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Modélisation Mathématique et Analyse Numérique

A NUMERICAL METHOD FOR SOLVING INVERSE EIGENVALUE PROBLEMS

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Abstract. Based on QR-like decomposition with column pivoting, a new and efficient numerical method for solving symmetric matrix inverse eigenvalue problems is proposed, which is suitable for both the distinct and multiple eigenvalue cases. A locally quadratic convergence analysis is given. Some numerical experiments are presented to illustrate our results.

Résumé. Basée sur la décomposition QR-type avec la colonne pivot, une nouvelle et efficace méthode numérique pour résoudre des problèmes inverses des valeurs propres des matrices symétriques est proposée, qui est convenable aux deux cas des valeurs propres distinctes et multiples. Une analyse de convergence localement quadratique de la méthode est donnée. Des expériences numériques sont présentées pour illustrer nos résultats.

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1. Introduction

Let $A(c)$ be the affine family

$$A(c) = A_0 + \sum_{i=1}^{n} c_i A_i \tag{1}$$

where $A_0, A_1, \cdots, A_n$ are real symmetric $n \times n$ matrices, and $c = (c_1, \cdots, c_n)^T \in \mathbb{R}^n$. We consider inverse eigenvalue problems (IEP) of the following form.

IEP. Given real numbers $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, find $c \in \mathbb{R}^n$ such that the eigenvalues $\lambda_1(c) \le \lambda_2(c) \le \cdots \le \lambda_n(c)$ of $A(c)$ satisfy

$$\lambda_i(c) = \lambda_i, \;\; i = 1, 2, \cdots, n. \tag{2}$$

The IEP are of great importance to many applications. A good collection of interesting applications where the IEP may arise is included in [13]. There is a large literature on conditions for existence of solutions to the IEP.

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See, for example [2, 9, 19, 20, 27-31, 33]. A special case of the IEP is obtained when the linear family (1) is defined by

$$A_i = e_i e_i^T, \quad i = 1, \cdots, n$$

where e_i is the ith unit vector, so that $A(c) = A_0 + D$, where $D = \text{diag}(c_1, \cdots, c_n)$. This problem is well known as the additive inverse eigenvalue problem. For decades there has been considerable discussion about the additive inverse eigenvalue problem. Some theoretical results and computational methods can be found, for example, in the articles [7, 8, 11, 12, 15, 17, 24, 32], and the book [33] and the references contained therein.

Numerical algorithms for solving the IEP can be found, for example, in [1, 3, 4, 6, 13, 16, 23, 33]. Friedland *et al.* [13] have surveyed four quadratically convergent numerical methods. One of the algorithms analyzed in [13] (also see [1, 4, 16]) is Newton's method for solving the nonlinear system (2). Each step in the numerical solution by Newton's method of the system (2) involves the solution of complete eigenproblem for the matrix $A(c)$. Two of the other methods analyzed in [13] are motivated as modifications to Newton's method in which computing time is saved by approximating the eigenvectors when the matrix $A(c)$ changes, rather than recomputing them. The fourth method considered in [13] is based on determinant evaluation and originated with Biegler-König [3], but it is not computationally attractive [13] for real symmetric matrices. When $\lambda_1, \cdots, \lambda_n$ include multiple eigenvalues, however, the eigenvalues $\lambda_1(c), \cdots, \lambda_n(c)$ of the matrix $A(c)$ are not, in general, differentiable at a solution c^*. Furthermore, the eigenvectors are not unique, and they cannot generally be defined to be continuous functions of c at c^*. The modification to the IEP has been considered in [13], but the number of the given eigenvalues and their multiplicities should be satisfied a certain condition in the modified problem. Based on the differentiability theory [21] of QR decomposition of a matrix depending on several variables, Li [23] presented a numerical method for solving inverse eigenvalue problems in the distinct eigenvalue case.

In this paper, we consider the formulation and local analysis of a quadratically convergent method for solving the IEP, assuming the existence of a solution. The paper is organized as follows. In Section 2 we recall some necessary differentiability theory for QR-like decomposition of a matrix dependent on several parameters. In Section 3 a new algorithm based on QR-like decomposition is proposed. It consists of extension of ideas developed by Li [22, 23], Dai and Lancaster [10], and is suitable for both the distinct and multiple eigenvalues cases. Its locally quadratic convergence analysis is given in Section 4. Finally in Section 5 some numerical experiments are presented to illustrate our results.

We shall use the following notation. A solution to the IEP will always be denoted by c^*. For the given eigenvalues $\lambda_1, \cdots, \lambda_n$, we write $\lambda^* = (\lambda_1, \cdots, \lambda_n)$. $\|.\|_2$ denotes the Euclidean vector norm or induced spectral norm, and $\|.\|_F$ the Frobenius matrix norm. For an $n \times m$ matrix $A = [a_1, \cdots, a_m]$, where a_i is the ith column vector of A, we define a vector col$A$ by $\text{col}A = [a_1{}^T, \cdots, a_m{}^T]^T$, and the norm $\|A\|_\psi := \max_{j=1,\cdots,m}(\|a_j\|_2)$. The symbol $\otimes$ denotes the Kronecker product of matrices.

2. QR-LIKE DECOMPOSITION AND DIFFERENTIABILITY

Let $A \in \mathbb{R}^{n\times n}$ and $m (1 \le m \le n)$ be an integer. Following Li [22], we define a QR-like decomposition of A with index m to be a factorization

$$A = QR, R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix} \tag{3}$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal, R_{11} is $(n-m)\times(n-m)$ upper triangular, and R_{22} is $m \times m$ square. When $m = 1$, this is a QR decomposition of A. Clearly, a QR-like decomposition of a matrix exists always. In fact, we need only construct a "partial" QR decomposition, see [14], for example. In general, however, it is not unique as the following theorem shows.

Theorem 2.1 (see [10,12]). *Let A be an $n \times n$ matrix whose first $n-m$ columns are linearly independent and let $A = QR$ be a QR-like decomposition with index m. Then $A = \widehat{Q}\widehat{R}$ is also a QR-like decomposition with* *index m if and only if*

$$Q = \widehat{Q}D, R = D^T\widehat{R} \tag{4}$$

where $D = \mathrm{diag}(D_{11}, D_{22})$, D_{11} is an orthogonal diagonal matrix, and D_{22} is an $m \times m$ orthogonal matrix.

Note that the linear independence hypothesis ensures that the R_{11} blocks of R and $\widehat{R}$ are nonsingular. In order to ensure that the submatrix R_{11} of such a decomposition is nonsingular, we admit a permutation of the columns of A. So the QR-like decomposition with column pivoting of $A \in \mathbb{R}^{n\times n}$ may be expressed as

$$AP = QR \tag{5}$$

where P is an $n \times n$ permutation matrix, and R is of the (3). If $\mathrm{rank}(A) = n - m$, then the permutation matrix P can be chosen such that the first $n - m$ columns of the matrix AP are linearly independent and $R = (r_{ij})$ satisfies

$$|r_{11}| \geq |r_{22}| \geq \cdots \geq |r_{n-m,n-m}| > 0, R_{22} = 0. \tag{6}$$

Now let $A(c) = (a_{ij}(c)) \in \mathbb{R}^{n\times n}$ be a continuously differentiable matrix-valued function of $c \in \mathbb{R}^n$. Here, the differentiability of $A(c)$ with respect to c means, for any $c^{(0)} \in \mathbb{R}^n$, we have

$$A(c) = A(c^{(0)}) + \sum_{j=1}^{n} \frac{\partial A(c^{(0)})}{\partial c_j}(c_j - c_j^{(0)}) + o(\|c - c^{(0)}\|_2) \tag{7}$$

where $c = (c_1, \cdots, c_n)^T, c^{(0)} = (c_1^{(0)}, \cdots, c_n^{(0)})^T$, and

$$\frac{\partial A(c^{(0)})}{\partial c_j} = (\frac{\partial a_{ij}(c)}{\partial c_j}|_{c=c^{(0)}}).$$

Note that if $A(c)$ is twice differentiable, then $o(\|c - c^{(0)}\|_2)$ in (7) may be replaced by $O(\|c - c^{(0)}\|_2^2)$. If $A(c)$ is of the form (1), then $A(c)$ is twice continuously differentiable, and

$$\frac{\partial A(c)}{\partial c_j} = A_j, \ \ j = 1, \cdots, n \tag{8}$$

$$A(c) = A(c^{(0)}) + \sum_{j=1}^{n} A_j(c_j - c_j^{(0)}). \tag{9}$$

We consider only hereafter the affine family $A(c)$. The next result which follows from Theorem 3.2 in [10] concerns the existence of a locally smooth QR-like decomposition of $A(c)$.

Theorem 2.2. *Let $A(c)$ be the affine family (1) and assume that* $\mathrm{rank}(A(c^{(0)})) \geq n - m$ *at $c^{(0)} \in \mathbb{R}^n$. Let P be a permutation matrix such that the first $n - m$ columns of $A(c^{(0)})P$ are linearly independent, and*

$$A(c^{(0)})P = Q^{(0)}R^{(0)}, R^{(0)} = \begin{pmatrix} R_{11}^{(0)} & R_{12}^{(0)} \\ 0 & R_{22}^{(0)} \end{pmatrix} \tag{10}$$

be a QR-like decomposition of $A(c^{(0)})P$ with index m. Then there exists a neighbourhood $\mathrm{N}(c^{(0)})$ of $c^{(0)}$ in $\mathbb{R}^n$ such that, for any $c \in \mathrm{N}(c^{(0)})$, there is a QR-like decomposition of $A(c)P$ with index m

$$A(c)P = Q(c)R(c), R(c) = \begin{pmatrix} R_{11}(c) & R_{12}(c) \\ 0 & R_{22}(c) \end{pmatrix} \tag{11}$$

with the following properties:

1. $Q(c^{(0)}) = Q^{(0)}, R(c^{(0)}) = R^{(0)}$.

2. All elements of $Q(c)$ *and* $R(c)$ *are continuous in* $\mathrm{N}(c^{(0)})$.

3. $R_{22}(c)$ *and the diagonal elements* $r_{jj}(c), j = 1, \cdots, n-m,$ *of* $R_{11}(c)$ *are continuously differentiable at* $c^{(0)}$. *Moreover, if we write*

$$Q^{(0)^T} A_j P = \begin{pmatrix} A_{j,11} & A_{j,12} \\ A_{j,21} & A_{j,22} \end{pmatrix}, \quad A_{j,11} \in \mathbb{R}^{(n-m)\times(n-m)}, \quad j = 1, \cdots, n \tag{12}$$

then

$$R_{22}(c) = R_{22}^{(0)} + \sum_{j=1}^{n} (A_{j,22} - A_{j,21} R_{11}^{(0)^{-1}} R_{12}^{(0)})(c_j - c_j^{(0)}) + O(\|c - c^{(0)}\|_2^2). \tag{13}$$

3. An algorithm based on QR-like decomposition

3.1. Formulation of the IEP

We now consider a new formulation of the IEP, which is an extension of ideas developed in [10, 22, 23]. For convenience, we assume that only the first eigenvalue is multiple, with multiplicity m, *i.e.*,

$$\lambda_1 = \cdots = \lambda_m < \lambda_{m+1} < \cdots < \lambda_n. \tag{14}$$

There is no difficulty in generalizing all our results to an arbitrary set of given eigenvalues.

Compute a QR-like decomposition with column pivoting of $A(c) - \lambda_1 I$ with index m

$$(A(c) - \lambda_1 I) P_1(c) = Q_1(c) R_1(c) \tag{15}$$

where

$$R_1(c) = \begin{pmatrix} R_{11}^{(1)}(c) & R_{12}^{(1)}(c) \\ 0 & R_{22}^{(1)}(c) \end{pmatrix}, \quad R_{11}^{(1)}(c) \in \mathbb{R}^{(n-m)\times(n-m)} \tag{16}$$

and QR decompositions with column pivoting of $A(c) - \lambda_i I (i = m+1, \cdots, n)$

$$(A(c) - \lambda_i I) P_i(c) = Q_i(c) R_i(c), i = m+1, \cdots, n \tag{17}$$

where

$$R_i(c) = \begin{pmatrix} R_{11}^{(i)}(c) & r_{12}^{(i)}(c) \\ 0 & r_{nn}^{(i)}(c) \end{pmatrix}, \quad R_{11}^{(i)}(c) \in \mathbb{R}^{(n-1)\times(n-1)} \tag{18}$$

and assume permutation matrices $P_i(c) \in \mathbb{R}^{n\times n} (i = 1, m+1, \cdots, n)$ are constant in a sufficiently small neighbourhood of c for each i, and are chosen such that

$$\det(R_{11}^{(1)}(c)) \neq 0, |e_1^T R_1(c) e_1| \geq |e_2^T R_1(c) e_2| \geq \cdots \geq |e_{n-m}^T R_1(c) e_{n-m}| \geq \|R_{22}^{(1)}(c)\|_\psi \tag{19}$$

and

$$|e_1^T R_i(c) e_1| \geq |e_2^T R_i(c) e_2| \geq \cdots \geq |e_n^T R_i(c) e_n| = |r_{nn}^{(i)}(c)|, i = m+1, \cdots, n. \tag{20}$$

Then the symmetric matrix $A(c)$ has the eigenvalues $\lambda_1, \lambda_{m+1}, \cdots, \lambda_n$ in which λ_1 is a multiple eigenvalue with multiplicity m if and only if

$$\begin{cases} R_{22}^{(1)}(c) = 0 \\ r_{nn}^{(i)}(c) = 0, i = m+1, \cdots, n. \end{cases} \tag{21}$$

We introduce a new formulation of the IEP:
Solve the following least squares problem

$$\min F(c) = \frac{1}{2}\{\|R_{22}^{(1)}(c)\|_F^2 + \sum_{i=m+1}^{n} (r_{nn}^{(i)}(c))^2\}. \tag{22}$$

If $m = 1$, *i.e.*, the given eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ are distinct, we may consider solving the nonlinear system

$$\begin{pmatrix} r_{nn}^{(1)}(c) \\ r_{nn}^{(2)}(c) \\ \cdots \\ r_{nn}^{(n)}(c) \end{pmatrix} = 0. \tag{23}$$

The formulation (23) has been studied by Li [23]. In fact, if $m = 1$ and a solution of the IEP exists, (22) and (23) are equivalent.

It is worth while mentioning that $F(c)$ may be not uniquely determined for any c because of the non-uniqueness of QR-like and QR decompositions. However, we shall show that such "flexibility" does not affect the effectiveness of our algorithm (see Lem. 4.1 below).

3.2. An algorithm

Let $c^{(k)}$ be sufficiently close to c^*. It follows from Theorem 2.2 that the matrix-valued function $R_{22}^{(1)}(c)$ and $n-m$ functions $r_{nn}^{(i)}(c)(i = m+1, \cdots, n)$ are continuously differentiable at $c^{(k)}$, and $R_{22}^{(1)}(c)$ and $r_{nn}^{(i)}(c)(i = m+1, \cdots, n)$ can be expressed as

$$\begin{aligned} R_{22}^{(1)}(c) &= R_{22}^{(1)}(c^{(k)}) + \sum_{j=1}^{n} \frac{\partial R_{22}^{(1)}(c^{(k)})}{\partial c_j}(c_j - c_j^{(k)}) + O(\|c - c^{(k)}\|_2^2) \\ &= R_{22}^{(1)}(c^{(k)}) + \sum_{j=1}^{n} [T_{j,22}^{(1)}(c^{(k)}) - T_{j,21}^{(1)}(c^{(k)}) R_{11}^{(1)^{-1}}(c^{(k)}) R_{12}^{(1)}(c^{(k)})](c_j - c_j^{(k)}) + O(\|c - c^{(k)}\|_2^2) \end{aligned} \tag{24}$$

$$\begin{aligned} r_{nn}^{(i)}(c) &= r_{nn}^{(i)}(c^{(k)}) + \sum_{j=1}^{n} \frac{\partial r_{nn}^{(i)}(c^{(k)})}{\partial c_j}(c_j - c_j^{(k)}) + O(\|c - c^{(k)}\|_2^2) \\ &= r_{nn}^{(i)}(c^{(k)}) + \sum_{j=1}^{n} [t_{j,22}^{(i)}(c^{(k)}) - t_{j,21}^{(i)}(c^{(k)}) R_{11}^{(i)^{-1}}(c^{(k)}) r_{12}^{(i)}(c^{(k)})](c_j - c_j^{(k)}) + O(\|c - c^{(k)}\|_2^2), \\ & \qquad i = m+1, \cdots, n \end{aligned} \tag{25}$$

where

$$\begin{cases} Q_1^T(c)(A_j - \lambda_1 I)P_1(c) = \begin{pmatrix} T_{j,11}^{(1)}(c) & T_{j,12}^{(1)}(c) \\ T_{j,21}^{(1)}(c) & T_{j,22}^{(1)}(c) \end{pmatrix}, & T_{j,11}^{(1)}(c) \in \mathbb{R}^{(n-m)\times(n-m)} \\ Q_i^T(c)(A_j - \lambda_i I)P_i(c) = \begin{pmatrix} T_{j,11}^{(i)}(c) & t_{j,12}^{(i)}(c) \\ t_{j,21}^{(i)}(c) & t_{j,22}^{(i)}(c) \end{pmatrix}, & T_{j,11}^{(i)}(c) \in \mathbb{R}^{(n-1)\times(n-1)}. \end{cases} \tag{26}$$

Let

$$f(c) = \begin{pmatrix} \mathrm{col} R_{22}^{(1)}(c) \\ r_{nn}^{(m+1)}(c) \\ \cdots \\ r_{nn}^{(n)}(c) \end{pmatrix}. \tag{27}$$

Then

$$F(c) = \frac{1}{2} f^T(c) f(c). \tag{28}$$

We apply the Gauss-Newton method (see [25]) to solve the least squares problem (22). By use of (24, 25), one step of Gauss-Newton method for the solution of (22) has the following form

$$J_f^T(c^{(k)}) J_f(c^{(k)})(c^{(k+1)} - c^{(k)}) = -J_f^T(c^{(k)}) f(c^{(k)}) \tag{29}$$

where

$$J_f(c) = \begin{pmatrix} \mathrm{col}\dfrac{\partial R_{22}^{(1)}(c)}{\partial c_1} & \cdots & \mathrm{col}\dfrac{\partial R_{22}^{(1)}(c)}{\partial c_n} \\ \dfrac{\partial r_{nn}^{(m+1)}(c)}{\partial c_1} & \cdots & \dfrac{\partial r_{nn}^{(m+1)}(c)}{\partial c_n} \\ \cdots & & \cdots \\ \dfrac{\partial r_{nn}^{(n)}(c)}{\partial c_1} & \cdots & \dfrac{\partial r_{nn}^{(n)}(c)}{\partial c_n} \end{pmatrix} \tag{30}$$

with

$$\begin{cases} \dfrac{\partial R_{22}^{(1)}(c)}{\partial c_j} = T_{j,22}^{(1)}(c) - T_{j,21}^{(1)}(c) R_{11}^{(1)^{-1}}(c) R_{12}^{(1)}(c) \\ \dfrac{\partial r_{nn}^{(i)}(c)}{\partial c_j} = t_{j,22}^{(i)}(c) - t_{j,21}^{(i)}(c) R_{11}^{(i)^{-1}}(c) r_{12}^{(i)}(c), i = m+1, \cdots, n. \end{cases} \tag{31}$$

Thus, our algorithm for solving the IEP has the following form:

Algorithm 3.1.

1. *Choose an initial approximation* $c^{(0)}$ *to* c^*, *and for* $k = 0, 1, 2, \cdots$
2. *Compute* $A(c^{(k)}) - \lambda_i I (i = 1, m+1, \cdots, n)$.

3. *Compute QR-like decomposition with column pivoting of* $A(c^{(k)}) - \lambda_1 I$ *with index* m:

$$(A(c^{(k)}) - \lambda_1 I)P_1(c^{(k)}) = Q_1(c^{(k)})R_1(c^{(k)}), R_1(c^{(k)}) = \begin{pmatrix} R_{11}^{(1)}(c^{(k)}) & R_{12}^{(1)}(c^{(k)}) \\ 0 & R_{22}^{(1)}(c^{(k)}) \end{pmatrix}$$

and QR decompositions with column pivoting of $A(c^{(k)}) - \lambda_i I (i = m+1, \cdots, n)$:

$$(A(c^{(k)}) - \lambda_i I)P_i(c^{(k)}) = Q_i(c^{(k)})R_i(c^{(k)}), R_i(c^{(k)}) = \begin{pmatrix} R_{11}^{(i)}(c^{(k)}) & r_{12}^{(i)}(c^{(k)}) \\ 0 & r_{nn}^{(i)}(c^{(k)}) \end{pmatrix}.$$

4. *If* $\sqrt{\|R_{22}^{(1)}(c^{(k)})\|_F^2 + \sum_{i=m+1}^{n} (r_{nn}^{(i)}(c^{(k)}))^2}$ *is small enough stop; otherwise:*
5. *Form* $f(c^{(k)})$ *and* $J_f(c^{(k)})$ *using (27, 30).*
6. *Find* $c^{(k+1)}$ *by solving linear system (29).*
7. *Go to* 2.

4. Convergence analysis

Because of the non-uniqueness of QR-like and QR decompositions described in Theorem 2.1 it is necessary to check that the iterates $c^{(k)}, k = 1, 2, \cdots$, generated by Algorithm 3.1 do not depend on the decompositions used in (15) and (17). We can prove the following result.

Lemma 4.1. *In Algorithm 3.1, for any fixed* k, *suppose*

$$\begin{aligned} (A(c^{(k)}) - \lambda_1 I)P_1(c^{(k)}) &= Q_1(c^{(k)})R_1(c^{(k)}) = \widetilde{Q}_1(c^{(k)})\widetilde{R}_1(c^{(k)}), \\ \widetilde{R}_1(c^{(k)}) &= \begin{pmatrix} \widetilde{R}_{11}^{(1)}(c^{(k)}) & \widetilde{R}_{12}^{(1)}(c^{(k)}) \\ 0 & \widetilde{R}_{22}^{(1)}(c^{(k)}) \end{pmatrix} \end{aligned} \tag{32}$$

are two (different) QR-like decompositions with column pivoting of $A(c^{(k)}) - \lambda_1 I$ *with index* m, *and*

$$\begin{aligned} (A(c^{(k)}) - \lambda_i I)P_i(c^{(k)}) &= Q_i(c^{(k)})R_i(c^{(k)}) = \widetilde{Q}_i(c^{(k)})\widetilde{R}_i(c^{(k)}), \\ \widetilde{R}_i(c^{(k)}) &= \begin{pmatrix} \widetilde{R}_{11}^{(i)}(c^{(k)}) & \widetilde{r}_{12}^{(i)}(c^{(k)}) \\ 0 & \widetilde{r}_{nn}^{(i)}(c^{(k)}) \end{pmatrix}, \quad i = m+1, \cdots, n \end{aligned} \tag{33}$$

are two (different) QR decompositions with column pivoting of $A(c^{(k)}) - \lambda_i I (i = m+1, \cdots, n)$, $J_f(c^{(k)}), f(c^{(k)})$ *and* $\widetilde{J}_f(c^{(k)}), \widetilde{f}(c^{(k)})$ *are obtained in Step 5 of Algorithm 3.1 corresponding to two different decompositions of (32) and (33). Then*

$$J_f^T(c^{(k)})J_f(c^{(k)}) = \widetilde{J}_f^T(c^{(k)})\widetilde{J}_f(c^{(k)}) \tag{34}$$

$$J_f^T(c^{(k)})f(c^{(k)}) = \widetilde{J}_f^T(c^{(k)})\widetilde{f}(c^{(k)}). \tag{35}$$

Proof. It follows from Theorem 2.1 that there exist a partitioned orthogonal matrix $D_1 = \mathrm{diag}(D_{11}, D_{22})$ where D_{11} is an orthogonal diagonal matrix, D_{22} is an $m \times m$ orthogonal matrix, and $n - m$ orthogonal diagonal matrices $D_i = \mathrm{diag}(\delta_1^{(i)}, \cdots, \delta_n^{(i)})$ where $\delta_j^{(i)} = \pm 1 (i = m+1, \cdots, n)$ such that

$$\left\{ \begin{array}{l} Q_1(c^{(k)}) = \widetilde{Q}_1(c^{(k)}) D_1, R_1(c^{(k)}) = D_1^T \widetilde{R}_1(c^{(k)}) \\ Q_i(c^{(k)}) = \widetilde{Q}_i(c^{(k)}) D_i, R_i(c^{(k)}) = D_i \widetilde{R}_i(c^{(k)}), i = m+1, \cdots, n. \end{array} \right. \tag{36}$$

By use of (26, 31, 36), we have

$$\left\{ \begin{array}{ll} R_{22}^{(1)}(c^{(k)}) = D_{22}^T \widetilde{R}_{22}^{(1)}(c^{(k)}), \dfrac{\partial R_{22}^{(1)}(c^{(k)})}{\partial c_j} = D_{22}^T \dfrac{\partial \widetilde{R}_{22}^{(1)}(c^{(k)})}{\partial c_j}, & j = 1, \cdots, n \\ r_{nn}^{(i)}(c^{(k)}) = \delta_n^{(i)} \widetilde{r}_{nn}^{(i)}(c^{(k)}), \dfrac{\partial r_{nn}^{(i)}(c^{(k)})}{\partial c_j} = \delta_n^{(i)} \dfrac{\partial \widetilde{r}_{nn}^{(i)}(c^{(k)})}{\partial c_j}, & i = m+1, \cdots, n. \end{array} \right. \tag{37}$$

From (37) and the properties of the Kronecker product (see [18]), we obtain

$$\left\{ \begin{array}{l} \mathrm{col} R_{22}^{(1)}(c^{(k)}) = (I \otimes D_{22}^T) \mathrm{col} \widetilde{R}_{22}^{(1)}(c^{(k)}) \\ \mathrm{col} \dfrac{\partial R_{22}^{(1)}(c^{(k)})}{\partial c_j} = (I \otimes D_{22}^T) \mathrm{col} \dfrac{\partial \widetilde{R}_{22}^{(1)}(c^{(k)})}{\partial c_j}, j = 1, \cdots, n. \end{array} \right. \tag{38}$$

From (27, 30, 37, 38), the matrices $J_f(c^{(k)})$, $\widetilde{J}_f(c^{(k)})$ and the vectors $f(c^{(k)})$, $\widetilde{f}(c^{(k)})$ obtained in Algorithm 3.1 satisfy

$$J_f(c^{(k)}) = \mathrm{diag}(I \otimes D_{22}^T, \delta_n^{(m+1)}, \cdots, \delta_n^{(n)}) \widetilde{J}_f(c^{(k)}) \tag{39}$$

$$f(c^{(k)}) = \mathrm{diag}(I \otimes D_{22}^T, \delta_n^{(m+1)}, \cdots, \delta_n^{(n)}) \widetilde{f}(c^{(k)}). \tag{40}$$

Hence (34) and (35) hold. □

Lemma 4.1 shows that the iterates $c^{(k)}$ generated by Algorithm 3.1 do not vary with different QR-like decompositions of $(A(c^{(k)}) - \lambda_1 I) P_1(c^{(k)})$ and different QR decompositions of $(A(c^{(k)}) - \lambda_i I) P_i(c^{(k)}) (i = m + 1, \cdots, n)$.

In order to analyse locally quadratic convergence of Algorithm 3.1, the following lemma on perturbation of QR-like decomposition is required.

Lemma 4.2 (see [10]). *Let $C_1 \in \mathbb{R}^{n \times n}$ have its first $n - m$ columns linearly independent and let $C_1 = Q_1 R_1$ be a QR-like decomposition with index m. Let $C_2 \in \mathbb{R}^{n \times n}$ be any matrix satisfying $\|C_1 - C_2\|_2 < \varepsilon$. Then, for sufficiently small ε, C_2 has a QR-like decomposition with index m, $C_2 = Q_2 R_2$, such that $\|Q_1 - Q_2\|_2 \leq \kappa_1 \varepsilon$ and $\|R_1 - R_2\|_2 \leq \kappa_2 \varepsilon$, where κ_1, κ_2 are constants independent on C_2.*

Theorem 4.1. *Suppose that the IEP have a solution c^*, and that in Algorithm 3.1 $P_i(c^{(k)}) = P_i(c^*) (i = 1, m + 1, \cdots, n)$ are independent on k when $\|c^* - c^{(k)}\|_2$ is sufficiently small. Assume also that $J_f(c^*) \in \mathbb{R}^{(m^2+n-m) \times n}$ corresponding to a QR-like decomposition of $(A(c^*) - \lambda_1 I) P_1(c^*)$ with index m and to QR decompositions of $(A(c^*) - \lambda_i I) P_i(c^*) (i = m + 1, \cdots, n)$ is of full rank. Then Algorithm 3.1 is locally quadratically convergent.*

Proof. First form the QR-like decomposition of $(A(c^*) - \lambda_1 I) P_1(c^*)$ with index m and $n - m$ QR decompositions of $(A(c^*) - \lambda_i I) P_i(c^*) (i = m + 1, \cdots, n)$

$$\left\{ \begin{array}{ll} (A(c^*) - \lambda_1 I) P_1(c^*) = Q_1(c^*) R_1(c^*) & \\ (A(c^*) - \lambda_i I) P_i(c^*) = Q_i(c^*) R_i(c^*), & i = m+1, \cdots, n. \end{array} \right. \tag{41}$$

Note that the matrix $J_f(c^*)$ corresponding to the decompositions (41), by assumption, has full rank, and that $J_f^T(c^*)J_f(c^*)$ is invertible.

Assuming that $\|c^* - c^{(k)}\|_2$ is sufficiently small, we can form a QR-like decomposition of $(A(c^{(k)}) - \lambda_1 I)P_1(c^*)$ with index m and $n-m$ QR decompositions of $(A(c^{(k)}) - \lambda_i I)P_i(c^*)(i = m+1, \cdots, n)$

$$\begin{cases} (A(c^{(k)}) - \lambda_1 I)P_1(c^*) = \widetilde{Q}_1(c^{(k)})\widetilde{R}_1(c^{(k)}) \\ (A(c^{(k)}) - \lambda_i I)P_i(c^*) = \widetilde{Q}_i(c^{(k)})\widetilde{R}_i(c^{(k)})(i = m+1, \cdots, n). \end{cases} \tag{42}$$

It follows from Lemma 4.2 that

$$\begin{cases} \|Q_i(c^*) - \widetilde{Q}_i(c^{(k)})\|_2 \le \kappa_1^{(i)}\varepsilon \\ \|R_i(c^*) - \widetilde{R}_i(c^{(k)})\|_2 \le \kappa_2^{(i)}\varepsilon, i = 1, m+1, \cdots, n \end{cases} \tag{43}$$

where $\varepsilon = \max\limits_{i=1,m+1,\cdots,n} \{\|(A(c^*) - A(c^{(k)}))P_i(c^*)\|_2\}$. Corresponding to the decompositions (42), we obtain a matrix $\widetilde{J}_f(c^{(k)}) \in \mathbb{R}^{(m^2+n-m)\times n}$. From the definition of $J_f(c)$ and (43) we know that $\|J_f(c^*) - \widetilde{J}_f(c^{(k)})\|_2$ is sufficiently small, and so is $\|J_f^T(c^*)J_f(c^*) - \widetilde{J}_f^T(c^{(k)})\widetilde{J}_f(c^{(k)})\|_2$ when $c^{(k)}$ is close to c^*. Therefore, $\widetilde{J}_f^T(c^{(k)})\widetilde{J}_f(c^{(k)})$ is invertible, $\widetilde{J}_f(c^{(k)})$ has full rank, and $\|\widetilde{J}_f(c^{(k)})\|_2$ is bounded.

The QR-like decomposition and $n-m$ QR decompositions obtained in Algorithm 3.1 at $c^{(k)}$ are not necessarily (42). Write them

$$\begin{cases} (A(c^{(k)}) - \lambda_1 I)P_1(c^*) = Q_1(c^{(k)})R_1(c^{(k)}) \\ (A(c^{(k)}) - \lambda_i I)P_i(c^*) = Q_i(c^{(k)})R_i(c^{(k)})(i = m+1, \cdots, n). \end{cases} \tag{44}$$

It follows from Lemma 4.1 and (39) that $J_f(c^{(k)})$ corresponding to the decompositions (44) also has full rank, $\|J_f^T(c^*)J_f(c^*) - J_f^T(c^{(k)})J_f(c^{(k)})\|_2$ is sufficiently small, and $\|J_f(c^{(k)})\|_2$ is bounded if $\|c^* - c^{(k)}\|_2$ is small enough. Using the perturbation theory for the inversion of a matrix (see, for example [26]), we have

$$\|[J_f^T(c^{(k)})J_f(c^{(k)})]^{-1}\|_2 \le \|[J_f^T(c^*)J_f(c^*)]^{-1}\|_2 + \omega(\varepsilon) \tag{45}$$

for the sufficiently small $\|c^* - c^{(k)}\|_2$, where $\omega(\varepsilon) \ge 0$ is a continuous function of ε and $\omega(0) = 0$.

Now we use Theorem 2.2 to extend smoothly the decompositions (44) to a neighbourhood of $c^{(k)}$ which may be assumed to include c^*. Then, abbreviating equations (24, 25)

$$R_{22}^{(1)}(c^*) = R_{22}^{(1)}(c^{(k)}) + \sum_{j=1}^{n} \frac{\partial R_{22}^{(1)}(c^{(k)})}{\partial c_j}(c_j^* - c_j^{(k)}) + O(\|c^* - c^{(k)}\|_2^2)$$

$$r_{nn}^{(i)}(c^*) = r_{nn}^{(i)}(c^{(k)}) + \sum_{j=1}^{n} \frac{\partial r_{nn}^{(i)}(c^{(k)})}{\partial c_j}(c_j^* - c_j^{(k)}) + O(\|c^* - c^{(k)}\|_2^2).$$

But $R_{22}^{(1)}(c^*) = 0$, $r_{nn}^{(i)}(c^*) = 0(i = m+1, \cdots, n)$, and we have

$$f(c^{(k)}) + J_f(c^{(k)})(c^* - c^{(k)}) = O(\|c^* - c^{(k)}\|_2^2).$$

Since $\|J_f(c^{(k)})\|_2$ is bounded, then

$$J_f^T(c^{(k)})J_f(c^{(k)})(c^* - c^{(k)}) = -J_f^T(c^{(k)})f(c^{(k)}) + O(\|c^* - c^{(k)}\|_2^2). \tag{46}$$

Comparing (46) with the equation (29) defining $c^{(k+1)}$, we have

$$J_f^T(c^{(k)})J_f(c^{(k)})(c^* - c^{(k+1)}) = O(\|c^* - c^{(k)}\|_2^2).$$

It follows from this and (45) that

$$\|c^* - c^{(k+1)}\|_2 = O(\|c^* - c^{(k)}\|_2^2)$$

as required. □

5. Numerical experiments

We first give some perturbation results which may be used to measure the difference between the given eigenvalues and those of $A(c^{(k)})$, where $c^{(k)}$ is an accepted approximation to c^*.

By use of QR-like decomposition (15) of $A(c^{(k)}) - \lambda_1 I$ with index m and QR decompositions (17) of $A(c^{(k)}) - \lambda_i I (i = m+1, \cdots, n)$, it is easily verified that

$$(A(c^{(k)}) - \lambda_1 I)Q_1(c^{(k)})E_m = P_1(c^{(k)})E_m R_{22}^{(1)^T}(c^{(k)}) \tag{47}$$

$$(A(c^{(k)}) - \lambda_i I)Q_i(c^{(k)})e_n = r_{nn}^{(i)}(c^{(k)})P_i(c^{(k)})e_n, i = m+1, \cdots, n \tag{48}$$

where $E_m = [e_{n-m+1}, \cdots, e_n]$. From the perturbation theory of eigenvalues (see [5, 26]), it follows that there exist m eigenvalues $\lambda_1(c^{(k)}), \cdots, \lambda_m(c^{(k)})$ of $A(c^{(k)})$ such that

$$|\lambda_i(c^{(k)}) - \lambda_1| \leq \|R_{22}^{(1)}(c^{(k)})\|_F, i = 1, \cdots, m \tag{49}$$

and that for any $i(m+1 \leq i \leq n)$ there is an eigenvalue λ of $A(c^{(k)})$ such that

$$|\lambda - \lambda_i| \leq |r_{nn}^{(i)}(c^{(k)})|, i = m+1, \cdots, n. \tag{50}$$

Since $\lambda_1 < \lambda_{m+1} < \cdots < \lambda_n$, the intervals $|z - \lambda_1| \leq \|R_{22}^{(1)}(c^{(k)})\|_F, |z - \lambda_i| \leq |r_{nn}^{(i)}(c^{(k)})| (i = m+1, \cdots, n)$ are disjoint from each other if $\|R_{22}^{(1)}(c^{(k)})\|_F, |r_{nn}^{(i)}(c^{(k)})|$ are sufficiently small. Therefore, the smallest m eigenvalues $\lambda_1(c^{(k)}), \cdots, \lambda_m(c^{(k)})$ of $A(c^{(k)})$ satisfy (49), and the remainder $n-m$ eigenvalues $\lambda_{m+1}(c^{(k)}), \cdots, \lambda_n(c^{(k)})$ are different from each other and satisfy

$$|\lambda_i(c^{(k)}) - \lambda_i| \leq |r_{nn}^{(i)}(c^{(k)})|, i = m+1, \cdots, n. \tag{51}$$

Now we present some of our numerical experiments with Algorithm 3.1, and also give a numerical comparison between Algorithm 3.1 and Method I in [13] for our examples. The following tests were made on a SUN workstation at CERFACS. Double precision arithmetic was used throughout. The iterations were stopped when the norm $\|f(c^{(k)})\|_2$ was less than 10^{-10}. For convenience, all vectors will be written as row-vectors.

TABLE 1.

Iteration	Algorithm 3.1		Method I in [13]	
	$\|c^* - c^{(k)}\|_2$	$\|f(c^{(k)})\|_2$	$\|c^* - c^{(k)}\|_2$	$\|\lambda^* - \lambda(c^{(k)})\|_2$
0	0.1020E+02	0.7064E+01	0.1020E+02	0.6401E+01
1	0.1627E+01	0.8234E+00	0.2064E+01	0.8931E+00
2	0.1360E+00	0.6400E−01	0.3070E+00	0.1031E+00
3	0.1419E−02	0.6335E−03	0.8195E−02	0.2725E−02
4	0.1576E−06	0.7023E−07	0.7170E−05	0.2316E−05
5	0.9010E−13	0.8392E−14	0.5139E−11	0.1690E−11

Example 5.1 (see [13]). This is an additive inverse eigenvalue problem with distinct eigenvalues. Let $n = 8$,

$$A_0 = \begin{pmatrix} 0 & 4 & -1 & 1 & 1 & 5 & -1 & 1 \\ 4 & 0 & -1 & 2 & 1 & 4 & -1 & 2 \\ -1 & -1 & 0 & 3 & 1 & 3 & -1 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & -1 & 4 \\ 1 & 1 & 1 & 1 & 0 & 1 & -1 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & -1 & 6 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \end{pmatrix}, \quad A_k = e_k e_k^T, k = 1, \cdots, 8$$

$$A(c) = A_0 + \sum_{i=1}^{n} c_i A_i.$$

The eigenvalues are prescribed to be

$$\lambda^* = (10, 20, 30, 40, 50, 60, 70, 80)$$

(i) with the starting point $c^{(0)} = (10, 20, 30, 40, 50, 60, 70, 80)$, Algorithm 3.1 converges to a solution

$$\begin{aligned} c^* = (&11.90787610, 19.70552151, 30.54549819, 40.06265749, \\ &51.58714029, 64.70213143, 70.17067582, 71.31849917) \end{aligned}$$

and the results are displayed in Table 1.

(ii) with a different starting point $c^{(0)} = (10, 80, 70, 50, 60, 30, 20, 40)$, Algorithm 3.1 also converges, but to a different solution

$$\begin{aligned} c^* = (&11.46135430, 78.88082936, 68.35339960, 49.87833041, \\ &59.16891783, 30.41047015, 24.83432401, 37.01237433). \end{aligned}$$

The nature of the convergence is illustrated in Table 2.

TABLE 2.

Iteration	Algorithm 3.1		Method I in [13]	
	$\|c^* - c^{(k)}\|_2$	$\|f(c^{(k)})\|_2$	$\|c^* - c^{(k)}\|_2$	$\|\lambda^* - \lambda(c^{(k)})\|_2$
0	0.6267E+01	0.4783E+01	0.6267E+01	0.4376E+01
1	0.5978E+00	0.3736E+00	0.8358E+00	0.4086E+00
2	0.1438E−01	0.8334E−02	0.3931E−01	0.1881E−01
3	0.9151E−05	0.5368E−05	0.9733E−04	0.4598E−04
4	0.3708E−11	0.2145E−11	0.6066E−09	0.2875E−09
5			0.1587E−14	0.4293E−13

TABLE 3.

Iteration	Algorithm 3.1		Method I in [13]	
	$\|c^* - c^{(k)}\|_2$	$\|f(c^{(k)})\|_2$	$\|c^* - c^{(k)}\|_2$	$\|\lambda^* - \lambda(c^{(k)})\|_2$
0	0.2828E−01	0.1025E+02	0.2828E−01	0.9327E−01
1	0.5689E−03	0.6087E−02	0.1466E−01	0.9630E−03
2	0.1348E−06	0.1087E−05	0.1844E−03	0.3045E−03
3	0.4210E−13	0.3746E−12	0.6129E−07	0.5262E−07
4			0.1314E−12	0.6573E−13

Example 5.2 (see [13]). An inverse eigenvalue problem with multiple eigenvalues is defined. Let $n = 8$,

$$V = \begin{pmatrix} 1 & -1 & -3 & -5 & -6 \\ 1 & 1 & -2 & -5 & -17 \\ 1 & -1 & -1 & 5 & 18 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & -1 \\ 2.5 & 0.2 & 0.3 & 0.5 & 0.6 \\ 2 & -0.2 & 0.3 & 0.5 & 0.8 \end{pmatrix}$$

and $B = I + VV^T$. We define the matrices $\{A_i\}$ as follows:

$$A_0 = 0, \quad A_k = \sum_{j=1}^{k-1} b_{kj}(e_k e_j^T + e_j e_k^T) + b_{kk} e_k e_k^T, k = 1, \cdots, 8, \quad A(c) = A_0 + \sum_{i=1}^{n} c_i A_i.$$

(i) We consider $c^* = (1, 1, 1, 1, 1, 1, 1, 1)$, then $A(c^*) = B$ and its eigenvalues are

$$\lambda^* = (1, 1, 1, 2.12075361, 9.21886818, 17.28136579, 35.70821864, 722.68079377).$$

With the starting point $c^{(0)} = (0.99, 0.99, 0.99, 0.99, 1.01, 1.01, 1.01, 1.01)$, Algorithm 3.1 converges and the results are displayed in Table 3.

We choose $\lambda^* = (1, 1, 1, 2.12075361, 9.21886818)$, and use the same starting point $c^{(0)}$, the locally unique solution found by Method I in [13] is also exactly c^*. The residual is also given in Table 3.

(ii) Now let us choose the target eigenvalues

$$\lambda^* = (1, 1, 1, 2.1, 9.0, 15.98788273, 34.43000675, 704.22223731).$$

TABLE 4.

Iteration	Algorithm 3.1		Method I in [13]	
	$\|c^* - c^{(k)}\|_2$	$\|f(c^{(k)})\|_2$	$\|c^* - c^{(k)}\|_2$	$\|\lambda^* - \lambda(c^{(k)})\|_2$
0	0.2444E+00	0.1667E+02	0.2444E+00	0.2096E+00
1	0.2683E−01	0.2269E+00	0.1421E+00	0.1925E+00
2	0.1167E−02	0.7393E−02	0.2205E+00	0.2042E+00
3	0.1919E−05	0.1619E−04	0.7226E−01	0.3231E−01
4	0.6633E−11	0.5174E−10	0.8662E−02	0.7108E−02
5			0.1983E−03	0.1444E−03
6			0.1086E−06	0.7892E−07
7			0.8441E−12	0.7171E−13

TABLE 5.

Iteration	Algorithm 3.1		Method I in [13]	
	$\|c^* - c^{(k)}\|_2$	$\|f(c^{(k)})\|_2$	$\|c^* - c^{(k)}\|_2$	$\|\lambda^* - \lambda(c^{(k)})\|_2$
0	0.2000E+00	0.3231E+00	0.2000E+00	0.1583E+00
1	0.4041E−01	0.4341E−01	0.9981E−01	0.2439E−01
2	0.7522E−03	0.6398E−03	0.3753E−02	0.1179E−02
3	0.3999E−06	0.4985E−06	0.6254E−06	0.5534E−06
4	0.2579E−13	0.1474E−13	0.6071E−12	0.1827E−12

Using the same starting point $c^{(0)}$ as above, Algorithm 3.1 converges quadratically to the locally unique solution c^*.

$$\begin{aligned} c^* = (&0.98336098, 0.97437047, 0.97531317, 1.05452291,\\ &0.85548596, 0.91177696, 0.92833105, 0.88800130). \end{aligned} \tag{52}$$

Table 4 gives the residual.

Choosing the target eigenvalues $\lambda^* = (1, 1, 1, 2.1, 9.0)$ and using the same starting point $c^{(0)}$, the locally unique solution found by Method I in [13] is also exactly (52). Table 4 displays the residual.

Example 5.3. Let $n = 4$,

$$A_0 = \mathrm{diag}(1.5, 1, 2, 1), \quad A_1 = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$A(c) = A_0 + \sum_{i=1}^{n} c_i A_i.$$

The eigenvalues are prescribed by

$$\lambda^* = (0, 2, 2, 4).$$

An exact solution of the IEP is $c^* = (1, 1, 1, 1)$. Algorithm 3.1 and Method I in [13] ($\lambda^* = (0, 2, 2)$) converge to the locally unique solution c^* from the starting point $c^{(0)} = (1.1, 0.9, 1.1, 0.9)$. The results are presented in Table 5.

These examples and our other numerical experiments with Algorithm 3.1 indicate that quadratic convergence indeed occurs in practice. We observed in most of our tests that Algorithm 3.1 took less iterations than Method I in [13].

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The proofs have not been corrected by the author.