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ON THE COMBINED EFFECT OF BOUNDARY APPROXIMATION AND NUMERICAL INTEGRATION ON MIXED FINITE ELEMENT SOLUTION OF 4TH ORDER ELLIPTIC PROBLEMS WITH VARIABLE COEFFICIENTS

PULIN K. BHATTACHARYYA¹ AND NEELA NATARAJ²

Abstract. Error estimates for the mixed finite element solution of 4th order elliptic problems with variable coefficients, which, in the particular case of aniso-/ortho-/isotropic plate bending problems, gives a direct, simultaneous approximation to bending moment tensor field $\Psi = (\psi_{ij})_{1 \leq i,j \leq 2}$ and displacement field $u$, have been developed considering the combined effect of boundary approximation and numerical integration.

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1. INTRODUCTION

In [5] a new mixed finite element method for 4th order elliptic partial differential equations with variable/constant coefficients defined in convex polygonal domain, from which the mixed method scheme of Hellan-Hermann-Miyoshi [15, 22, 23, 28] for the biharmonic problem in convex polygonal domain can be retrieved as a particular case with a proper choice of coefficients $a_{ijkl}$ of the equation [see (2.2)], was developed with all details of mathematical analysis of convergence. This mixed finite element method found its application in the mixed method analysis of shell problems in [31] and also specific mention in [33]. But for the same isotropic plate bending problem, the mixed method scheme of [5] and that of Hellan-Hermann-Miyoshi are different. Error estimates of order $O(h^{m-1})$ have been obtained in [5] under the assumption that an exact integration of the integrals of the bilinear forms is possible, the domain being a convex polygonal one (i.e. no approximation of the boundary is necessary), the convexity of the polygonal domain (in all papers) being a requirement for the regularity [21, 24] of the solution on which the proof of the existence of solution of the continuous mixed variational problem and error estimates are based. But in many practical situations both approximation of the curved boundary of the convex domain by a polygonal one or some other suitable curved boundary and numerical integration for the evaluation of bilinear forms are to be performed. In such situations an estimate for the combined effect of the numerical integration and approximation of the curved boundary of the convex domain on the mixed finite element solution of the problem is essential. Such estimates for classical finite element methods of solution of second order problems have been obtained in [17, 19, 32, 35, 36, 38-40], and of fourth

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order problems in [8, 27], but to our knowledge such results for mixed finite element methods for fourth order problems are conspicuous by their absence in published research literature. Moreover, construction of estimates for these combined effects on mixed method solution for fourth order problems is associated with mathematical difficulties. The present paper contains new, original results in this direction. For other mixed/hybrid schemes for this fourth order elliptic problem, we refer to [6, 7, 10–12, 29].

2. MIXED VARIATIONAL PROBLEM

Let \( \Omega \) be an open, convex, bounded domain in \( \mathbb{R}^2 \) with Lipschitz-continuous curved boundary \( \Gamma \), piecewise of \( C^m \) class [1, 17, 21, 32, 38] \( m \geq 3 \), in which we consider the boundary value problem (P): for given \( f \in L^2(\Omega) \), find \( u \) such that:

\[
(P) : \quad Au = f \text{ in } \Omega, \quad u|_{\Gamma} = (\frac{\partial u}{\partial n})|_{\Gamma} = 0,
\]

where

\[
(Au)(x) = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial^2}{\partial x_k \partial x_l} (a_{ijkl} \frac{\partial^2 u}{\partial x_i \partial x_j})(x) = (a_{ijkl} u_{ij}, kl)(x) \quad \forall x \in \bar{\Omega}.
\]

(In (2.2) and also in the sequel, Einstein's summation convention with respect to twice repeated indices \( 1 \leq i, j, k, l \leq 2 \) has been followed), coefficients \( a_{ijkl} \) satisfy the following conditions [5]:

\[
\begin{align*}
& (A1) \quad a_{ijkl} \in C^0(\bar{\Omega}); \quad a_{ijkl} \geq 0; \quad a_{ijkl}(x) = a_{klji}(x) = a_{jikl}(x) = a_{jlk,i}(x) \quad \forall x \in \bar{\Omega}; \\
& (A2) \quad \exists \alpha_0 > 0 \text{ such that } \forall \xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathbb{R}^4 \text{ with } \xi_{21} = \xi_{12}, \quad a_{ijkl}(x)\xi_{ij}\xi_{kl} \geq \alpha_0 \|\xi\|_2^2 \quad \forall x \in \bar{\Omega}.
\end{align*}
\]

Then, under (A1–A2), the corresponding Galerkin variational problem (PG):

For given \( f \in L^2(\Omega) \), find \( u \in H^2_0(\Omega) \) [1, 17, 21, 26, 32] such that

\[
(P_G) : \quad a(u, v) = l(v) \quad \forall v \in H^2_0(\Omega),
\]

where

\[
\begin{align*}
a(u, v) &= (Au, v)_{0, \Omega} = \int_\Omega a_{ijkl} u_{ij} v_{kl} d\Omega = a(v, u) \quad \forall u, v \in H^2_0(\Omega); \\
l(v) &= (f, v)_{0, \Omega} = \int_\Omega f v d\Omega \quad \forall v \in H^2_0(\Omega)
\end{align*}
\]

has a unique solution [4, 20].

Introducing Hilbert spaces \( \mathbf{H} \) and \( \mathbf{V} \) of admissible tensor-valued functions:

- \( \mathbf{H} = \{ \Phi : \Phi = (\phi_{ij})_{i,j=1,2}; \quad \phi_{ij} = \phi_{ji} \in L^2(\Omega) \quad \forall i, j = 1, 2 \} \)

with

\[
\|\Phi\|_{\mathbf{H}}^2 = \|\Phi\|_{0, \Omega}^2 = \|\phi_{11}\|_{0, \Omega}^2 + 2\|\phi_{12}\|_{0, \Omega}^2 + \|\phi_{22}\|_{0, \Omega}^2 \quad \forall \Phi \in \mathbf{H};
\]

\[
\|\Phi\|_{\mathbf{V}'}^2 = \|\Phi\|_{0', \Omega}^2 = \|\phi_{11}\|_{0', \Omega}^2 + 2\|\phi_{12}\|_{0', \Omega}^2 + \|\phi_{22}\|_{0', \Omega}^2 \quad \forall \Phi \in \mathbf{V}';
\]

\[
\|\Phi\|_{\mathbf{V}'}^2 = \|\Phi\|_{0', \Omega}^2 = \|\phi_{11}\|_{0', \Omega}^2 + 2\|\phi_{12}\|_{0', \Omega}^2 + \|\phi_{22}\|_{0', \Omega}^2 \quad \forall \Phi \in \mathbf{V}';
\]
\[ V = \{ \Phi : \Phi \in H, \phi_{ij} \in H^1(\Omega) \forall i, j = 1, 2 \} \subset H \]

with

\[ \| \Phi \|^2 = \| \Phi \|^2_\Omega = \| \phi_{11} \|^2_\Omega + 2\| \phi_{12} \|^2_\Omega + \| \phi_{22} \|^2_\Omega \quad \forall \Phi \in V, V \hookrightarrow H; \]

and

\[ W \equiv H^1_0(\Omega) \quad \text{with} \quad \| \chi \|_W = \| \chi \|_\Omega, \forall \chi \in W, \]

we associate to \((P_G)\), the continuous Mixed Variational Problem \((Q)\) developed in [5] as follows: For given \( f \in L^2(\Omega) \), find \((\Psi, u) \in V \times W\) such that

\[
(Q) : \begin{cases} 
A(\Psi, \Phi) + b(\Phi, u) = 0 & \forall \Phi \in V, \\
- b(\Psi, v) = \langle f, v \rangle_{0, \Omega} & \forall v \in W,
\end{cases}
\]

where \(A(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}, b(\cdot, \cdot) : V \times W \rightarrow \mathbb{R}\) are continuous bilinear forms defined by:

\[
A(\Psi, \Phi) = \int_\Omega A_{ijkl} \psi_{ij} \phi_{kl} \ d\Omega = A(\Phi, \Psi) \quad \forall \Psi, \Phi \in V \subset H; \quad (2.11)
\]

\[
b(\Phi, \chi) = \int_\Omega \phi_{ij} \chi_i \ d\Omega \quad \forall \Phi \in V, \forall \chi \in W; \quad (2.12)
\]

coefficients \(A_{ijkl} = A_{ijkl}(x)\) are defined in terms of \(a_{ijkl}\) satisfying the following properties [5]:

- \(A_{ijkl} \in C^0(\overline{\Omega})\), \(A_{ijkl}(x) = A_{klij}(x) = A_{lkij}(x) = A_{lkji}(x) \forall i, j, k, l = 1, 2, \forall x \in \overline{\Omega}, \quad (2.13)\)

- \(\exists \alpha_0 > 0 \text{ such that } \forall \xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathbb{R}^4 \text{ with } \xi_{21} = \xi_{12}, A_{ijkl}(x) \xi_{ij} \xi_{kl} \geq \alpha_0 \| \xi \|^2_{\mathbb{R}^4} \forall \xi \in \overline{\Omega}. \quad (2.14)\)

- \(\forall x \in \overline{\Omega}, \forall \zeta = (\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}) \in \mathbb{R}^4 \text{ with } \zeta_{21} = \zeta_{12}, \forall \zeta = (\zeta_{11}, \zeta_{12}, \zeta_{21}, \zeta_{22}) \in \mathbb{R}^4 \) with \(\zeta_{21} = \zeta_{12}, \quad A_{ijkl}(x) a_{ijmn}(x) \zeta_{mn} \zeta_{kl} = \xi_{ij} \xi_{kl}. \quad (2.15)\)

**Proposition 2.1.** [5]

(i) \(\exists \alpha > 0 \text{ such that } \quad A(\Phi, \Phi) \geq \alpha\| \Phi \|^2_H \forall \Phi \in V \hookrightarrow H. \quad (2.16)\)

(ii) \(\exists \beta > 0 \text{ such that } \quad \sup_{\Phi \in V \setminus \{0\}} \frac{|b(\Phi, \chi)|}{\| \Phi \|_V} \geq \beta\| \chi \|_1, \Omega \forall \chi \in W \quad (2.17)\)

(iii) \((Q)\) has at most one solution \((\Psi, u) \in V \times W.\)

**Remark 2.1.** (2.17) is Babuška-Brezzi condition [2,13,14,30].
Since \( A(\cdot, \cdot) \) is not \( V \)-elliptic, (Q) is not well-posed a priori in general. But we have

**Theorem 2.1.** [5] If the solution \( u \in H_0^3(\Omega) \) of Galerkin variational problem \( (P_G) \) belongs to \( H^3(\Omega) \cap H_0^2(\Omega) \) and \( \psi_{ij} = a_{ijkl} u_{,kl} \in H^1(\Omega), \forall i, j = 1, 2 \), then (Q) has a unique solution \( (\Psi, u) \in V \times W \).

Conversely, if (Q) has a solution \( (\Psi, u) \in V \times W \), (which will be a unique one by virtue of Proposition 2.1), the second component \( u \) will be the unique solution of \( (P_G) \) and

\[
\Psi = (\psi_{ij})_{i,j=1,2} \text{ with } \psi_{ij} = a_{ijkl} u_{,kl} \text{ and } u_{,ij} = A_{ijkl} \psi_{kl} \text{ \forall } i, j = 1, 2. \tag{2.18}
\]

**Examples.**

1. **Biharmonic problem**

For \( a_{ijkl} \) defined by: \( a_{1111} = 1; a_{1121} = a_{2121} = a_{2112} = a_{1221} = 1/2; a_{ijkl} = 0 \) otherwise, which satisfy the assumptions \((A1-A2)\) we get the Dirichlet problem of the biharmonic operator \( \Lambda = \Delta^2 \). The coefficients \( A_{ijkl} \) are defined by: \( A_{1111} = 1; A_{1121} = A_{2121} = A_{2112} = A_{1221} = 1/2; A_{ijkl} = 0 \) otherwise.

Then, the corresponding bilinear form \( A(\cdot, \cdot) \) in (Q) is as follows:

\[
A(\Psi, \Phi) = \int_{\Omega} \psi_{ij} \phi_{ij} d\Omega \quad \forall \Psi = (\psi_{ij})_{i,j=1,2}, \Phi = (\phi_{ij})_{i,j=1,2} \in V. \tag{2.19}
\]

In this particular case, the algorithm (Q) reduces to the Hellan-Hermann-Miyoshi (H-H-M) algorithm [15,28] for the biharmonic equation, i.e. the solution \( (\Psi, u) \in V \times W \) of the problem (Q):

\[
\int_{\Omega} \psi_{ij} \phi_{ij} d\Omega + \int_{\Omega} \phi_{ij,j} u_{,i} d\Omega = 0 \quad \forall \Phi \in V, \tag{2.20}
\]

\[
\int_{\Omega} \psi_{ij,j} v_{,i} d\Omega = -(f,v)_{0,\Omega} \quad \forall v \in W, \tag{2.21}
\]

is given by: \( u, \Psi = (\psi_{ij})_{i,j=1,2} \) with \( \psi_{ij} = a_{ijkl} u_{,kl} = u_{,ij} \) \( \forall i, j = 1, 2 \), where \( u \in H_0^2(\Omega) \cap H^3(\Omega) \) is the solution of the problem \( (P_G) \) corresponding to the biharmonic equation.

**Remark 2.2.** If \( u \) is the deflection of the bent elastic plate, then \( \psi_{ij} = u_{,ij} \) \( (i, j = 1, 2) \) denote the components of the change in curvature tensor, but not the bending and twisting moments in the plate in general.

2. **Plate bending problems**

(i) Anisotropic case [4,25]:

\[
a_{1111} = D_{11}, a_{1121} = a_{1221} = a_{2112} = a_{2121} = a_{2211} = a_{2121} = a_{1211} = a_{1121} = D_{16}, \]

\[
a_{1222} = a_{2222} = a_{2212} = a_{2221} = D_{26}, a_{2211} = a_{1122} = D_{12} \tag{2.22}
\]

where \( D_{ij} = D_{ij}(x_1, x_2) \) \( \forall (x_1, x_2) \in \bar{\Omega} \) denote rigidities [25] defined by \( D_{ij} = B_{ij} t^3/12 \) \( (i = 1, 2; j = 1, 2, 6) \), the \( B_{ij} \)'s being expressions in terms of elastic constants of the generalized Hooke's Law for the anisotropic material of the thin plate, \( t = t(x_1, x_2) \) being the thickness of the plate at the point \((x_1, x_2) \in \bar{\Omega} \), such that

\[
D_{11}, D_{22}, D_{66} > 0, D_{12} = \nu_1 D_{22} = \nu_2 D_{11} (0 \leq \nu_i < 1/2),
\]

\[
0 \leq D_{16} < (1 - \nu_j)D_{ii} (i \neq j), 1 \leq i,j \leq 2, D_{16} + D_{26} < D_{66}. \tag{2.23}
\]
Define $A_{ijkl} = A_{ijkl}(x)$ $\forall x = (x_1, x_2) \in \bar{\Omega}$ $\forall i, j, k, l = 1, 2$ with the help of $a_{ijkl}$ as follows:

\begin{align*}
A_{iii} &= 4(D_{jj} D_{66} - D_{j6}^2)/|A(\cdot)| \quad (i \neq j); \quad A_{1112} = (D_{11} D_{22} - D_{21}^2)/|A(\cdot)|; \\
A_{1112} &= 2(D_{12} D_{26} - D_{16} D_{22})/|A(\cdot)|; \quad A_{1122} = 4(D_{16} D_{26} - D_{12} D_{66})/|A(\cdot)|; \\
A_{1222} &= 2(D_{12} D_{16} - D_{11} D_{26})/|A(\cdot)|
\end{align*}

(2.24)

with $|A(\cdot)|$ defined by

\[|A(x)| = 4(D_{11} D_{22} D_{66} - D_{11} D_{26}^2 - D_{66} D_{12}^2 - D_{22} D_{16}^2 + D_{12} D_{16} D_{66})(x),\]

(2.25)

and other $A_{ijkl}$ are determined with the symmetry property in (2.13). The corresponding bilinear form $A(\cdot, \cdot)$ in (Q) is given by:

\begin{align*}
A(\Psi, \Phi) &= \int_\Omega \frac{4}{|A(x)|} \left[ \{(D_{22} D_{66} - D_{26}^2)\psi_{11} + (D_{16} D_{26} - D_{12} D_{66})\psi_{22} + (D_{12} D_{26} - D_{16} D_{22})\psi_{12}\}\phi_{11} \\
&\quad \{(D_{16} D_{26} - D_{12} D_{66})\psi_{11} + (D_{11} D_{66} - D_{16}^2)\psi_{22} + (D_{16} D_{12} - D_{11} D_{26})\psi_{12}\}\phi_{22} \\
&\quad \{(D_{12} D_{26} - D_{16} D_{22})\psi_{11} + (D_{16} D_{12} - D_{11} D_{26})\psi_{22} + (D_{11} D_{22} - D_{12}^2)\psi_{12}\}\phi_{12} \right] \, d\Omega \\
&\quad \forall \Psi, \Phi \in V;
\end{align*}

(2.26)

$b(\cdot, \cdot)$ being the same bilinear form in (2.12).

The solution $(\Psi, u) \in V \times W$ of (Q) is characterized by: $u$ is the deflection of the bent plate, $\Psi = (\psi_{ij})_{1 \leq i, j \leq 2}$ is the bending moment tensor with bending moments $\psi_{ii}$ in the $x_i$-direction ($i = 1, 2$) and twisting moment $\psi_{12} = \psi_{21}$, i.e., one obtains directly and simultaneously $u'$ and $\psi_{ij}'s$.

(ii) The orthotropic case [4, 25, 37] can be obtained from the anisotropic case (i) by putting in (2.22–2.26),

\begin{align*}
a_{iii} &= D_i; \quad a_{1122} = a_{2211} = D_1 = \nu_1 D_2 = \nu_2 D_1; \\
a_{1212} &= a_{2121} = a_{2112} = a_{1221} = D_t; \quad a_{ijkl} = 0 \quad \text{otherwise},
\end{align*}

(2.27)

where $D_i = E_i t^3/(12(1 - \nu_1 \nu_2)) > 0$, ($i = 1, 2$); $D_t = G t^3/12 > 0$, $H = D_1 \nu_2 + 2D_t$, $G = E_1 E_2/(E_1 + (1 + 2\nu_1)E_2) > 0$, $E_1 \nu_2 = E_2 \nu_1$, $E_i$ and $\nu_i$, $i = 1, 2$ being the Young's moduli and Poisson's coefficients respectively, and the thickness function $t \in C^0(\bar{\Omega})$ is such that $0 < t_0 \leq t(x_1, x_2) \leq t_1$, $\forall (x_1, x_2) \in \bar{\Omega}$. Then

\begin{align*}
A(\Psi, \Phi) &= \int_\Omega \left[ \frac{1}{D_1(1 - \nu_1 \nu_2)}(\psi_{11} - \nu_1 \psi_{22})\phi_{11} + \frac{1}{D_2(1 - \nu_1 \nu_2)}(-\nu_2 \psi_{11} + \psi_{22})\phi_{22} + \frac{1}{D_t} \psi_{12}\phi_{12} \right] \, d\Omega \quad \forall \Psi, \Phi \in V,
\end{align*}

(2.28)

and the solution $(\Psi, u) \in V \times W$ of (Q) is such that $u$ is the deflection of the bent plate, $\Psi = (\psi_{ij})_{i,j=1,2}$ with $\psi_{ij} = a_{ijkl} u_{ijkl}$ $\forall i, j = 1, 2$ giving the bending and twisting moments in the plate, i.e., $\psi_{11} = D_1(u_{11} + \nu_2 u_{22})$, $\psi_{22} = D_2(u_{11} + \nu_1 u_{22})$ are the bending moments in the $x_1$ and $x_2$ directions, the twisting moment being $\psi_{12} = \psi_{21} = 2D_t u_{12}$.

(iii) The isotropic case is obtained from the orthotropic case by putting $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$ and consequently, $D_1 = D_2 = D$ in all formulae in (ii) for the orthotropic plate. In this case also, $u$ is the deflection of the bent plate; $\psi_{11} = D(u_{11} + \nu u_{22})$, $\psi_{22} = D(\nu u_{11} + u_{22})$, $\psi_{12} = \psi_{21} = D(1 - \nu)u_{12}$ are the bending moments in the $x_1$ and $x_2$ directions and twisting moment respectively.

Remark 2.3. For $D = 1$, $\nu = 0$, we get H-H-M mixed scheme in (2.20–2.21) [15].
3. MIXED FINITE ELEMENT PROBLEM \((Q_h)\) WITH APPROXIMATION OF THE CURVED BOUNDARY \(\Gamma\) AND NUMERICAL INTEGRATION

3.1. Triangulations \(\tau_h\) and \(\tau_h^{\text{exact}}\)

Let \(\Gamma_h\) be a (straight) polygonal boundary approximating \(\Gamma\) such that

\[ \Gamma_h \subset \Omega, \ \Gamma_h \cap \Gamma = \{P_i\}_{i=1}^{N_{C}} \cup \{P_j\}_{j=1}^{N(\Gamma_h)} = V(\Gamma_h) \]  \(\text{(3.1)}\)

where \(V(\Gamma_h)\) is the set of all vertices (corner points) of \(\Gamma_h\) with \(\text{Card}(V(\Gamma_h)) = N(\Gamma_h)\), the set of all corner points \(\{P_i\}_{i=1}^{N_{C}}\), at which \(C^m\)-smoothness \((m \geq 3)\) does not hold, being its proper subset.

Let \(\Omega_h \subset \mathbb{R}^2\) the domain interior to \(\Gamma_h\) such that

\[ \bar{\Omega}_h = \Omega_h \cup \Gamma_h \subset \bar{\Omega} \]  \(\text{(3.2)}\)

is the closed convex polygonal domain contained in \(\bar{\Omega}\) (see Fig. 3.1).

Let \(\tau_h\) be an exact, admissible, regular, quasi-uniform \([3,17]\) triangulation of \(\bar{\Omega}_h\) such that

\[ \bar{\Omega}_h = \bigcup_{T \in \tau_h} T \subset \bar{\Omega} \text{ with } \tau_h = \tau_h^b \cup \tau_h^0, \]  \(\text{(3.3)}\)

where

\[ \tau_h^b = \{ T : T \in \tau_h, \ exactly \ two \ vertices \ a_1,T \text{ and } a_2,T \text{ of } T \text{ lie on } \Gamma_h \cap \Gamma \} \]  \(\text{(3.4)}\)

\[ = \text{ set of all boundary triangles of } \tau_h; \]

\[ \tau_h^0 = \{ T : T \in \tau_h \text{ is an interior triangle } i.e. \text{ atmost one of its vertices lie on } \Gamma_h \} \]  \(\text{(3.5)}\)

\[ = \text{ set of all interior triangles of } \tau_h. \]
Let \( \tilde{T} \) denote the set of all curved boundary triangles \( \tilde{T} \) obtained from the boundary triangles \( T \in \tau_h \) by replacing the straight boundary side of \( T \) by a part of \( \Gamma \) joining the two boundary vertices on \( \Gamma_h \cap \Gamma \), the other two sides being the same ones of the corresponding boundary triangle \( T \in \tau_h^b \). (See Fig. 3.2.)

Then, \( \tau_h^{\text{exact}} = \tau_h^b \cup \tau_h^0 \), \( \tau_h^0 \subset \tau_h \) being the set of all interior triangles defined in (3.5), denotes an exact triangulation of \( \Omega = \Omega \cup \Gamma \). i.e.

\[
\tilde{\Omega} = \bigcup_{T \in \tau_h^{\text{exact}}} \tilde{T}, \quad \tilde{\Omega}_h = \bigcup_{T \in \tau_h} T, \quad \text{Card}(\tau_h^{\text{exact}}) = \text{Card}(\tau_h^b).
\]

### 3.2. Reference triangle \( \hat{T} \) and affine mapping \( F_T : \hat{T} \rightarrow T \)

Let \( \hat{T} \) be the reference triangle with vertices \( \hat{a}_1 = (1,0), \hat{a}_2 = (0,1), \hat{a}_3 = (0,0) \) and \( \forall T \in \tau_h \), \( F_T : \hat{T} \rightarrow T \) be an invertible affine mapping from \( \hat{T} \) onto \( T \in \tau_h \) defined by:

\[
\forall \hat{x} \in \hat{T}, \quad F_T(\hat{x}) = B_T \hat{x} + b_T = x \in T,
\]

such that

\[
F_T(\hat{a}_i) = a_{i,T}, \quad 1 \leq i \leq 3, \quad \{a_{i,T}\}_{i=1}^3 \text{ being the vertices of } T \in \tau_h,
\]

\[
[J(F_T)] = B_T \text{ is the invertible } 2 \times 2 \text{ Jacobian matrix and Jacobian } J(F_T) = \det B_T > 0,
\]

\( \forall \hat{\phi} \in P_m(\hat{T}), \exists \phi \in P_m(T), P_m(K) \) being the linear space of polynomials of degree \( \leq m \) defined on \( K = \hat{T} \) or \( T \), such that \( \forall x \in T \) with \( x = F_T(\hat{x}) \),

\[
\phi(x) = \hat{\phi} \cdot F_T(\hat{x}) = \hat{\phi}(\hat{x}) \text{ with } \hat{\phi} = \phi \cdot F_T, \quad \phi = \hat{\phi} \cdot F_T^{-1}.
\]

Thus, under the affine mapping \( F_T \) defined in (3.7), \( \tau_h \) is affine-equivalent to \( \hat{T} \), i.e. \( \tau_h \) is an affine family of triangles and hence, an exact triangulation of \( \Omega_h = \Omega_h \cup \Gamma_h \).

### 3.3. Numerical integration formulae

Let

\[
\int_{\hat{T}} \hat{\phi}(\hat{x}) \, d\hat{x} \approx \sum_{n=1}^{N_t} \hat{w}_n \hat{\phi}(\hat{b}_n^i) \quad (i = 1, 2)
\]

### Reference Triangle and Affine Mapping

Let \( T \) be the reference triangle with vertices \( a_1 = (1,0), a_2 = (0,1), a_3 = (0,0) \) and \( \forall T \in \tau_h \), \( F_T : \hat{T} \rightarrow T \) be an invertible affine mapping from \( \hat{T} \) onto \( T \) defined by:

\[
\forall \hat{x} \in \hat{T}, \quad F_T(\hat{x}) = B_T \hat{x} + b_T = x \in T,
\]

such that

\[
F_T(\hat{a}_i) = a_{i,T}, \quad 1 \leq i \leq 3, \quad \{a_{i,T}\}_{i=1}^3 \text{ being the vertices of } T \in \tau_h,
\]

\[
[J(F_T)] = B_T \text{ is the invertible } 2 \times 2 \text{ Jacobian matrix and Jacobian } J(F_T) = \det B_T > 0,
\]

\( \forall \hat{\phi} \in P_m(\hat{T}), \exists \phi \in P_m(T), P_m(K) \) being the linear space of polynomials of degree \( \leq m \) defined on \( K = \hat{T} \) or \( T \), such that \( \forall x \in T \) with \( x = F_T(\hat{x}) \),

\[
\phi(x) = \hat{\phi} \cdot F_T(\hat{x}) = \hat{\phi}(\hat{x}) \text{ with } \hat{\phi} = \phi \cdot F_T, \quad \phi = \hat{\phi} \cdot F_T^{-1}.
\]

Thus, under the affine mapping \( F_T \) defined in (3.7), \( \tau_h \) is affine-equivalent to \( \hat{T} \), i.e. \( \tau_h \) is an affine family of triangles and hence, an exact triangulation of \( \Omega_h = \Omega_h \cup \Gamma_h \).

### Numerical Integration Formulae

Let

\[
\int_{\hat{T}} \hat{\phi}(\hat{x}) \, d\hat{x} \approx \sum_{n=1}^{N_t} \hat{w}_n \hat{\phi}(\hat{b}_n^i) \quad (i = 1, 2)
\]
be two quadrature schemes with positive weights \( w_n^i > 0 \) and evaluation points \( b_n^i \in \hat{T} \) \((i = 1, 2, 1 \leq n \leq N_i)\). The quadrature scheme (3.11) exact for \( P_i(T) \) for \( i = 1 \) (resp. \( P_2(T) \) for \( i = 2 \)) will be used in the evaluation of the bilinear forms of the mixed finite element problem in the sequel. Then,

\[
\int_T \phi(x) dx = \int_{\hat{T}} \hat{\phi}(\hat{x}) \det(B_T) d\hat{x} \approx \sum_{n=1}^{N_i} w_{n,T}^i \phi(b_{n,T}^i)
\]

with \( w_{n,T}^i = \det(B_T) w_n^i > 0, \ b_{n,T}^i = F_T(b_n^i) \in T, \ 1 \leq n \leq N_i, \ i = 1, 2 \), is obtained from (3.11) under invertible affine mapping \( F_T \) in (3.7–3.10).

To each \( \Omega_h \), we associate auxiliary infinite dimensional Hilbert spaces \( V(\Omega_h) \) and \( H^1_0(\Omega_h) \) defined by:

- \( V(\Omega_h) = \{ \Phi : \Phi = (\phi_{ij})_{i,j=1,2}, \phi_{ij} = \phi_{ji} \in H^1(\Omega_h) \ \forall i,j = 1,2 \} \)

with

\[
\| \Phi \|_{V(\Omega_h)}^2 = \| \Phi \|_{1,\Omega_h}^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} \| \phi_{ij} \|_{1,\Omega_h}^2;
\]

- \( H^1_0(\Omega_h) = \{ v : v \in H^1(\Omega_h), v|_{\Gamma_h} = 0 \} \) with \( \| v \|_{H^1_0(\Omega_h)} = \| v \|_{1,\Omega_h} \),

and the auxiliary continuous bilinear forms

\[
\hat{A}(\cdot, \cdot) : V(\Omega_h) \times V(\Omega_h) \rightarrow \mathbb{R}, \quad \hat{b}(\cdot, \cdot) : V(\Omega_h) \times H^1(\Omega_h) \rightarrow \mathbb{R}
\]

defined by:

\[
\hat{A}(\Phi, \Psi) = \int_{\Omega_h} A_{ijkl} \phi_{ij} \psi_{kl} d\Omega_h \text{ with } |\hat{A}(\Psi, \Phi)| \leq M \| \Psi \|_{0,\Omega_h} \| \Phi \|_{0,\Omega_h} \quad \forall \Phi, \Psi \in V(\Omega_h);
\]

\[
\hat{b}(\Phi, \chi) = \int_{\Omega_h} \phi_{ij} \chi_i d\Omega_h \text{ with } |\hat{b}(\Psi, \chi)| \leq M \| \Phi \|_{1,\Omega_h} \| \chi \|_{1,\Omega_h} \quad \forall \Phi \in V(\Omega_h), \forall \chi \in H^1(\Omega_h).
\]

And to each \( \tau_h \) of \( \Omega_h \), we associate the following finite dimensional subspaces:

- \( X_h = \{ \phi_h : \phi_h \in C^0(\Omega_h), \phi_h \downarrow_T \in P_2(T) \ \forall T \in \tau_h \} \subset H^1(\Omega_h) \);

- \( V_h = \{ \Phi_h : \Phi_h = (\phi_{hij})_{i,j=1,2}, \phi_{hij} = \phi_{hji} \in X_h \ \forall i,j = 1,2 \} \subset V(\Omega_h) \)

with

\[
\| \Phi_h \|_{V_h} = \| \Phi_h \|_{V(\Omega_h)};
\]

- \( W_h = \{ \chi_h : \chi_h \in X_h, \chi_h \downarrow_{\Gamma_h} = 0 \} \subset H^1_0(\Omega_h) \) with \( \| \chi_h \|_{W_h} = \| \chi_h \|_{1,\Omega_h} \),

in which we have replaced the essential boundary condition \( \chi \downarrow_T \) in the definition of \( W \) in (2.9) by the boundary condition \( \chi_h \downarrow_{\Gamma_h} \) in (3.19).
3.4. Extensions

Let $\tilde{T} \in \tau_h^b \subset \tau_h^{\text{exact}}$ be a curved boundary triangle containing the corresponding boundary triangle $T \in \tau_h^b \subset \tau_h$ with $T \subset \tilde{T}$ (see Fig. 3.2). For $\phi_T = \phi_h \downarrow_T \in P_2(T)$ with $\phi_h \in X_h$, $\tilde{\phi}$ is the natural (polynomial) extension to $\tilde{T}$ of the polynomial $\phi_T \in P_2(T)$ defined by: $\tilde{\phi} \in P_2(\tilde{T})$ with $\tilde{\phi} \downarrow_{\tilde{T}} = \phi_T \in P_2(T)$.

Then, to $X_h$ we associate $\tilde{X}_h$ as the linear space of natural (piecewise polynomial) extensions to $\tilde{\Omega}$ of functions $\phi_h \in X_h$ defined in $\Omega_h$:

- $\tilde{X}_h = \{\tilde{\phi}_h : \tilde{\phi}_h \in C^0(\tilde{\Omega}), \tilde{\phi}_h \downarrow_{\tilde{\Omega}} = \phi_h \in X_h, \tilde{\phi}_h \downarrow_T \in P_2(\tilde{T}) \forall \tilde{T} \in \tau_h^b \subset \tau_h^{\text{exact}}\} \subset H^1(\Omega); \quad (3.20)$
- $\tilde{V}_h = \{\tilde{\Phi}_h : \tilde{\Phi}_h = (\tilde{\phi}_{h12})_{i,j=1,2}$ with $\tilde{\phi}_{h12} = \tilde{\phi}_{h21}$ such that $\tilde{\phi}_{hij} \in \tilde{X}_h \forall i,j = 1,2\}; \quad (3.21)$
- $\tilde{W}_h = \{\tilde{\chi}_h : \tilde{\chi}_h \downarrow_{\tilde{\Omega}} \in W_h, \tilde{\chi}_h \downarrow_{\tilde{\Omega} - \tilde{\Omega}_h} = 0\} \subset H^1_0(\Omega). \quad (3.22)$

With the help of numerical integration formulae in (3.12), we define new continuous, bilinear forms

$$A^N(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R}, \quad b^N(\cdot, \cdot) : V_h \times W_h \rightarrow \mathbb{R}$$

by

$$A^N_h(\Phi_h, \Psi_h) = \sum_{T \in \tau_h} \sum_{n=1}^{N_1} w^1_{n,T}(A_{ijkl} \phi_{hij} \psi_{hkl})(b^1_{n,T}) = A^N_h(\Psi_h, \Phi_h) \quad \forall \Phi_h, \Psi_h \in V_h, \quad (3.23)$$

and $\exists M_0 > 0$ such that

$$|A^N_h(\Psi_h, \Phi_h)| \leq M_0 \|\Psi_h\|_{0,\Omega_h} \|\Phi_h\|_{0,\Omega_h} \quad \forall \Psi_h, \Phi_h \in V_h;$$

and $\exists m_0 > 0$ such that

$$|b^N_h(\Phi_h, \chi_h)| \leq m_0 \|\Phi_h\|_{1,\Omega_h} \|\chi_h\|_{1,\Omega_h} \quad \forall \Phi_h \in V_h, \chi_h \in W_h. \quad (3.24)$$

Now, to the problem (Q) in (2.10), we associate the following ‘Affine’ Mixed Finite Element Problem (Q_h) as follows: Find $(\Psi_h, u_h) \in V_h \times W_h$ such that

$$\begin{align*}
(Q_h) : & A^N_h(\Psi_h, \Phi_h) + b^N_h(\Phi_h, u_h) = 0 \quad \forall \Phi_h \in V_h, \\
& -b^N_h(\Psi_h, \chi_h) = \langle f, \chi_h \rangle_{0,\Omega_h} \quad \forall \chi_h \in W_h,
\end{align*} \quad (3.25)$$

where $A^N_h(\cdot, \cdot), b^N_h(\cdot, \cdot)$ are defined by (3.23) and (3.24) respectively,

$$\langle f, \chi_h \rangle_{0,\Omega_h} = \int_{\Omega_h} f \chi_h \, d\Omega_h \quad \forall \chi_h \in W_h. \quad (3.26)$$

Remark 3.1. We are considering the important situations in which exact integration of (3.26) is possible.
Lemma 3.1. Let the quadrature schemes (3.11) with \( i = 1 \) and 2 correspond to the definitions of \( A_h^{NI} (\cdot, \cdot) \) and \( b_h^{NI} (\cdot, \cdot) \) in (3.23) and (3.24) respectively.

Then, (a) \( \exists \alpha_0 > 0 \), independent of \( h \), such that

\[
A_h^{NI}(\Phi_h, \Phi_h) \geq \alpha_0 \|\Phi_h\|^2_{0, \Omega_h} \quad \forall \Phi_h \in V_h;
\]  

(b) \( \exists \beta_1 > 0 \), independent of \( h \), such that

\[
\sup_{\Phi_h \in V_h - \{0\}} \frac{|b_h^{NI}(\Phi_h, \chi_h)|}{\|\Phi_h\|_{V_h}} \geq \beta_1 \|\chi_h\|_{W_h} \quad \forall \chi_h \in W_h.
\]

Proof. (a) For \( i = 1 \), the quadrature scheme (3.11) used in (3.23) is exact for \( P^1(\hat{T}) \). Then, using (2.14), we have:

\[
\forall T \in \tau_h, \quad \sum_{n=1}^{N_1} w_{n,T}^1 (A_{ijkl}(\Phi_{hij}\Phi_{hkl})(b_{n,T}^1) \geq \alpha_0 \sum_{n=1}^{N_1} w_{n,T}^1 (\phi_{h11}^2(b_{n,T}^1) + 2\phi_{h12}^2(b_{n,T}^1) + \phi_{h22}^2(b_{n,T}^1)),
\]

\[
= \alpha_0 \sum_{n=1}^{N_1} w_{n,T}^1 (det B_T)(\phi_{h11}^2(b_{n}^1) + 2\phi_{h12}^2(b_{n}^1) + \phi_{h22}^2(b_{n}^1))
\]

\[
= \alpha_0 \int (\phi_{h11}^2 + 2\phi_{h12}^2 + \phi_{h22}^2) dT = \alpha_0 \|\Phi_h\|^2_{0,T}.
\]

(b) For \( i = 2 \), the quadrature scheme (3.11) used in (3.24) is exact for \( P_2(\hat{T}) \). Choose \( \Phi_h^* = (\chi_h, 0, 0, \chi_h) \) with \( \chi_h \in W_h \).

Then

\[
\Phi_h^* \in V_h \quad \text{with} \quad \|\Phi_h^*\|_{1, \Omega_h} = \sqrt{2}\|\chi_h\|_{1, \Omega_h}
\]

and

\[
\sup_{\Phi_h \in V_h - \{0\}} \frac{|b_h^{NI}(\Phi_h, \chi_h)|}{\|\Phi_h\|_{1, \Omega_h}} \geq \frac{b_h^{NI}(\Phi_h^*, \chi_h)}{\|\Phi_h^*\|_{1, \Omega_h}} = \frac{b_h^{NI}(\Phi_h^*, \chi_h)}{\sqrt{2}\|\chi_h\|_{1, \Omega_h}} \quad \text{[using (3.29)]}
\]

where

\[
b_h^{NI}(\Phi_h^*, \chi_h) = \sum_{T \in \tau_h} \sum_{n=1}^{N_2} w_{n,T}^r (\chi_{h1}^2 + (\chi_{h2})^2) (b_{n,T}^2) \geq \gamma \sum_{T \in \tau_h} |\chi_h|^2_{1,T} \quad \text{with} \quad \gamma > 0 \quad \text{(17)}.
\]

\[
\Rightarrow b_h^{NI}(\Phi_h^*, \chi_h) \geq \gamma \sum_{T \in \tau_h} |\chi_h|^2_{1,T} = \gamma |\chi_h|^2_{1,\Omega_h} = \gamma |\chi_h|^2_{1,\Omega_h}.
\]

Applying Friedrichs' inequality in (3.31), we have

\[
b_h^{NI}(\Phi_h^*, \chi_h) \geq \gamma C(\Omega)\|\chi_h\|^2_{1, \Omega} = \gamma C(\Omega)\|\chi_h\|^2_{1, \Omega_h}.
\]

From (3.30) and (3.32), we get

\[
\sup_{\Phi_h \in V_h - \{0\}} \frac{|b_h^{NI}(\Phi_h, \chi_h)|}{\|\Phi_h\|_{V_h}} \geq \beta_1 \|\chi_h\|_{1, \Omega_h} \quad \forall \chi_h \in W_h \quad \text{with} \quad \beta_1 = \gamma C(\Omega)/\sqrt{2} > 0.
\]
Remark 3.2. The inequality (3.28) is the discrete Babuška-Brezzi condition [13,14,30].

**Theorem 3.1.** The 'affine' mixed finite element problem \((Q_h)\) defined by (3.25) has a unique solution \((\Psi_h, u_h) \in V_h \times W_h\).

**Proof.** Since the linear problem \((Q_h)\) is defined on \(V_h \times W_h\) which is a finite dimensional vector space, the uniqueness of its solution in \(V_h \times W_h\) implies its existence in \(V_h \times W_h^*\).

The homogeneous problem corresponding to \((Q_h)\):

\[
\begin{align*}
A_h^{NI}(\Psi_h, \Phi_h) + b_h^{NI}(\Phi_h, u_h) &= 0 & \forall \Phi_h \in V_h, \\
-b_h^{NI}(\Psi_h, \chi_h) &= 0 & \forall \chi_h \in W_h,
\end{align*}
\]

has a unique solution \(\Psi_h = 0, u_h = 0\) by virtue of (3.27) and (3.28), from which the result follows.

### 4. Error estimates

#### 4.1. Auxiliary interpolation operator \(\mathcal{P}_h\)

Since functions in \(H^s(\Omega) \cap H_0^1(\Omega)\) with \(s \geq 2\) are continuous in \(\tilde{\Omega}\) with \(\tilde{\Omega}_h \subset \tilde{\Omega}\) and \(\Gamma \cap \Gamma_h = V(\Gamma_h) = \text{set of boundary vertices of } \tau_h = \{a_i,T\}_{T \in T_h} \) [see (3.1)], we can define an auxiliary interpolation operator \(\mathcal{P}_h\) as follows: \(\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega), s = 2,3,\)

\[
\mathcal{P}_h \chi \in C^0(\tilde{\Omega}_h), \quad \mathcal{P}_h \chi \downarrow_T \in P_2(T), \quad \mathcal{P}_h \chi(a_i,T) = \chi(a_i,T), \quad 1 \leq i \leq 6, \quad \forall T \in \tau_h,
\]

where \(\{a_i,T\}_T\) and \(\{a_i,T\}_T^6\) being the vertices and midside nodes of \(T \in \tau_h\) respectively such that \(\partial T_1 = [a_1,T, a_2,T]\) is the boundary side of \(T \in \tau_h^b\). Then, from (4.1) it follows that \(\forall \mathcal{P}_h \chi(a_i,T) = \chi(a_i,T) \neq 0\) in general for \(a_4,T = (a_1,T + a_2,T)/2\). Hence, \(\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega), s = 2,3,\)

\[
\mathcal{P}_h \chi \in \{\chi_h : \chi_h \in H^1(\Omega_h) \cap C^0(\tilde{\Omega}_h), \quad \chi_h(a_i,T) = 0 \quad \forall T \in \tau_h^b, \quad i = 1,2,\}
\]

and the classical estimate [17] holds: \(\exists C > 0, \text{ independent of } h,\) such that

\[
\|\chi - \mathcal{P}_h \chi\|_{r,\Omega_h} \leq Ch^{s-r}\|\chi\|_{s,\Omega_h} \quad (s = 2,3; r = 0,1).
\]

(In (4.3) and also in the sequel the same \(C\) has been used to denote a generic strictly positive constant, independent of \(h\), having different values at different steps of the proofs.)

But \(\mathcal{P}_h \chi \notin W_h \subset H_0^1(\Omega_h)\). Hence, we introduce \(W_h\)-interpolation operator \(\mathcal{P}_{0h}\) defined by:

\[
\begin{align*}
\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega), \quad s = 2,3, & \quad \mathcal{P}_{0h} \chi \in C^0(\tilde{\Omega}_h), \quad \mathcal{P}_{0h} \chi \downarrow_T \in P_2(T) \quad \forall T \in \tau_h, \\
\mathcal{P}_{0h} \chi(a_i,T) = \chi(a_i,T) & \quad \forall \text{ interior node } a_i,T \in \Omega_h, \quad \mathcal{P}_{0h} \chi \downarrow_{\Omega_h} = 0.
\end{align*}
\]

From (4.4), it follows that \(\mathcal{P}_{0h} \chi \in W_h \subset H_0^1(\Omega_h)\) and we have

**Proposition 4.1.** Let \(\tau_h = \tau_h^b \cup \tau_h^0\) be the triangulation defined in (3.1-3.5). \(\forall \chi \in H^s(\Omega) \cap H_0^1(\Omega), s = 2,3,\) let \(\mathcal{P}_{0h} \chi \in W_h\) be defined by (4.4). Then, the following estimates hold:

**For** \(s = 2,\)

\[
\|\chi - \mathcal{P}_{0h} \chi\|_{r,\Omega_h} \leq Ch^{2-r}\|\chi\|_{2,\Omega} \quad (r = 0,1);
\]

**For** \(s = 3,\)

\[
\|\chi - \mathcal{P}_{0h} \chi\|_{r,\Omega_h} \leq Ch^{3-r-1/2}\|\chi\|_{3,\Omega} \quad (r = 0,1).
\]
Proof. \( \forall \chi \in H^s(\Omega) \cap H^1_0(\Omega), \ s = 2, 3, \)
\[
\| \chi - P_{oh} \chi \|_{0,\Omega_h} \leq \| \chi - P_h \chi \|_{0,\Omega_h} + \| P_h \chi - P_{oh} \chi \|_{0,\Omega_h} \quad (4.7)
\]
and
\[
\| \chi - P_{oh} \chi \|_{1,\Omega_h} \leq \| \chi - P_h \chi \|_{1,\Omega_h} + \| P_h \chi - P_{oh} \chi \|_{1,\Omega_h}, \quad (4.8)
\]
where \( P_h \chi \) is defined by (4.1). Then, from (4.3),
\[
\| \chi - P_h \chi \|_{0,\Omega_h} \leq Ch^s|\chi|_{s,\Omega} , \quad \| \chi - P_h \chi \|_{1,\Omega_h} \leq Ch^{s-1}|\chi|_{s,\Omega_h}. \quad (4.9)
\]
From (4.1, 4.3, 4.4), we have: \( \forall \) interior triangle \( T \in \tau^0_h, \) \( (P_h \chi - P_{oh} \chi) \uparrow T = 0, \) and \( \forall \) boundary triangle \( T \in \tau^b_h, \) \( (P_h \chi - P_{oh} \chi) \uparrow T = \chi(a_{4,T})\phi_{4,T} \) with \( \phi_{4,T} \in P_2(T), \) \( \phi_{4,T}(a_{4,T}) = 1, \phi_{4,T}(a_{4,T}) = 0, \) \( 1 \leq i \neq 4 \leq 6, \) \( a_{4,T} = (a_{1,T} + a_{2,T})/2 \) being the midpoint of the boundary side \( \partial T_1 \) of \( T \in \tau^b_h. \)
Hence,
\[
\| P_h \chi - P_{oh} \chi \|^2_{0,\Omega_h} = \sum_{T \in \tau^b_h} \| P_h \chi - P_{oh} \chi \|^2_{0,T} = \sum_{T \in \tau^b_h} |\chi(a_{4,T})|^2 \| \phi_{4,T} \|^2_{0,T} \quad (4.10)
\]
and
\[
| P_h \chi - P_{oh} \chi |^2_{1,\Omega_h} = \sum_{T \in \tau^b_h} |\chi(a_{4,T})|^2 \| \phi_{4,T} \|^2_{1,T}. \quad (4.11)
\]
But
\[
\| \phi_{4,T} \|^2_{0,T} \leq Ch^2_2 \| \phi_{4,T} \|^2_{1,T} \leq Ch^2_2; \quad \| \phi_{4,T} \|^2_{1,T} \leq C|\tilde{\phi}_{4,T}|^2 \quad (4.12)
\]
Now, we will find estimate for \( |\chi(a_{4,T})| \) in (4.10) and (4.11), for which we are to consider the cases \( s = 2 \) and \( s = 3 \) separately.
Case \( s = 2. \) From imbedding results \([1]\) \( H^2(\Omega) \hookrightarrow C^{0,\lambda}(\bar{\Omega}) \) with \( \lambda \in [0,1[, \) \( C^{0,\lambda}(\bar{\Omega}) \) being the linear space of \( \lambda \)-Holder continuous functions in \( \Omega. \) Hence, \( \forall \chi \in H^2(\Omega) \cap H^1_0(\Omega) \subset C^{0,\lambda}(\bar{\Omega}), \) \( \chi(a_{4,T}) - \chi(\tilde{a}_{4,T}) \leq C\| a_{4,T} - \tilde{a}_{4,T} \|_{2,\Omega}, \) \( \chi(\tilde{a}_{4,T}) \) is the point of intersection of the perpendicular bisector of the boundary side \( \partial T_1 = [a_{1,T}, a_{2,T}] \) of \( T \in \tau^b_h \) with the boundary \( \partial T \cap \Gamma \) such that \( \| a_{4,T} - \tilde{a}_{4,T} \|_{2,\Omega} \leq Ch^2_2 \) and \( \chi(\tilde{a}_{4,T}) = 0. \)
Hence
\[
|\chi(a_{4,T})| \leq Ch^2_2 \| \chi \|_{2,\Omega} \quad \forall \lambda \subset [0,1[. \quad (4.13)
\]
From (4.10) and (4.12), for \( h = \max_{T \in \tau_h} \{ h_T \}, \) we have
\[
\| P_h \chi - P_{oh} \chi \|^2_{0,\Omega_h} \leq Ch^{2\lambda} \| \chi \|^2_{2,\Omega} \quad (\sum_{T \in \tau^b_h} h_T)^{\lambda} |\chi|_{2,\Omega} \leq Ch^{2\lambda + 1} |\chi|_{2,\Omega} \leq Ch^{\lambda + 1} \text{meas}(\Gamma_h) |\chi|_{2,\Omega} \leq Ch^{\lambda + 1} |\chi|_{2,\Omega} \leq Ch^{\lambda + 1} |\chi|_{2,\Omega},
\]
where \( \sum_{T \in \tau^b_h} h_T \leq C \text{meas}(\Gamma_h) \) for some \( C > 0, \) independent of \( 'h', \) since \( \tau_h \) is a regular triangulation.
\[
\Rightarrow \| P_h \chi - P_{oh} \chi \|_{0,\Omega_h} \leq Ch^{2\lambda + 1/2} |\chi|_{2,\Omega} \quad \text{with } \lambda \in [0,1[. \quad (4.14)
\]
Similarly, from (4.11–4.13), we get

\[ |P_h x - P_{0h} x|_{1,\Omega_h}^2 \leq C \sum_{T \in T_h^k} h_T^4 \| x \|^2_{2,\Omega} \]

\[ \leq C h^{4\lambda-1} \left( \sum_{T \in T_h^k} h_T \right) \| x \|^2_{2,\Omega} \]

\[ \leq C h^{4\lambda-1} \text{meas} (\Gamma) \| x \|^2_{2,\Omega} \text{ with } \lambda \in ]1/4, 1[ \]

\[ \implies |P_h x - P_{0h} x|_{1,\Omega} \leq C h^{2\lambda-1/2} \| x \|_{2,\Omega} \text{ with } \lambda \in ]1/4, 1[. \quad (4.15) \]

Hence, from (4.7–4.9, 4.14, 4.15), we get: for \( \lambda \in ]3/4, 1[ \),

\[ \| x - P_{0h} x \|_{0,\Omega_h} \leq C \left[ h^2 \| x \|_{2,\Omega_h} + h^{2\lambda+1/2} \| x \|_{2,\Omega} \right] \leq C h^2 \| x \|_{2,\Omega} ; \]

\[ \| x - P_{0h} x \|_{1,\Omega_h} \leq C \left[ h \| x \|_{2,\Omega_h} + h^{2\lambda-1/2} \| x \|_{2,\Omega} \right] \leq C h \| x \|_{2,\Omega} \]

and

\[ \| x - P_{0h} x \|^2_{1,\Omega_h} = |x - P_{0h} x|_{1,\Omega_h}^2 + \| x - P_{0h} x \|^2_{0,\Omega_h} \leq C h^2 \| x \|^2_{2,\Omega} \]

which implies the result.

Case \( s = 3 \). Since \( H^3(\Omega) \hookrightarrow C^1(\bar{\Omega}) \equiv C^{0,1}(\bar{\Omega}) \), we have \( \| x \|_{1,\infty,\Omega} \leq C \| x \|_{3,\Omega} \forall x \in H^3(\Omega) \cap H^1_0(\Omega) \). Since \( \| a_{4,T} - \tilde{a}_{4,T} \|_{L^2} \leq C h^2 \), using the mean-value theorem along the line segment \([a_{4,T}, a_{4,T}]\) we have:

\[ \forall x \in H^3(\Omega) \cap H^1_0(\Omega), \ |x(a_{4,T})| \leq C h^2 \sup_{\xi \in [a_{4,T}, a_{4,T}]} \left| \frac{\partial x}{\partial \xi}(\xi) \right| \leq C h^2 \| x \|_{1,\infty,\Omega} \leq C h^2 \| x \|_{3,\Omega}. \quad (4.16) \]

Hence, from (4.10–4.12, 4.16),

\[ \| P_h x - P_{0h} x \|^2_{0,\Omega_h} \leq C \left[ \sum_{T \in T_h^k} h_T^4 \| x \|^2_{3,\Omega} h_T^2 \right] \leq C h^5 \left( \sum_{T \in T_h^k} h_T \right) \| x \|^2_{3,\Omega} \]

\[ \implies \| P_h x - P_{0h} x \|_{0,\Omega_h} \leq C h^{5/2} \text{meas} (\Gamma)^{1/2} \| x \|_{3,\Omega} \leq C h^{5/2} \| x \|_{3,\Omega} \quad (4.17) \]

and

\[ |P_h x - P_{0h} x|_{1,\Omega_h}^2 \leq C \left[ \sum_{T \in T_h^k} h_T^4 \| x \|^2_{3,\Omega} \right] \leq C h^3 \left( \sum_{T \in T_h^k} h_T \right) \| x \|^2_{3,\Omega} \]

\[ \implies |P_h x - P_{0h} x|_{1,\Omega} \leq C h^{3/2} \text{meas} (\Gamma)^{1/2} \| x \|_{3,\Omega} \leq C h^{3/2} \| x \|_{3,\Omega}. \quad (4.18) \]
Thus, from (4.7-4.9) and (4.17-4.18), we get (4.5-4.6):
\[
\|x - \mathcal{P}_h x\|_{0, \Omega_h} \leq C \left[ h^3 \|x\|_{3, \Omega_h} + h^{5/2} \|x\|_{3, \Omega} \right] \leq C h^{5/2} \|x\|_{3, \Omega};
\]
\[
\|x - \mathcal{P}_h x\|_{1, \Omega_h} \leq C \left[ h^2 \|x\|_{3, \Omega_h} + h^{3/2} \|x\|_{3, \Omega} \right] \leq C h^{3/2} \|x\|_{3, \Omega}\]
(4.19)
and
\[
\|x - \mathcal{P}_h x\|_{1, \Omega_h}^2 = \|x - \mathcal{P}_h x\|_{1, \Omega_h}^2 + \|x - \mathcal{P}_h x\|_{0, \Omega_h}^2 \leq C h^3 \|x\|_{3, \Omega}^2,
\]
and we get (4.6).

**Remark 4.1.** There is a loss of exponent of $h$ by $1/2$ in (4.6) due to a 'crude' polygonal approximation of the curved boundary $\Gamma$. Moreover, from the proof of the Case $s = 3$, we find that it cannot be improved upon even by assuming additional regularity of $X$ i.e. $\|x - \mathcal{P}_h x\|_{r, \Omega_h} \leq C h^{3-r-1/2} \|x\|_{3, \Omega}$, where $X \in H^s(\Omega) \cap H^1_0(\Omega)$ with $s > 3$. Hence it suggests to improve the boundary approximation, for example, by isoparametric mapping \[9\].

We will need the inverse inequalities [14,17,18]: $\forall \phi \in X_h$ (resp. $\Phi_h \in V_h$), $\exists \gamma^* > 0$ (resp. $\exists \gamma > 0$) independent of $h$, such that
\[
|\phi|_{1, \Omega_h} \leq \gamma^* \|\phi\|_{0, \Omega_h} \quad (\text{resp. } |\Phi_h|_{1, \Omega_h} \leq \gamma \|\Phi_h\|_{0, \Omega_h})
(4.20)
\]
and the following important well known estimates:

**Proposition 4.2.** [38] For domains $\Omega$ and $\Omega_h$ defined earlier such that $\omega_h = \Omega - \Omega_h$ with $h \in [0, h_0]$, $0 < h_0 < 1$, $\forall \chi \in H^1(\Omega)$,
\[
\|\chi\|_{0, \omega_h} \leq C h \|\chi\|_{1, \Omega} \quad \text{for some } C > 0.
(4.21)
\]

**Lemma 4.1** (p. 199 [36]). Let $T \in \tau_h$ and $\tilde{T} \in \tilde{\tau}_h^\text{exact}$ be any pair of boundary triangles such that $T \subset \tilde{T}$, $\tilde{T} \in \tilde{\tau}^\text{exact}_h$ being the curved boundary triangle constructed from the boundary triangle $T \in \tau_h$ [see (3.1-3.6)]. Suppose that $\rho = \meas(\tilde{T} - T)/\meas T$. Let $\bar{p}$ be a polynomial on $\tilde{T}$, which is a natural (polynomial) extension to $\tilde{T}$ of the polynomial 'p' defined on $T$. Then, $\exists \gamma > 0$, depending only on the degree of $p$, such that
\[
\|\bar{p}\|_{1, \tilde{T} - T}^2 \leq C \rho(T) \|p\|_{1, T}^2 \quad \forall T \in \tau_h^h \subset \tau_h.
(4.22)
\]

**Corollary 4.1.** Let $\bar{\phi}_h \in \bar{X}_h$ be the natural extension to $\bar{\Omega}$ of the function $\phi_h \in X_h$ defined in (3.20). Then, $\exists \gamma > 0$, independent of $h$, such that
\[
\|\bar{\phi}_h\|_{1, \omega_h}^2 = \sum_{T \subset \tilde{T} \in \tilde{\tau}_h^h} \|\bar{\phi}_h\|_{1, \tilde{T} - T}^2 \leq C h \|\phi_h\|_{1, \Omega_h}^2 \forall \phi_h \in X_h \text{ with } \bar{\phi}_h \in \bar{X}_h
(4.23)
\]
and
\[
\omega_h = \Omega - \Omega_h, \ \meas(\omega_h) = O(h^2) \quad [38].
(4.24)
\]

**Proof.** The result (4.23) is obtained from (4.22) by summing over all boundary triangles $T \subset \tilde{T} \in \tilde{\tau}_h^h$ and increasing the right-hand side to include all interior triangles $T \in \tau_h^h$ and considering the fact that $\rho = O(h)$ $\forall \tau_h$ [36].
Proposition 4.3.  

- Let $A_{ijkl} \in W^{1,\infty}(\Omega)$, $\forall i, j, k, l = 1, 2$.  

- Let the quadrature scheme (3.11) with $i = 1$, which is exact for $P_4(\hat{T})$, correspond to the definition (3.23) of $A_{hl}^{N1}(\cdot, \cdot)$. Then, $\exists C > 0$, independent of $h$, such that $\forall \varphi_h, \Phi_h \in \nabla_h$,

$$|\tilde{A}(\varphi_h, \Phi_h) - A_{hl}^{N1}(\varphi_h, \Phi_h)| \leq C h \|A\|_{1,\infty, \Omega} \|\varphi_h\|_{0, \Omega} \|\Phi_h\|_{0, \Omega},$$  

(4.26)

where $\tilde{A}(-, -)$ and $A_{hl}^{N1}(-, -)$ are defined by (3.15) and (3.23) respectively,

$$\|A\|_{1,\infty, \Omega} = \sup_{T \in \tau_h^{\text{exact}}} \sum_{i,j,k,l=1}^{2} \|A_{ijkl}\|_{1,\infty, \hat{T}}.$$  

(4.27)

Proof. For fixed $i, j, k, l = 1, 2$ (i.e. no summation is to be understood with respect to twice repeated indices $i, j, k, l$), $\forall T \in \tau_h$, set

$$E_T(A_{ijkl}\sigma_{lij}\phi_{kl}) = \int_{T} A_{ijkl}\sigma_{lij}\phi_{kl} dT - \sum_{n=1}^{N_1} w_{n,T}^{1}(A_{ijkl}\sigma_{lij}\phi_{kl})(b_{n,T}^{1}),$$  

(4.28)

$$\dot{E}(\dot{A}_{ijkl}\sigma_{lij}\dot{\phi}_{kl}) = \int_{T} \dot{A}_{ijkl}(\dot{x})\sigma_{lij}(\dot{x})\dot{\phi}_{kl}(\dot{x}) dT - \sum_{n=1}^{N_1} \dot{w}_{n,T}^{1}(\dot{A}_{ijkl}\sigma_{lij}\dot{\phi}_{kl})(\dot{b}_{n,T}^{1})$$  

(4.29)

with

$$E_T(A_{ijkl}\sigma_{lij}\phi_{kl}) = (\det B_T) \dot{E}(\dot{A}_{ijkl}\sigma_{lij}\dot{\phi}_{kl}), \quad b_{n,T}^{1} = F_T(\dot{b}_{n,T}^{1}).$$  

(4.30)

Then

$$|\tilde{A}(\varphi_h, \Phi_h) - A_{hl}^{N1}(\varphi_h, \Phi_h)| = \left| \int_{\Omega_h} A_{ijkl}\sigma_{lij}\phi_{kl} d\Omega_h - \sum_{T \in \tau_h} \sum_{n=1}^{N_1} w_{n,T}^{1}(A_{ijkl}\sigma_{lij}\phi_{kl})(b_{n,T}^{1}) \right|$$

$$\leq \sum_{T \in \tau_h} \sum_{i,j,k,l=1}^{2} |E_T(A_{ijkl}\sigma_{lij}\phi_{kl})|. \quad \text{(4.31)}$$

$\forall$ fixed $i, j, k, l = 1, 2$, $\sigma_{lij}, \phi_{kl} \in P_2(\hat{T})$, $\dot{A}_{ijkl} \in W^{1,\infty}(\hat{T})$ and hence

$$|\dot{E}(\dot{A}_{ijkl}\sigma_{lij}\dot{\phi}_{kl})| \leq C\|A_{ijkl}\sigma_{lij}\phi_{kl}\|_{0,\infty, \hat{T}} \leq C\|\dot{A}_{ijkl}\|_{0,\infty, \hat{T}} \|\sigma_{lij}\phi_{kl}\|_{0,\infty, \hat{T}}$$

$$\leq C\|\dot{A}_{ijkl}\|_{1,\infty, \hat{T}} \|\sigma_{lij}\phi_{kl}\|_{0,\infty, \hat{T}}. \quad \text{(4.32)}$$

$\forall$ fixed $i, j, k, l = 1, 2$, and for fixed $\sigma_{lij}, \phi_{kl} \in P_2(\hat{T})$, define

$$\dot{E}(\cdot) : W^{1,\infty}(\hat{T}) \rightarrow \mathbb{R} \text{ by } \dot{E}(\dot{A}_{ijkl}) = \dot{E}(\dot{A}_{ijkl}\sigma_{lij}\dot{\phi}_{kl}). \quad \text{(4.33)}$$

From (4.32), (4.33) $\dot{E}(\cdot)$ is a linear bounded functional on $W^{1,\infty}(\hat{T})$ with $\|\dot{E}(\cdot)\| \leq C\|\sigma_{lij}\phi_{kl}\|_{0,\infty, \hat{T}}$ and $\dot{E}(\dot{p}_0) = \dot{E}(p_0\sigma_{lij}\phi_{kl}) = 0 \ \forall p_0 \in P_0(\hat{T})$, since the quadrature formula in (4.29) is exact for $P_4(\hat{T})$.  


Hence, by Bramble-Hilbert lemma, we have: \( \forall \) fixed \( i, j, k, l = 1, 2 \) (with no summation),

\[
|\dot{E}(A_{ijkl})| = |\dot{E}(\overline{A_{ijkl}}\sigma_{ij}\Phi_k)-highlightation| 
\leq C||\sigma_{ij}\Phi_k||_{0,\infty,T}|A_{ijkl}|_{1,\infty,T}. \tag{4.34}
\]

But \( \forall \) fixed \( T \in \tau_h \),

\[
|\overline{A_{ijkl}}|_{1,\infty,T} \leq C_T||A_{ijkl}||_{1,\infty,T} \forall i, j, k, l = 1, 2 \tag{4.35}
\]

and

\[
||\sigma_{ij}\Phi_k||_{0,\infty,T} \leq ||\sigma_{ij}||_{0,\infty,T}||\Phi_k||_{0,\infty,T} \leq C||\sigma_{ij}||_{0,T}||\Phi_k||_{0,T} \quad \text{(norm equivalence in a f.d.v.s.)}
\leq C(detB_T)^{-1}||\sigma_{ij}||_{0,T}||\Phi_k||_{0,T}[16]. \tag{4.36}
\]

Hence, \( \forall \) fixed \( i, j, k, l = 1, 2 \),

\[
|\dot{E}(\overline{A_{ijkl}})(\sigma_{ij}\Phi_k))| \leq C_T(detB_T)^{-1}||\sigma_{ij}||_{0,T}||\Phi_k||_{0,T}||A_{ijkl}||_{1,\infty,T} \forall T \in \tau_h. \tag{4.37}
\]

\[
\implies |E_T(A_{ijkl}\sigma_{ij}\Phi_k))| \leq C_T||\sigma_{ij}||_{0,T}||\Phi_k||_{0,T}||A_{ijkl}||_{1,\infty,T} \forall T \in \tau_h \quad \text{[using (4.30)]} \tag{4.38}
\]

\[
\implies \sum_{T \in \tau_h} \sum_{i,j,k,l=1}^2 |E_T(A_{ijkl}\sigma_{ij}\Phi_k))| \leq \sum_{T \in \tau_h} \sum_{i,j,k,l=1}^2 C_T||\sigma_{ij}||_{0,T}||\Phi_k||_{0,T}||A_{ijkl}||_{1,\infty,T}
\leq C_T\left(\sum_{T \in \tau_h} \sum_{i,j,k,l=1}^2 ||A_{ijkl}||_{1,\infty,T}\right)\left(\sum_{T \in \tau_h} \sum_{i,j,k,l=1}^2 ||\sigma_{ij}||_{0,T}||\Phi_k||_{0,T}\right)
\leq C_T||A||_{1,\infty}\sigma_{hk}||_{0,\Omega_h}||\Phi_h||_{0,\Omega_h} \tag{4.39}
\]

where

\[
||A||_{1,\infty,\Omega} \geq ||A||_{1,\infty,\Omega_h} = \sup_{T \in \tau_h} \sum_{i,j,k,l=1}^2 ||A_{ijkl}||_{1,\infty,T} \tag{4.40}
\]

and the result (4.26) follows from (4.31, 4.39, 4.40).

**Proposition 4.4.** Suppose that the conditions of Theorem 2.1 hold. Then,

\[
|\tilde{A}(\Psi, \Phi_h) + \tilde{b}(\Phi_h, u)| \leq Ch^3(1 + \sqrt{h})||u||_{3,\Omega}||\Phi_h||_{1,\Omega_h}. \tag{4.41}
\]

where \( \tilde{A}(\cdot, \cdot) \) and \( \tilde{b}(\cdot, \cdot) \) are defined by (3.15) and (3.16) respectively.

**Proof.** From the conditions of Theorem 2.1, \( u \in H^3(\Omega) \cap H^2_0(\Omega) \) is the solution of (PG) with \( \Psi = (\psi_{ij})_{i,j=1,2} \), \( \psi_{ij} = a_{ijkl}u_{kl} \in H^1(\Omega) \forall i, j = 1, 2 \).
Then, \( \forall \Phi_h \in V_h \)

\[
\left| \tilde{A}(\Psi, \Phi_h) + \tilde{b}(\Phi_h, u) \right| = \left| \int_{\Omega_h} A_{ijkl} \psi_{ij} \phi_{hkld} \, dx + \int_{\Omega_h} \phi_{hij} u_i \, dx \right|
\]

\[
= \left| \int_{\Omega_h} A_{ijkl} \psi_{ij} \phi_{hkld} \, dx + \int_{\Omega_h} \phi_{hij} u_i \, dx - \int_{\Omega} A_{ijkl} \psi_{ij} \phi_{hkld} \, dx - \int_{\Omega} \phi_{hij} u_i \, dx \right|
\]

[by virtue of (2.10)]

\[
\leq \left| \int_{\omega_h} A_{ijkl} \psi_{ij} \phi_{hkld} \, dx \right| + \left| \int_{\omega_h} \phi_{hij} u_i \, dx \right| \quad \text{with} \quad \omega_h = \Omega - \Omega_h,
\]

(4.42)

where \( \tilde{\Phi}_h = (\tilde{\phi}_{hij})_{i,j=1,2} \in \tilde{V}_h \) is a natural extension to \( \Omega \) of \( \Phi_h \in V_h \) defined in (3.21).

* Estimate for the first term on the right-hand side of (4.42)

\[
\left| \int_{\omega_h} A_{ijkl} \psi_{ij} \phi_{hkld} \, dx \right| \leq \sum_{i,j,k,l=1}^{2} \left| \int_{\omega_h} A_{ijkl} \psi_{ij} \phi_{hkld} \, dx \right| = \sum_{k,l=1}^{2} \left| \int_{\omega_h} u_{kl} \phi_{hkld} \, dx \right|
\]

since \( A_{ijkl} \psi_{ij} = A_{ijkl} \delta_{im} \delta_{jn} = \delta_{km} \delta_{ln} \delta_{mn} = u_{kl} \) [see (2.15)].

Then, since \( u \in H^2(\Omega), \tilde{\phi}_{hkld} \in H^1(\Omega) \), we can use (4.21).

Hence, for fixed \( k, l = 1, 2 \)

\[
\left| \int_{\omega_h} u_{kl} \phi_{hkld} \, dx \right| \leq \|u_{kl}\|_{0,\omega_h} \|\phi_{hkld}\|_{0,\omega_h}
\]

\[
\leq (C_h \|u_{kl}\|_{1,\Omega})(C_h \|\phi_{hkld}\|_{1,\Omega}) \leq C h^2 \|u\|_{3,\Omega} \|\phi_{hkld}\|_{1,\Omega} \leq C h^2 \|u\|_{3,\Omega} \|\tilde{\Phi}_h\|_{1,\Omega}
\]

\[
\implies \left| \int_{\omega_h} A_{ijkl} \psi_{ij} \phi_{hkld} \, dx \right| \leq C h^2 \|u\|_{3,\Omega} \|\tilde{\Phi}_h\|_{1,\Omega}.
\]

(4.43)

* Estimate for the second term on the right-hand side of (4.42)

For fixed \( i, j = 1, 2 \),

\[
\left| \int_{\omega_h} \phi_{hij} u_i \, dx \right| \leq (\text{meas } \omega_h)^{\frac{1}{2}} \|u_i\|_{1,\omega_h} \|\phi_{hij}\|_{0,\omega_h} \leq C h \|u_i\|_{1,\omega_h} \|\phi_{hij}\|_{0,\omega_h}
\]

(4.44)

[since \( u_i \in H^2(\Omega) \implies C^0(\Omega) \), \( \phi_{hij} \in L^2(\Omega) \implies u_i \phi_{hij} \in L^2(\Omega) \) and \( \text{meas}(\omega_h) = O(h^2) \) (see (4.24))].

But for fixed \( i, j = 1, 2 \)

\[
\|u_i \phi_{hij}\|_{0,\omega_h}^2 \leq \left( \max_{x \in \Omega} |u_i(x)| \right)^2 \|\phi_{hij}\|_{0,\omega_h}^2 \leq \|u_i\|_{0,\omega_h}^2 \|\phi_{hij}\|_{0,\omega_h}^2
\]

\[
\leq C \|u_i\|_{2,\Omega}^2 \|\phi_{hij}\|_{1,\omega_h}^2 \leq C \|u_i\|_{2,\Omega}^2 \|\phi_{hij}\|_{1,\omega_h}^2 \leq C h \|u_i\|_{2,\Omega}^2 \|\phi_{hij}\|_{1,\Omega}^2,
\]

(4.45)
since the third inequality in (4.45) follows from $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and the last inequality follows from (4.23). Then from (4.44) and (4.45),

$$\sum_{i,j=1}^{2} \left| \int_{\Omega} \phi_{ij} f_{ij} \, dx \right| \leq \sum_{i,j=1}^{2} Ch^2 \|\phi_{ij}\|_{1,\Omega_h} \| u \|_{3,\Omega} \leq Ch^2 \| u \|_{3,\Omega} \| \Phi_h \|_{1,\Omega_h}. \quad (4.46)$$

Finally, from (4.42, 4.43, 4.46), we get the result (4.41).

**Lemma 4.2.** Let the quadrature scheme (3.11) with $i = 2$, which is exact for $P_2(\bar{T})$, correspond to the definition (3.24) of $b^n_l(\cdot, \cdot)$ and $\tilde{b}(\cdot, \cdot)$ be defined by (3.16).

Then, $\forall \Phi \in V(\Omega_h)$, $\exists$ a tensor-valued function $\Theta_h \in V_h$, for which the following hold:

$$\tilde{b}(\Phi, \chi_h) = b^n_l(\Theta_h, \chi_h) \quad \forall \chi_h \in W_h \subset H^1_0(\Omega_h) \quad (4.47)$$

and $\exists C > 0$, independent of $h$, such that

$$\| \Phi - \Theta_h \|_{r, \Omega_h} \leq Ch^{1-r} \| \Phi \|_{1, \Omega}, \quad (r = 0, 1). \quad (4.48)$$

**Proof.** For $\phi \in H^1(\Omega_h)$, we can associate a $\phi_h \in X_h$ such that

$$\| \phi - \phi_h \|_{r, \Omega_h} \leq Ch^{1-r} \| \phi \|_{1, \Omega_h} \quad (r = 0, 1) \quad (4.49)$$

for some $C > 0$ independent of $h$.

$\Rightarrow \forall \Phi = (\phi_{ij})_{i,j=1,2} \in V(\Omega_h)$, $\exists \Phi_h \in V_h$ such that

$$\| \Phi - \Phi_h \|_{r, \Omega_h}^2 = \sum_{i,j=1}^{2} \| \phi_{ij} - \phi_{ij} \|_{r, \Omega_h}^2 \leq \sum_{i,j=1}^{2} C^2 h^{2-2r} \| \phi_{ij} \|_{1, \Omega_h}^2 = C^2 h^{2-2r} \| \Phi \|_{1, \Omega_h}^2 \quad (r = 0, 1) \quad (4.50)$$

Define an auxiliary bilinear form $B_h(\cdot, \cdot) : W_h \times W_h \rightarrow R$ by:

$$B_h(z_h, \mu_h) = \sum_{T \in \mathcal{T}_h} \sum_{n=1}^{N_2} w_n^2(T)(\nabla z_h \cdot \nabla \mu_h)(b^n_{l,T}) \quad \forall z_h, \mu_h \in W_h \quad (4.51)$$

with $w_n^2(T) > 0$, $1 \leq n \leq N_2$, which corresponds to the quadrature scheme 3.11) with $i = 2$ exact for $P_2(\bar{T})$, and a linear form $I_h(\cdot) : W_h \rightarrow R$ by:

$$I_h(\mu_h) = \tilde{b}(\Phi, \mu_h) - b^n_l(\Phi_h, \mu_h) \quad \forall \mu_h \in W_h \quad (4.52)$$

for fixed elements $\Phi \in V(\Omega_h)$, $\Phi_h \in V_h$ satisfying (4.50).

$B_h(\cdot, \cdot)$ is continuous on $W_h \times W_h$ and $W_h$-elliptic.
In fact,

\[
B_h(\mu_h, \mu_h) = \sum_{T \in \tau_h} \sum_{n=1}^{N_T} u_{n,T}^2 (\nabla \mu_h \cdot \nabla \mu_h)(b_{n,T}^2)
\]

\[
\geq C|\mu_h|_{1,\Omega}^2 = C|\mu_h|_{1,\Omega}^2 \quad \text{(since } \mu_h = 0 \text{ outside } \Omega_h) \]

\[
\geq C(\Omega)\|\mu_h\|_{1,\Omega}^2 \quad \text{(by virtue of Friedrichs' inequality)}
\]

\[
= C(\Omega)\|\mu_h\|_{1,\Omega}^2 \quad \text{(since } \bar{\mu}_h = \mu_h \text{ in } \Omega_h, \bar{\mu}_h = 0 \text{ outside } \Omega_h)
\]

\[
\implies B_h(\mu_h, \mu_h) \geq C(\Omega)\|\mu_h\|_{1,\Omega}^2 \quad \forall \mu_h \in W_h.
\] (4.53)

\(l_h(\cdot)\) is continuous on \(W_h\).

Hence, from Lax-Milgram lemma, \(\exists\) a unique \(z_h \in W_h\) such that

\[
B_h(z_h, \mu_h) = \tilde{b}(\Phi, \mu_h) - b_h^{NI}(\Phi_h, \mu_h) \quad \forall \mu_h \in W_h
\] (4.54)

for fixed \(\Phi \in V(\Omega_h)\) and \(\Phi_h \in V_h\) satisfying (4.50).

Choose \(\varphi_h = (z_h\delta_{ij})_{i,j=1,2} \) with \(z_h \in W_h\). Then

\[
\varphi_h \in V_h \text{ with } \|\varphi_h\|_{1,\Omega} = \sqrt{2}\|z_h\|_{1,\Omega},
\] (4.55)

and

\[
b_h^{NI}(\varphi_h, \mu_h) = B_h(z_h, \mu_h) = \tilde{b}(\Phi, \mu_h) - b_h^{NI}(\Phi_h, \mu_h) \quad \text{[using (4.54)]}
\]

\[
\implies b_h^{NI}(\varphi_h, \mu_h) + \tilde{b}(\Phi, \mu_h) = b_h^{NI}(\varphi_h + \Phi, \mu_h) = \tilde{b}(\Phi, \mu_h) \quad \forall \mu_h \in W_h
\]

\[
\implies \text{the result (4.47) holds with } \Theta_h = (\varphi_h + \Phi_h) \in V_h, \Phi_h \text{ satisfying (4.50).}
\] (4.56)

- **Estimate for** \(\|\Phi - \Theta_h\|_{1,\Omega_h}\)

\(\forall\) fixed elements \(\Phi \in V(\Omega_h), \Phi_h \in V_h\) satisfying (4.50), we get from (4.53, 4.54) and the continuity of \(l_h(\cdot)\):

\[
C\|z_h\|_{1,\Omega}^2 \leq B_h(z_h, z_h) \leq M\|z_h\|_{1,\Omega} (\|\Phi\|_{1,\Omega} + \|\Phi_h\|_{1,\Omega})
\]

\[
\leq M\|z_h\|_{1,\Omega} (2\|\Phi\|_{1,\Omega} + \|\Phi - \Phi_h\|_{1,\Omega})
\]

\[
\leq CM\|z_h\|_{1,\Omega} \|\Phi\|_{1,\Omega} \quad \text{[by virtue of (4.50)]}
\]

\[
\implies \|z_h\|_{1,\Omega} \leq C\|\Phi\|_{1,\Omega}.
\] (4.57)

Hence from (4.55) and (4.57), and the definition of \(\Theta_h\), we have

\[
\|\varphi_h\|_{1,\Omega} = \sqrt{2}\|z_h\|_{1,\Omega} \leq C\|\Phi\|_{1,\Omega}.
\] (4.58)

\[
\implies \|\Theta_h - \Phi_h\|_{1,\Omega} \leq C\|\Phi\|_{1,\Omega}.
\] (4.59)

\[
\implies \|\Phi - \Theta_h\|_{1,\Omega} \leq \|\Phi - \Phi_h\|_{1,\Omega} + \|\Phi_h - \Theta_h\|_{1,\Omega}
\]

\[
\leq C\|\Phi\|_{1,\Omega} \leq C\|\Phi\|_{1,\Omega}.
\] (4.60)

- **Estimate for** \(\|\Phi - \Theta_h\|_{0,\Omega_h}\)

Since \(\Omega\) is convex, \(\forall g \in L^2(\Omega)\), define \(\chi \in H^2(\Omega) \cap H^1_0(\Omega)\) as the unique solution of:

\[
-\Delta \chi = g \text{ in } \Omega, \quad \chi|_\Gamma = 0 \quad \text{with } \|\chi\|_{2,\Omega} \leq C\|g\|_{0,\Omega}.
\] (4.61)
\( \forall z_h \in W_h \text{ with } \tilde{z}_h \in \tilde{W}_h, \text{ we have} \)

\[
\|z_h\|_{0, \Omega_h} = \|\tilde{z}_h\|_{0, \Omega} = \sup_{g \in L^2(\Omega)} \frac{\int_{\Omega} \tilde{z}_h g d\Omega}{\|g\|_{0, \Omega}}.
\]  

(4.62)

Then from (4.61),

\[
- \int_{\Omega} (\Delta \chi) \tilde{z}_h \ d\Omega = \int_{\Omega} g \tilde{z}_h \ d\Omega \quad \forall \tilde{z}_h \in \tilde{W}_h \subset H^1_0(\Omega)
\]

\[
\implies \int_{\Omega_h} (\nabla \chi) \cdot \nabla z_h \ d\Omega_h = \int_{\Omega_h} g z_h \ d\Omega_h \quad \forall z_h \in W_h, \text{ since } \tilde{z}_h = 0 \text{ in } \Omega - \Omega_h.
\]  

(4.63)

Hence, using (4.54), we get:

\[
\left| \int_{\Omega_h} g z_h \ d\Omega_h \right| = \left| \int_{\Omega_h} \nabla \chi \cdot \nabla z_h \ d\Omega_h \right| \leq \left| \int_{\Omega_h} \nabla (\chi - \chi_h) \cdot \nabla z_h \ d\Omega_h \right| + \left| \int_{\Omega_h} \nabla \chi_h \cdot \nabla z_h \ d\Omega_h \right|
\]

\[
- \sum_{T \in \tau_n} \sum_{n=1}^{N_2} w_{n,T}^2 (\nabla \chi_h \cdot \nabla z_h) (b_{n,T}^2) + |\hat{b}(\Phi, \chi_h) - b_{h}^{NI}(\Phi_h, \chi_h)|
\]

\( \forall \chi_h \in W_h \text{ and for fixed elements } \Phi \in V(\Omega_h), \Phi_h \in V_h \text{ satisfying (4.50).} \)

Then, for \( \chi_h = P_{0h} \chi \in W_h \) with \( \chi \in H^2(\Omega) \cap H^1_0(\Omega) \) defined in (4.4), we have:

\[
\int_{\Omega_h} \nabla (P_{0h} \chi) \cdot \nabla z_h \ d\Omega_h = \sum_{T \in \tau_n} \sum_{n=1}^{N_2} w_{n,T}^2 (\nabla P_{0h} \chi \cdot \nabla z_h) (b_{n,T}^2),
\]

and consequently,

\[
\left| \int_{\Omega_h} g z_h \ d\Omega_h \right| \leq \left| \int_{\Omega_h} \nabla (\chi - P_{0h} \chi) \cdot \nabla z_h \ d\Omega_h \right| + |\hat{b}(\Phi - \Phi_h, P_{0h} \chi - \chi)|
\]

\[
+ |\hat{b}(\Phi - \Phi_h, \chi) - b(\Phi - \Phi_h, \chi)| + b(\Phi - \Phi_h, \chi) |.
\]  

(4.64)

- **Estimate for the first term on the right-hand side of (4.64)**

Using (4.5) and (4.57),

\[
\left| \int_{\Omega_h} \nabla (\chi - P_{0h} \chi) \cdot \nabla z_h \ d\Omega_h \right| \leq |\chi - P_{0h} \chi|_{1, \Omega_h} |z_h|_{1, \Omega_h} \leq C_h \|g\|_{0, \Omega} \|z_h\|_{1, \Omega_h}
\]

\[
\leq C_h \|g\|_{0, \Omega} \|\Phi\|_{1, \Omega_h} \text{ [by (4.61)].}
\]  

(4.65)

- **Estimate for the second term on the right-hand side of (4.64)**

Using the continuity of \( \hat{b}(\cdot, \cdot) \), (4.5, 4.50) and (4.61), we have

\[
|\hat{b}(\Phi - \Phi_h, P_{0h} \chi - \chi)| \leq \tilde{m} \|\Phi - \Phi_h\|_{1, \Omega_h} \|P_{0h} \chi - \chi\|_{1, \Omega_h} \leq C_h \|\Phi\|_{1, \Omega_h} \|g\|_{0, \Omega}.
\]  

(4.66)
• Estimate for the third term on the right-hand side of (4.64)

\[
|\tilde{b}(\Phi - \Phi_h, \chi) - b(\Phi - \tilde{\Phi}_h, \chi)| = \left| \int_{\Omega_h} (\phi_{ij} - \phi_{hij}) \chi \, d\Omega_h - \int_{\Omega} (\phi_{ij} - \tilde{\phi}_{hij}) \chi \, d\Omega \right|
\leq \left| \int_{\omega_h} (\phi_{ij} - \tilde{\phi}_{hij}) \chi \, dx \right| \leq \| (\phi_{ij} - \tilde{\phi}_{hij}) \|_{0,\omega_h} \| \chi \|_{0,\omega_h}
\leq Ch\|\chi\|_{2,\Omega} \sum_{i,j=1}^2 \| (\phi_{ij} - \tilde{\phi}_{hij}) \|_{0,\omega_h} \quad \text{[using (4.21)]}.
\]  

(4.67)

For fixed \( i, j = 1, 2 \) \( \| (\phi_{ij} - \tilde{\phi}_{hij}) \|_{0,\omega_h} \leq \| \phi_{ij} - \tilde{\phi}_{hij} \|_{1,\omega_h} \leq \| \phi_{ij} \|_{1,\omega_h} + \| \tilde{\phi}_{hij} \|_{1,\omega_h}. \)

From (4.23), \( \forall i, j = 1, 2, \tilde{\phi}_{hij} \in \tilde{X}_h \) defined in (3.20),

\[
\| \tilde{\phi}_{hij} \|_{1,\omega_h} \leq Ch^{1/2}\| \phi_{hij} \|_{1,\Omega_h} \leq Ch^{1/2}\| \Phi_h \|_{1,\Omega_h}.
\]  

(4.68)

Then, using (4.50),

\[
\| \tilde{\phi}_{hij} \|_{1,\omega_h} \leq Ch^{1/2}(\| \Phi - \Phi_h \|_{1,\Omega_h} + \| \Phi \|_{1,\Omega_h}) \leq Ch^{1/2}(\| \Phi \|_{1,\Omega_h} + \| \Phi \|_{1,\Omega_h}) \leq Ch^{1/2}|\Phi|_{1,\Omega}.
\]  

(4.69)

From (4.67–4.69), we have

\[
\sum_{i,j=1}^2 \| (\phi_{ij} - \tilde{\phi}_{hij}) \|_{0,\omega_h} \leq \| \phi \|_{1,\Omega} + Ch^{1/2}|\Phi|_{1,\Omega} \leq C|\Phi|_{1,\Omega}.
\]  

(4.70)

Hence, from (4.67),

\[
|\tilde{b}(\Phi - \Phi_h, \chi) - b(\Phi - \tilde{\Phi}_h, \chi)| \leq Ch\|\chi\|_{2,\Omega} |\Phi|_{1,\Omega} \leq Ch\| g \|_{0,\Omega} |\Phi|_{1,\Omega}.
\]  

(4.71)

• Estimate for the fourth term on the right-hand side of (4.64)

For \( \chi \in H^2(\Omega) \cap H_0^\delta(\Omega), \)

\[
|b(\Phi - \tilde{\Phi}_h, \chi)| \leq C\| \Phi - \tilde{\Phi}_h \|_{0,\Omega} |\chi|_{2,\Omega}
\]  

(4.72)

with

\[
\| \Phi - \tilde{\Phi}_h \|_{0,\Omega}^2 = \| \Phi - \Phi_h \|_{0,\Omega_h}^2 + \| \Phi - \Phi_h \|_{0,\omega_h}^2 \quad (\omega_h = \Omega - \Omega_h).
\]  

(4.73)
For fixed $i, j = 1, 2$, $(\phi_{ij} - \tilde{\phi}_{hij}) \in H^1(\Omega)$ and hence from (4.21),
\[
\|\phi_{ij} - \tilde{\phi}_{hij}\|_{0,\omega_h} \leq C^2 h^2 \|\phi_{ij} - \tilde{\phi}_{hij}\|_{1,\Omega}^2. \tag{4.74}
\]

But
\[
\|\phi_{ij} - \tilde{\phi}_{hij}\|_{1,\Omega}^2 = \|\phi_{ij} - \phi_{hij}\|_{1,\Omega}^2 + \|\phi_{ij} - \tilde{\phi}_{hij}\|_{1,\omega_h}^2 \\
\leq C\|\phi_{ij}\|_{1,\Omega}^2 + (\|\phi_{ij}\|_{1,\omega_h} + \|\phi_{hij}\|_{1,\omega_h})^2 \leq C\left[\|\phi_{ij}\|_{1,\Omega}^2 + \|\phi_{ij}\|_{1,\omega_h}^2 + \|\phi_{hij}\|_{1,\omega_h}^2\right] \\
\Rightarrow \|\phi_{ij} - \tilde{\phi}_{hij}\|_{1,\Omega}^2 \leq C\left[\|\phi_{ij}\|_{1,\Omega}^2 + Ch\|\bar{\Phi}\|_{1,\Omega}^2\right] \quad \text{[using (4.23)]}
\Rightarrow \sum_{i,j=1}^{2} \|\phi_{ij} - \tilde{\phi}_{hij}\|_{0,\omega_h} \leq C\|\bar{\Phi}\|_{1,\Omega} \Rightarrow \|\bar{\Phi} - \tilde{\Phi}_{h}\|_{1,\Omega} \leq C\|\bar{\Phi}\|_{1,\Omega}.

Hence
\[
\|\Phi - \tilde{\Phi}_{h}\|_{0,\omega_h} \leq Ch\|\bar{\Phi}\|_{1,\Omega} \quad \text{[from (4.74)].}
\]
Then from (4.73)
\[
\|\Phi - \tilde{\Phi}_{h}\|_{0,\omega}^2 \leq Ch^2\|\bar{\Phi}\|_{1,\Omega}^2 + C^2 h^2 \|\bar{\Phi}\|_{1,\Omega}^2 \leq C^2 h^2 \|\bar{\Phi}\|_{1,\Omega}^2.
\]

Hence,
\[
|b(\Phi - \tilde{\Phi}_{h}, \chi)| \leq Ch\|\bar{\Phi}\|_{1,\Omega}\|\chi\|_{2,\Omega} \leq Ch\|g\|_{0,\omega}\|\bar{\Phi}\|_{1,\Omega} \quad \text{[using (4.61)].} \tag{4.75}
\]
Substituting the estimates (4.65–4.66, 4.71, 4.75) in (4.64) and using the result \(\|\Phi\|_{1,\omega_h} \leq \|\Phi\|_{1,\Omega}\), we have
\[
\left|\int_{\Omega} g\tilde{z}_h \, d\Omega\right| - \left|\int_{\Omega} g z_h \, d\Omega\right| \leq Ch\|g\|_{0,\omega}(\|\bar{\Phi}\|_{1,\Omega} + \|\bar{\Phi}\|_{1,\Omega} + \|\bar{\Phi}\|_{1,\Omega}) \\
\leq C h\|g\|_{0,\omega}\|\bar{\Phi}\|_{1,\Omega}. \tag{4.76}
\]
Then, from (4.62, 4.76), we have
\[
\|z_h\|_{0,\omega_h} = \|\tilde{z}_h\|_{0,\Omega} = \sup_{g \in L^2(\Omega)} \frac{\left|\int_{\Omega} \tilde{z}_h g \, d\Omega\right|}{\|g\|_{0,\Omega}} \leq C_h\|\bar{\Phi}\|_{1,\Omega}.
\]
Since, \(\varphi_h = (z_h, \delta_{ij})\) with \(z_h \in W_h\) and \(\varphi_h = \varphi_h + \Phi_h \in V_h\), [see (4.56)], we have
\[
\|\varphi_h - \tilde{\Phi}_{h}\|_{0,\omega_h} = \|\varphi_h\|_{0,\omega_h} = \sqrt{2}\|z_h\|_{0,\omega_h} \leq Ch\|\bar{\Phi}\|_{1,\Omega}. \tag{4.77}
\]
Hence
\[
\|\Phi - \varphi_h\|_{0,\omega_h} \leq \|\Phi - \Phi_h\|_{0,\omega_h} + \|\varphi_h - \varphi_h\|_{0,\omega_h} \\
\leq Ch\|\Phi\|_{1,\Omega} + Ch\|\Phi\|_{1,\Omega} \quad \text{[from (4.50) and (4.77)]} \\
\Rightarrow \|\Phi - \varphi_h\|_{0,\omega_h} \leq Ch\|\bar{\Phi}\|_{1,\Omega}. \tag{4.78}
\]
Thus, (4.60) and (4.78) establish the result (4.48).

**Theorem 4.1.**

- Suppose that the assumptions of Theorem 2.1, Propositions 4.3 and 4.4 hold.

- Let \( \{ \tau_h \} \) (resp. \( \{ \tau_h^{\text{exact}} \} \)) be a family of quasi-uniform, regular, admissible triangulations [17] of \( \Omega_h = \Omega_h \cup \Gamma_h \) (resp. \( \Omega = \Omega \cup \Gamma \)) defined in (3.3) with \( 0 < h < h_0, \ h_0 \in ]0,1[ \).

- Let the quadrature scheme (3.11) with \( i = 1 \) (resp. \( i = 2 \)), which is exact for \( P_4(\hat{T}) \) (resp. \( P_5(\hat{T}) \)) correspond to the definition (3.23) of \( A_h^{NI}(\cdot,\cdot) \) (resp. (3.24) of \( b_h^{NI}(\cdot,\cdot) \)).

Then, \( \exists C > 0 \), independent of \( h \), such that

\[
||\Psi - \Psi_h||_{0,\Omega_h} \leq C\sqrt{h} \left[ ||u||_{3,\Omega} + h^{1/2}||\Psi||_{1,\Omega} \right]; \tag{4.79}
\]

\[
||u - u_h||_{1,\Omega_h} \leq C\sqrt{h} \left[ ||u||_{3,\Omega} + h^{1/2}||\Psi||_{1,\Omega} \right], \tag{4.80}
\]

where \((\Psi, u) \in V \times W \) [resp. \((\Psi_h, u_h) \in V_h \times W_h \)] is the unique solution of \((Q) \) [resp. \((Q_h) \)].

**Proof.** Since \( \Psi \downarrow \Omega_h \in V(\Omega_h) \), from Lemma 4.2, \( \exists \Theta_h \in V_h \) such that

\[
b_h^{NI}(\Theta_h, \chi_h) = \tilde{b}(\Psi, \chi_h) = \int_{\Omega_h} \psi_{i,j} \chi_{h,i} \ d\Omega_h \ \forall \chi_h \in W_h \tag{4.81}
\]

and

\[
||\Psi - \Theta_h||_{r,\Omega_h} \leq C h^{1-r} ||\Psi||_{1,\Omega} \ (r = 0,1). \tag{4.82}
\]

Then, from (4.81), the definition of \( \tilde{W}_h \), and the second equation of (2.10),

\[
b_h^{NI}(\Theta_h, \chi_h) = \int_{\Omega} \psi_{i,j} \tilde{\chi}_{h,i} \ d\Omega = \tilde{b}(\Psi, \tilde{\chi}_h) = -(f, \tilde{\chi}_h)_{0,\Omega} = -(f, \chi_h)_{0,\Omega_h} \ \forall \chi_h \in W_h \text{ with } \tilde{\chi}_h \in \tilde{W}_h. \tag{4.83}
\]

Hence using the second equation of (3.25) and (4.83), we have

\[
b_h^{NI}(\Psi_h - \Theta_h, \chi_h) = -(f, \chi_h)_{0,\Omega_h} + (f, \chi_h)_{0,\Omega_h} = 0 \ \forall \chi_h \in W_h. \tag{4.84}
\]
From the ellipticity of $A_h^{NI}(\cdot, \cdot)$ in (3.27), we have for $\Theta_h \in \mathbf{V}_h$ corresponding to $\Psi \in \mathbf{V}(\Omega_h)$ satisfying (4.81-4.83),

$$
\alpha \|\Psi_h - \Theta_h\|_{0,\Omega_h} \leq A_h^{NI}(\Psi_h - \Theta_h, \Psi_h - \Theta_h) = \left[ A(\Psi - \Theta_h, \Psi_h - \Theta_h) - A(\Psi, \Psi_h - \Theta_h) \right] \\
+ \left[ A(\Theta_h, \Psi_h - \Theta_h) - A_h^{NI}(\Theta_h, \Psi_h - \Theta_h) \right] - b_h^{NI}(\Psi_h - \Theta_h, u_h) \quad \text{[using the first equation of (3.25)]}
$$

which has been obtained by using (4.84) and the definition (4.4) of $\mathbf{V}_h$. Since the quadrature scheme (3.11) with $i = 2$ corresponding to the definition of $b_h^{NI}(\cdot, \cdot)$ is exact for $P_2(\bar{T})$,

$$
\delta(\Psi_h - \Theta_h, \mathbf{P}_0 u_h) - b_h^{NI}(\Psi_h - \Theta_h, \mathbf{P}_0 u_h) = 0.
$$

Hence, applying the triangular inequality, the continuity of $A(\cdot, \cdot)$ and $\delta(\cdot, \cdot)$ and finally dividing both sides by $\alpha \|\Psi_h - \Theta_h\|_{0,\Omega_h}$, we get from (4.85-4.86):

$$
\left\| \Psi_h - \Theta_h \right\|_{0,\Omega_h} \leq C \left\{ \left[ \frac{\bar{\delta}(\Psi - \Theta_h, \Psi_h - \Theta_h)}{\|\Psi_h - \Theta_h\|_{0,\Omega_h}} + \frac{\bar{\delta}(\Psi_h - \Theta_h, u_h)}{\|\Psi_h - \Theta_h\|_{0,\Omega_h}} \right] + \left[ \frac{\|\Psi_h - \Theta_h\|_{1,\Omega_h}}{\|\Psi_h - \Theta_h\|_{0,\Omega_h}} \right] \right\}.
$$

Using $\|u - \mathbf{P}_0 u\|_{1,\Omega_h} \leq C h^3/2 \|u\|_{3,\Omega}$ for $u \in H^3(\Omega) \cap H^3_0(\Omega)$, [from (4.6)],

$$
\|\Psi_h - \Theta_h\|_{1,\Omega_h} \leq C \frac{h^3}{n} \|\Psi_h - \Theta_h\|_{0,\Omega_h}, \quad \text{[from (4.20)]},
$$

and [from Propositions 4.3 and 4.4],

$$
|\bar{A}(\Theta_h, \Psi_h - \Theta_h) - A_h^{NI}(\Theta_h, \Psi_h - \Theta_h)| \leq C h \|A\|_{1,\infty,\Omega_h} \|\Theta_h\|_{0,\Omega_h} \|\Psi_h - \Theta_h\|_{0,\Omega_h}
$$

and

$$
|\bar{A}(\Psi_h - \Theta_h) + \delta(\Psi_h - \Theta_h, u)| \leq C h^3 (1 + \sqrt{h}) \|u\|_{3,\Omega} \|\Psi_h - \Theta_h\|_{1,\Omega_h}.
$$
we have
\[
\| \Psi_h - \Theta_h \|_{0, \Omega_h} \leq C \left[ h \| \Psi \|_{1, \Omega} + \sqrt{h} \| u \|_{3, \Omega} + \sqrt{h} (1 + \sqrt{h}) \| u \|_{3, \Omega} + h \| A \|_{1, \infty, \Omega} \| \Theta \|_{0, \Omega_h} \right]
\]
(4.91)

with \( C > 0 \), independent of \( h \).

Since \( \| \Theta_h \|_{0, \Omega_h} \leq \| \Psi - \Theta_h \|_{0, \Omega_h} + \| \Psi \|_{0, \Omega_h} \leq C h \| \Psi \|_{1, \Omega} + \| \Psi \|_{1, \Omega} \leq C \| \Psi \|_{1, \Omega} \), we get from (4.91):
\[
\| \Psi_h - \Theta_h \|_{0, \Omega_h} \leq C \left[ h^{1/2} (2 + \sqrt{h}) \| u \|_{3, \Omega} + h \| \Psi \|_{1, \Omega} \right] \leq Ch^{1/2} \left[ \| u \|_{3, \Omega} + h^{1/2} \| \Psi \|_{1, \Omega} \right].
\]

Hence
\[
\| \Psi - \Psi_h \|_{0, \Omega_h} \leq \| \Psi - \Theta_h \|_{0, \Omega_h} + \| \Psi_h - \Theta_h \|_{0, \Omega_h}
\leq C \left[ h \| \Psi \|_{1, \Omega} + h^{1/2} (\| u \|_{3, \Omega} + h^{1/2} \| \Psi \|_{1, \Omega}) \right]
\]
\[
\Rightarrow \| \Psi - \Psi_h \|_{0, \Omega_h} \leq Ch^{1/2} \left[ \| u \|_{3, \Omega} + h^{1/2} \| \Psi \|_{1, \Omega} \right] \text{ with } C > 0, \text{ independent of } h.
\]
(4.92)

Now, we will prove (4.80).

• **Estimate for** \( \| u - u_h \|_{1, \Omega_h} \)

From the discrete Brezzi-Babuska condition (3.28) for \( b_h^{NI}(\cdot, \cdot) \), we have
\[
\beta_1 \| u_h - P_{0h} u \|_{1, \Omega_h} \leq \sup_{\Phi_h \in \mathbf{V}_h \setminus \{0\}} \frac{|b_h^{NI}(\Phi_h, u_h - P_{0h} u)|}{\| \Phi_h \|_{1, \Omega_h}}. \]
(4.93)

But
\[
b_h^{NI}(\Phi_h, u_h - P_{0h} u) = \tilde{b}(\Phi_h, u - P_{0h} u) + b_h^{NI}(\Phi_h, u_h) - \tilde{b}(\Phi_h, u) + \left[ \tilde{b}(\Phi_h, P_{0h} u) - b_h^{NI}(\Phi_h, P_{0h} u) \right]
+ \tilde{A}(\Psi, \Phi_h) - \tilde{A}(\Psi, \Phi_h),
\]
where
\[
\tilde{b}(\Phi_h, P_{0h} u) - b_h^{NI}(\Phi_h, P_{0h} u) = 0 \quad \forall \Phi_h \in \mathbf{V}_h
\]
\[
\Rightarrow b_h^{NI}(\Phi_h, u_h - P_{0h} u) = \tilde{b}(\Phi_h, u - P_{0h} u) - A_h^{NI}(\Psi_h, \Phi_h) - \left[ \tilde{A}(\Psi, \Phi_h) + \tilde{b}(\Phi_h, u) \right]
+ \tilde{A}(\Psi - \Psi_h, \Phi_h) + \tilde{A}(\Psi_h, \Phi_h) \quad [\text{using (3.25)}]
\]
\[
\Rightarrow b_h^{NI}(\Phi_h, u_h - P_{0h} u) = \tilde{A}(\Psi - \Psi_h, \Phi_h) + \tilde{b}(\Phi_h, u - P_{0h} u) + \left[ \tilde{A}(\Psi_h, \Phi_h) - A_h^{NI}(\Psi_h, \Phi_h) \right]
- \left[ \tilde{A}(\Psi, \Phi_h) + \tilde{b}(\Phi_h, u) \right].
\]
(4.94)
Applying the triangular inequality and the continuity of the bilinear forms \( \tilde{A}(\cdot, \cdot) \) and \( \tilde{b}(\cdot, \cdot) \) in (4.94), we get from (4.93):

\[
\|u_h - P_{0h}u\|_{1,\Omega_h} \leq \frac{1}{\beta} \left\{ e_{H} \left[ \| \tilde{A}(\mu - \phi, \phi)_{0,\Omega_h} + \tilde{b}(\mu, u) - P_{0h}u\|_{1,\Omega_h} \right] + \left[ \sup_{\phi_h \in V_h - \{0\}} \frac{\| \tilde{A}(\phi_h, \phi_h) - \tilde{A}^N(\phi_h, \phi_h) \|}{\| \phi_h \|_{1,\Omega_h}} \right] \right\}.
\]  

(4.95)

Then, applying (4.92) and Propositions 4.3 and 4.4, we have

\[
\|u_h - P_{0h}u\|_{1,\Omega_h} \leq C \left[ h^{1/2}(\|u\|_{3,\Omega} + \sqrt{h}\|\Psi\|_{1,\Omega}) + h^{3/2}\|u\|_{3,\Omega} + h\|A\|_{1,\infty,\Omega}\|\Psi\|_{0,\Omega} + A_h^{3/2}(1 + \sqrt{h})\|u\|_{3,\Omega} \right].
\]  

(4.96)

But

\[
\|\Psi_h\|_{0,\Omega_h} \leq \|\mu - \phi\|_{0,\Omega_h} + \|\mu\|_{0,\Omega_h} \leq C h^{1/2}(\|u\|_{3,\Omega} + \sqrt{h}\|\Psi\|_{1,\Omega}) + \|\mu\|_{0,\Omega}\]  

(4.97)

From (4.96) and (4.97),

\[
\|u_h - P_{0h}u\|_{1,\Omega_h} \leq C \left[ h^{1/2}(\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega}) + (h^{3/2} + h + h\sqrt{h}(1 + \sqrt{h}))\|u\|_{3,\Omega} + h\|\Psi\|_{1,\Omega} \right]
\]  

\[
\leq C h^{1/2} \left[ \|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega} \right].
\]  

(4.98)

Then,

\[
\|u - u_h\|_{1,\Omega_h} \leq \|u - P_{0h}u\|_{1,\Omega_h} + \|P_{0h}u - u_h\|_{1,\Omega_h} \leq C(h^{3/2}\|u\|_{3,\Omega} + h^{1/2}\|u\|_{3,\Omega} + h\|\Psi\|_{1,\Omega})
\]

\[
\implies \|u - u_h\|_{1,\Omega_h} \leq C h^{1/2}(\|u\|_{3,\Omega} + h^{1/2}\|\Psi\|_{1,\Omega})
\]  

(4.99)

with \( C > 0 \) independent of \( h \).

** Remark 4.2.** Error estimate (4.79) [resp. (4.80)] depends on the estimates of the three terms occurring due to the errors involved with

(i) interpolation;

(ii) approximation of the curved boundary \( \Gamma \) by the polygon \( \Gamma_h \);

(iii) non-exact integration

i.e. the terms in the second and third square brackets on the right-hand side of (4.87) (resp. (4.95)) correspond to (ii) and (iii) respectively, and the terms in the first square bracket correspond to (i) and also indirectly to (ii) [see (4.6)]. Hence, it will be interesting to study the two particular cases:

Case 1: there is no approximation of boundary, in other words \( \Gamma \) is a polygon, but numerical integration is performed, i.e. error due to (iii) is present, but an error due to (ii) is absent;

Case 2: polygonal boundary approximation is made, but no numerical integration is necessary and hence, it is not performed i.e. error due to (ii) is present, but an error due to (iii) is absent.
Case 1. \( \Gamma \) is a (straight) polygonal boundary of the convex polygonal domain \( \Omega \) which is considered in all papers \([4,5,15,28,33]\) etc., i.e.

\[
\Gamma = \Gamma_h, \, \Omega = \Omega_h, \, \bar{\Omega} = \bar{\Omega}_h = \bigcup_{T \in \tau_h} T \quad \forall h > 0
\]  

(4.100)

\( \implies \) error due to (ii) is absent. Moreover, using higher order elements i.e. \( P_m \)-elements with \( m > 2 \), to construct finite element spaces, a remarkable improvement in the error estimates, i.e. \( \| \Psi - \Psi_h \|_{0, \Omega} = O(h^{m-1}) \), \( \| u - u_h \|_{1, \Omega} = O(h^{m-1}) \), \( m \geq 2 \) can be obtained under some additional assumptions on the regularity of solution and the use of quadrature schemes with higher degree of accuracy. In fact, (4.100) holds, and \( P_m \)-elements with \( m \geq 2 \) can be used to define \( X_h \subset H^1(\Omega) \), \( V_h \subset V \), \( W_h \subset H^1_0(\Omega) \), i.e.

\[
\begin{align*}
X_h &= \{ \chi_h : \chi_h \in C^0(\Omega), \chi_h \downarrow_T \in P_m(T) \quad \forall T \in \tau_h \} \subset H^1(\Omega), \\
V_h &= \{ \Phi_h : \Phi_h = (\phi_{hij})_{1 \leq i,j \leq 2} \text{ with } \phi_{hij} = \phi_{hji} \in X_h \} \subset V, \\
W_h &= \{ \chi_h : \chi_h \in X_h, \chi_h \downarrow_T = 0 \} \subset H^1_0(\Omega).
\end{align*}
\]  

(4.101)

Then, we use quadrature schemes (3.11) with higher degrees of accuracy:

(A3) \( A_h^{NI}(\cdot, \cdot) \) (resp. \( b_h^{NI}(\cdot, \cdot) \)) defined by (3.23) [resp. (3.24)] corresponds to the quadrature scheme (3.11) with \( i = 1 \) [resp. \( i = 2 \)] which is exact for \( P_{3m-2}(\bar{T}) \) [resp. \( P_{2m-2}(\bar{T}) \)].

Following the steps of the proofs of (3.27) and (3.28), we have: \( \exists \alpha_0 > 0 \), independent of \( h \) such that

\[
A_h^{NI}(\Phi_h, \Phi_h) \geq \alpha_0 \| \Phi_h \|^2_{0, \Omega} \quad \forall \Phi_h \in V_h \subset V;
\]  

(4.102)

\( \exists \beta_1 > 0 \), independent of \( h \) such that

\[
\sup_{\Phi_h \in V_h - \{0\}} \frac{|b_h^{NI}(\Phi_h, \chi_h)|}{\| \Phi_h \|_V} \geq \beta_1 \| \chi_h \|_{1, \Omega} \quad \forall \chi_h \in W_h \subset H^1_0(\Omega)
\]  

(4.103)

and the corresponding \( (Q_h) \) has a unique solution \( (\Psi_h, u_h) \in V_h \times W_h \).

Moreover,

\[
(4.100) \implies \tilde{A}(\cdot, \cdot) = A(\cdot, \cdot), \quad \tilde{b}(\cdot, \cdot) = b(\cdot, \cdot)
\]  

(4.104)

and

\[
|\tilde{A}(\Psi, \Phi_h) + \tilde{b}(\Phi_h, u)| = |A(\Psi, \Phi_h) + b(\Phi_h, u)| = 0 \quad \forall \Phi_h \in V_h \subset V
\]  

(4.105)

in Proposition 4.4 by virtue of the first equation (2.10);

\[
|b(\Phi_h, \chi_h) - b_h^{NI}(\Phi_h, \chi_h)| = 0 \quad \forall \Phi_h \in V_h, \forall \chi_h \in W_h
\]  

(4.106)

with \( V_h \) and \( W_h \) defined by (4.101), since the quadrature scheme used in \( b_h^{NI}(\cdot, \cdot) \) is exact for \( P_{2m-2}(\bar{T}) \), \( m \geq 2 \). Proposition 4.3 is replaced by the following result, whose proof is analogous.

Under (A3), for \( A_{ijkl} \in W^{m-1, \infty}(\Omega) \) with \( m \geq 2 \), \( \forall i, j, k, l = 1, 2 \), we have

\[
|A(\Phi_h, \sigma_h) - A_h^{NI}(\Phi_h, \sigma_h)| \leq C h^{m-1} \| \Phi_h \|_{0, \Omega} \| \sigma_h \|_{0, \Omega} \quad \forall \Phi_h, \sigma_h \in V_h \subset V.
\]  

(4.107)

By virtue of (4.100), the interpolation operators \( P_h \) and \( P_{0h} \) defined in (4.1) and (4.4) respectively are now identical, i.e.

\[
P_h \chi = P_{0h} \chi \in W_h \subset H^1_0(\Omega) \quad \forall \chi \in H^s(\Omega) \cap H^1_0(\Omega) \quad \text{with} \ s \geq 2,
\]  

(4.108)
and estimates (4.5) and (4.6) are replaced by the classical estimates:

\[ \forall \chi \in H^s(\Omega) \cap H_0^1(\Omega) \text{ with } s \geq 2, \quad \| \chi - \mathcal{P}_{oh} \chi \|_{r, \Omega} \leq C h^{s-r} \| \chi \|_{s, \Omega} \quad (s \geq 2; r = 0, 1). \]  

(4.109)

Lemma 4.2 holds with \( b_h^{NI}(\cdot, \cdot) \) corresponding to the quadrature scheme (3.11) with \( i = 2 \), which is exact for \( P_{2m-2}(T) \). Then, \( \forall \Phi = (\phi_{ij})_{1 \leq i, j \leq 2} \) with \( \phi_{ij} = \phi_{ji} \in H^{m-1}(\Omega), \ m \geq 2 \), \( \exists \Theta_h \in V_h \subset V \) satisfying

\[ b(\Phi, \chi_h) = b_h^{NI}(\Theta_h, \chi_h) \quad \forall \chi_h \in W_h \subset H_0^1(\Omega) \quad \text{such that} \]

\[ \| \Phi - \Theta_h \|_{r, \Omega} \leq C h^{m-r-1} \| \Phi \|_{m-1, \Omega} \quad (r = 0, 1). \]  

(4.110)

Now, following the steps of the proof of Theorem 4.1, using assumption (A3) and (4.100–4.110), assuming that \( u \in H^{m+1}(\Omega) \cap H_0^2(\Omega) \) with \( m \geq 2 \) is the unique solution of \((P_G)\) in (2.3–2.4) and \((\Psi, u)\) is the unique solution of \((Q)\) with \( \Psi = (\psi_{ij})_{1 \leq i, j \leq 2}, \ \psi_{ij} = \psi_{ji} \in H^{m-1}(\Omega) \quad \forall i, j = 1, 2, \)

\[ \| \Psi - \Psi_h \|_{0, \Omega} \leq C h^{m-1} \left[ \| u \|_{m+1, \Omega} + \| \Psi \|_{m-1, \Omega} \right], \]  

(4.111)

\[ \| u - u_h \|_{1, \Omega} \leq C h^{m-1} \left[ \| u \|_{m+1, \Omega} + \| \Psi \|_{m-1, \Omega} \right]. \]  

(4.112)

Estimates (4.111) and (4.112) are of the same order \( O(h^{m-1}), m \geq 2 \) as obtained in [5] (resp. for \( H-H-M \) mixed scheme for the biharmonic problem (2.20–2.21) by Brezzi-Raviart in [15], pages 16–17) under the same regularity assumptions, when errors due to (ii) and (iii) are absent, i.e. when \( \Gamma \) is a polygon and exact integration is performed.

For \( m = 1 \), neither the estimates of [5] nor those of Brezzi-Raviart in [15] hold (see Remark 2 of [15], page 20), but Miyoshi obtained estimates of order \( O(h^{1/2}) \) for \( m = 1 \) in [28], in which the elegant, systematic mixed method analysis of Babuška-Brezzi-Raviart has not been followed (see also [34])!

Hence, based on Babuška-Brezzi-Raviart mixed method analysis, best available error estimates for this problem using \( P_2 \)-elements are of order \( O(h) \) [5], when errors due to (ii) and (iii) are absent. Moreover, when quadrature schemes with higher degrees of accuracy \( P_{3m-2}(T) \) for \( A_h^{NI}(\cdot, \cdot) \) and \( P_{2m-2}(T) \) for \( b_h^{NI}(\cdot, \cdot) \) (\( m \geq 2 \)) are used, the error due to only (iii) is of the order \( O(h^{m-1}) \) [see (4.107)].

Case 2. Curved boundary \( \Gamma \) is approximated by a polygon \( \Gamma_h \) as in (3.1–3.6), but exact integration is possible and performed i.e. only error due to (ii) is present. Since exact integration is performed, \( \tilde{A}(\cdot, \cdot) = A_h^{NI}(\cdot, \cdot), \ \tilde{b}(\cdot, \cdot) = b_h^{NI}(\cdot, \cdot) \) and the term in the third square bracket on the right-hand side of (4.87) [resp. (4.95)] vanishes.

For polygonal approximation \( \Gamma_h \) to \( \Gamma \), we have \( \text{meas}(\omega_h) = O(h^2) \) with \( \omega_h = \Omega - \Omega_h \). Consequently, from (4.42, 4.44, 4.45) in the proof of Proposition 4.4,

\[ |\tilde{A}(\Psi, \Psi_h - \Phi_h) + \tilde{b}(\Psi_h - \Phi_h, u)| = O(h^{3/2}), \]  

(4.113)

even if \( P_m \)-elements with \( m \geq 2 \) are used to define finite element spaces \( V_h \) and \( W_h \) [see (4.101)]. In other words, by using \( P_m \)-elements with \( m > 2 \), the estimate (4.113) can not be improved unless better approximation of \( \Gamma \) is made. Incidentally, this is exactly the reason for using \( P_2 \)-elements in the definition of \( V_h \) and \( W_h \) in (3.18) and (3.19) respectively.

Again, from Remark 4.1, we find that for

\[ u \in H^{m+1}(\Omega) \cap H_0^2(\Omega) \subset H^{m+1}(\Omega) \cap H_0^1(\Omega), \quad \| u - \mathcal{P}_{oh} u \|_{1, \Omega_h} = O(h^{3/2}) \quad \forall m \geq 2, \]  

(4.114)
which can not be improved upon unless boundary approximation is improved. Hence, for polygonal boundary approximation, we find from (4.113) and (4.114) that the estimates of order $O(h^{3/2})$ can not be improved upon by any choice of $m > 2$

(i) in the definition of $V_h$ and $W_h$ in (4.101) and

(ii) in the regularity of solution $u \in H^{m+1}(\Omega) \cap H_0^2(\Omega)$ of $(P_G)$, i.e. the optimal case is $m = 2$.

Finally, the use of the inverse inequality (4.88) in (4.91) is necessary (see [5,14,15,18]) and gives the estimate:

$$\|\Psi - \Psi_h\|_{0,\Omega_h} = O(h^{1/2}),$$

which is used to get the estimate:

$$\|u - u_h\|_{1,\Omega_h} = O(h^{1/2}).$$

(4.116)

Thus, for this crude but most important and commonly used polygonal approximation $\Gamma_h$ to $\Gamma$, there is a loss in the exponent of $h$ by only ‘$1/2$’ in the estimates (4.115–4.116), the best available estimates [5], [15] based on Babuška-Brezzi-Raviart mixed method analysis being $\|\Psi - \Psi_h\|_{0,\Omega} = O(h)$, $\|u - u_h\|_{1,\Omega} = O(h)$ for $m = 2$, when there is neither boundary approximation nor non-exact integration (see also Case 1 above for $m = 2$). In fact, in [9], the estimates $\|\Psi - \Psi_h\|_{0,\Omega} = O(h)$, $\|u - u_h\|_{1,\Omega} = O(h)$ have been obtained when $\Gamma$ has been approximated by a curved boundary $\Gamma_h$ constructed with the help of isoparametric mapping, for which $\Omega_h \not\subset \bar{\Omega}$, $\bar{\Omega} \not\subset \Omega_h$ and $\Omega_h$ is no longer convex in general. Consequently, a completely different, independent analysis has been developed in [9].

Hence, it is obvious from the facts explained above that for polygonal approximation $\Gamma_h$, the estimates $\|\Psi - \Psi_h\|_{0,\Omega_h} = O(h^{1/2})$ and $\|u - u_h\|_{1,\Omega_h} = O(h^{1/2})$ are the ‘best’ ones based on Babuška-Brezzi-Raviart mixed method analysis for fourth order problems.

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REFERENCES