have*

$$\sum_{p\le x}\frac{\phi(p-1)}{p} = A\,\mathrm{li}(x) + O\left(\frac{x}{(\log x)^D}\right) . \qquad (6)$$

We define

$$c(\chi) = \frac{1}{(p-1)}\sum_a{}' \chi(a)$$

where $\sum_a{}'$ runs over all the primitive roots in an interval of length p . Then

$$\sum_{\chi \pmod p} c(\chi)\,\chi(a) = \begin{cases} 1 & \text{if } a \text{ is a primitive root mod } p , \\ 0 & \text{else.}\end{cases}$$

It is clear that

$$c(\chi_0) = \phi(p-1)/(p-1) .$$

If $\chi \ne \chi_0$, then we may express the sum form $c(\chi)$ in terms of Ramanujan sums to obtain

$$|c(\chi)| \le 1/\mathrm{ord}\ \chi ,$$

where ord χ is the least integer d for which $\chi^d = \chi_0$.
Then we have

$$\begin{aligned}\frac{1}{N}\sum_{a\le N} N_a(x) &= \frac{1}{N}\sum_{p\le x} M_p(N)\\ &= \frac{1}{N}\sum_{p\le x}\sum_{a\le N}\sum_{\chi(\mathrm{mod}\ p)} c(\chi)\,\chi(a)\\ &= \frac{1}{N}\sum_{p\le x}\sum_{\chi(\mathrm{mod}\ p)} c(\chi)\sum_{a\le N}\chi(a)\\ &= \frac{1}{N}\sum_{p\le x}\frac{\phi(p-1)}{(p-1)}\{[N]-[N/p]\}\\ &\quad + O\Big(\frac{1}{N}\sum_{p\le x}\sum_{\substack{\chi(\mathrm{mod}\ p)\\ \chi\ne\chi_0}}\frac{1}{\mathrm{ord}\ \chi}\Big|\sum_{a\le N}\chi(a)\Big|\Big)\\ &= \sum_{p\le x}\frac{\phi(p-1)}{p} + O\Big(\frac{x}{N\log x}\Big) + O\Big(\frac{S_1}{N}\Big), \qquad (7)\end{aligned}$$

where

$$S_1 = \sum_{p\le x}\sum_{\substack{\chi(\mathrm{mod}\ p)\\ \chi\ne\chi_0}}\frac{1}{\mathrm{ord}\ \chi}\Big|\sum_{a\le N}\chi(a)\Big| .$$

Since any non-principal character mod p is primitive, we have

$$S_1 = \sum_{p\le x}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\ p)}\frac{1}{\mathrm{ord}\ \chi}\Big|\sum_{a\le N}\chi(a)\Big|$$

where $\sum^*$ denotes summation over primitive characters. We now use Holder's inequality to obtain

$$S_1^{2r} \le \Big\{\sum_{p\le x}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\ p)}\Big(\frac{1}{\mathrm{ord}\ \chi}\Big)^{2r/(2r-1)}\Big\}^{2r-1}\sum_{p\le x}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\ p)}\Big|\sum_{a\le N}\chi(a)\Big|^{2r} . \qquad (8)$$

The first sum on the right hand side of this inequality is easily estimated to be

$$\ll (c_2\, x)^{2r-1} . \qquad (9)$$

The second sum is

$$\begin{aligned}S_2 &= \sum_{p\le x}\sideset{}{^*}\sum_{\chi(\mathrm{mod}\ p)}\Big|\sum_{a\le N}\chi(a)\Big|^{2r}\\ &= \sum_{p\le x}\sum_{\chi(\mathrm{mod}\ p)}\Big|\sum_{a\le N^r}\tau'_r(a)\,\chi(a)\Big|^2\end{aligned}$$

where $\tau'_r(a)$ is the number of ways of writing a as the product of r factors each of which is less than N. If we define $\tau_r(a)$ to be the number of ways of writing a as the product of r factors, then $\tau'_r(a) \le \tau_r(a)$. Then large sieve inequality (see Gallagher [2]) gives

$$S_2 \ll (x^2 + N^r) \sum_{a \le N^r} \{\tau'_r(a)\}^2$$

$$\ll (x^2 + N^r) \sum_{a \le N^r} \tau_r^2(a) . \qquad (10)$$

Mardjanichvili [3] estimated the sum on the right hand side of this inequality to be

$$\ll N^r \{\log(e\, N^r)\}^{r^2-1} . \qquad (11)$$

We thus have from (8) , (9) , (10) and (11) that

$$S_1^{2r} \ll (c_2 x)^{2r-1} (x^2+N^r)\, N^r \{\log (e\, N^r)\}^{r^2-1} .$$

By choosing $r = [\frac{2 \log x}{\log N}] + 1$, we deduce

$$S_1 \ll x^{1-1/2r}\, N \{\log(e\, x^2)\}^{(r^2-1)/2r}$$

or,

$$S_1/N \ll x / (\log x)^D .$$

This together with (6) and (7) yields (5).

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REFERENCES

[1] P.J. STEPHENS. - "An average result for Artin's conjecture", Mathematika, 16 (1969), 178-188.

[2] P.X. GALLAGHER. - "The large sieve", Mathematika 14, (1967), 14-20.

[3] C. MARDJANICHVILI. - Doklady Akad. Nauk SSSR, 22 (1939), n° 7.

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