ESAIM: MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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ESAIM: Modélisation mathématique et analyse numérique, tome 34, nº 2 (2000), p. 315-335
<http://www.numdam.org/item?id=M2AN_2000__34_2_315_0>

Mathematical Modelling and Numerical Analysis M2AN, Vol. 34, N° 2, 2000, pp. 315–335
Modélisation Mathématique et Analyse Numérique

FLUIDS WITH ANISOTROPIC VISCOSITY*

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Abstract. Motivated by rotating fluids, we study incompressible fluids with anisotropic viscosity. We use anisotropic spaces that enable us to prove existence theorems for less regular initial data than usual. In the case of rotating fluids, in the whole space, we prove Strichartz-type anisotropic, dispersive estimates which allow us to prove global wellposedness for fast enough rotation.

Mathematics Subject Classification. 35Q30, 35Q35, 76U05.

Received: December 7,1999.

INTRODUCTION

The aim of this paper is the study of the following system

$$(NS_h)\left\{\begin{array}{rcl} \partial_t v + v\cdot\nabla v - \nu\Delta_h v - \nu_V\partial_3^2 v & = & -\nabla p \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3\\ \operatorname{div} v & = & 0\\ v_{|t=0} & = & v_0\,, \end{array}\right.$$

where Δ_h denotes the horizontal Laplacian, *i.e.* the operator $\partial_{x_1}^2 + \partial_{x_2}^2$. The constants $\nu > 0$ and $\nu_V \geq 0$ represent respectively the horizontal and vertical viscosities. Obviously, this system is locally well posed when the initial data v_0 belongs to a Sobolev space $H^{\frac{5}{2}+\eta}(\mathbb{R}^3)$ for any positive real number η simply because the energy estimates are better than the ones for Euler (*i.e.* $\nu = \nu_V = 0$). On the other hand, it is well known that if ν and ν_V are positive, and if v_0 is only in $L^2(\mathbb{R}^3)$, then a global weak solution exists (that is Leray's theorem, see [16]). Finally, if v_0 belongs to the homogeneous Sobolev space $\dot H^{\frac12}(\mathbb{R}^3)$, then a unique local solution exists (that is Fujita and Kato's theorem, see [10]). Moreover, if $\|v_0\|_{\dot H^{\frac12}}$ is small enough with respect to $\min\{\nu,\nu_V\}$, then the system is globally well posed. Let us note that the homogeneous space $\dot H^{\frac12}(\mathbb{R}^3)$ is well adapted to the Navier–Stokes system, in the sense that it is invariant under the scaling of the equation: if v is a solution of the Navier–Stokes equation, with data v_0, then the same goes for v_λ defined by $v_\lambda(t,x) \stackrel{\text{def}}{=} \lambda v(\lambda^2 t, \lambda x)$, with

Keywords and phrases. Navier-Stokes equations, Rotating fluids, Strichartz estimates.

* *Dedicated to Roger Temam for his 60th birthday*

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data $v_{0,\lambda}(x) \stackrel{\text{def}}{=} \lambda v_0(\lambda x)$; it is easy to see that

$$\|v_{0,\lambda}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = \|v_0\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}.$$

What we want to investigate here is the case when ν_V is 0 or converges to 0. We want to prove existence and if possible uniqueness results as close as possible to the case when the two viscosities are positive and fixed.

Let us begin by saying something about the motivation of this problem. It is well known since a series a work devoted to Ekman boundary layers for rotating fluids (see for instance [13], [7] and [8]) that it makes sense to consider anisotropic viscosities and more precisely viscosities of the type

$$-\nu\Delta_h - \beta\varepsilon\partial_3^2.$$

That is obviously a particular case of the system (NS_h) stated at the beginning of this introduction.

Considering the anisotropy of the problem, we shall have to use spaces of functions that take into account that anisotropy. All we shall do here is strongly related to energy estimates, so the tool will be anisotropic Sobolev spaces. Such spaces have been introduced by D. Iftimie in [15] for the study of incompressible Navier-Stokes equations in thin domains.

Let us recall the definition of those spaces. It requires an anisotropic dyadic decomposition of the Fourier space, so let us start by recalling the definition of the following operators of localization in Fourier space:

$$\begin{aligned}
\Delta_j^h a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi_h|)\widehat{a}) \quad \text{for} \quad j \in \mathbb{Z}\\
\Delta_k^v a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_3|)\widehat{a}) \quad \text{for} \quad k \in \mathbb{N}\\
\Delta_{-1}^v a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\psi(|\xi_3|)\widehat{a})\\
\Delta_k^v a &\stackrel{\text{def}}{=} 0 \quad \text{for} \quad k \leq -2,
\end{aligned}$$

where $\mathcal{F}a$ and $\widehat{a}$ denote the Fourier transform of any function a. The functions φ and ψ are smooth, compactly supported functions, with support respectively in a fixed ring of $\mathbb{R}$ far from the origin, and in a fixed ball containing the origin and such that

$$\forall t \in \mathbb{R}\setminus\{0\}\,,\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}t) = 1 \quad \text{and} \quad \forall t \in \mathbb{R}\,,\ \psi(t) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}t) = 1.$$

We refer to [3] for a precise construction of an isotropic decomposition of the Fourier space; let us note that there exists an integer N_0 such that if $|j-j'| \geq N_0$, then $\operatorname{supp}\varphi(2^{-j}\cdot) \cap \operatorname{supp}\varphi(2^{-j'}\cdot) = \emptyset$.

Let us remark that we consider a homogeneous decomposition in the horizontal variable and an inhomogeneous decomposition in the vertical variable. The associate Sobolev spaces are defined as follows.

Definition 1. *Let s and s' be two real numbers and a a tempered distribution. Let*

$$\|a\|_{s,s'} \stackrel{\text{def}}{=} \left(\sum_{j,k} 2^{2(js+s'k)}\|\Delta_j^h\Delta_k^v a\|_{L^2}^2\right)^{\frac{1}{2}}.$$

The space $H^{s,s'}$ is the closure of $\mathcal{D}(\mathbb{R}^3)$ for the above (semi-)norm.

Let us remark that, as usual when homogeneous Sobolev spaces are involved, $H^{s,s'}$ is a Hilbert space if and only if $s<1$. Spaces of this type have been used by J. Rauch and M. Reed in [18] and by M. Sablé-Tougeron in [19].

Let us state the three main theorems of this paper.

Theorem 2 (Existence). *Let $s_0 > 1/2$ be a real number, and let $v_0 \in H^{0,s_0}$. Then a positive time T and a solution v of (NS_h) defined on $[0,T] \times \mathbb{R}^3$ exist such that*

$$v \in L^\infty([0,T]; H^{0,s_0}) \cap L^2([0,T]; H^{1,s_0}).$$

Furthermore, there exists a constant c such that if $\|v_0\|_{H^{0,s_0}}$ is less than $c\nu$, then we can choose $T = +\infty$.

Theorem 3 (Uniqueness). *Let s_0 and s be two real numbers, greater respectively than $1/2$ and $3/2$, and let v_0 be an initial data in $H^{0,s}$. Then at most one solution v of (NS_h) exists in the space*

$$v \in L^\infty(\mathbb{R}^+; H^{0,s}) \cap L^2(\mathbb{R}^+; H^{1,s}).$$

Let us denote by $T^\star$ the maximal time of existence; if $T^\star$ is finite, then

$$\int_0^{T^\star} \|v(t)\|^4_{H^{1/2,s_0}} \mathrm{d}t + \int_0^{T^\star} \|v(t)\|^2_{H^{1,s_0}} \mathrm{d}t = +\infty.$$

Those statements deserve a few comments. The initial data in Theorem 2 have only L^2 regularity in the horizontal variable, and $H^{\frac{1}{2}+\eta}$ in the vertical variable; for positive viscosities, it is proved by D. Iftimie in [15] that the problem is well posed. Here, with that regularity (which is very close to the scaling of the equation), we obtain short time existence for large data and global existence for small data as in Fujita-Kato's Theorem. Uniqueness requires one additional vertical derivative. That comes from the fact that the lack of vertical viscosity prevents us from gaining vertical regularity.

The third result of this paper concerns the rotating fluid equations, obtained from (NS_h) by adding the Coriolis force $\dfrac{1}{\varepsilon} v \times e_3$, where $e_3 \stackrel{\text{def}}{=} (0,0,1)$. Here $\varepsilon > 0$ represents the Rossby number; we refer to [12] and [17] for a physical interpretation of that system. Note that the skew–symmetry of the operator $Lv \stackrel{\text{def}}{=} v \times e_3$ implies that Theorems 2 and 3 hold for the rotating fluid equations: their proof only involves energy estimates. Moreover, the rotation induces a dispersive effect which leads to the following result.

Theorem 4 (Rotating fluids). *Let v_0 be an initial data in $H^{0,s}$ with s strictly greater than $3/2$. Let $\nu > 0$ and $\nu_V \geq 0$ be two real numbers. Then a positive real number ε_0 exists, depending only on ν and v_0, such that, for any $\varepsilon \leq \varepsilon_0$, the system*

$$\left\{ \begin{array}{rcl} \partial_t v + v \cdot \nabla v - \nu \Delta_h v - \nu_V \partial_3^2 v + \dfrac{1}{\varepsilon} v \times e_3 & = & -\nabla p \\ \operatorname{div} v & = & 0 \\ v_{|t=0} & = & v_0 \, , \end{array} \right.$$

has a unique global solution in the space $L^\infty([0,T]; H^{0,s}) \cap L^2([0,T]; H^{1,s})$.

Let us notice that this global regularity result is different from those proved in the case of periodic boundary conditions (see for instance [1] and [11] in those cases, the global regularity is due to non resonance conditions).

The structure of the paper is the following:

- the first section is devoted to some basic properties of the spaces $H^{s,s'}$;
- the second section, the core of the proof of Theorems 2 and 3, is devoted to the proof of an energy estimate in anisotropic Sobolev spaces for the convective term $v \cdot \nabla$. The fact that the vector field v is divergence free will allow us to avoid bad terms containing $\partial_3 v$ which are not compensated by viscosity;
- the third section is nothing but an application of that energy estimate through classical contraction arguments, in order to prove Theorems 2 and 3;
- the fourth section consists in showing a dispersive effect for rotating fluids, as pointed out in [6], and applying it in this context in order to prove Theorem 4.

1. Algebraic properties of spaces $H^{s,s'}$

Before stating the theorem we want to prove, let us give a few properties concerning those spaces, which we shall use throughout the study. The main results are essentially contained in [18], in [19] and in [15]. They concern product rules in the spaces $H^{s,s'}$; let us start by recalling the classical product rules in isotropic Sobolev spaces: if a and b are two tempered distributions, then for any $d \geq 1$,

$$\forall s,t < \frac{d}{2},\ s+t > 0, \quad \|ab\|_{\dot H^{s+t-\frac{d}{2}}(\mathbb{R}^d)} \leq C_{s,t}\|a\|_{\dot H^s(\mathbb{R}^d)}\|b\|_{\dot H^t(\mathbb{R}^d)}, \tag{1.1}$$

where the $\dot H^s$ spaces are the usual homogeneous Sobolev spaces, defined by the norm $\|u\|_{\dot H^s} \stackrel{\text{def}}{=} \| \, |\cdot|^s \, |\widehat{u}(\cdot)| \, \|_{L^2}$.

Furthermore, in the final section, we shall be working with bidimensional functions as well as tridimensional functions, and in particular taking the product of such functions. We shall be in need of the following result, whose proof can be found in [11] (in the case of periodic boundary conditions, but it is identical in the case of the whole space):

$$\forall s,t < 1,\ s+t > 0, \quad \|ab\|_{\dot H^{s+t-1}(\mathbb{R}^3)} \leq C_{s,t}\|a\|_{\dot H^s(\mathbb{R}^2)}\|b\|_{\dot H^t(\mathbb{R}^3)}. \tag{1.2}$$

The same results hold for inhomogeneous Sobolev spaces.

Now we shall prove the following product rule in our anisotropic Sobolev spaces.

Lemma 1. *Let σ and σ' be two real numbers, smaller than 1, such that $\sigma+\sigma' > 0$, and let $s_0 > 1/2$ and s be such that $s + s_0 > 0$; a constant C exists such that the following holds. Let a and b be two tempered distributions. Then*

$$\|ab\|_{H^{\sigma+\sigma'-1,s}} \leq C(\|a\|_{H^{\sigma,s_0}}\|b\|_{H^{\sigma',s}} + \|a\|_{H^{\sigma,s}}\|b\|_{H^{\sigma',s_0}}).$$

Proof of the lemma. By definition of $H^{\sigma,s}$, we have

$$\begin{aligned}\|a\|^2_{H^{\sigma,s}} &= \sum_{j,k} 2^{2(j\sigma+ks)} \int |\Delta_j^h \Delta_k^v a(x_h,x_3)|^2 \, dx_h dx_3 \\ &= \sum_k \int 2^{2ks} \|\Delta_k^v a(\cdot,x_3)\|^2_{H^\sigma(\mathbb{R}^2)} \, dx_3,\end{aligned}$$

so we can write

$$\|a\|_{H^{\sigma,s}} = \left\| 2^{-ks} \|\Delta_k^v a\|_{L^2(\mathbb{R}_{x_3},H^\sigma(\mathbb{R}^2))} \right\|_{\ell^2(\mathbf{Z})}. \tag{1.3}$$

Then we just have to write the paraproduct algorithm of J.-M. Bony (see [2]), in the vertical direction, which reads

$$\Delta_k^v(ab) = \Delta_k^v(T_a^v b + T_b^v a + R^v(a,b)),$$

with

$$T_a^v b \stackrel{\text{def}}{=} \sum_{k'} S_{k'}^v a \, \Delta_{k'}^v b \quad \text{and} \quad R^v(a,b) \stackrel{\text{def}}{=} \sum_{k'} \sum_{j\in\{-1,0,1\}} \Delta_{k'}^v a \Delta_{k'-j}^v b.$$

Let us state the following lemma, whose proof is obvious.

Lemma 2. *For any $s > 1/2$ and any $\sigma \in \mathbb{R}$, a constant C exists such that for any function a,*

$$\|a\|_{L^\infty(\mathbb{R}_{x_3},H^\sigma(\mathbb{R}^2))} \leq C\|a\|_{H^{\sigma,s}}.$$

Then we have, by the bidimensional product rules recalled in (1.1),

$$\begin{aligned}\|S_k^v a \Delta_k^v b\|_{L^2(\mathbb{R}_{x_3};H^{\sigma+\sigma'-1})} &\leq \|S_k^v a\|_{L^\infty(\mathbb{R}_{x_3};H^{\sigma})}\|\Delta_k^v b\|_{L^2(\mathbb{R}_{x_3};H^{\sigma'})}\\ &\leq c_k 2^{-ks}\|a\|_{H^{\sigma,s_0}}\|b\|_{H^{\sigma',s}},\end{aligned}$$

where $c_k \in \ell^2(\mathbf{Z})$, according to Lemma 2. Similarly we have

$$\begin{aligned}\|\Delta_{k'}^v a \Delta_{k'-j}^v b\|_{L^1(\mathbb{R}_{x_3};H^{\sigma+\sigma'-1})} &\leq \|\Delta_{k'} a\|_{L^2(\mathbb{R}_{x_3};H^{\sigma})}\|\Delta_{k'-j}^v b\|_{L^2(\mathbb{R}_{x_3};H^{\sigma'})}\\ &\leq c_{k'}^2 2^{-k'(s+s_0)}\|a\|_{H^{\sigma,s_0}}\|b\|_{H^{\sigma',s}}.\end{aligned}$$

Then Bernstein's inequality yields

$$\|\Delta_k^v(\Delta_{k'}^v a \Delta_{k'-j}^v b)\|_{L^2(\mathbb{R}_{x_3};H^{\sigma+\sigma'-1})} \leq c_{k'}^2 2^{-k'(s+s_0)+\frac{k}{2}}\|a\|_{H^{\sigma,s_0}}\|b\|_{H^{\sigma',s}}.$$

This implies that

$$\|\Delta_k^v R(a,b)\|_{L^2(\mathbb{R}_{x_3};H^{\sigma+\sigma'-1})} \leq c_k 2^{-k(s+s_0-\frac{1}{2})}\|a\|_{H^{\sigma,s_0}}\|\|b\|_{H^{\sigma',s}}\|,$$

and Lemma 1 is proved. □

2. Anisotropic energy estimates

We shall prove the following lemma, which is the core of the proof of Theorem 2.

Lemma 3. *For any real numbers $s_0 > 1/2$ and $s \geq s_0$, a constant C exists such that for any vector fields a and b, with a divergence free,*

$$\begin{aligned}|(\Delta_k^v(a\cdot\nabla b)|\Delta_k^v b)_{L^2}| \leq C d_k 2^{-2ks}\|b\|_{H^{\frac{1}{2},s}}(&\|a\|_{H^{\frac{1}{2},s_0}}\|\nabla_h b\|_{H^{0,s}} + \|a\|_{H^{\frac{1}{2},s}}\|\nabla_h b\|_{H^{0,s_0}}\\ &+\|\nabla_h a\|_{H^{0,s_0}}\|b\|_{H^{\frac{1}{2},s}} + \|\nabla_h a\|_{H^{0,s}}\|b\|_{H^{\frac{1}{2},s_0}}) \quad \text{with} \quad \sum_{k\in\mathbf{Z}} d_k = 1.\end{aligned}$$

Proof of the lemma. Let us define

$$F_k^h \stackrel{\text{def}}{=} \Delta_k^v(a^h\cdot\nabla_h b) \quad \text{and} \quad F_k^v \stackrel{\text{def}}{=} \Delta_k^v(a^3\partial_3 b).$$

The terms $(F_k^h|b_k)_{L^2}$ and $(F_k^v|b_k)$ are estimated in two different ways. The reason why is that the term F_k^h involves only horizontal derivatives which can be compensated by the horizontal viscosity as in the case of the usual Navier-Stokes system.

Let us start by proving the result for F_k^h. We have, using similar computations to those leading to Lemma 1,

$$
\begin{aligned}
A_k &\stackrel{\text{def}}{=} \|\Delta_k^v(a^h\cdot\nabla_h b)\|_{L^2(\mathbb{R}_{x_3};H^{-\frac12}(\mathbb{R}^2))}\\
&\leq \sum_{i=1}^{2}\|\Delta_k^v(T^v_{a^i}\partial_i b)+T^v_{\partial_i b}a^i+R^v(a^i,\partial_i b)\|_{L^2(\mathbb{R}_{x_3};H^{-\frac12}(\mathbb{R}^2))}\\
&\leq c_k 2^{-ks}\|S_k^v a^h\|_{L^\infty(\mathbb{R}_{x_3};H^{\frac12}(\mathbb{R}^2))}\|\nabla_h b\|_{L^2(\mathbb{R}_{x_3};L^2(\mathbb{R}^2))}\\
&\quad + c_k 2^{-ks}\|S_k^v\nabla_h b\|_{L^\infty(\mathbb{R}_{x_3};H^{\frac12}(\mathbb{R}^2))}\|a\|_{L^2(\mathbb{R}_{x_3};L^2(\mathbb{R}^2))}\\
&\quad + \sum_{\substack{|k_1-k_2|\leq 1\\ |k_1-k|\leq N_0}}\|\Delta_{k_1}^v a\|_{L^\infty(\mathbb{R}_{x_3};H^{\frac12}(\mathbb{R}^2))}\|\Delta_{k_2}^v\nabla_h b\|_{L^2(\mathbb{R}_{x_3};L^2(\mathbb{R}^2))}\\
&\leq c_k 2^{-ks}(\|a\|_{H^{\frac12,s_0}}\|\nabla_h b\|_{H^{0,s}}+\|a\|_{H^{\frac12,s}}\|\nabla_h b\|_{H^{0,s_0}}).
\end{aligned}
$$

Finally, we have

$$
|(\Delta_k^v(a^h\cdot\nabla_h b)|b_k)_{L^2(\mathbb{R}^3)}|\leq Cd_k 2^{-2ks}\|b\|_{H^{\frac12,s}}(\|a\|_{H^{\frac12,s_0}}\|\nabla_h b\|_{H^{0,s}}+\|a\|_{H^{\frac12,s}}\|\nabla_h b\|_{H^{0,s_0}}),
$$

so the result is proved for the term $(F_k^h|b_k)_{L^2}$. To estimate the term $(F_k^v|b_k)_{L^2}$, let us first use J.-M. Bony's decomposition in the vertical variable: we can write

$$
\begin{aligned}
\Delta_k^v(a^3\partial_3 b) &= F_k^{v,1}+F_k^{v,2}\quad\text{with}\\
F_k^{v,1} &\stackrel{\text{def}}{=} \Delta_k^v\sum_{k'\geq k-N_0}S^v_{k'+2}(\partial_3 b)\Delta^v_{k'}a^3\quad\text{and}\\
F_k^{v,2} &\stackrel{\text{def}}{=} \Delta_k^v\sum_{|k'-k|\leq N_0}S^v_{k'-1}a^3\partial_3\Delta^v_{k'}b.
\end{aligned}
$$

We clearly have

$$
|(F_k^{v,1}|b_k)_{L^2(\mathbb{R}^3)}|\leq Cc_k 2^{-ks}\|b\|_{H^{\frac12,s}}\|F_k^{v,1}\|_{L^2(\mathbb{R}_{x_3};H^{-\frac12}(\mathbb{R}^2))},
$$

and Bernstein's inequality, along with bidimensional product rules recalled in (1.1), yields

$$
\|F_k^{v,1}\|_{L^2(\mathbb{R}_{x_3};H^{-\frac12}(\mathbb{R}^2))}\leq C2^{\frac{k}{2}}\sum_{k'\geq k-N_0}\|S_{k'+2}\partial_3 b\|_{L^2(\mathbb{R}_{x_3};H^{\frac12}(\mathbb{R}^2))}\|\Delta_{k'}a^3\|_{L^2(\mathbb{R}^3)}.
$$

But it is clear that

$$
\|S_{k'+2}\partial_3 b\|_{L^2(\mathbb{R}_{x_3};H^{\frac12}(\mathbb{R}^2))}\leq C2^{k'(1-s)}\|b\|_{H^{\frac12,s}},
$$

so

$$
\|F_k^{v,1}\|_{L^2(\mathbb{R}_{x_3};H^{-\frac12}(\mathbb{R}^2))}\leq C2^{\frac{k}{2}}\sum_{k'\geq k-N_0}2^{k'(1-s)}\|\Delta_{k'}a^3\|_{L^2(\mathbb{R}^3)}\|b\|_{H^{\frac12,s}}.
$$

Here we are going to use in a crucial way the fact that a is divergence free: we have

$$
\partial_3 a^3 = -\mathrm{div}_h a^h.
$$

So, we infer that

$$
\begin{aligned}
\|\Delta^v_{k'}a^3\|_{L^2(\mathbb{R}^3)} &\leq C2^{-k'}\|\Delta^v_{k'}\partial_3 a^3\|_{L^2(\mathbb{R}^3)}\\
&\leq C2^{-k'}\|\Delta^v_{k'}\,\mathrm{div}_h\, a^h\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
$$

That implies that

$$
\begin{aligned}
\|F_k^{v,1}\|_{L^2(\mathbb{R}_{x_3};H^{-\frac12}(\mathbb{R}^2))} &\leq C2^{\frac{k}{2}} \sum_{k'\geq k-N_0} 2^{k'(1-s)}2^{-k'(1+s_0)}c_{k'}\|\nabla_h a\|_{H^{0,s_0}}\|b\|_{H^{\frac12,s}}\\
&\leq Cc_k 2^{-ks}\|\nabla_h a\|_{H^{0,s_0}}\|b\|_{H^{\frac12,s}},
\end{aligned}
$$

since $s_0 > 1/2$.

Up to that point, we did not actually use the energy estimate, but only laws of product or Sobolev embeddings. The estimate of the term $(F_k^{v,2}|b_k)_{L^2}$ will use in a crucial way the structure of the nonlinearity.

First of all, following a computation done in [5], we get that

$$
\begin{aligned}
(F_k^{v,2}|b_k)_{L^2} &= (S_k^v(a^3)\partial_3 b_k|b_k)_{L^2} + R_k(a,b) \quad \text{with} \\
R_k(a,b) &\stackrel{\text{def}}{=} \sum_{|k'-k|\leq N_0} \big([\Delta_k^v, S_{k'-1}^v a^3]\partial_3\Delta_{k'}^v b|b_k\big)_{L^2} + \sum_{|k'-k|\leq N_0} \big((S_k^v - S_{k'-1}^v)a^3\partial_3\Delta_{k'}^v b|b_k\big)_{L^2}.
\end{aligned}
\tag{2.4}
$$

Then, using an integration by parts, we infer that

$$
(S_k^v(a^3)\partial_3 b_k|b_k)_{L^2} = -\frac12 (S_k^v(\partial_3 a^3)b_k|b_k)_{L^2}.
$$

Thanks to the fact that a is divergence free, we deduce that

$$
(F_k^{v,2}|b_k)_{L^2} = (b_k S_k^v \mathrm{div}_h a^h|b_k)_{L^2} + R_k(a,b).
$$

The term $(b_k S_k^v\mathrm{div}_h a^h|b_k)_{L^2}$ can be estimated as the term $(F_k^h|b_k)$ that appeared above. To estimate the term $([\Delta_k^v, S_{k'-1}^v a^3]\partial_3\Delta_{k'}^v b|b_k)_{L^2}$, we write that, for any function ω,

$$
([\Delta_k^v, S_{k'-1}^v a^3]\omega)(x_h,x_3) = 2^k\int h(2^k y_3)(S_{k'-1}^v a^3(x_h,x_3) - S_{k'-1}^v a^3(x_h,x_3-y_3)))\omega(x_h,x_3-y_3))\mathrm{d}y_3.
$$

Writing a Taylor formula, we get that

$$
([\Delta_k^v, S_{k'-1}^v a^3]\omega)(x_h,x_3) = \int_{\mathbb{R}\times[0,1]} h_1(2^k(y_3)(S_{k'-1}^v\partial_3 a^3)(x_h,x_3+t(x_3-y_3))\omega(x_h,x_3-y_3)\mathrm{d}y_3\mathrm{d}t
$$

with $h_1(z) = zh(z)$. Thanks to the fact that a is divergence free, we get

$$
([\Delta_k^v, S_{k'-1}^v a^3]\omega)(x_h,x_3) = -\int_{\mathbb{R}\times[0,1]} h_1(2^k(y_3)(S_{k'-1}^v\,\mathrm{div}_h\, v^h)(x_h,x_3+t(x_3-y_3))\omega(x_h,x_3-y_3))\mathrm{d}y_3\mathrm{d}t.
$$

Now, let us apply the rules of product of Sobolev spaces on $\mathbb{R}^2$; this implies that

$$
\|([\Delta_k^v, S_{k'-1}^v a^3]\omega)(\cdot,x_3)\|_{H^{-\frac12}(\mathbb{R}^2)} \leq C\int |h_1(2^k y_3)|\,\|\mathrm{div}_h\, v^h\|_{L^\infty(\mathbb{R}_{x_3};L^2(\mathbb{R}^2))}\|\omega(\cdot,x_3-y_3)\|_{H^{\frac12}(\mathbb{R}^2)}\mathrm{d}y_3.
$$

From this, we deduce that

$$
\|([\Delta_k^v, S_{k'-1}^v a^3]\omega)(\cdot,x_3)\|_{L^2(\mathbb{R}_{x_3};H^{-\frac12}(\mathbb{R}^2))} \leq C2^{-k}\|S_{k'-1}^v\,\mathrm{div}_h\, a_h\|_{L^\infty(\mathbb{R}_{x_3};L^2(\mathbb{R}^2))}\|\omega\|_{L^2(\mathbb{R}^2;H^{\frac12}(\mathbb{R}^2))}.
$$

So we get

$$([\Delta_k^v, S_{k'-1}^v a^3]\partial_3 b_{k'}|b_k)_{L^2} \le \|S_{k'-1}^v \mathrm{div}_h a^h\|_{L^\infty(\mathbb{R}_{x_3};L^2(\mathbb{R}^2))} \times 2^{k'-k}\|b_{k'}\|_{L^2(\mathbb{R}_{x_3};H^{\frac12}(\mathbb{R}^2))}\|b_k\|_{L^2(\mathbb{R}_{x_3};H^{\frac12}(\mathbb{R}^2))}.$$

Finally, with Lemma 2 joint with the fact that $|k'-k|\le N_0$, we infer

$$([\Delta_{k'}^v, S_{k'-1}^v a^3]\partial_3 b_{k'}|b_k)_{L^2} \le C d_k \|\nabla_h a\|_{H^{0,s_0}} 2^{-2ks}\|b\|^2_{H^{1/2,s}}.$$

The term $(S_k^v - S_{k'-1}^v)a^3\partial_3\Delta_{k'}^v b$ can be estimated exactly as the term $F_k^{v,1}$. That concludes the proof of Lemma 3. □

Remark 5. We are unable to estimate the term $(F_k^v|b_k)_{L^2(\mathbb{R}_{x_3};H^s(\mathbb{R}^2))}$, for any $s\neq 0$: that is due to the fact that the estimate of $(S_k^v(a^3)\partial_3 b_k|b_k)_{L^2(\mathbb{R}_{x_3};H^s(\mathbb{R}^2))}$ which appears in (2.4) would lead to terms of the type $([\Delta_j^h, S_k^v(a^3)]\partial_3 b_k|b_k)_{L^2(\mathbb{R}^3)}$. We do not see here any way of recovering the vertical derivative of b_k because of the lack of vertical viscosity.

3. Proof of Theorems 2 and 3

Proof of Theorem 2. It is obvious that for smooth initial data, smooth solutions exist (of course only for short times); we shall prove *a priori* estimates for such smooth solutions.

As it occurs in the classical Navier-Stokes system, global results for small initial data are easier to prove than local results for large data. In any case, the proof relies strongly on Lemma 3 proved in the previous section; let us start by deducing from that lemma the global part of Theorem 2. Applying the operator Δ_k^v and using an L^2 energy estimate gives, considering the fact that $\nu_V \ge 0$,

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|v_k(t)\|^2_{L^2} + \nu\|\nabla_h v_k(t)\|^2_{L^2} \le (\Delta_k^v(v\cdot\nabla v)|v_k)_{L^2},$$

where we denote by v_k the term $\Delta_k^v v$. So using the above Lemma 3 with $s=s_0$ and $a=b=v$, we get

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|v_k(t)\|^2_{L^2} + \nu\|\nabla_h v_k(t)\|^2_{L^2} \le C d_k 2^{-2ks_0}\|v\|^2_{H^{\frac12,s_0}}\|\nabla_h v\|_{H^{0,s_0}}. \tag{3.5}$$

Now multiplying this inequality by 2^{2ks_0} and taking the sum over k gives

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{H^{0,s_0}} + \nu\|\nabla_h v(t)\|^2_{H^{0,s_0}} \le C\|v\|^2_{H^{\frac12,s_0}}\|\nabla_h v\|_{H^{0,s_0}}.$$

An obvious interpolation inequality tells us that

$$\|v\|^2_{H^{\frac12,s_0}} \le \|v\|_{H^{0,s_0}}\|\nabla_h v\|_{H^{0,s_0}}.$$

So we infer that

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{H^{0,s_0}} + \nu\|\nabla_h v(t)\|^2_{H^{0,s_0}} \le C\|v(t)\|_{H^{0,s_0}}\|\nabla_h v\|^2_{H^{0,s_0}}.$$

So, if $\|v_0\|_{H^{0,s_0}}$ is small enough, the function $\|v(t)\|^2_{H^{0,s_0}}$ decreases; more precisely, if

$$\|v(t)\|_{H^{0,s_0}} \le \frac{\nu}{2C},$$

then we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{H^{0,s_0}}+\frac{\nu}{2}\|\nabla_h v(t)\|^2_{H^{0,s_0}}\leq 0.$$

So, we get that, for any positive t,

$$\|v(t)\|^2_{H^{0,s_0}}+\nu\int_0^t\|\nabla_h v(t')\|^2_{H^{0,s_0}}\mathrm{d}t'\leq\|v_0\|^2_{H^{0,s_0}}.$$

Now standard compactness arguments give the global existence result.

The proof of the local part of Theorem 2 consists in several parts, considering differently the high vertical frequencies and the low ones.

The case of high vertical frequencies is treated thanks to the following lemma.

Lemma 4. *For any $s_0>1/2$, a constant C exists such that for any smooth solution v of (NS_h) on $[0,T]\times\mathbb{R}^3$, for any integer N, we have*

$$\begin{aligned}\|(\mathrm{Id}-S_N^v)v\|_{L_T^\infty(H^{0,s_0})}&+2\nu\|\nabla_h(\mathrm{Id}-S_N^v)v\|^2_{L_T^2(H^{0,s_0})}\\&\leq C\|(\mathrm{Id}-S_N^v)v_0\|^2_{H^{0,s_0}}+C\int_0^T\|v(t)\|^2_{H^{\frac12,s_0}}\|\nabla_h v(t)\|_{H^{0,s_0}}\mathrm{d}t.\end{aligned}$$

Proof of the lemma. We just have to multiply inequality (3.5) by 2^{2ks_0} and take the sum for k bigger than $N-1$; we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k\geq N-1}2^{2ks_0}\|v_k(t)\|^2_{L^2}\right)+\nu\sum_{k\geq N-1}2^{2ks_0}\|\nabla_h v_k(t)\|^2_{L^2}\leq C\|v(t)\|^2_{H^{\frac12,s_0}}\|\nabla_h v(t)\|_{H^{0,s_0}}.$$

Then Lemma 4 follows by integration in time. □

Let us now consider the low vertical frequencies and high horizontal frequencies.

Lemma 5. *For any $s_0>1/2$, a constant C exists such that we have the following properties. Let M and N be two integers such that $M\geq N$, and let us define $v_{M,N}\stackrel{def}{=}(\mathrm{Id}-S_M^h)S_N^v v$. Then*

$$\|v_{M,N}\|^2_{L_T^\infty(H^{0,s_0})}+2\nu\|\nabla_h v_{M,N}\|^2_{L_T^2(H^{0,s_0})}\leq C\|v_{M,N|t=0}\|^2_{H^{0,s_0}}+C\int_0^T\|v(t)\|^2_{H^{1/2,s_0}}\|\nabla_h v(t)\|_{H^{0,s_0}}\mathrm{d}t.$$

Proof. Again by standard energy estimates, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{M,N}(t)\|^2_{L^2}+\nu\|\nabla_h v_{M,N}(t)\|^2_{L^2}&=(F^h_{M,N}|v_{M,N})_{L^2}+(F^v_{M,N}|v_{M,N})_{L^2},\quad\text{with}\\ F^h_{M,N}&\stackrel{\text{def}}{=}(\mathrm{Id}-S_M^h)S_N^v(v^h\nabla_h v)\quad\text{and}\\ F^v_{M,N}&\stackrel{\text{def}}{=}(\mathrm{Id}-S_M^h)S_N^v(v^3\partial_3 v).\end{aligned}$$

The term $F^h_{M,N}$ is estimated easily, using the product rules given in Lemma 1. Using the law of product, we have

$$\|F_{M,N}(\cdot,x_3)\|_{H^{-\frac12}(\mathbb{R}^2)}\leq C\|v^h(\cdot,x_3)\|_{H^{\frac12}(\mathbb{R}^2)}\|\nabla_h v(\cdot,x_3)\|_{L^2(\mathbb{R}^2)}.$$

As $s_0 > 1/2$, we get

$$\|F^h_{M,N}\|_{L^2(\mathbb{R}_{x_3};H^{-\frac{1}{2}}(\mathbb{R}^2))} \leq C\|v^h\|_{H^{1/2,s_0}}\|\nabla_h v\|_{H^{0,s_0}},$$

hence

$$|(F^h_{M,N}|v_{M,N})_{L^2(\mathbb{R}^3)}| \leq C\|v\|^2_{H^{1/2,s_0}}\|\nabla_h v\|_{H^{0,s_0}}.$$

To estimate $\|F^v_{M,N}\|_{L^2(\mathbb{R}_{x_3};H^{-\frac{1}{2}}(\mathbb{R}^2))}$, we use the fact that v is divergence free: we can write that

$$F^v_{M,N} = (\mathrm{Id} - S^h_M)S^v_N(\partial_3(v^3 v) + v\,\mathrm{div}_h v^h).$$

The term $(\mathrm{Id} - S^h_M)S^v_N(v\,\mathrm{div}_h v^h)$ can be estimated exactly as $F^h_{M,N}$. As to the other term, it is basic Fourier analysis to observe that

$$\mathrm{Id} - S^h_M = 2^{-M}(\mathrm{Id} - S^h_M)(\chi_1(D)\partial_1 + \chi_2(D)\partial_2),$$

where χ_i are smooth, homogeneous functions of degree 0. Then it is clear that when $M \geq N$,

$$\begin{aligned}\big((\mathrm{Id} - S^h_M)S^v_N\partial_3(v^3 v)|v_{M,N}\big)_{L^2(\mathbb{R}^3)} &\leq \|v^3 v\|_{L^2(\mathbb{R}^3)}2^{N-M}\|\nabla_h v_{M,N}\|_{L^2(\mathbb{R}^3)}\\ &\leq C\|v\|^2_{H^{1/2,s_0}}\|\nabla_h v_{M,N}\|_{H^{0,s_0}}\\ &\leq C\|v\|^2_{H^{1/2,s_0}}\|\nabla_h v\|_{H^{0,s_0}}.\end{aligned}$$

So finally

$$(F_{M,N}|v_{M,N})_{L^2(\mathbb{R}^3)} \leq C\|v\|^2_{H^{1/2,s_0}}\|\nabla_h v\|_{H^{0,s_0}},$$

which proves the lemma in an obvious way. □

Let us conclude the proof of the local existence part of Theorem 2. Let us observe that v_0 is in $L^2(\mathbb{R}^3)$, so we have the basic L^2 energy estimate

$$\|v(t)\|^2_{L^2} + 2\nu\int_0^t \|\nabla_h v(t')\|^2_{L^2}\mathrm{d}t' \leq \|v_0\|^2_{L^2}.$$

So it is obvious that

$$\|S^v_N S^h_M v(t)\|^2_{H^{0,s_0}} + 2\nu\int_0^t \|\nabla_h S^v_N S^h_M v(t')\|^2_{H^{0,s_0}}\mathrm{d}t' \leq C2^{2M}2^{2Ns_0}T\|v_0\|^2_{L^2}.$$

So using an interpolation inequality and Lemmas 4 and 5, we infer that

$$\begin{aligned}(2\nu)^{\frac{1}{2}}\|v\|^2_{L^4_T(H^{\frac{1}{2},s_0})} + 2\nu\|\nabla_h v\|^2_{L^2_T(H^{0,s_0})} \leq C\Big(&\|(\mathrm{Id} - S^v_N)v_0\|^2_{H^{0,s_0}}\\ &+ \|S^v_N(\mathrm{Id} - S^h_M)v_0\|^2_{H^{0,s_0}} + 2^{2M}2^{2Ns_0}T\|v_0\|^2_{L^2} + \int_0^T \|v(t')\|^2_{H^{\frac{1}{2},s_0}}\|v(t')\|_{H^{0,s_0}}\mathrm{d}t'\Big).\end{aligned}$$

So we deduce that

$$\begin{aligned}\nu^{\frac{1}{2}}\|v\|^2_{L^4_T(H^{\frac{1}{2},s_0})} + \nu\|\nabla_h v\|^2_{L^2_T(H^{0,s_0})} \leq C\Big(&\|(\mathrm{Id} - S^v_N)v_0\|^2_{H^{0,s_0}}\\ &+ \|S^v_N(\mathrm{Id} - S^h_M)v_0\|^2_{H^{0,s_0}} + 2^{2M}2^{2Ns_0}T\|v_0\|^2_{L^2} + \frac{C}{\nu}\|v\|^4_{L^4_T(H^{\frac{1}{2},s_0})}\Big).\end{aligned}$$

Then choosing first an integer N such that

$$\|(\mathrm{Id} - S^v_N)v_0\|^2_{H^{0,s_0}} \leq \eta,$$

then an integer M greater than N such that

$$\|S_N^v(\mathrm{Id}-S_M^h)v_0\|_{H^{0,s_0}}^2 \leq \eta,$$

and finally a strictly positive real, number T_0 such that

$$2^{2M}2^{2Ns_0}T\|v_0\|_{L^2}^2 \leq \eta,$$

we obtain that, for any $T \leq T_0$,

$$\nu^{\frac{1}{2}}\|v\|^2_{L^4_T(H^{\frac{1}{2},s_0})} + \nu\|\nabla_h v\|^2_{L^2_T(H^{0,s_0})} \leq C\eta + \frac{C}{\nu}\|v\|^4_{L^4_T(H^{\frac{1}{2},s_0})}.$$

So, for any positive real number η, a positive real number T_1 exists so that

$$\nu^{\frac{1}{2}}\|v\|^2_{L^4_{T_1}(H^{\frac{1}{2},s_0})} + \nu\|\nabla_h v\|^2_{L^2_{T_1}(H^{0,s_0})} \leq \eta.$$

That yields the result, and proves Theorem 2. □

Proof of Theorem 3. Let us start by proving the propagation result stated in Theorem 3, considering $v_0 \in H^{0,s}$, with $s > s_0$; we just have to follow the computations leading to the global existence result with data in H^{0,s_0}. We have, as for (3.5) and using also Lemma 3,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_k(t)\|^2_{L^2} + \nu\|\nabla_h v_k(t)\|^2_{L^2} \leq Cd_k 2^{-2ks}\|v\|_{H^{\frac{1}{2},s}}(\|v\|_{H^{\frac{1}{2},s}}\|\nabla_h v\|_{H^{0,s_0}} + \|v\|_{H^{\frac{1}{2},s_0}}\|\nabla_h v\|_{H^{0,s}}),$$

so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{H^{0,s}} + \nu\|\nabla_h v(t)\|^2_{H^{0,s}} \leq \frac{C}{\nu^2}\|v\|^4_{H^{\frac{1}{2},s_0}}\|v\|^2_{H^{0,s}} + \frac{C}{\nu}\|\nabla_h v\|^2_{H^{0,s_0}}\|v\|^2_{H^{0,s}}.$$

By integration, we find that

$$\|v\|^2_{L^\infty_T(H^{0,s})} + 2\nu\|\nabla v\|^2_{L^2_T(H^{0,s})} \leq \exp\left(\frac{C}{\nu^2}\int_0^T \|v(t)\|^4_{H^{1/2,s_0}}\mathrm{d}t + \frac{C}{\nu}\int_0^T \|\nabla_h v(t)\|^2_{H^{0,s_0}}\mathrm{d}t\right).$$

So we find that the life span of the solution is controlled by the following norm:

$$T < +\infty \quad \Rightarrow \quad \int_0^T \|v(t)\|^4_{H^{1/2,s_0}}\mathrm{d}t + \int_0^T \|\nabla_h v(t)\|^2_{H^{0,s_0}}\mathrm{d}t = +\infty.$$

Now we are left with the proof of the uniqueness of the solutions when $s > 3/2$. So let v_1 and v_2 be two solutions of (NS_h), with initial data v_0, such that

$$\forall i \in \{1,2\}, \quad v_i \in L^\infty([0,T],H^{0,s}) \cap L^2([0,T],H^{1,s}), \quad s > \frac{3}{2},$$

and let $w \overset{\mathrm{def}}{=} v_1 - v_2$. We have

$$\partial_t w - \nu\Delta_h w + v_1 \cdot \nabla w + w \cdot \nabla v_2 = -\nabla p,$$

hence using Lemma 3 and interpolation inequalities, we can write

$$
\begin{aligned}
(v_1 \cdot \nabla w | w)_{H^{0,s}} &\leq C\Big(\|\nabla_h v_1\|_{H^{0,s-1}} \|w\|^2_{H^{1/2,s-1}} + \|v_1\|_{H^{1/2,s-1}} \|w\|_{H^{1/2,s-1}} \|\nabla_h w\|_{H^{0,s-1}}\Big) \\
&\leq \Big(\|\nabla_h v_1\|_{H^{0,s-1}} \|w\|_{H^{0,s-1}} \|\nabla_h w\|_{H^{0,s-1}} \\
&\qquad\qquad + \|v_1\|_{H^{1/2,s-1}} \|w\|^{\frac{1}{2}}_{H^{0,s-1}} \|\nabla_h w\|^{\frac{3}{2}}_{H^{0,s-1}}\Big) \\
&\leq \frac{\nu}{4} \|\nabla_h w\|^2_{H^{0,s-1}} + \frac{C}{\nu}\Big(\|\nabla_h v_1\|^2_{H^{1/2,s-1}} + \frac{1}{\nu^2}\|v_1\|^4_{H^{1/2,s-1}}\Big)\|w\|_{H^{0,s-1}}.
\end{aligned}
$$

But with Lemma 1, we can write that

$$
\big|(\Delta_k^v(w^h \cdot \nabla_h v_2)|w_k)\big| \leq C2^{-2k(s-1)} d_k \|w\|^2_{H^{\frac{1}{2},s-1}} \|\nabla_h v_2\|_{H^{0,s-1}}
$$

and

$$
\big|(\Delta_k^v(w^3 \partial_3 v_2)|w_k)\big| \leq C2^{-2k(s-1)} d_k \|w\|_{H^{\frac{1}{2},s-1}} \|w\|_{H^{0,s-1}} \|v_2\|_{H^{\frac{1}{2},s}}.
$$

It follows that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{H^{0,s-1}} + \nu\|\nabla_h w\|^2_{H^{0,s-1}} &\leq \frac{\nu}{4}\|\nabla_h w\|^2_{H^{0,s-1}} \\
&+ \frac{C}{\nu}\Big(\|\nabla_h v_1\|^2_{H^{1/2,s-1}} + \|\nabla_h v_2\|^2_{H^{1/2,s-1}} + \frac{1}{\nu^2}\Big(\|v_1\|^4_{H^{1/2,s-1}} + \|v_2\|^4_{H^{1/2,s-1}}\Big)\Big)\|w\|_{H^{0,s-1}}.
\end{aligned}
$$

The result follows by time integration, and Theorem 3 is proved. □

4. The case of rotating fluids: proof of Theorem 4

In this final section, we are going to prove Theorem 4. We shall in fact prove a more precise result that stated in Theorem 4, and in order to do so, we need to introduce some notation. We shall consider the following rotating fluid system:

$$
(RF^\varepsilon) \qquad \left\{ \begin{aligned} \partial_t v - \nu\Delta_h v - \nu_V \partial_3^2 v + v\cdot\nabla v + \frac{1}{\varepsilon} v \times e_3 &= -\nabla p \\ v_{|t=0} &= v_0\,. \end{aligned} \right.
$$

The elements of the kernel of the operator $Lv \stackrel{\text{def}}{=} P(v \times e_3)$, where P is the projector onto divergence–free vector fields, are bidimensional vector fields; so it is natural to introduce the solution of the bidimensional Navier–Stokes equation:

$$
\left\{ \begin{aligned} \partial_t \bar{u} - \nu\Delta_h \bar{u} + \bar{u}\cdot\nabla\bar{u} &= -\nabla\bar{p} \\ \bar{u}_{|t=0} &= \bar{u}_0\,, \end{aligned} \right.
$$

where $\bar{u}_0$ is a bidimensional, divergence–free vector field.

Finally in order to state the following theorem, we need to introduce the solution to the free wave equation associated with (RF^ε).

$$
\left\{ \begin{aligned} \partial_t w_F - \nu\Delta_h w_F - \nu_V \partial_3^2 w_F + \frac{1}{\varepsilon} w_F \times e_3 &= -\nabla p_F \\ \operatorname{div} w_F &= 0 \\ w_{F|t=0} &= w_0\,. \end{aligned} \right.
$$

We shall prove the following result.

Theorem 6. *Let $\nu > 0$ and $\nu_V \geq 0$ be two real numbers. Let $v_0 = \bar{u}_0 + w_0$, where $\bar{u}_0 \in L^2(\mathbb{R}^2)$ is a bidimensional vector field, and where $w_0 \in H^{0,s}$, with $s > 1/2$. We suppose both fields are divergence free. Then there exists ε_0, independent of ν_V, such that for $\varepsilon \leq \varepsilon_0$, there is a global solution to the system (RF^ε), bounded in $L^\infty(\mathbb{R}^+, H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s})$. Moreover, if $s > 3/2$, then the solution is unique. Finally when ε goes to zero, v satisfies*

$$v - \bar{u} - w_F \to 0 \quad in \quad L^\infty(\mathbb{R}^+, H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s}).$$

The proof of that theorem mixes the results obtained in the previous sections with anisotropic Strichartz estimates on w_F, in the spirit of [6]: let us define

$$\mathcal{F}\left(A^\varepsilon_{RF} f\right)(\xi) \stackrel{\text{def}}{=} e^{i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi_h|^2 - \nu_V t\xi_3^2}\mathcal{F}f(\xi).$$

We have the following estimate.

Theorem 7. *For any vector field f we have, for any $p \in [1, \infty]$,*

$$\|\Delta^h_j \Delta^v_k A^\varepsilon_{RF} f\|_{L^p_T(L^{\infty,2}_{x_h,x_3})} \leq C_\nu 2^{j(1-\frac{2}{p})} \min\left(1, (\varepsilon 2^{2j})^{\frac{1}{4p}} 2^{\frac{1}{2p}|j-k|}\right) \|\Delta^h_j \Delta^v_k f\|_{L^2(\mathbb{R}^3)}. \tag{4.6}$$

Proof of Theorem 7. We follow exactly the same lines as in [6]. For the convenience of the reader, we present here a self contained proof. Like all Strichartz–type inequalities, (4.6) is the consequence of dispersive estimates. Writing

$$A_{RF}(\tau, \theta, \widetilde{\theta}) f \stackrel{\text{def}}{=} \mathcal{F}^{-1}\left(e^{i\tau\frac{\xi_3}{|\xi|} - \theta|\xi_h|^2 - \widetilde{\theta}\xi_3^2}\widehat{f}(\xi)\right),$$

we have the following dispersive lemma.

Lemma 6. *There exists a constant C such that the following holds. For any function f, we have*

$$\|\Delta^h_j \Delta^v_k A_{RF}(\tau, \theta, \widetilde{\theta}) f\|_{L^{\infty,2}_{x_h,x_3}} \leq C \min\left(1, \tau^{-\frac{1}{2}} 2^{|j-k|}\right) 2^{2j} e^{-C\theta 2^{2j}} \|f\|_{L^{1,2}_{x_h,x_3}}. \tag{4.7}$$

Proof of the lemma. Let us define

$$I_{j,k}(\tau, \theta, x', \xi_3) \stackrel{\text{def}}{=} \varphi(2^{-k}\xi_3) \int_{\mathbb{R}^2} e^{ix_h\cdot\xi_h + i\tau\frac{\xi_3}{|\xi|} - \theta|\xi_h|^2} \varphi(2^{-j}|\xi_h|)\, d\xi_h.$$

Then we have

$$\|\Delta^h_j \Delta^v_k A_{RF}(\tau, \theta, \widetilde{\theta}) f\|_{L^{\infty,2}_{x_h,x_3}} \leq Ce^{-C\widetilde{\theta} 2^{2k}} \|I_{j,k}(\tau, \theta, \cdot, \cdot)\|_{L^{\infty,\infty}_{x_h,\xi_3}} \|f\|_{L^{1,2}_{x_h,x_3}},$$

where we have used a Young inequality in the horizontal variables, and Hölder's inequality in the vertical variable, associated with Plancherel's formula.

Now all we need is an estimate of $I_{j,k}(\tau, \theta, \cdot, \cdot)$ in $L^{\infty,\infty}_{x_h,\xi_3}$: we can write

$$I_{j,k}(\tau, \theta, x_h, \xi_3) = 2^{2k}\widetilde{I}_{j,k}(\tau, 2^{2k}\theta, 2^k x_h, 2^{-k}\xi_3)$$

with

$$\widetilde{I}_{j,k}(\tau, \eta, y', \zeta_3) \stackrel{\text{def}}{=} \varphi(\zeta_3) \int_{\mathbb{R}^2} e^{iy'\cdot\zeta' + i\tau\frac{\zeta_3}{|\zeta|} - \eta|\zeta_h|^2} \varphi(2^{k-j}|\zeta_h|)\, d\zeta_h.$$

So we shall integrate by parts in the following integral:

$$\Phi(\zeta_1, \zeta_3) \stackrel{\text{def}}{=} \int_{\mathbb{R}} e^{i\tau\frac{\zeta_3}{|\zeta|} - \eta|\zeta_h|^2} \varphi(|\zeta_h|)\, d\zeta_2.$$

Let us introduce the operator $\mathcal{L}$ defined as

$$\forall a, \quad \mathcal{L}a = \frac{1}{1+\tau\alpha^2(\zeta)}\left(a - i\alpha(\zeta)\partial_{\zeta_2}a\right) \quad \text{with} \quad \alpha(\zeta) = \partial_{\zeta_2}\left(\frac{\zeta_3}{|\zeta|}\right).$$

An easy computation yields

$$\mathcal{L}(e^{i\tau\frac{\zeta_3}{|\zeta|}}) = e^{i\tau\frac{\zeta_3}{|\zeta|}},$$

which implies immediately that

$$\Phi(\zeta_1,\zeta_3) = \int_{\mathbb{R}} e^{i\tau\frac{\zeta_3}{|\zeta|}}\ {}^t\mathcal{L}\left(e^{-\eta|\zeta_h|^2}\varphi(|\zeta|)\right)\,\mathrm{d}\zeta_2.$$

But we can compute, for any function Ψ,

$${}^t\mathcal{L}\left(\Psi(\zeta)\right) = \left(\frac{1}{1+\alpha^2\tau} - i(\partial_{\zeta_2}\alpha)\frac{1-\tau\alpha^2}{(1+\tau\alpha^2)^2}\right)\Psi(\zeta) - \frac{i\alpha}{1+\tau\alpha^2}\partial_{\zeta_2}\Psi(\zeta),$$

so using the fact that ζ is in a fixed ring of $\mathbb{R}^3$ and that $\varphi \in \mathcal{D}(\mathbb{R})$, we get after elementary computations

$$\left|{}^t\mathcal{L}\left(e^{-\eta|\zeta_h|^2}\varphi(2^{k-j}|\zeta_h|)\right)\right| \leq \frac{C\min(2^{k-j},1)}{1+\tau\zeta_2^2\zeta_3^2}e^{-\eta|\zeta_h|^2}.$$

It follows that

$$\begin{aligned}
\|\widetilde{I}_{j,k}(\tau,\eta,y',\zeta_3)\|_{L^{\infty,\infty}_{y',\zeta_3}} &\leq Ce^{-C\eta}\left\|\varphi(\zeta_3)\int_{|\zeta_2|\leq C2^{j-k}}\min(2^{k-j},1)\frac{d\zeta_2}{1+\tau\zeta_2^2\zeta_3^2}\right\|_{L^\infty_{\zeta_3}}\\
&\leq Ce^{-C\eta}\tau^{-\frac12}\min(2^{j-k},1)\|\varphi(\zeta_3)\zeta_3^{-1}\|_{L^\infty}\\
&\leq Ce^{-C\eta}\tau^{-\frac12}\min(2^{j-k},1).
\end{aligned}$$

So finally we have obtained that

$$\|\Delta_j^h\Delta_k^v A_{RF}(\tau,\theta)f\|_{L^{\infty,2}_{x_h,x_3}} \leq Ce^{-C\widetilde{\theta}2^{2k}}e^{-C\theta 2^{2j}}2^{2k}\min(2^{j-k},1)\tau^{-\frac12}\|f\|_{L^{1,2}_{x_h,x_3}}. \tag{4.8}$$

Now to conclude to (4.7), we note that by Sobolev embeddings,

$$\begin{aligned}
\|\Delta_j^h\Delta_k^v A_{RF}(\tau,\theta,\widetilde{\theta})f\|_{L^{\infty,2}_{x_h,x_3}} &\leq C2^j e^{-C\widetilde{\theta}2^{2k}}e^{-C\theta2^{2j}}\|\Delta_j^h\Delta_k^v f\|_{L^2}\\
&\leq C2^{2j}e^{-C\widetilde{\theta}2^{2k}}e^{-C\theta2^{2j}}\|\Delta_j^h\Delta_k^v f\|_{L^{1,2}_{x_h,x_3}},
\end{aligned}$$

which with (4.8) implies that

$$\|\Delta_j^h\Delta_k^v A_{RF}(\tau,\theta)f\|_{L^{\infty,2}_{x_h,x_3}} \leq Ce^{-C\widetilde{\theta}2^{2k}}e^{-C\theta2^{2j}}2^{2j}\min\left(\frac{1}{\tau^{\frac12}}2^{|j-k|},1\right)\|f\|_{L^{1,2}_{x_h,x_3}}.$$

The lemma is proved. □

End of the proof of Theorem 7. To prove the result, we are going to use duality arguments, based on the anisotropic estimate (4.7). In the following, we shall call

$$\mathcal{B} \stackrel{\text{def}}{=} \left\{\Psi \in \mathcal{D}(\mathbb{R}^3),\ \|\Psi\|_{L^\infty_T(L^{1,2}_{x_h,x_3})} \leq 1\right\}. \tag{4.9}$$

The following set of equalities is standard:

$$
\begin{aligned}
\|\Delta_j^h\Delta_k^v A^\varepsilon_{RF} f\|_{L^1_T(L^{\infty,2}_{x_h,x_3})} &= \sup_{\Psi\in\mathcal{B}} \int_{\mathbb{R}^+} < \Delta_j A^\varepsilon_{RF}(t) f, \Psi(t) > \,\mathrm{d}t \\
&= \sup_{\Psi\in\mathcal{B}} (2\pi)^{-3} \int_{\mathbb{R}^+\times\mathbb{R}^3} \varphi(2^{-j}|\xi|) e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi_h|^2 - \nu_V \xi_3^2} \widehat{f}(\xi)\widehat{\Psi}(t,\xi)\,\mathrm{d}t\mathrm{d}\xi \\
&= \sup_{\Psi\in\mathcal{B}} (2\pi)^{-3} \sum_{k\le j} \int_{\mathbb{R}^+\times\mathbb{R}^3} \widehat{f}_{j,k}(\xi)\widehat{\Psi}_{j,k}(t,\xi) e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi_h|^2 - \nu_V t\xi_3^2}\,\mathrm{d}t\mathrm{d}\xi,
\end{aligned}
$$

where we have defined $f_{j,k} = \Delta_j^h\Delta_k^v f$ and $\Psi_{j,k} = \Delta_j^h\Delta_k^v\Psi$. So it follows that

$$
\|\Delta_j A^\varepsilon_{\nu,n} f\|_{L^1_T(L^{\infty,2}_{x_h,x_3})} \le \sup_{\Psi\in\mathcal{B}} (2\pi)^{-3} \sum_{k\le j} \|f_{j,k}\|_{L^2} \left\| \int_{\mathbb{R}^+} e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi_h|^2 - \nu_V t\xi_3^2} \widehat{\Psi}_{j,k}(t,\xi)\,\mathrm{d}t \right\|_{L^2(\mathbb{R}^3)}.
$$

But we have

$$
\begin{aligned}
&\left\| \int_{\mathbb{R}^+} e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi_h|^2 - \nu_V t\xi_3^2} \widehat{\Psi}_{j,k}(t,\xi)\,\mathrm{d}t \right\|^2_{L^2(\mathbb{R}^3)} \\
&= \int_{(\mathbb{R}^+)^2\times\mathbb{R}^3} e^{-i\frac{t}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu t|\xi_h|^2 - \nu_V t\xi_3^2} \widehat{\Psi}_{j,k}(t,\xi) e^{i\frac{s}{\varepsilon}\frac{\xi_3}{|\xi|} - \nu s|\xi_h|^2 - \nu_V s\xi_3^2} \overline{\widehat{\Psi}}_{j,k}(s,\xi)\,\mathrm{d}t\mathrm{d}s\mathrm{d}\xi \\
&\le \int_{(\mathbb{R}^+)^2} \|\Psi_{j,k}(s,\cdot)\|_{L^{1,2}_{x_h,x_3}} \|\Psi_{j,k}(t,\cdot)\|_{L^{1,2}_{x_h,x_3}} \min\left(1, \frac{\varepsilon^{\frac12} 2^{|j-k|}}{|t-s|^{\frac12}}\right) 2^{2j} e^{-\nu(t+s)2^{2j}}\,\mathrm{d}t\mathrm{d}s.
\end{aligned}
\tag{4.10}
$$

Then we just need to notice that

$$
\int_{(\mathbb{R}^+)^2} \frac{e^{-\nu(t+s)2^{2j}}}{|t-s|^{\frac12}}\,\mathrm{d}t\mathrm{d}s = 2^{-3j} \int_{(\mathbb{R}^+)^2} \frac{e^{-\nu(t+s)}}{|t-s|^{\frac12}}\,\mathrm{d}t\mathrm{d}s,
$$

and that

$$
\int_{(\mathbb{R}^+)^2} e^{-\nu(t+s)2^{2j}}\,\mathrm{d}t\mathrm{d}s = \frac{2^{-2j}}{\nu^2},
$$

so (4.10) yields finally

$$
\|\Delta_j^h\Delta_k^v A^\varepsilon_{RF} f\|_{L^1_T(L^{\infty,2}_{x_h,x_3})} \le C_\nu \min\left(1, (\varepsilon 2^{2j})^{\frac14} 2^{\frac{|j-k|}{2}}\right) 2^{-j} \|\Delta_j^h\Delta_k^v f\|_{L^2(\mathbb{R}^3)}.
$$

But by Sobolev embeddings and energy estimates, we have

$$
\|\Delta_j^h\Delta_k^v A^\varepsilon_{RF} f\|_{L^1_T(L^{\infty,2}_{x_h,x_3})} \le C 2^j \|\Delta_j^h\Delta_k^v f\|_{L^2(\mathbb{R}^3)},
$$

and the theorem is proved by interpolation.

□

Proof of Theorem 6. The proof is going to use Theorem 7 proved above. Such an estimate is best used when the horizontal and vertical frequencies of w_F are fixed, and it is possible to do so for the following reason: it is clear that, as w_F satisfies a linear equation,

$$
\lim_{\substack{N,N_3\to-\infty\\ N',N_3'\to+\infty}} (\mathrm{Id}-S^h_N)S^h_{N'}(\mathrm{Id}-S^v_{N_3})S^v_{N_3'}\, w_F = w_F \quad \text{in} \quad H^{\sigma,\sigma'}, \quad \forall\sigma,\sigma'. \tag{4.11}
$$

So for any $\eta > 0$, there exist integers N, N', N_3, N_3' such that

$$\|w_F - w_{F,m}\|_{L^\infty(\mathbb{R}^+, H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s})} \leq \eta, \tag{4.12}$$

where we have noted

$$w_{F,m} \stackrel{\text{def}}{=} (\mathrm{Id} - S_N^h) S_{N'}^h (\mathrm{Id} - S_{N_3}^v) S_{N_3'}^v w_F.$$

Then (4.6) becomes simply

$$\forall p \in [1, \infty], \quad \|w_{F,m}\|_{L_T^p(L^{\infty,2}_{x_h, x_3})} \leq C_\eta \varepsilon^{1/4p} \|w_{0,m}\|_{L^2(\mathbb{R}^3)}, \tag{4.13}$$

where here and in the following, we have written C_η for any constant depending on N, N', N_3 and N_3', such that (4.12) is satisfied.

Let us recall the following classical estimates on the bidimensional Navier–Stokes equation, and on linear parabolic equations: we have

$$\forall t \geq 0, \quad \|\bar{u}(t)\|^2_{L^2(\mathbb{R}^2)} + 2\nu \int_0^t \|\nabla_h \bar{u}(t')\|^2_{L^2(\mathbb{R}^2)} \, \mathrm{d}t' = \|\bar{u}_0\|^2_{L^2(\mathbb{R}^2)}, \tag{4.14}$$

$$\text{and} \quad \forall t \geq 0, \quad \|w_F(t)\|^2_{H^{0,s}} + 2\nu \int_0^t \|\nabla_h w_F(t')\|^2_{H^{0,s}} \, \mathrm{d}t' = \|w_0\|^2_{H^{0,s}}. \tag{4.15}$$

Now if we define $w \stackrel{\text{def}}{=} v - \bar{u} - w_{F,m}$, it is enough to prove that w is unique, and goes to zero in $L^\infty(\mathbb{R}^+, H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s})$. The function w satisfies the following equation:

$$\partial_t w - \nu \Delta_h w - \nu_V \partial_3^2 w + \frac{1}{\varepsilon} w \times e_3 = -\nabla \tilde{p} + \sum_{j=1}^{6} F_j \tag{4.16}$$

with

$$\begin{aligned}
F_1 &\stackrel{\text{def}}{=} -w \cdot \nabla w \\
F_2 &\stackrel{\text{def}}{=} -\bar{u} \cdot \nabla w - w \cdot \nabla \bar{u} \\
F_3 &\stackrel{\text{def}}{=} -w_{F,m} \cdot \nabla w \\
F_4 &\stackrel{\text{def}}{=} -w \cdot \nabla w_{F,m} \\
F_5^j &\stackrel{\text{def}}{=} -\mathrm{div}_h(w^j_{F,m} \bar{u}^h) - \mathrm{div}_h(\bar{u}^j w^h_{F,m}) - \mathrm{div}_h(w^j_{F,m} w^h_{F,m}) \\
F_6^j &\stackrel{\text{def}}{=} -\bar{u}^3 \partial_3 w^j_{F,m} - \bar{u}^j \partial_3 w^3_{F,m} - \partial_3(w^j_{F,m} w^3_{F,m}).
\end{aligned}$$

and

$$w_{|t=0} = \left(\mathrm{Id} - (\mathrm{Id} - S_N^h) S_{N'}^h (\mathrm{Id} - S_{N_3}^v) S_{N_3'}^v\right) w_0.$$

Note that, according to (4.11), we have

$$\forall \eta > 0, \quad \exists (N, N', N_3, N_3'), \quad \|w_{|t=0}\|_{H^{0,s}} \leq \eta. \tag{4.17}$$

The uniqueness result follows simply from Theorem 3, since the skew–symmetric term $\frac{1}{\varepsilon} w \times e_3$ does not interfere in the energy estimates. So let us prove that for ε small enough, there is a global solution w to equation (4.16), which goes to zero in $L^\infty(\mathbb{R}^+, H^{0,s}) \cap L^2(\mathbb{R}^+, H^{1,s})$. The proof will use estimates proved in the previous section, as well as the Strichartz estimate proved above.

First, using Lemma 3 proved in Section 2, we get

$$(\Delta_k^v(w \cdot \nabla w)|\Delta_k^v w)_{L^2(\mathbb{R}^3)} \leq C d_k 2^{-2ks} \|w\|_{H^{1/2,s}}^2 \|\nabla_h w\|_{H^{0,s}}.$$

Using the interpolation between $H^{0,s}$ and $H^{1,s}$, and taking the sum over k, we get

$$(F_1|w)_{H^{0,s}} \leq C\|w\|_{H^{0,s}} \|\nabla_h w\|_{H^{0,s}}^2. \tag{4.18}$$

Now, let us estimate $(F_2|w)_{H^{0,s}}$. Since $\bar{u}$ does not depend on the third variable, we have

$$\begin{aligned}(\Delta_k^v(\bar{u} \cdot \nabla w)|\Delta_k^v w)_{L^2(\mathbb{R}^3)} &= (\bar{u} \cdot \nabla \Delta_k^v w)|\Delta_k^v w)_{L^2(\mathbb{R}^3)} \\ &= -\frac{1}{2}\int_{\mathbb{R}^3} \operatorname{div}_h \bar{u}^h |\Delta_k^v w|^2 \, dx,\end{aligned}$$

so as $H^{1/2}(\mathbb{R}^2)$ is embedded in $L^4(\mathbb{R}^2)$, we get

$$|(\Delta_k^v(\bar{u} \cdot \nabla w)|\Delta_k^v w)_{L^2(\mathbb{R}^3)}| \leq C\|\nabla_h \bar{u}\|_{L^2(\mathbb{R}^2)} \|\Delta_k^v w\|_{L^2(\mathbb{R}_{x_3};H^{1/2}(\mathbb{R}^2))}^2.$$

By interpolation between $L^2(\mathbb{R}^2)$ and $\dot{H}^1(\mathbb{R}^2)$, that yields

$$|(\Delta_k^v(\bar{u} \cdot \nabla w)|\Delta_k^v w)_{L^2(\mathbb{R}^3)}| \leq \frac{C}{\nu}\|\nabla_h \bar{u}\|_{L^2(\mathbb{R}^2)}^2 \|\Delta_k^v w\|_{L^2(\mathbb{R}^3)}^2 + \frac{\nu}{16}\|\nabla_h \Delta_k^v w\|_{L^2(\mathbb{R}^3)}^2,$$

and multiplying by 2^{2ks} and summing over k, we get finally

$$|(\bar{u} \cdot \nabla w|w)_{H^{0,s}}| \leq \frac{C}{\nu}\|\nabla_h \bar{u}\|_{L^2(\mathbb{R}^2)}^2 \|w\|_{H^{0,s}}^2 + \frac{\nu}{16}\|\nabla_h w\|_{H^{0,s}}^2. \tag{4.19}$$

Moreover, we have

$$|(\Delta_k^v(w \cdot \nabla \bar{u})|\Delta_k^v w)_{L^2(\mathbb{R}^3)}| \leq C\|\Delta_k^v w\|_{L^2(\mathbb{R}_{x_3};H^{1/2}(\mathbb{R}^2))}^2 \|\nabla_h \bar{u}\|_{L^2(\mathbb{R}^2)},$$

so similarly

$$|(w \cdot \nabla \bar{u}|w)_{H^{0,s}}| \leq \frac{C}{\nu}\|\nabla_h \bar{u}\|_{L^2(\mathbb{R}^2)}^2 \|w\|_{H^{0,s}}^2 + \frac{\nu}{16}\|\nabla_h w\|_{H^{0,s}}^2. \tag{4.20}$$

So, putting estimates (4.19) and (4.20), we get that

$$|(F_2|w)_{H^{0,s}}| \leq \frac{C}{\nu}\|\nabla_h \bar{u}\|_{L^2(\mathbb{R}^2)}^2 \|w\|_{H^{0,s}}^2 + \frac{\nu}{8}\|\nabla_h w\|_{H^{0,s}}^2. \tag{4.21}$$

In order to estimate the term $(F_3|w)_{H^{0,s}}$, we use Lemma 3 proved in Section 2, which yields

$$\left|(\Delta_k^v(w_{F,m} \cdot \nabla w)|\Delta_k^v w)_{L^2(\mathbb{R}^3)}\right| \leq C d_k 2^{-2ks} \|w\|_{H^{1/2,s}} \times \left(\|w_{F,m}\|_{H^{1/2,s}} \|\nabla_h w\|_{H^{0,s}} + \|w\|_{H^{1/2,s}} \|\nabla_h w_{F,m}\|_{H^{0,s}}\right),$$

which in turn implies that

$$|(F_3|w)_{H^{0,s}}| \leq \frac{\nu}{16}\|\nabla_h w\|_{H^{0,s}}^2 + \frac{C}{\nu}\left(\|w_{F,m}\|_{H^{1/2,s}}^2 \|w\|_{H^{1/2,s}}^2 + \|w\|_{H^{0,s}}^2 \|\nabla_h w_{F,m}\|_{H^{0,s}}^2\right).$$

So finally, by interpolation, we infer

$$|(F_3|w)_{H^{0,s}}| \leq \frac{\nu}{8}\|\nabla_h w\|_{H^{0,s}}^2 + \frac{C}{\nu}\|w\|_{H^{0,s}}^2 \left(\frac{C}{\nu^2}\|w_{F,m}\|_{H^{1/2,s}}^4 + \|\nabla_h w_{F,m}\|_{H^{0,s}}^2\right). \tag{4.22}$$

Moreover, we can write

$$\begin{aligned}
|(F_4|w)_{H^{0,s}}| &= |(w\cdot\nabla w_{F,m}|w)_{H^{0,s}}| \\
&\leq \int_{\mathbb{R}^2}\left|\Big(w(x_h,\cdot)\cdot\nabla w_{F,m}(x_h,\cdot)|w(x_h,\cdot)\Big)_{H^s(\mathbb{R}_{x_3})}\right| \mathrm{d}x_h \\
&\leq \int_{\mathbb{R}^2}\|w(x_h,\cdot)\|_{H^s(\mathbb{R}_{x_3})}^2\|\nabla w_{F,m}(x_h,\cdot)\|_{H^s(\mathbb{R}_{x_3})}\,\mathrm{d}x_h \\
&\leq \|w\|_{H^{0,s}}^2\|\nabla w_{F,m}\|_{L^\infty(\mathbb{R}^2;H^s(\mathbb{R}_{x_3}))}.
\end{aligned}$$

But the spectral localization of $w_{F,m}$ implies that, for any η, a constant C_η exists such that

$$\|\nabla w_{F,m}\|_{L^\infty(\mathbb{R}^2;H^s(\mathbb{R}_{x_3}))} \leq C_\eta\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}},$$

so finally

$$|(F_4|w)_{H^{0,s}}| \leq C_\eta\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}\|w\|_{H^{0,s}}^2. \tag{4.23}$$

To estimate $|(F_5|w)_{H^{0,s}}|$, let us write that

$$\begin{aligned}
|(F_5|w)_{H^{0,s}}| &\leq \|F_5\|_{H^{-1,s}}\|\nabla_h w\|_{H^{0,s}} \\
&\leq \frac{C}{\nu}\|F_5\|_{H^{-1,s}}^2 + \frac{\nu}{8}\|\nabla_h w\|_{H^{0,s}}^2.
\end{aligned}$$

By definition of F_5, we have

$$\|F_5\|_{H^{-1,s}}^2 \leq C(\|w_{F,m}\bar{u}\|_{H^{0,s}}^2 + \|w_{F,m}w_{F,m}\|_{H^{0,s}}^2).$$

As the Fourier transform of $w_{F,m}$ has its support in the vertical frequencies included in the ball of radius N_3', we have

$$\begin{aligned}
\|F_5\|_{H^{-1,s}}^2 &\leq CN_3'^{2s}(\|w_{F,m}\bar{u}\|_{L^2(\mathbb{R}^3)}^2 + \|w_{F,m}w_{F,m}\|_{L^2(\mathbb{R}^3)}^2) \\
&\leq CN_3'^{2s}(\|\bar{u}\|_{L^2(\mathbb{R}^2)}^2\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}^2 + \|w_{F,m}\|_{L^{2,\infty}_{x_h,x_3}}^2\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}^2).
\end{aligned}$$

Using again the spectral localization in the vertical variable and the energy estimate of the free rotating system, we get that

$$\begin{aligned}
\|w_{F,m}\|_{L^{2,\infty}_{x_h,x_3}}^2 &\leq CN_3'\|w_{F,m}\|_{L^2(\mathbb{R}^3)}^2 \\
&\leq CN_3'\|w_{0,m}\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}$$

So, using now the energy estimate in the $2-D$ Navier-Stokes equation, we get that

$$\|F_5\|_{H^{-1,s}}^2 \leq CN_3'^{2s}(\|\bar{u}_0\|_{L^2(\mathbb{R}^2)}^2 + N_3'\|w_{0,m}\|_{L^2(\mathbb{R}^3)})\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}^2.$$

As in all that follows, let us denote by C_η any constant depending only on η and on the initial data $\bar{u}_0$ and v_0; we can rewrite the above estimate as

$$|(F_5|w)_{H^{0,s}}| \leq C_\eta \|w_{F,m}\|^2_{L^{\infty,2}_{x_h,x_3}} + \frac{\nu}{8}\|\nabla_h w\|_{H^{0,s}} \tag{4.24}$$

To estimate $|(F_6|w)_{H^{0,s}}|$, let us write that

$$|(F_6|w)_{H^{0,s}}| \leq \|F_6\|_{H^{0,s}}\|w\|_{H^{0,s}}.$$

Using again the spectral localization in the vertical variable of $w_{F,m}$, the energy estimate in the $2-D$ Navier-Stokes equation and the energy conservation for free rotating system, we get that

$$\|F_6\|_{H^{0,s}} \leq C{N'_3}^{s+1}\Big(\|\bar{u}_0\|_{L^2(\mathbb{R}^2)} + {N'_3}^{\frac{1}{2}}\|w_{0,m}\|_{L^2(\mathbb{R}^3)}\Big)\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}.$$

As above, this can be written as

$$|(F_6|w)_{H^{0,s}}| \leq C_\eta \|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}\|w\|_{H^{0,s}}. \tag{4.25}$$

An energy estimate in $H^{0,s}$ on equation (4.16) yields, writing $F \stackrel{\text{def}}{=} \sum_{j=1}^{6} F_j$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{H^{0,s}} + \nu\|\nabla_h w(t)\|^2_{H^{0,s}} = |(F(t)|w(t))_{H^{0,s}}|.$$

So using (4.18) and (4.21–4.25) yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{H^{0,s}} + \nu\|\nabla_h w(t)\|^2_{H^{0,s}} &\leq \Phi(t)\|w(t)\|^2_{H^{0,s}} + G_\eta(t)\\ &\quad + \left(\frac{\nu}{4} + C\|w(t)\|_{H^{0,s}}\Big(1+\frac{1}{\nu}\|w(t)\|_{H^{0,s}}\Big)\right)\|\nabla_h w(t)\|^2_{H^{0,s}}\end{aligned}$$

with

$$\begin{aligned} G_\eta(t) &\stackrel{\text{def}}{=} C_\eta\|w_{F,m}\|^2_{L^{\infty,2}_{x_h,x_3}} + C_\eta\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}\|w(t)\|_{H^{0,s}} \quad \text{and}\\ \Phi(t) &\stackrel{\text{def}}{=} \frac{C}{\nu}\left(\|\nabla_h\bar{u}(t)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla_h w_{F,m}(t)\|^2_{H^{0,s}} + \frac{C}{\nu^2}\|w_{F,m}(t)\|^4_{H^{\frac{1}{2},s}} + \nu C_\eta\|w_{F,m}(t)\|_{L^{\infty,2}_{x_h,x_3}}\right).\end{aligned}$$

Note that according to (4.14), (4.15) and (4.13), the function Φ is in $L^1(\mathbb{R}^+)$ and

$$\|\Phi\|_{L^1(\mathbb{R}^+)} \leq \frac{C}{\nu}\left(\frac{2}{\nu}\|\bar{u}_0\|^2_{L^2} + \|w_0\|^2_{H^{0,s}}\Big(1+\frac{C}{\nu}\|w_0\|^2_{H^{0,s}}\Big) + C_\eta\|w_{0,m}\|_{L^2(\mathbb{R}^3)}\right). \tag{4.26}$$

Let us fix η such that

$$C\|w(0)\|_{H^{0,s}}\Big(1+\frac{1}{\nu}\|w(0)\|_{H^{0,s}}\Big) \leq \frac{\nu}{16}\cdot \tag{4.27}$$

Let us consider the supremum $T^\star$ of the set of the real numbers T such that

$$\forall t \leq T,\ C\|w(t)\|_{H^{0,s}}\Big(1+\frac{1}{\nu}\|w(t)\|_{H^{0,s}}\Big) \leq \frac{\nu}{8}\cdot \tag{4.28}$$

As long as the above condition (4.28) is satisfied, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2_{H^{0,s}} + \frac{\nu}{2}\|\nabla_h w(t)\|^2_{H^{0,s}} \leq \Phi(t)\|w(t)\|^2_{H^{0,s}} + C_\eta\|w_{F,m}\|^2_{L^{\infty,2}_{x_h,x_3}} + C_\eta\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}.$$

Let us define

$$\widetilde{w}(t) \stackrel{\mathrm{def}}{=} w(t)\exp\left(-\int_0^t \Phi(t')\mathrm{d}t'\right).$$

We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{w}(t)\|^2_{H^{0,s}} + \frac{\nu}{2}\|\nabla_h \widetilde{w}(t)\|^2_{H^{0,s}} \leq \|w_{|t=0}\|^2_{H^{0,s}} + C_\eta\|w_{F,m}\|^2_{L^{\infty,2}_{x_h,x_3}} + C_\eta\|w_{F,m}\|_{L^{\infty,2}_{x_h,x_3}}.$$

So by integration, we get, for any $t \leq T^\star$,

$$\|\widetilde{w}(t)\|^2_{H^{0,s}} + \frac{\nu}{2}\int_0^t \|\nabla_h\widetilde{w}(t')\|^2_{H^{0,s}}\mathrm{d}t' \leq \|w_{|t=0}\|^2_{H^{0,s}} + C_\eta\int_0^t\left(\|w_{F,m}(t')\|^2_{L^{\infty,2}_{x_h,x_3}} + C_\eta\|w_{F,m}(t')\|_{L^{\infty,2}_{x_h,x_3}}\right)\mathrm{d}t'.$$

From now on, we shall use the Strichartz' type estimates. Using the estimate (4.13), we get

$$C_\eta\int_0^t\left(\|w_{F,m}(t')\|^2_{L^{\infty,2}_{x_h,x_3}} + C_\eta\|w_{F,m}(t')\|_{L^{\infty,2}_{x_h,x_3}}\right)\mathrm{d}t' \leq C_\eta(\varepsilon)$$

where $C_\eta(\varepsilon)$ is a function of ε such that, for fixed η, $\lim\limits_{\varepsilon\to 0} C_\eta(\varepsilon) = 0$.

So finally

$$\|\widetilde{w}(t)\|^2_{H^{0,s}} + \frac{\nu}{2}\int_0^t \|\nabla_h\widetilde{w}(t')\|^2_{H^{0,s}}\mathrm{d}t' \leq \|w_{|t=0}\|^2_{H^{0,s}} + C_\eta(\varepsilon),$$

hence

$$\|w(t)\|^2_{H^{0,s}} + \frac{\nu}{2}\int_0^t \|\nabla_h\widetilde{w}(t')\|^2_{H^{0,s}}\mathrm{d}t' \leq \left(\|w_{|t=0}\|^2_{H^{0,s}} + C_\eta(\varepsilon)\right)\exp\int_0^t \Phi(t')\,\mathrm{d}t',$$

and the theorem is proved using (4.26). □

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