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Modélisation Mathématique et Analyse Numérique

EXISTENCE OF A SOLUTION FOR A NONLINEARLY ELASTIC PLANE MEMBRANE "UNDER TENSION"

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Abstract. A justification of the two-dimensional nonlinear "membrane" equations for a plate made of a Saint Venant-Kirchhoff material has been given by Fox *et al.* [9] by means of the method of formal asymptotic expansions applied to the three-dimensional equations of nonlinear elasticity. This model, which retains the material-frame indifference of the original three dimensional problem in the sense that its energy density is invariant under the rotations of $\mathbb{R}^3$, is equivalent to finding the critical points of a functional whose nonlinear part depends on the first fundamental form of the unknown deformed surface. We establish here an existence result for these equations in the case of the membrane submitted to a boundary condition of "tension", and we show that the solution found in our analysis is injective and is the unique minimizer of the nonlinear membrane functional, which is not sequentially weakly lower semi-continuous. We also analyze the behaviour of the membrane when the "tension" goes to infinity and we conclude that a "well-extended" membrane may undergo large loadings.

Résumé. Une justification des équations bidimensionnelles non linéaires "en membrane" d'une plaque constituée d'un matériau de Saint Venant-Kirchhoff a été fournie par Fox *et al.* [9] par la méthode des développements asymptotiques formels appliquée aux équations de l'élasticité tridimensionnelle non linéaire. Ce modèle, qui conserve la propriété d'indifférence matérielle du problème tridimensionnel non linéaire en ce sens que sa densité d'énergie est invariante par les rotations de $\mathbb{R}^3$, s'écrit sous la forme d'un problème de point critique pour une fonctionnelle dont la partie non linéaire dépend de la première forme fondamentale de la surface déformée inconnue. On établit ici un résultat d'existence pour ces équations dans le cas d'une plaque membranaire soumise à une condition au bord de "tension", et on montre que la solution mise en évidence par notre analyse est injective et est l'unique minimiseur de la fonctionnelle membranaire non linéaire, qui n'est pas faiblement séquentiellement semi-continue inférieurement. L'analyse du comportement de la membrane lorsque la "tension" tend vers l'infini nous permet de conclure qu'une membrane convenablement "étirée" est en mesure de supporter des forces importantes.

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1. Introduction

The classical two-dimensional equations of a *nonlinearly elastic "membrane" plate* and those of a *nonlinearly elastic "flexural" plate* have been identified and justified as they appear in the mechanical literature (see [11] for instance) by Fox *et al.* [9] by means of the *method of formal asymptotic expansions* applied to the three-dimensional equations of nonlinear elasticity for a Saint Venant-Kirchhoff material.

Those two nonlinear models present two remarkable common features. In both cases, the scalings for the displacements set in the analysis of Fox *et al.* [9] are of order $O(1)$ with respect to the thickness ε of the plate and their energy density is invariant under the rotations of $\mathbb{R}^3$ as the original three-dimensional energy. For these reasons, they are called "large displacements" and frame-indifferent theories. In consequence, they must be distinguished from the more familiar nonlinear Kirchhoff-Love theory justified, again by a formal asymptotic analysis, by Ciarlet and Destuynder [4]; see in this respect the extensive presentation given in [3].

Another approach has been developed by Le Dret and Raoult [12] who have justified *another nonlinear "membrane" plate model.* By using Γ-convergence theory, they give a convergence result, as the thickness tends to zero, of *quasi-minimizers* of the three-dimensional energies towards a minimizer of a two-dimensional "membrane" energy. The existence of a minimizer to this energy is thus *de facto* established.

The equations found by Fox *et al.* [9] take the form of *critical point problems* for the associated energies, which in both cases are expressed in terms of the geometry of the unknown deformed surface.

The energy density of a "membrane" plate is a quadratic and positive definite expression (*via* the two-dimensional elasticity tensor of the plate) in terms of the exact difference between the *metric tensor* of the unknown surface and that of the reference configuration.

The stored energy of a "flexural" plate is a quadratic and positive definite expression (again *via* the two-dimensional elasticity tensor of the plate) in terms of the exact difference between the *curvature tensor* of the unknown surface and that of the reference configuration. Another specific feature of the "flexural" model is that the critical point problem is formulated over a manifold of *admissible deformations* which are those that preserve the metric of the undeformed plate and satisfy boundary conditions of clamping or of simple support. The existence of a minimizer to the nonlinear "flexural" plate functional is established in [5].

The purpose of this paper is to establish existence results for the nonlinear membrane plate equations and to give some properties of the solutions.

In Section 2, we describe the problem, in terms of a system of partial differential equations or equivalently as a critical point problem. The difficulties inherent to these two formulations are of two kinds. First, the boundary value problem system is *quasilinear* and not *semi-linear* as in the case of the nonlinear Kirchhoff-Love theory for instance. Second, as already noted by Fox *et al.* [9], the functional energy associated to the nonlinear membrane model is coercive but *not* sequentially weakly lower semi-continuous, which forbids to apply the classical theorem of the calculus of variations. For these reasons, the mathematical analysis of these equations is very delicate.

In Section 3, *via* the inverse function theorem, we establish as announced in [6] the existence of an *injective* solution to the nonlinear membrane problem when the plate is submitted to a *boundary condition of place of "tension"*, introduced by Fox *et al.* [9], and to *"small enough" forces.* This result holds without restriction on the direction of the forces in contrast to the case of the clamped plate for which the forces were assumed to be parallel to the plane of the plate (see [7]). Furthermore, we show that the solution found in this fashion possesses the remarkable feature of being the *unique* minimizer to the associated membrane functional over the "whole" affine space of admissible deformations. Thus, we establish in an indirect way, an existence and uniqueness result for a minimization problem in a case where the standard method of the calculus of variation cannot be applied. For the case of the clamped plate, the solution was only a *local* minimizer in an "optimal" affine space strictly included in the set of admissible deformations (see [7]).

In Section 4, we prove that when the "tension" goes to infinity, the radius of the ball containing the forces for which we can associate an injective solution may also go to infinity, in a cubic fashion. As a consequence of this result, we can assert that a "well extended" plate can undergo important loadings. For any given force, we also give an asymptotic estimate for the solutions when the "tension" goes to infinity.

2. The nonlinear membrane plate model

Greek indices and exponents take their values in the set $\{1,2\}$, Latin indices take their values in the set $\{1,2,3\}$, and the summation convention with respect to repeated indices is used. Vectors of $\mathbb{R}^2$ or $\mathbb{R}^3$ and vector valued functions are written in boldface letters. The Euclidean inner product, the exterior product, and the Euclidean norm of vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$ are denoted $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\wedge\mathbf{b}$ and $|\ \mathbf{a}\ |$. The standard Euclidean distance between x and y, points of $\mathbb{R}^2$ is denoted by $d(x,y)$.

Let ω be an open, bounded, and connected subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, the set ω being locally on one side of γ. Let γ_0 be a subset of γ whose length is > 0.

The usual norm of the Sobolev space $\mathbf{W}^{m,p}(\omega;\mathbb{R}^3), m\in\mathbb{N}, p\in]0,+\infty[$, is noted $||\ .\ ||_{m,p,\omega}$.

The ball of $\mathbf{W}^{m,p}(\omega;\mathbb{R}^3)$ $(m\in\mathbb{N}, p\in]0,+\infty[)$ centered at $\mathbf{0}$ and with radius $R{\geq}0$ is denoted $B_{m,p}(\mathbf{0},R)$.

Let $\boldsymbol{\varphi}_0 \in \mathbf{W}^{\frac{3}{4},4}(\gamma;\mathbb{R}^3)$ be a given mapping defining the boundary condition of place and let $\boldsymbol{f}\in\mathbf{L}^2(\omega;\mathbb{R}^3)$ be the density of forces acting on the plate. Then the asymptotic analysis of Fox *et al.* [9] justifies the nonlinear membrane plate model as a critical point problem for the functional

$$I_M(\boldsymbol{f}) = I_M - L(\boldsymbol{f}),$$

where I_M and $L(\boldsymbol{f})$ are defined on the affine space

$$\{\boldsymbol{\varphi}\in\mathbf{W}^{1,4}(\omega;\mathbb{R}^3), \boldsymbol{\varphi}=\boldsymbol{\varphi}_0 \text{ on } \gamma_0\},$$

by

$$I_M(\boldsymbol{\varphi}) = \int_\omega \Big\{\frac{\lambda\,\mu}{\lambda+2\mu}\,\tilde{a}_{\sigma\sigma}(\boldsymbol{\varphi})\tilde{a}_{\tau\tau}(\boldsymbol{\varphi}) + \mu\,\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi})\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi})\Big\}\mathrm{d}\omega,$$

$$L(\boldsymbol{f})(\boldsymbol{\varphi}) = \int_\omega \boldsymbol{f}.\boldsymbol{\varphi}\mathrm{d}\omega,$$

where

$$\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi}) = \partial_\alpha\boldsymbol{\varphi}\cdot\partial_\beta\boldsymbol{\varphi} - \delta_{\alpha\beta}$$

are the components of the change of metric tensor between the unknown deformed surface and the undeformed one.

As noted by Fox*et al.* [9], the functional I_M is not sequentially weakly lower semi-continuous, which forbids the use of the standard method of the calculus of variations.

Equivalently, this problem can be written under the form of the following boundary value problem: Find $\boldsymbol{\varphi}\in\mathbf{W}^{1,4}(\omega;\mathbb{R}^3)$ such that, in the distributional sense:

$$\begin{cases} -\partial_\alpha\Big\{\Big(\dfrac{2\lambda\mu}{\lambda+2\mu}\,\delta_{\alpha\beta}\,\tilde{a}_{\sigma\sigma}(\boldsymbol{\varphi}) + 2\mu\,\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi})\Big)\,\partial_\beta\,\boldsymbol{\varphi}\Big\} = \boldsymbol{f} & \text{in } \omega,\\ \boldsymbol{\varphi}=\boldsymbol{\varphi}_0 \qquad \text{on } \gamma_0, & \\ \Big(\dfrac{2\lambda\mu}{\lambda+2\mu}\,\delta_{\alpha\beta}\,\tilde{a}_{\sigma\sigma}(\boldsymbol{\varphi}) + 2\mu\,\tilde{a}_{\alpha\beta}(\boldsymbol{\varphi})\Big)\,\nu_\alpha\,\partial_\beta\,\boldsymbol{\varphi} = \mathbf{0} & \text{on } \gamma-\gamma_0, \end{cases}$$

where $\delta_{\alpha\beta}$ denote the Kronecker symbol. Note that in this formulation, the unknown $\boldsymbol{\varphi}$ is the *deformation* of the plate, *i.e.* the position taken by the plate under the action of the applied forces.

In the next section, we establish the existence of an injective solution when the plane membrane is submitted to a boundary condition of place of "tension" *via* the inverse function theorem. We also show that the solution found in this fashion has the remarkable property of being the *unique* minimizer of the associated membrane functional - which is *not* sequentially weakly lower semi-continuous - over the whole set of admissible deformations.

3. The membrane submitted to a boundary condition of "tension"

In the sequel, we denote by $\boldsymbol{\iota}$ the mapping defined from $\bar{\omega}$ into $\mathbb{R}^3$ by

$$\boldsymbol{\iota}(x_1, x_2) = (x_1, x_2, 0), \quad \text{for all } (x_1, x_2) \in \bar{\omega},$$

and by **id** the restriction of the identity map of $\mathbb{R}^2$ to $\bar{\omega}$.

In the case where $\gamma_0 = \gamma$ and the boundary condition of place is of the form

$$\varphi = k\ \boldsymbol{\iota} \ \text{ on } \gamma, \tag{3.1}$$

where $k > 1$ is given, we have the following *existence result*, which shows that a nonlinear plane membrane "under tension", and submitted to "small enough" forces , has a solution:

Theorem 3.1. *Assume that the boundary γ is of class $\mathcal{C}^2$. For any $p > 2$, there exists a neighborhood $\mathbf{F}_k^p$ of the origin in $\mathbf{L}^p(\omega;\mathbb{R}^3)$ and a neighborhood $\mathbf{U}_k^p$ of the origin in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ such that for each $\boldsymbol{f} \in \mathbf{F}_k^p$, there is a unique $\boldsymbol{u} \in \mathbf{U}_k^p$ such that $\varphi(\boldsymbol{u}) = \boldsymbol{u} + k\ \boldsymbol{\iota}$ is a solution to the nonlinear membrane boundary value problem. Furthermore, the mapping implicitly defined in this fashion is a $\mathcal{C}^\infty-$diffeomorphism between $\{k\ \iota + \mathbf{U}_k^p\}$ and $\mathbf{F}_k^p$.*

Proof. The proof is broken into five steps.

Step 1. Consider the nonlinear operator $\mathbf{T}_k$ defined from the space $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ into $\mathbf{L}^p(\omega;\mathbb{R}^3)$ by:

$$\mathbf{T}_k(\boldsymbol{u}) = -\partial_\alpha\Big\{\Big(\frac{2\lambda\ \mu}{\lambda+2\mu}\ \delta_{\alpha\beta}\ \tilde{a}_{\sigma\sigma}(\varphi(\boldsymbol{u})) + 2\mu\ \tilde{a}_{\alpha\beta}(\varphi(\boldsymbol{u}))\Big)\partial_\beta\varphi(\boldsymbol{u})\Big\}, \tag{3.2}$$

where $\varphi(\boldsymbol{u})$ is the element of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)$ defined by

$$\varphi(\boldsymbol{u}) = k\ \boldsymbol{\iota} + \boldsymbol{u}, \tag{3.3}$$

for all $\boldsymbol{u} \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$.

As already noted by Fox *et al.* [9], the linearization of $\mathbf{T}_k$ around $\mathbf{0}$ gives the usual linear Poisson equation for the vertical component.

More precisely, since $\mathbf{W}^{1,p}(\omega)$ is a Banach algebra for $p > 2$, this operator is of class $\mathcal{C}^\infty$ between $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ and $\mathbf{L}^p(\omega;\mathbb{R}^3)$, and its differential at the origin is given by

$$\mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{u}) = -2\mu(k^2-1)\frac{3\lambda+2\mu}{\lambda+2\mu}\ \triangle\boldsymbol{u} - k^2\ (\partial_\alpha n_{\alpha 1}(\nabla\boldsymbol{u}), \partial_\alpha n_{\alpha 2}(\nabla\boldsymbol{u}), 0), \tag{3.4}$$

where

$$n_{\alpha\beta}(\nabla\boldsymbol{u}) = \frac{4\lambda\ \mu}{\lambda+2\mu}\ \mathrm{e}_{\sigma\sigma}(\boldsymbol{u})\delta_{\alpha\beta} + 4\mu\ \mathrm{e}_{\alpha\beta}(\boldsymbol{u}). \tag{3.5}$$

Step 2. We next show that the linear operator $\mathrm{d}\mathbf{T}_k(\mathbf{0})$ associated to boundary conditions of Dirichlet type satisfies the specific assumptions of [1].

For conciseness, let us introduce the following quantities:

$$a_1 = 2\mu(k^2-1)\frac{3\lambda+2\mu}{\lambda+2\mu} > 0, \quad a_2 = \frac{4\lambda\mu}{\lambda+2\mu}k^2 > 0, \quad a_3 = 4\mu k^2 > 0.$$

Using the same notations as in [1], we have for $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $|\,\boldsymbol{\xi}\,|^2 = {\xi_1}^2 + {\xi_2}^2$:

$$\begin{aligned} l'_{\alpha\beta}(\boldsymbol{\xi}) &= \Big(a_1 + \frac{a_3}{2}\Big)|\,\boldsymbol{\xi}\,|^2\,\delta_{\alpha\beta} + \Big(a_2 + \frac{a_3}{2}\Big)\xi_\alpha\xi_\beta, \quad (\alpha,\beta) \in \{1,2\}^2,\\ l'_{33}(\boldsymbol{\xi}) &= a_1|\,\boldsymbol{\xi}\,|^2,\\ l'_{3\alpha} &= l'_{\alpha 3} = 0, \qquad\qquad \alpha = 1,2. \end{aligned}$$

(i) First, we establish that the system is *uniformly elliptic* in the sense that there exists $c \geq 1$ such that:

$$\forall \boldsymbol{\xi} \in \mathbb{R}^2, \qquad c^{-1}|\,\boldsymbol{\xi}\,|^6 \leq L(\boldsymbol{\xi}) \leq c|\,\boldsymbol{\xi}\,|^6,$$

where $L(\boldsymbol{\xi}) = \det(l'_{ij}(\boldsymbol{\xi}))$.

A simple computation gives us

$$L(\boldsymbol{\xi}) = a_1\Big(a_1 + \frac{a_3}{2}\Big)(a_1 + a_2 + a_3)|\,\boldsymbol{\xi}\,|^6,$$

which establishes the uniform ellipticity of the system.

(ii) Next, we have to verify that the system satisfies the *supplementary condition*, namely that for each pair of linearly independent vectors $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ of $\mathbb{R}^2$, the polynomial

$$\tau \in \mathbb{C} \to \mathbb{L}(\boldsymbol{\xi} + \tau\boldsymbol{\xi}') \in \mathbb{C}$$

has exactly $m = 3$ roots with positive imaginary part, where $2m = 6$ denotes the degree of the polynomial

$$\xi \in \mathbb{R}^2 \to L(\xi) \in \mathbb{R}.$$

Let $\boldsymbol{\xi}$ and $\boldsymbol{\xi}$' be a given pair of linearly independent vectors of $\mathbb{R}^2$. Then, for $\tau \in \mathbb{C}$, we have:

$$L(\boldsymbol{\xi} + \tau\boldsymbol{\xi}') = a_1\Big(a_1 + \frac{a_3}{2}\Big)(a_1 + a_2 + a_3)(\tau^2|\,\boldsymbol{\xi}'\,|^2 + 2\tau\boldsymbol{\xi}\cdot\boldsymbol{\xi}' + |\,\boldsymbol{\xi}\,|^2)^3.$$

Since the two vectors are supposed to be independent, we have:

$$(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')^2 < |\,\boldsymbol{\xi}\,|^2|\,\boldsymbol{\xi}'\,|^2.$$

Hence, the polynomial

$$\tau \in \mathbb{C} \to \tau^2|\,\boldsymbol{\xi}'\,|^2 + 2\tau\boldsymbol{\xi}\cdot\boldsymbol{\xi}' + |\,\boldsymbol{\xi}\,|^2 \in \mathbb{C}$$

has two non real complex conjugate roots. We deduce that the polynomial of degree $2m = 6$:

$$\tau \in \mathbb{C} \to a_1\Big(a_1 + \frac{a_3}{2}\Big)(a_1 + a_2 + a_3)\Big(\tau^2|\,\boldsymbol{\xi}'\,|^2 + 2\tau\boldsymbol{\xi}\cdot\boldsymbol{\xi}' + |\,\boldsymbol{\xi}\,|^2\Big)^3 \in \mathbb{C}$$

has exactly $m = 3$ roots with positive imaginary part. Therefore, the *supplementary condition* is satisfied.

(iii) Now, we establish the *strong ellipticity* of the system in the sense that there exists $c > 0$ such that

$$\forall \boldsymbol{\xi} \in \mathbb{R}^2, \forall \boldsymbol{\eta} \in \mathbb{C}^3, \qquad \Re(l'_{ij}(\boldsymbol{\xi})\eta_i\bar{\eta}_j) \geq c|\,\boldsymbol{\xi}\,|^2(|\,\eta_1\,|^2 + |\,\eta_2\,|^2 + |\,\eta_3\,|^2).$$

Since the $l'_{ij}(\boldsymbol{\xi})$ are real, it suffices to show that:

$$\forall \boldsymbol{\xi} \in \mathbb{R}^2, \forall \boldsymbol{\eta} \in \mathbb{R}^3, \qquad l'_{ij}(\boldsymbol{\xi})\eta_i\eta_j \geq c \mid \boldsymbol{\xi} \mid^2 ({\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2).$$

In our case, we have

$$l'_{ij}(\boldsymbol{\xi})\eta_i\eta_j = a_1 \mid \boldsymbol{\xi} \mid^2 ({\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2) + \frac{a_3}{2} \mid \boldsymbol{\xi} \mid^2 ({\eta_1}^2 + {\eta_2}^2) + (a_2 + \frac{a_3}{2}) \mid \boldsymbol{\xi} \mid^2 (\xi_1\eta_1 + \xi_2\eta_2)^2,$$

which gives the *strong ellipticity* of the system:

$$l'_{ij}(\boldsymbol{\xi})\eta_i\eta_j \geq a_1 \mid \boldsymbol{\xi} \mid^2 ({\eta_1}^2 + {\eta_2}^2 + {\eta_3}^2).$$

(iv) Finally, the *complementary boundary condition* also holds, since we know from [1] that this is the case for any system verifying the *strong ellipticity property* and associated to a boundary condition of Dirichlet type.

Step 3. From the strong ellipticity of the system established in Step 2, from the $\mathbf{H}^1_0(\omega;\mathbb{R}^3)$-ellipticity of the corresponding bilinear form (which is a consequence of the strong ellipticity of the system for the boundary condition considered here), we deduce from a result of Nečas [13] that $d\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$, considered as an operator from $\mathbf{H}^2(\omega;\mathbb{R}^3) \cap \mathbf{H}^1_0(\omega;\mathbb{R}^3)$ into $\mathbf{L}^2(\omega;\mathbb{R}^3)$, defines an isomorphism between those spaces.

Step 4. The results of Step 2 allow us to use a result of Geymonat [10] to deduce that $d\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$, seen as an operator from $\mathbf{V}^q(\omega;\mathbb{R}^3)$ into $\mathbf{L}^q(\omega;\mathbb{R}^3)$ where $\mathbf{V}^q(\omega;\mathbb{R}^3) = \{\boldsymbol{\psi} \in \mathbf{W}^{2,q}(\omega;\mathbb{R}^3); \boldsymbol{\psi} = \mathbf{0} \text{ on } \gamma\}$, has an index independent of $q \in]1,+\infty[$.

From Step 3, we then know that this index is equal to 0.

From the imbedding of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)$ into $\mathbf{H}^2(\omega;\mathbb{R}^3) \cap \mathbf{H}^1_0(\omega;\mathbb{R}^3) (p > 2)$, and from the injectivity of $d\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ on $\mathbf{H}^2(\omega;\mathbb{R}^3) \cap \mathbf{H}^1_0(\omega;\mathbb{R}^3)$, it follows that $d\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ is injective on the space $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)$. The nullity of the index then shows the surjectivity of the linear operator $d\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ from the space $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)$ into $\mathbf{L}^p(\omega;\mathbb{R}^3)$.

From the open mapping theorem, we deduce that $d\mathbf{T}_{\bar{\boldsymbol{\varphi}}}(\mathbf{0})$ is an isomorphism between $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)$ and $\mathbf{L}^p(\omega;\mathbb{R}^3)$.

Step 5. Combined with the results of Steps 1 and 4 and the relation $\mathbf{T}_k(\mathbf{0}) = \mathbf{0}$, the inverse function theorem provides the existence of a neighborhood $\mathbf{F}^p_k$ of the origin in $\mathbf{L}^p(\omega;\mathbb{R}^3)$ and of a neighborhood $\mathbf{U}^p_k$ of the origin in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)$ such that $\mathbf{T}_k$ defines a $\mathcal{C}^\infty$ diffeomorphism between $\mathbf{U}^p_k$ and $\mathbf{F}^p_k$. This establishes the theorem.

We can also establish the injectivity of the deformation:

Theorem 3.2. *With the assumptions and notations of Theorem 3.1, there exists a neighborhood* $\tilde{\mathbf{F}}^p_k$ *of the origin in* $\mathbf{L}^p(\omega;\mathbb{R}^3)$ *contained in* $\mathbf{F}^p_k$ *such that the unique solution in* $\{k\,\boldsymbol{\iota} + \mathbf{U}^p_k\}$ *(Theorem 3.1) to the nonlinear membrane "under tension" problem associated to any element of* $\tilde{\mathbf{F}}^p_k$*, is injective in* $\bar{\omega}$.

Proof. In fact, we prove a stronger result: The mapping defined by the two first components of the deformation is already injective in a neighborhood included in $\mathbf{F}^p_k$.

From Theorem 3.1 and the continuity of the imbedding of $\mathrm{W}^{2,p}(\omega)$ into $C^1(\bar{\omega})$, we infer the existence of a neighborhood $\tilde{\mathbf{F}}^p_k \subset \mathbf{F}^p_k$ such that to any $\boldsymbol{p} \in \tilde{\mathbf{F}}^p_k$, the element $\boldsymbol{u}$ associated by Theorem 3.1 satisfies:

$$\det(\nabla \boldsymbol{\varphi}_H) > 0 \text{ in } \bar{\omega} \text{ where } \boldsymbol{\varphi}_H = k\ \mathbf{id} + (u_1, u_2).$$

Moreover, $\boldsymbol{\varphi}_H \in C^1(\bar{\omega}, \mathbb{R}^2)$ satisfies:

$$\boldsymbol{\varphi}_H = k\ \mathbf{id} \text{ on } \gamma.$$

The application k **id** being injective, a classical result using topological degree arguments (see for instance Th. 5.5-2 in [2]) gives the injectivity of the mapping φ_H.

Since the associated deformation (Th. 3.1) satisfies $\boldsymbol{\varphi}(\boldsymbol{u}) = (\varphi_H, u_3)$, the injectivity of the deformation follows from that of φ_H. □

In the next paragraph, we establish that the solution given by Theorem 3.1 is also the *unique* minimizer of the functional associated to the problem over the *whole* space, and not only a *local* minimizer as in the case of the clamped plate [7]. Thus, we will have shown, in an indirect way, both the *existence and uniqueness* of the solution to a minimization problem whose functional is *not* sequentially weakly-lower semi-continuous.

Theorem 3.3. *Under the assumptions and notations of Theorem 3.1, there exists a neighborhood* $\check{\mathbf{F}}^p_k$ *of the origin in* $\mathbf{L}^p(\omega;\mathbb{R}^3)$ *contained in* $\mathbf{F}^p_k$*, such that the unique solution to the nonlinear membrane "under tension" problem in the set* $\{k\,\boldsymbol{\iota} + \mathbf{U}^p_k\}$ *associated by Theorem 3.1 to any element* $\boldsymbol{f}$ *of* $\check{\mathbf{F}}^p_k$ *is the unique minimizer of the functional* $I_M(\boldsymbol{f})$ *over the whole affine space* $\{k\,\boldsymbol{\iota} + \mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)\}$.

Proof. The proof is divided into two steps.

Step 1. For any $\boldsymbol{f}$ in $\mathbf{F}^p_k$, consider the functional $I_k(\boldsymbol{f})$ of class C^∞ defined from $\mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)$ into $\mathbb{R}$ by

$$I_k(\boldsymbol{f})(\boldsymbol{u}) = I_M(\boldsymbol{f})(k\,\boldsymbol{\iota} + \boldsymbol{u}) \tag{3.6}$$

for all $\boldsymbol{u} \in \mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)$. From the definition, we see that

$$\begin{aligned} I_k(\boldsymbol{f})(\boldsymbol{v}) = {}& \frac{\lambda\,\mu}{\lambda+2\mu}\int_\omega \Big(\sum_\alpha (\partial_\alpha \boldsymbol{v}\cdot\partial_\alpha \boldsymbol{v} + 2\,k\,\partial_\alpha v_\alpha + k^2 - 1)\Big)^2 \mathrm{d}\omega \\ & + \mu\sum_{\alpha,\beta}\int_\omega \Big(\partial_\alpha \boldsymbol{v}\cdot\partial_\beta \boldsymbol{v} + k(\partial_\alpha v_\beta + \partial_\beta v_\alpha) + (k^2-1)\delta_{\alpha\beta}\Big)^2 \mathrm{d}\omega - \int_\omega \boldsymbol{f}\cdot(k\,\boldsymbol{\iota} + \boldsymbol{v})\mathrm{d}\omega. \end{aligned} \tag{3.7}$$

[Note that in (3.7) we don't use the summation convention with respect to the repeated indices. We do the same in (3.8)].

Let $\boldsymbol{\varphi}(\boldsymbol{u}) = \boldsymbol{u} + k\,\boldsymbol{\iota}$ be the deformation associated by Theorem 3.1 in the set $\{k\,\boldsymbol{\iota} + \mathbf{U}^p_k\}$. From the definition of $I_k(\boldsymbol{f})$, it follows that $\boldsymbol{u}$ is a critical point of the functional $I_k(\boldsymbol{f})$.

Step 2. We next establish that, provided $\boldsymbol{f}$ belongs to an appropriate neighborhood $\check{\mathbf{F}}^p_k$ of the origin, included in $\mathbf{F}^p_k$, this critical point is also the unique minimizer of $I_k(\boldsymbol{f})$ over the space $\mathbf{W}^{1,4}_0(\omega;\mathbb{R}^3)$.

To this end, let $\boldsymbol{w}$ be any element of $\mathbf{W}^{1,4}_0(\omega,\mathbb{R}^3)$. Then, from (3.7), a computation shows that:

$$\begin{aligned} I_k(\boldsymbol{f})(\boldsymbol{u}+\boldsymbol{w}) = {}& I_k(\boldsymbol{f})(\boldsymbol{u}) + \mathrm{d}I_k(\boldsymbol{f})(\boldsymbol{u})(\boldsymbol{w}) \\ & + \frac{\lambda\,\mu}{\lambda+2\mu}\int_\omega \Big(\sum_\alpha(\partial_\alpha \boldsymbol{w}\cdot\partial_\alpha \boldsymbol{w} + 2\,k\,\partial_\alpha w_\alpha + 2\,\partial_\alpha \boldsymbol{u}\cdot\partial_\alpha \boldsymbol{w})\Big)^2 \mathrm{d}\omega \\ & + \mu\sum_{\alpha,\beta}\int_\omega \Big(\partial_\alpha \boldsymbol{w}\cdot\partial_\beta \boldsymbol{w} + k(\partial_\alpha w_\beta + \partial_\beta w_\alpha) + \partial_\alpha \boldsymbol{u}\cdot\partial_\beta \boldsymbol{w} + \partial_\beta \boldsymbol{u}\cdot\partial_\alpha \boldsymbol{w}\Big)^2 \mathrm{d}\omega \\ & + \frac{2\,\lambda\,\mu}{\lambda+2\mu}\int_\omega \Big(\sum_\alpha (\partial_\alpha \boldsymbol{u}\cdot\partial_\alpha \boldsymbol{u} + 2\,k\,\partial_\alpha u_\alpha + k^2 - 1)\Big)\Big(\sum_\alpha \partial_\alpha \boldsymbol{w}\cdot\partial_\alpha \boldsymbol{w}\Big)\mathrm{d}\omega \\ & + 2\,\mu\sum_{\alpha,\beta}\int_\omega (\partial_\alpha \boldsymbol{u}\cdot\partial_\beta \boldsymbol{u} + k\,(\partial_\alpha u_\beta + \partial_\beta u_\alpha) + (k^2-1)\delta_{\alpha\beta})\,\partial_\alpha \boldsymbol{w}\cdot\partial_\beta \boldsymbol{w}\,\mathrm{d}\omega. \end{aligned} \tag{3.8}$$

Let

$$a_k = 2\ \mu\ (k^2 - 1)\Big(\frac{3\lambda + 2\mu}{\lambda + 2\mu}\Big) > 0. \tag{3.9}$$

Then it follows that

$$\begin{aligned} I_k(\boldsymbol{f})(\boldsymbol{u}+\boldsymbol{w}) \geq I_k(\boldsymbol{f})(\boldsymbol{u}) &+ \mathrm{d}I_k(\boldsymbol{f})(\boldsymbol{u})(\boldsymbol{w}) + a_k \int_\omega \partial_\alpha \boldsymbol{w}\cdot\partial_\alpha \boldsymbol{w}\ \mathrm{d}\omega \\ &+ \frac{2\ \lambda\ \mu}{\lambda+2\mu}\int_\omega (\partial_\sigma \boldsymbol{u}\cdot\partial_\sigma \boldsymbol{u} + 2\ k\ \partial_\sigma u_\sigma)\ (\partial_\tau \boldsymbol{w}\cdot\partial_\tau \boldsymbol{w})\ \mathrm{d}\omega \\ &+ 2\mu \int_\omega (\partial_\alpha \boldsymbol{u}\cdot\partial_\beta \boldsymbol{u} + k\ (\partial_\alpha u_\beta + \partial_\beta u_\alpha))\ \partial_\alpha \boldsymbol{w}\cdot\partial_\beta \boldsymbol{w}\ \mathrm{d}\omega. \end{aligned}$$

Since $\boldsymbol{u}$ is a critical point of the functional $I_k(\boldsymbol{f})$, this inequality becomes

$$\begin{aligned} I_k(\boldsymbol{f})(\boldsymbol{u}+\boldsymbol{w}) \geq I_k(\boldsymbol{f})(\boldsymbol{u}) &+ a_k \int_\omega \partial_\alpha \boldsymbol{w}\cdot\partial_\alpha \boldsymbol{w}\mathrm{d}\omega \\ &+ \frac{2\ \lambda\ \mu}{\lambda+2\mu}\int_\omega (\partial_\sigma \boldsymbol{u}\cdot\partial_\sigma \boldsymbol{u} + 2\ k\ \partial_\sigma u_\sigma)\ (\partial_\tau \boldsymbol{w}\cdot\partial_\tau \boldsymbol{w})\ \mathrm{d}\omega \\ &+ 2\mu \int_\omega (\partial_\alpha \boldsymbol{u}\cdot\partial_\beta \boldsymbol{u} + k\ (\partial_\alpha u_\beta + \partial_\beta u_\alpha))\ \partial_\alpha \boldsymbol{w}\cdot\partial_\beta \boldsymbol{w}\ \mathrm{d}\omega. \end{aligned} \tag{3.10}$$

From Theorem 3.1 and the continuity of the imbedding of $\mathrm{W}^{2,p}(\omega)$ in $\mathcal{C}^1(\bar\omega)$, we infer the existence of a neighborhood $\tilde{\mathbf{F}}_k^p$ contained in $\mathbf{F}_k^p$ such that the solution $\boldsymbol{u}$ associated in $\mathbf{U}_k^p$ satisfies the following inequality for all $\boldsymbol{w} \in \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$:

$$I_k(\boldsymbol{f})(\boldsymbol{u}+\boldsymbol{w}) \geq I_k(\boldsymbol{f})(\boldsymbol{u}) + a'_k \int_\omega \partial_\alpha \boldsymbol{w}\cdot\partial_\alpha \boldsymbol{w}\mathrm{d}\omega,$$

where the constant a'_k satisfies

$$0 < a'_k < a_k.$$

This last inequality shows that, for any element $\boldsymbol{v}$ of $\mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ different from $\boldsymbol{u}$, we have

$$I_k(\boldsymbol{f})(\boldsymbol{v}) > I_k(\boldsymbol{f})(\boldsymbol{u}).$$

From the definition of $I_k(\boldsymbol{f})$, we have then established that

$$I_M(\boldsymbol{f})(k\boldsymbol{\iota}+\boldsymbol{v}) > I_M(\boldsymbol{f})(k\boldsymbol{\iota}+\boldsymbol{u}),$$

which establishes the theorem. □

In the next Section, we study the behaviour of the plane membrane as the "tension" goes to infinity *i.e.* as k tends to infinity. We establish that as k tends to infinity the neighborhood $\mathbf{F}_k^p$ of Theorem 3.1 contains a ball centered at $\mathbf{0}$ in $\mathbf{L}^p(\omega;\mathbb{R}^3)$ whose radius grows proportionally to k^3. For any given force we also prove a convergence result for the solutions as k tends to infinity.

4. Behaviour of the membrane as the "tension" goes to infinity

We first need two preliminary results, for which the same assumptions as in Theorem 3.1 are made. In the following κ denotes a given real satisfying $0 < \kappa < 1$.

Lemma 4.1. *There exist $\delta_0 > 0$ and $k_0 > 0$ such that if $0 < \delta < \delta_0$ and $k > k_0$, then for any $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ satisfying*

$$\|\boldsymbol{u}\|_{2,p,\omega} < \delta\, k, \quad \|\boldsymbol{v}\|_{2,p,\omega} < \delta\, k,$$

$$\mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{w}-\boldsymbol{v}) = \mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{v}-\boldsymbol{u}) - (\mathbf{T}_k(\boldsymbol{v}) - \mathbf{T}_k(\boldsymbol{u})),$$

the following inequality holds:

$$\|\boldsymbol{w}-\boldsymbol{v}\|_{2,p,\omega} \leq \kappa \|\boldsymbol{u}-\boldsymbol{v}\|_{2,p,\omega}\,.$$

Proof. (i) Let $k > 0$, $\delta > 0$ be given. Let $\boldsymbol{u}$, $\boldsymbol{v}$,$\boldsymbol{w}$ be given elements of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ satisfying

$$\|\boldsymbol{u}\|_{2,p,\omega} < \delta\, k, \|\boldsymbol{v}\|_{2,p,\omega} < \delta\, k, \tag{4.1}$$

$$\mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{w}-\boldsymbol{v}) = \mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{v}-\boldsymbol{u}) - (\mathbf{T}_k(\boldsymbol{v}) - \mathbf{T}_k(\boldsymbol{u})). \tag{4.2}$$

Then, (4.2) also reads

$$k^2\, \mathbf{T}(\boldsymbol{w}-\boldsymbol{v}) + 2\mu\frac{3\lambda+2\mu}{\lambda+2\mu}\, \Delta(\boldsymbol{w}-\boldsymbol{v}) = k\, (\mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u})) + (\mathbf{B}(\boldsymbol{v}) - \mathbf{B}(\boldsymbol{u})), \tag{4.3}$$

where $\mathbf{T}$, $\mathbf{A}$ and $\mathbf{B}$ are the operators defined from $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ into $\mathbf{L}^p(\omega;\mathbb{R}^3)$ by:

$$\mathbf{T}(\boldsymbol{\eta}) = -2\mu\frac{3\lambda+2\mu}{\lambda+2\mu}\, \Delta\boldsymbol{\eta} - \, (\partial_\alpha n_{\alpha 1}(\nabla\boldsymbol{\eta}), \partial_\alpha n_{\alpha 2}(\nabla\boldsymbol{\eta}), 0), \tag{4.4}$$

$$\begin{aligned}\mathbf{A}(\boldsymbol{\eta}) = &-\partial_\alpha\Big\{ \Big(\frac{2\lambda\,\mu}{\lambda+2\mu}\, \delta_{\alpha\beta}\, (2\, \partial_\sigma\eta_\sigma) + 2\mu\, (\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha)\Big)\, \partial_\beta\boldsymbol{\eta}\Big\} \\ &- \partial_\alpha\Big\{ \Big(\frac{2\lambda\,\mu}{\lambda+2\mu}\, \delta_{\alpha\beta}\, (\partial_\sigma\boldsymbol{\eta}\cdot\partial_\sigma\boldsymbol{\eta}) + 2\mu\, \partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}\Big)\, \partial_\beta\boldsymbol{\iota}\Big\},\end{aligned} \tag{4.5}$$

$$\mathbf{B}(\boldsymbol{\eta}) = -\partial_\alpha\Big\{ \Big(\frac{2\lambda\,\mu}{\lambda+2\mu}\, \delta_{\alpha\beta}\, \partial_\sigma\boldsymbol{\eta}\cdot\partial_\sigma\boldsymbol{\eta} + 2\mu\, \partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\boldsymbol{\eta}\Big)\, \partial_\beta\boldsymbol{\eta}\Big\}, \tag{4.6}$$

for all $\eta \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$.

(ii) Since $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)$ is a Banach algebra ($p > 2$), we infer that the operators $\mathbf{A}$ and $\mathbf{B}$ are of class $\mathcal{C}^\infty$.

It can be seen that there exists a constant $C_1 > 0$ such that for each $R > 0$ and any $\boldsymbol{\eta} \in B_{2,p}(\mathbf{0}, R)$ the following inequalities hold

$$\begin{aligned}&\|\mathrm{d}\mathbf{A}(\boldsymbol{\eta})(\boldsymbol{w})\|_{0,p,\omega} \leq C_1\, R\, \|\boldsymbol{w}\|_{2,p,\omega}, \\ &\|\mathrm{d}\mathbf{B}(\boldsymbol{\eta})(\boldsymbol{w})\|_{0,p,\omega} \leq C_1\, R^2\, \|\boldsymbol{w}\|_{2,p,\omega},\end{aligned}$$

for all $\boldsymbol{w} \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$.

Since $\mathbf{L}^p(\omega;\mathbb{R}^3)$ is a reflexive Banach space ($1 < 2 < p < +\infty$), it has the Radon-Nikodým property with respect to the Bochner integral (see for instance [8]). Let $R > 0$ be given and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be any elements in $B_{2,p}(\mathbf{0}, R)$. From the Radon-Nikodým property we have

$$\mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u}) = \int_0^1 \mathrm{d}\mathbf{A}(\boldsymbol{u} + t(\boldsymbol{v} - \boldsymbol{u}))(\boldsymbol{v} - \boldsymbol{u}) \; \mathrm{d}t.$$

From the usual properties of the Bochner integral, we deduce

$$\| \mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u}) \|_{0,p,\omega} \leq \int_0^1 \| \mathrm{d}\mathbf{A}(\boldsymbol{u} + t(\boldsymbol{v} - \boldsymbol{u}))(\boldsymbol{v} - \boldsymbol{u}) \|_{0,p,\omega} \; \mathrm{d}t,$$

and then that

$$\| \mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u}) \|_{0,p,\omega} \leq C_1 \; R \; \| \boldsymbol{v} - \boldsymbol{u} \|_{2,p,\omega}.$$

In the same way, we have

$$\| \mathbf{B}(\boldsymbol{v}) - \mathbf{B}(\boldsymbol{u}) \|_{0,p,\omega} \leq C_1 \; R^2 \; \| \boldsymbol{v} - \boldsymbol{u} \|_{2,p,\omega}.$$

Moreover, exactly as for $\mathrm{d}\mathbf{T}_k(\mathbf{0})$, it can be proved that the linear operator $\mathbf{T}$ defines an isomorphism between $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ and $\mathbf{L}^p(\omega;\mathbb{R}^3)$. Let then $C > 0$ be such that

$$\forall \boldsymbol{f} \in \mathbf{L}^p(\omega, \mathbb{R}^3) \; , \quad \| \mathbf{T}^{-1}(\boldsymbol{f}) \|_{2,p,\omega} \leq C \;\; \| \boldsymbol{f} \|_{0,p,\omega},$$

and let

$$c_1 = 2\mu \frac{3\lambda + 2\mu}{\lambda + 2\mu}.$$

(iii) Noting that (4.3) can be written as:

$$k^2 \; (\boldsymbol{w} - \boldsymbol{v}) = \mathbf{T}^{-1}\Big(- c_1 \Delta(\boldsymbol{w} - \boldsymbol{v}) + k \; (\mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u})) + \mathbf{B}(\boldsymbol{v}) - \mathbf{B}(\boldsymbol{u})\Big),$$

we infer that:

$$k^2 \; \| \boldsymbol{w} - \boldsymbol{v} \|_{2,p,\omega} \leq C \; (c_1 \;\; \| \boldsymbol{w} - \boldsymbol{v} \|_{2,p,\omega} + k \;\; \| \mathbf{A}(\boldsymbol{v}) - \mathbf{A}(\boldsymbol{u}) \|_{0,p,\omega} + \| \mathbf{B}(\boldsymbol{v}) - \mathbf{B}(\boldsymbol{u}) \|_{0,p,\omega}).$$

Hence,

$$(k^2 - C \; c_1) \; \| \boldsymbol{w} - \boldsymbol{v} \|_{2,p,\omega} \leq C \; C_1(\delta + \delta^2) \; k^2 \; \| \boldsymbol{v} - \boldsymbol{u} \|_{2,p,\omega}. \tag{4.7}$$

Let $\delta_0 > 0$ be given such that $\kappa - C \; C_1 \; (\delta_0^2 + \delta_0) > 0$, and let

$$k_0 = \max\Big(\sqrt{\frac{\kappa \; C \; c_1}{\kappa - C \; C_1 \; (\delta_0^2 + \delta_0)}}, 1 + \sqrt{C \; c_1}\Big) > 0.$$

Then, for $0 < \delta < \delta_0$ and $k > k_0$, we have

$$\frac{C \; C_1(\delta + \delta^2) \; k^2}{(k^2 - C \; c_1)} < \kappa,$$

and (4.7) shows that

$$\| \boldsymbol{w} - \boldsymbol{v} \|_{2,p,\omega} \leq \kappa \| \boldsymbol{u} - \boldsymbol{v} \|_{2,p,\omega},$$

which establishes the lemma.

Lemma 4.2. *Let C, c_1, k_0 and δ_0 be as in Lemma 4.1, let*

$$M = \frac{1-\kappa}{C} \ (1 - \frac{C\ c_1}{k_0^2}) > 0.$$

For any $0 < \delta < \delta_0$ *and* $0 < k_0 < k$ *if*

$$\| \boldsymbol{f} \|_{0,p,\omega} < M \ \delta \ k^3,$$

then the sequence of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ *recursively defined by*

$$\begin{cases} \boldsymbol{u}_0 = \mathbf{0}, \\ d\mathbf{T}_k(\mathbf{0})(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) = f - \mathbf{T}_k(\boldsymbol{u}_n) \ , n \in \mathbb{N}, \end{cases}$$

satisfies

$$\begin{cases} \| \boldsymbol{u}_n \|_{2,p,\omega} < \delta \ k, \\ \| \boldsymbol{u}_{n+2} - \boldsymbol{u}_{n+1} \|_{2,p,\omega} \leq \kappa \ \| \boldsymbol{u}_{n+1} - \boldsymbol{u}_n \|_{2,p,\omega} \ , \ n \in \mathbb{N}. \end{cases}$$

Proof. (i) Let $0 < \delta < \delta_0$ and $k > k_0$ be given. Let $\boldsymbol{f} \in \mathbf{L}^p(\omega;\mathbb{R}^3)$ such that $\| \boldsymbol{f} \|_{0,p,\omega} < M \ \delta \ k^3$. By definition, $\mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{u}_1) = \boldsymbol{f}$. With the notations of Lemma 4.1, we also have:

$$k^2 \ L(\boldsymbol{u}_1) = -c_1 \ \Delta \boldsymbol{u}_1 + \boldsymbol{f}.$$

Then

$$\| \boldsymbol{u}_1 \|_{2,p,\omega} \leq \frac{C}{k^2 - C\ c_1} \ \| \boldsymbol{f} \|_{0,p,\omega}.$$

From the definition of M, we infer that

$$\| \boldsymbol{u}_1 \|_{2,p,\omega} \leq \delta \ (1-\kappa) \ k. \tag{4.8}$$

(ii) Since

$$\mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{u}_2 - \boldsymbol{u}_1) = \mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{u}_1 - \boldsymbol{u}_0) - \mathbf{T}_k(\boldsymbol{u}_1) + \mathbf{T}_k(\boldsymbol{u}_0),$$

and $\| \boldsymbol{u}_0 \|_{2,p,\omega} < \delta \ k$, $\| \boldsymbol{u}_1 \|_{2,p,\omega} < \delta \ k$, we deduce from Lemma 4.1 that

$$\| \boldsymbol{u}_2 - \boldsymbol{u}_1 \|_{2,p,\omega} \leq \kappa \ \| \boldsymbol{u}_1 - \boldsymbol{u}_0 \|_{2,p,\omega} = \kappa \ \| \boldsymbol{u}_1 \|_{2,p,\omega}.$$

Then

$$\| \boldsymbol{u}_2 \|_{2,p,\omega} \ \leq \ \| \boldsymbol{u}_2 - \boldsymbol{u}_1 \|_{2,p,\omega} + \| \boldsymbol{u}_1 \|_{2,p,\omega} \ \leq (1+\kappa) \ \| \boldsymbol{u}_1 \|_{2,p,\omega},$$

which shows, from (4.8), that

$$\| \boldsymbol{u}_2 \|_{2,p,\omega} \leq (1+\kappa)(1-\kappa)\, \delta\, k < \delta\, k.$$

(iii) Assume now that for a given $n \geq 0$, we have:

$$\begin{cases} \forall\, l \leq n+2\ ,\ \| \boldsymbol{u}_l \|_{2,p,\omega} < \delta\, k, \\ \forall\, p \leq n\ ,\ \| \boldsymbol{u}_{l+2} - \boldsymbol{u}_{l+1} \|_{2,p,\omega} \leq \kappa\ \| \boldsymbol{u}_{l+1} - \boldsymbol{u}_l \|_{2,p,\omega} \leq \kappa^l\ \| \boldsymbol{u}_1 \|_{2,p,\omega}. \end{cases}$$

Since

$$\mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{u}_{n+3} - \boldsymbol{u}_{n+2}) = \mathrm{d}\mathbf{T}_k(\mathbf{0})(\boldsymbol{u}_{n+2} - \boldsymbol{u}_{n+1}) - \mathbf{T}_k(\boldsymbol{u}_{n+2}) + \mathbf{T}_k(\boldsymbol{u}_{n+1}),$$

and $\| \boldsymbol{u}_{n+2} \|_{2,p,\omega} < \delta\, k$, $\| \boldsymbol{u}_{n+1} \|_{2,p,\omega} < \delta\, k$, we deduce from Lemma 4.1 that

$$\| \boldsymbol{u}_{n+3} - \boldsymbol{u}_{n+2} \|_{2,p,\omega} \leq \kappa\ \| \boldsymbol{u}_{n+2} - \boldsymbol{u}_{n+1} \|_{2,p,\omega} \leq \kappa^{n+2}\ \| \boldsymbol{u}_1 \|_{2,p,\omega}.$$

Moreover, we have

$$\| \boldsymbol{u}_{n+3} \|_{2,p,\omega} \leq \sum_{l=0}^{n+2} \| \boldsymbol{u}_{l+1} - \boldsymbol{u}_l \|_{2,p,\omega} \leq \sum_{l=0}^{n+2} \kappa^l\ \| \boldsymbol{u}_1 \|_{2,p,\omega},$$

so that, from (4.18),

$$\| \boldsymbol{u}_{n+3} \|_{2,p,\omega} < \delta\, k.$$

By induction we have then proved the lemma. □

Now, by concluding as in the proof of the inverse function theorem based on the Banach contraction principle, we infer from Lemmas 4.1 and 4.2 that $\mathbf{T}_k$ defines a $\mathcal{C}^\infty$-diffeomorphism between a neighborhood of $\mathbf{0}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ contained in the ball $B_{2,p}(\mathbf{0}, \delta\, k)$ and the ball $B_{0,p}(\mathbf{0}, M\, \delta\, k^3)$ of $\mathbf{L}^p(\omega;\mathbb{R}^3)$. This establishes the following result:

Theorem 4.1. *We make the same assumptions as in Theorem 3.1 concerning γ and p. Then there exist reals $\delta_0 > 0$ and $k_0 > 0$ such that if $0 < \delta < \delta_0$ and $k > k_0$, then the neighborhood $\mathbf{F}_k^p$ of Theorem 3.1 contains a ball centered at $\mathbf{0}$ in $\mathbf{L}^p(\omega;\mathbb{R}^3)$ of radius $M\, \delta\, k^3$, where $M > 0$ is a constant independent of δ. Furthermore, the element $\boldsymbol{u}$ such that $\mathbf{T}_k(\boldsymbol{u}) = \boldsymbol{f}$, associated by Theorem 3.1 to any element $\boldsymbol{f}$ of this ball, is situated in the ball centered at $\mathbf{0}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ of radius $\delta\, k$.*

We can also establish further properties of the solution given by the preceding theorem as k tends to $+\infty$.

Theorem 4.2. *With the same assumptions and notations as in Theorem 4.1, there exists $0 < \bar{\delta}_0 < \delta_0$ such that if $0 < \delta < \bar{\delta}_0$ and $k > k_0$, then the unique solution in $\{k\, \boldsymbol{\iota} + B_{2,p}(\mathbf{0}, \delta\, k)\}$ to the nonlinear membrane "under tension" problem associated to any element $\boldsymbol{f}$ of $B_{0,p}(\mathbf{0}, M\, \delta\, k^3)$ is injective and is the unique minimizer of the functional $I_M(\boldsymbol{f})$ over the whole affine space $\{k\, \boldsymbol{\iota} + \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)\}$.*

Proof. We only sketch the proof. Let k be any real in $]k_0, +\infty[$.

(i) To prove injectivity, we just adapt the proof of Theorem 3.2. By writing the solution as $\{k\boldsymbol{\iota} + \delta k\, \boldsymbol{u}\}$, with $\boldsymbol{u}$ in the unit ball of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$, we see that there exists $0 < \bar{\delta}_0 < \delta_0$ such that if $0 < \delta < \bar{\delta}_0$, then:

$$\det(k\boldsymbol{\iota} + \delta k\, u) > 0 \text{ in } \omega.$$

(ii) Concerning the minimizing property, still writing the solution as $\{k\iota + \delta k\ \boldsymbol{u}\}$, with $\boldsymbol{u}$ in the unit ball of $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$, from (3.10), we see that for each $\boldsymbol{w} \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$, the following inequality holds:

$$\begin{aligned} I_k(\boldsymbol{f})(k\boldsymbol{u}+\boldsymbol{w}) \geq I_k(\boldsymbol{f})(k\ \boldsymbol{u}) &+ 2\mu\ (k^2-1)\frac{3\lambda+2\mu}{\lambda+2\mu}\int_\omega \partial_\alpha \boldsymbol{w}\cdot\partial_\alpha \boldsymbol{w}\ \mathrm{d}\omega \\ &+ \frac{2\ \lambda\ \mu}{\lambda+2\mu}\int_\omega (\delta^2\ k^2\ \partial_\sigma \boldsymbol{u}\cdot\partial_\sigma \boldsymbol{u} + 2\ \delta\ k^2\ \partial_\sigma u_\sigma)\ (\partial_\tau \boldsymbol{w}\cdot\partial_\tau \boldsymbol{w})\ \mathrm{d}\omega \\ &+ 2\mu\int_\omega (\delta^2\ k^2\ \partial_\alpha \boldsymbol{u}\cdot\partial_\beta \boldsymbol{u} + \delta\ k^2\ (\partial_\alpha u_\beta + \partial_\beta u_\alpha))\ \partial_\alpha \boldsymbol{w}\cdot\partial_\beta \boldsymbol{w}\ \mathrm{d}\omega. \end{aligned}$$

Then we see as in the proof of Theorem 3.3 that there exists $0<\bar{\delta}_0<\delta_0$ such that if $0<\delta<\bar{\delta}_0$ and if $\boldsymbol{w}$ is different from $k\boldsymbol{u}$, we have:

$$I_M(\boldsymbol{f})(k\ \boldsymbol{\iota}+\boldsymbol{w}) > I_M(\boldsymbol{f})(k\ \boldsymbol{\iota}+k\ \boldsymbol{u}),$$

which establishes the theorem. □

Now, for a given density of forces, we have the following convergence result as k tends to $+\infty$.

Theorem 4.3. *Let $R>0$ be given. With the same notations and assumptions as in Theorem 4.1, there exists $k_0(R)\geq k_0$ and $\tilde{C}>0$ such that if $\boldsymbol{f}\in B_{0,p}(\mathbf{0},R)$ and $k>k_0(R)$ then the element $\boldsymbol{u}_k \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ associated by Theorem 4.1 to $\boldsymbol{f}$ satisfies the following estimates:*

$$\|\ k^2\ \mathbf{T}(\boldsymbol{u}_k)-\boldsymbol{f}\ \|_{0,p,\omega} \leq \frac{\tilde{C}}{k^2}\ max(R,R^3),\ \ and\ \|\ \boldsymbol{u}_k - \frac{1}{k^2}\ \mathbf{T}^{-1}(\boldsymbol{f})\ \|_{2,p,\omega} \leq \frac{\tilde{C}}{k^4}\ max(R,R^3).$$

Proof. Set

$$k_0(R) = \max\Big(k_0, \Big(\frac{R}{M\ \delta_0}\Big)^{\frac{1}{3}}\Big)\geq k_0,$$

and let $k>k_0(R)$ be given. Then set

$$\delta = \frac{R}{M\ k^3}. \tag{4.9}$$

Since $k>k_0(R)$, we have

$$0<\delta<\delta_0. \tag{4.10}$$

From Theorem 4.1 we infer that $\mathbf{T}_k$ defines a $\mathcal{C}^\infty$-diffeomorphism between a neighborhood of $\mathbf{0}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ contained in the ball $B_{2,p}(\mathbf{0},\delta\ k)$ and the ball $B_{0,p}(\mathbf{0},M\ \delta\ k^3)$ of $\mathbf{L}^p(\omega;\mathbb{R}^3)$. This shows that $\mathbf{T}_k$ defines a $\mathcal{C}^\infty$-diffeomorphism between a neighborhood of $\mathbf{0}$ in $\mathbf{W}^{2,p}(\omega;\mathbb{R}^3)\cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3)$ contained in the ball $B_{2,p}(\mathbf{0},R/Mk^2)$ and the ball $B_{0,p}(\mathbf{0},R)$ of $\mathbf{L}^p(\omega;\mathbb{R}^3)$.

Now, let $\boldsymbol{f}$ be any element in the ball $B_{0,p}(\mathbf{0},R)$ of $\mathbf{L}^p(\omega;\mathbb{R}^3)$. Then, the element $\boldsymbol{u}_k$ associated by Theorem 4.1 satisfies

$$\|\ \boldsymbol{u}_k\ \|_{2,p,\omega} \leq \frac{R}{M\ k^2}. \tag{4.11}$$

But the equation

$$\mathbf{T}_k(\boldsymbol{u}_k) = \boldsymbol{f}$$

also reads

$$k^2\ \mathbf{T}(\boldsymbol{u}_k) + c_1\ \Delta \boldsymbol{u}_k = k\ \mathbf{A}(\boldsymbol{u}_k) + \mathbf{B}(\boldsymbol{u}_k)\ + \boldsymbol{f}.$$

Thanks to the estimate (4.11), we obtain from the expressions of $\mathbf{A}$ and $\mathbf{B}$, given in (4.5, 4.6), that there exists a constant $C_2 > 0$, such that

$$||\ k^2\ \mathbf{T}(\boldsymbol{u}_k) - \boldsymbol{f}\ ||_{0,p,\omega} \leq C_2\ \Big(\frac{R}{k^2} + \frac{R^2}{k^3} + \frac{R^3}{k^6}\Big).$$

Since $k > k_0 > 1$, we have

$$||\ k^2\ \mathbf{T}(\boldsymbol{u}_k) - \boldsymbol{f}\ ||_{0,p,\omega} \leq \frac{3\ C_2}{k^2}\ \max(R, R^3). \tag{4.12}$$

We have already seen in Lemma 4.1 that there exists $C > 0$ such that

$$||\ \boldsymbol{u}\ ||_{2,p,\omega} \leq C\ \ ||\ \mathbf{T}(\boldsymbol{u})\ ||_{0,p,\omega}\ ,\ \text{ for all } \boldsymbol{u} \in \mathbf{W}^{2,p}(\omega;\mathbb{R}^3) \cap \mathbf{W}_0^{1,4}(\omega;\mathbb{R}^3).$$

From (4.12), we deduce

$$||\ \boldsymbol{u}_k - \frac{1}{k^2}\ \mathbf{T}^{-1}(\boldsymbol{f})\ ||_{2,p,\omega} \leq \frac{3\ C\ C_2}{k^4}\ \max(R, R^3).$$

Then, the theorem is proved, with $k_0(R)$ and $\tilde{C} = \max(3\ C_2, 3\ C\ C_2)$. □

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