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ASYMPTOTIC ANALYSIS OF MAGNETIC INDUCTION WITH HIGH FREQUENCY FOR SOLID CONDUCTORS (*)

Olivier COULAUD ([1])

Abstract. — *In this paper we describe the behaviour both in time and in space of an induction field created by an imposed high frequency alternating current around a solid conductor. To do this, we introduce two time scales and we decompose the induction field in a mean field and an oscillating field. With the help of singular perturbations theory and multiple scales method we obtain two uncoupled models; one for the large time scale and the other for the high frequencies. For a cross section of a solid column of metal, we build the two first terms of the asymptotic expansion of the induction field. Moreover, we justify the classical harmonic approximation used in such configuration. Finally in the case of a cylinder column, we apply the previous results by computing numerically the induction for different values of the frequency. © Elsevier, Paris*

Key words : multiple time scales, singular perturbation, Maxwell's equations, high frequency, eddy current, boundary layer, time average.

Résumé. — *Dans ce papier nous décrivons le comportement en temps et en espace du champ d'induction magnétique créé par un courant alternatif à haute fréquence imposé autour d'un conducteur solide. Pour cela, nous introduisons deux échelles de temps et nous décomposons le champ d'induction en un champ moyen et un champ oscillant. À l'aide de la théorie des perturbations singulières et des méthodes d'échelles multiples nous obtenons deux modèles découplés, un pour les grandes échelles de temps et l'autre pour les hautres fréquences. Dans le cas d'une coupe d'une colonne de métal, nous construisons les deux premiers termes du développement asymptotique du champ d'induction. De plus, pour un courant harmonique mono-fréquence nous justifions l'approximation harmonique pour les temps grands. Finalement, nous appliquons les résultats obtenus dans le cas d'une colonne cylindrique, en calculant numériquement le champ d'induction pour différentes fréquences. © Elsevier, Paris*

Contents

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1. INTRODUCTION

Nowadays a lot of industrial processes based on magnetic induction are used in industry and in laboratories (see [12]) The main feature of magnetic induction is that it can act at distance So Thermal or mechanical energy is brought to material without any contact When an alternating current flows in the inductors, it generates induced current in conductors The induced current in liquids induces a magnetic field which creates Lorentz forces These forces may be used to confine the metal or to maintain it in levitation Furthermore, these forces create a motion which may be turbulent Applications concerned by the mechanical effects generated by Lorentz Forces are for example electromagnetic stirring, electromagnetic continuous casting for aluminium [1], guiding or confining jets of liquid metals [13], electromagnetic levitation [10], [14], [9], and so on In all these processes, the frequency of the imposed alternating current in the inductors is an important parameter In fact, with this parameter we can adjust the depth of the penetration of the magnetic field inside the conductors and by this the layer where we give thermal or mechanical energy

Here we are interesting in frequency at magnitude 10^4 to 10^5 Hz At such frequencies the magnetic field is concentrated in a thin layer near the surface of the conductor This phenomena is the so called "skin-effect", this means that we have a boundary layer near the surface of the conductor The thickness of this layer is proportional to the inverse of the square root of the frequency ω of the imposed alternating current For very high frequencies like 10^4 Hz or more, due to the fact that the magnetic field does not penetrate into the liquid, we can consider that the eddy currents are only on the surface

Because we consider high frequency current we have a small time scale which is given by oscillation of the imposed current Now the magnetic field penetrate by diffusion inside the conductor The diffusion characteristic time is independent of the frequency, it depends only on the feature of the material Therefore this is a slow time scale compared with the previous one Then, our problem has two time scales

Our goal is for high frequency to describe the most important terms To do it, we will use asymptotic expansion with respect to the inverse of the frequency This requires to consider two time scales

In this paper we consider only solid conductors (see [4] for liquid conductors) to understand the coupling in multiple time scales methods and singular perturbation methods Moreover to construct the expansion at the second order in the boundary layer we will restrict for simplicity our calculus to a bidimensional geometry

With the knowledge of the behaviour in space of the induction we will be able to build efficient scheme to catch the boundary layer effect The idea is to use informations obtained by singular perturbation analysis in the scheme of domain decomposition methods (see [7]) This aspect will not be dealt in this paper

In the first section we give the natural hypotheses for those applications to obtain Maxwell's equations rewritten only with the magnetic induction field (b-formulation) In their non dimensional forms Maxwell equations in the conductor have a small parameter denoted by ϵ In section 2, we introduce two time scales in the system A small one given by the frequency of the imposed current density called the magnetic time scale and a large one define by $\tau = \epsilon t$ Moreover we introduce the decomposition of the induction in mean value field and a fluctuating one By a singular perturbation method the first order term of the induction field is obtained Then in the third section we apply the previous result in the case of guiding jet Firstly for a harmonic current and a general cross section of the column we construct first and second order term of the induction and the induced current Finally for a cylinder cross section some numerical results are given

2 MODELIZATION AND EQUATIONS

In this section we give the hypotheses and we establish the nondimensional set of equations

We denote by K a compact set in $\mathbb{R}^3$ (corresponding to the inductors), Ω an open set in $\mathbb{R}^3$ with compact closure $\bar{\Omega}$ (corresponding to the solid conductors) We assume that K and $\bar{\Omega}$ do not intersect We denote by Ω_0 the complement of $\bar{\Omega} \cup K$ (which corresponds to the vacuum or the air) and by Ω_{out} the exterior of $\bar{\Omega}$ (see 2 1)

We assume that the following hypotheses hold

- The Maxwell equations are valid in the whole space
- The air is non conductor

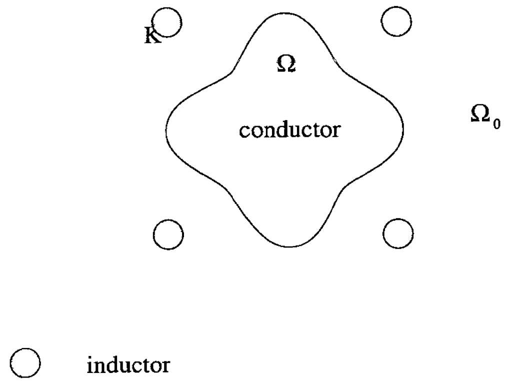

Figure 2.1. — Section of a solid column.

- The media have a linear behaviour and Ohm's law is valid.
- The current displacement can be neglected in Ampere's equation.
- There is no induced currents inside the inductors.

All these assumptions are natural in our range of applications.
The total given current density outside the conductor is given by

$$\mathbf{J}(\mathbf{x}, t) = \begin{cases} \mathbf{J}(\mathbf{x}, t) & \text{in } K, \\ 0 & \text{in } \Omega_0. \end{cases}$$

and satisfies

$$\operatorname{div} \mathbf{J} = 0 \quad \text{in } \Omega_{out} \quad \text{and} \quad \int_{\Omega_{out}} \mathbf{J}(\mathbf{x}, t) = 0 .$$

Under these hypotheses the induction field $\mathbf{B}$ and the electric field $\mathbf{E}$ satisfy (see [8] for more details)

$$\begin{cases} \operatorname{div} \mathbf{B} = 0 & \text{in } \mathbb{R}^d, \\ \operatorname{rot} \mu^{-1} \mathbf{B} = \mathbf{J} & \text{in } \Omega_{out}, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \dfrac{1}{\sigma\mu} \operatorname{rot} \operatorname{rot} \mathbf{B} = 0 & \text{in } \Omega, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \operatorname{rot} \mathbf{E} = 0 & \text{in } \Omega_{out}, \end{cases}$$

where $\mu := \mu(\mathbf{x})$ is the magnetic permeability and σ the electric conductivity of Ω.
The boundary conditions at the interface between the conductor and the vacuum are

$$[\mathbf{B} \,.\, \mathbf{n}] = 0, \quad [\mu^{-1} \mathbf{B} \wedge \mathbf{n}] = 0 \quad \text{and} \quad [\mathbf{E} \wedge \mathbf{n}] = 0$$

where $[\ .\ ]$ denotes the jump across the interface.
At infinity we impose

$$\lim_{\|\mathbf{x}\| \to +\infty} \|\mathbf{B}\| = 0 .$$

DIMENSIONLESS EQUATIONS

To study our problem we introduce dimensionless quantities. The time is normalised by the inverse of the frequency ω of the imposed current density, the space coordinates by L, a characteristic length, the induction field by $J_d \mu_0 L$, the electric field by J_d / σ, where J_d is the maximum of the imposed current and $\mu_0 = 4 \pi 10^{-7}\ \mathrm{Hm}^{-1}$ the vacuum magnetic permeability. We define the relative magnetic permeability χ by

$$\chi(\mathbf{x}) := \begin{cases} \chi_\Omega = \dfrac{\mu_\Omega}{\mu_0} & \text{if } x \in \Omega\,, \\ \chi_0 = 1 & \text{if } x \in \Omega_0, \\ \chi_K = \dfrac{\mu_K}{\mu_0} & \text{if } x \in K\,, \end{cases}$$

and we introduce the parameter ϵ defined by

$$\epsilon = \frac{1}{\mu_0\, \sigma L^2\, \omega}\,. \tag{2.1}$$

Because we are interested in high frequencies, ϵ is a small parameter. This parameter ϵ is the inverse of the so-called "screen parameter". Let us define $\omega_d = \mu_0\, \sigma L^2$, the parameter $\omega_d L$ is the speed of diffusion of the magnetic field inside the conductor. Also the parameter ϵ can be consider as the ratio of the frequency of the characteristic magnetic diffusion time in the conductor to ω the given frequency of the imposed current density. This means that when ϵ is small the speed of diffusion of the magnetic field inside the conductor is small too and then the magnetic field does not "propagate" inside the conductor. Through this paper we will see the importance of this parameter on the behaviour of the induction field.

With the same notations for $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{J}$, the dimensionless set of equations is

$$\begin{cases} \operatorname{div} \mathbf{B} & = 0 \ \text{ in } \mathbb{R}^d, \\ \operatorname{rot} \chi^{-1} \mathbf{B} & = \mathbf{J} \ \text{ in } \Omega_{out}, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \epsilon \chi^{-1} \operatorname{rot} \operatorname{rot} \mathbf{B} & = 0 \ \text{ in } \Omega\,, \\ \dfrac{\partial \mathbf{B}}{\partial t} + \epsilon \operatorname{rot} \mathbf{E} & = 0 \ \text{ in } \Omega_{out}, \end{cases} \tag{2.2}$$

with the boundary conditions

$$[\mathbf{B}\,.\,\mathbf{n}] = 0, \quad [\chi^{-1} \mathbf{B} \wedge \mathbf{n}] = 0, \quad [\mathbf{E} \wedge \mathbf{n}] = 0 \quad \text{and} \quad \lim_{\|x\| \to +\infty} \|\mathbf{B}\| = 0\,. \tag{2.3}$$

LEMMA 1: *The system (2.2)-(2.3) has at last one solution.*

Remark 1: We will not consider in this paper the question of existence but rather look at expansions of the given regular solution with respect to the small parameter.

Proof of the lemma: By linearity, we have to prove that $\mathbf{J} = 0$ implies $\mathbf{B} = 0$. We denote by $\mathbf{B}_{out}$ (resp. $\mathbf{B}_s$) the value of $\mathbf{B}$ outside (resp. inside) the conductor Ω and the same for $\mathbf{E}_{out}$, $\mathbf{E}_s$. We multiply by $\mathbf{B}$ the equation in Ω in system (2.2). Then by integrating on Ω we obtain

$$\frac{1}{2} \frac{d}{dt} |\mathbf{B}_s|^2_{L^2(\Omega)} + \epsilon \chi_\Omega^{-1} |\operatorname{rot} \mathbf{B}_s|^2_{L^2(\Omega)} + \epsilon \chi_\Omega^{-1} \int_{\partial\Omega} \operatorname{rot} B_s \wedge \mathbf{n}\,.\,\mathbf{B}_s\, ds = 0\,.$$

With the help of Ampere's equation and of the Ohm law the condition on the electric field $[\mathbf{E} \wedge \mathbf{n}] = 0$ on $\partial\Omega$ may be rewritten as follows

$$\chi_\Omega^{-1} \operatorname{rot} \mathbf{B}_s \wedge \mathbf{n} = \mathbf{E}_{out} \wedge \mathbf{n} .$$

Because $\mathbf{B}_s \,.\, \mathbf{n} = \mathbf{B}_{out} \,.\, \mathbf{n}$ on $\partial\Omega$ and with the above condition we have

$$\chi_\Omega^{-1}(\operatorname{rot} \mathbf{B}_s \wedge \mathbf{n}) \,.\, \mathbf{B}_s = \chi_\Omega(\mathbf{E}_{out} \wedge \mathbf{n}) \,.\, \mathbf{B}_{out}$$

and by (2.2) the boundary integral writes

$$\epsilon\chi_\Omega^{-1} \int_{\partial\Omega} (\operatorname{rot} \mathbf{B}_s \wedge \mathbf{n}) \,.\, \mathbf{B}_s \, ds = \frac{\chi_\Omega}{2} \frac{d}{dt} |\mathbf{B}_{out}|^2_{L^2(\Omega_{out})} .$$

Finally we have

$$\chi_\Omega \frac{d}{dt} |\mathbf{B}|^2_{L^2(\Omega_{out})} + \frac{d}{dt} |\mathbf{B}|^2_{L^2(\Omega)} + 2\, \epsilon\chi_\Omega^{-1} |\operatorname{rot} \mathbf{B}|^2_{L^2(\Omega)} = 0 .$$

This implies that $|\mathbf{B}_t(t)|^2 = 0\; p.p.$ in t and completes the proof.

Because we are only interested in the induction field, in the rest of the paper we do not consider the equation and the condition on the electric field.

3. ASYMPTOTIC ANALYSIS

In this section we study the behaviour in ϵ of the induction. We introduce two time scales and a corresponding time decomposition for the various quantities. The induction is written as the sum of a mean value and of an oscillating term. As we will see after only the system of the oscillating terms has a small parameter also we will concentrate on this system through this part. We will give a complete scheme to construct all terms of the singular expansion.

3.1. Two time scales and time decomposition

Because we have the factor ϵ^{-1} in front of all the time derivatives in (2.2) as we have shown by asymptotic analysis in [3], all the quantities are stationary at the first order in the metal at the magnetic time scale. In many applications the magnetic time is closed to the millisecond whereas the diffusion time is closed to the second. Typically ϵ is closed to 10^{-4}.

The idea here is to use multiple-scale analysis (see [11]) to overcome these difficulties. We introduce a slow time τ defined by

$$\tau := \epsilon t$$

and we consider the quantities, $\mathbf{B}$ and $\mathbf{J}$, as functions of the three independent variables $\mathbf{x}$, t and τ as follow

$$\mathbf{B}(\mathbf{x}, \tau, t) = \mathbf{B}(\mathbf{x}, \epsilon t, t) := \mathbf{B}(\mathbf{x}, t) .$$

If we come back to the dimensional quantities, the nondimensional slow time τ is defined by $\tau = (\mu_0 \sigma L^2)\, t_r = \omega_d\, t_r$, where t_r is the actual time and ω_d is defined in the previous section. This shows that τ is the good nondimensional time to follow the diffusion in the conductor.

We plug the relation

$$\frac{\partial}{\partial t} := \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}$$

in the system (2.2) to obtain the new set of equations:

$$\operatorname{div} \mathbf{B} = 0 \quad \text{in } \mathbb{R}^d, \tag{3.1}$$

$$\operatorname{rot} \chi^{-1} \mathbf{B} = \mathbf{J}(\mathbf{x}, \tau, t) \quad \text{in } \Omega_{out}, \tag{3.2}$$

$$\epsilon^{-1} \frac{\partial \mathbf{B}}{\partial t} + \frac{\partial \mathbf{B}}{\partial \tau} + \chi^{-1} \operatorname{rot} \operatorname{rot} \mathbf{B} = 0 \quad \text{in } \Omega, \tag{3.3}$$

with the boundary conditions

$$[\mathbf{B} \,.\, \mathbf{n}] = [\chi^{-1} \mathbf{B} \wedge \mathbf{n}] = 0, \tag{3.4}$$

$$\lim_{\|\mathbf{x}\| \to +\infty} \|\mathbf{B}(\mathbf{x}, \tau, t)\| = 0, \tag{3.5}$$

and the initial value

$$\mathbf{B}(\mathbf{x}, 0, 0) = \phi_b(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Omega. \tag{3.6}$$

Due to the choice of the dimensionless time the given current density $\mathbf{J}(\mathbf{x}, \tau, t)$ is a 2π-periodic function in t.

We decompose the given current density and the induction into their mean value and an oscillating term as follows

$$\begin{cases} \mathbf{J}(\mathbf{x}, \tau, t) := \tilde{\mathbf{J}}(\mathbf{x}, \tau) + \mathbf{j}(\mathbf{x}, \tau, t), \\ \mathbf{B}(\mathbf{x}, \tau, t) := \tilde{\mathbf{B}}(\mathbf{x}, \tau) + \mathbf{b}(\mathbf{x}, \tau, t), \end{cases} \tag{3.7}$$

where $\tilde{\cdot}$, is the mean operator defined by the average over a period

$$\tilde{\mathbf{B}}(\mathbf{x}, \tau) := \frac{1}{2\pi} \int_0^{2\pi} \mathbf{B}(\mathbf{x}, \tau, s)\, ds,$$

and the oscillating induction terms $\mathbf{j}$ and $\mathbf{b}$ are 2π-periodic in t.

Remark 2: With the definition of the mean operator we have $\tilde{\mathbf{b}} = 0$.

Classically, we obtain the system satisfied by the mean field which is

$$\begin{cases} \operatorname{div} \tilde{\mathbf{B}} = 0 & \text{in } \mathbb{R}^d, \\ \operatorname{rot} \chi^{-1} \tilde{\mathbf{B}} = \tilde{\mathbf{J}}(\mathbf{x}, \tau) & \text{in } \Omega_{out}, \\ \dfrac{\partial \tilde{\mathbf{B}}}{\partial \tau} + \chi^{-1} \operatorname{rot} \operatorname{rot} \tilde{\mathbf{B}} = 0 & \text{on } \Omega, \\ [\tilde{\mathbf{B}} \,.\, \mathbf{n}] = 0 & \text{in } \partial\Omega, \\ [\chi^{-1} \tilde{\mathbf{B}} \wedge \mathbf{n}] = 0 & \text{on } \partial\Omega, \\ \lim\limits_{\|\mathbf{x}\| \to +\infty} \|\tilde{\mathbf{B}}(\mathbf{x}, \tau)\| = 0, & \\ \tilde{\mathbf{B}}(\mathbf{x}, 0) = \phi_b(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega. \end{cases} \tag{3.8}$$

To construct the system satisfied by the oscillating terms we substract (3.8) to (3.1)-(3.6) and we obtain

$$\begin{cases} \operatorname{div} \mathbf{b} = 0 & \text{in } \mathbb{R}^d , \\ \operatorname{rot} \chi^{-1} \mathbf{b} = \mathbf{j}(\mathbf{x}, \tau, t) & \text{in } \Omega_{out}, \\ \epsilon^{-1} \dfrac{\partial \mathbf{b}}{\partial t} + \dfrac{\partial \mathbf{b}}{\partial \tau} + \chi^{-1} \operatorname{rot} \operatorname{rot} \mathbf{b} = 0 & \text{in } \Omega , \\ [\mathbf{b} \,.\, \mathbf{n}] = 0 , \\ [\chi^{-1} \mathbf{b} \wedge \mathbf{n}] = 0 , \\ \lim\limits_{\|\mathbf{x}\| \to +\infty} \|\mathbf{b}(\mathbf{x}, \tau, t)\| = 0. \end{cases} \tag{3.9}$$

Let us emphasise the following features of this decomposition

- Because the system (3.1)-(3.6) is linear the system for the oscillating terms (3.9) and the system for the mean value (3-8) terms are uncoupled.
- Only the oscillating system (3.9) has a small parameter. So we only have to expand $\mathbf{b}$ in term of ϵ.

3.2. Asymptotic behaviour of the oscillating terms

As shown in [3], we expect here a boundary layer near $\partial\Omega$. Indeed, it is easy to get a contradiction when assuming regular expansion for the magnetic field. In fact, the transmission conditions at the interface are over determined.

3.2.1. *Asymptotic expansions*

We use the boundary function method (see [6], [15]) to solve this singular perturbation problem (3.9). The perturbation term of the induction is decomposed as follows for all integer N

$$\mathbf{b} = \begin{cases} \mathbf{E}_N^s \mathbf{b} + \mathbf{E}_N^{bl} b & \text{inside the conductor} \\ \mathbf{E}_N^{out} \mathbf{b} = \sum\limits_{i=0}^{2N} \epsilon^{i/2} \mathbf{b}_i^{out} & \text{outside the conductor} \end{cases} + o(\epsilon^N) ,$$

where $\mathbf{E}_N^s \mathbf{b}$ is the regular part of the expansion, $\mathbf{E}_N^{bl} \mathbf{b}$ is the corrector term given by

$$\mathbf{E}_N^s \mathbf{b} = \sum_{i=0}^{2N} \epsilon^{i/2} \mathbf{b}_i^s \quad \text{and} \quad \mathbf{E}_N^{bl} \mathbf{b} = \sum_{i=0}^{2N} \epsilon^{i/2} \mathbf{b}_i^{bl} , \tag{3.10}$$

with $\mathbf{b}_i^s$, $\mathbf{b}_i^{out}$ and $\mathbf{b}_i^{bl}$ 2π-periodic in t.

Notation: The superscript s, out (resp. bl) is used for fields inside, outside the conductor (resp. for fields inside the boundary layer).

The above decomposition looks spatially like in the following picture. We see that the boundary layer is at the junction of two "regular domains". This means that the boundary layer is actually a transition layer. This implies that we cannot construct independently the two regular expansions $\mathbf{E}_N^{out} \mathbf{b}$ and $\mathbf{E}_N^s \mathbf{b}$.

We expand the given oscillating current density $\mathbf{j}$ as follows

$$\mathbf{j}(\mathbf{x}, \tau, t) = \sum_{i=0}^{2N} \epsilon^{i/2} \mathbf{j}_i (\mathbf{x}, \tau, t) + o(\epsilon^N) . \tag{3.11}$$

In many applications $\mathbf{j}$ is only given by $\mathbf{j}(\mathbf{x}, \tau, t) = \mathbf{j}_0 (\mathbf{x}, t)$. This case will be studied in detail in section 4.

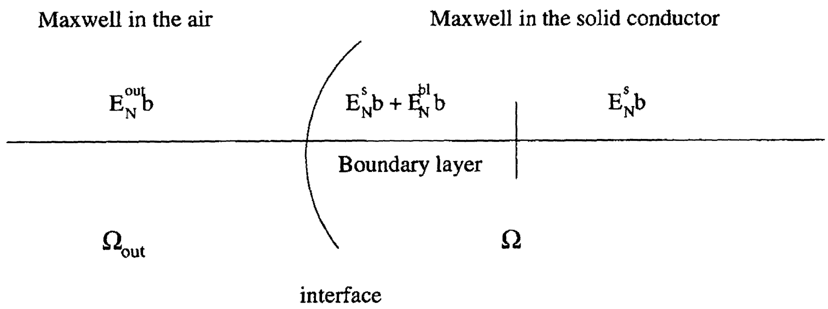

Figure 3.2. — The transition layer.

Now we construct the set of equations satisfied by the functions appearing in the regular part of the two expansions of the induction. By using (3.10)-(3.11) in (3.9) we obtain for each power of ϵ that the induction terms satisfy the two systems.

Outside Ω

$$\begin{cases} \operatorname{div} \mathbf{b}_i^{out} = 0 & \text{in } \Omega_{out}, \\ \operatorname{rot} \chi^{-1} \mathbf{b}_i^{out} = \mathbf{j}_i(\mathbf{x}, \tau, t) & \text{in } \Omega_{out}, \quad \forall i \geq 0 . \\ \lim\limits_{\|x\| \to +\infty} \|\mathbf{b}_i^{out}(\mathbf{x}, \tau, t)\| = 0 . \end{cases} \tag{3.12}$$

Inside Ω

$$\begin{cases} \dfrac{\partial \mathbf{b}_0^s}{\partial t} = \dfrac{\partial \mathbf{b}_1^s}{\partial t} = 0 , \\ \dfrac{\partial \mathbf{b}_i^s}{\partial t} = - \dfrac{\partial \mathbf{b}_{i-2}^s}{\partial \tau} + \chi^{-1} \operatorname{rot} \operatorname{rot} \mathbf{b}_{i-2}^s & i \geq 2 , \\ \operatorname{div} \mathbf{b}_i^s = 0 & i \geq 0 . \end{cases} \tag{3.13}$$

To obtain the boundary conditions of the above systems we have to introduce the boundary layer terms.

3.2.2. *Expansion in the bidimensional boundary layer*

In the boundary layer the corrector $\mathbf{E}_N^{bl} \mathbf{b} = \mathbf{b} - \mathbf{E}_N^s \mathbf{b}$ satisfies

$$\epsilon^{-1} \frac{\partial \mathbf{E}_N^{bl} \mathbf{b}}{\partial t} + \frac{\partial \mathbf{E}_N^{bl} \mathbf{b}}{\partial \tau} + \chi^{-1} \operatorname{rot} \operatorname{rot} \mathbf{E}_N^{bl} \mathbf{b} = \epsilon^N \left(\frac{\partial \mathbf{b}_N^s}{\partial \tau} + \chi^{-1} \operatorname{rot} \operatorname{rot} \mathbf{b}_N^s \right) \quad \text{in } \Omega . \tag{3.14}$$

For simplicity **we now assume that Ω is 2-d.** The 3-d case gives us only a more complicated equations rewritten with the local variables of the surface.

We rewrite (3.14) in curvilinear coordinates. We refer the reader to *figure 6.4* in Appendix for details. We assume that the boundary, $\partial\Omega$, is smooth. This implies that the local curvature is such that

$$\rho(s) = O(1) .$$

We stretch the normal direction of the curve, r by $r = \sqrt{\epsilon}\xi$. We write the equations as a series in ϵ and by identification we obtain the systems

$$\begin{cases} D_0\, \mathbf{b}_0^{bl} = 0 \\ \left(\frac{\partial}{\partial t} - \chi^{-1} A_0\right) \mathbf{b}_0^{bl} = 0 \end{cases}$$

$$\begin{cases} D_0\, \mathbf{b}_k^{bl} = -D_1\, \mathbf{b}_{k-1}^{bl} \\ \left(\frac{\partial}{\partial t} - \chi^{-1} A_0\right) \mathbf{b}_k^{bl} = -a_k \frac{\partial}{\partial \tau} \mathbf{b}_{k-2}^{bl} + \chi^{-1} \sum_{i=0}^{k-1} A_{k-i}\, \mathbf{b}_i^{bl} \end{cases} \quad \text{for } k > 0\,, \tag{3.15}$$

where the operators D_i and A_i are defined in Appendix and the coefficients a_k by $a_0 = a_1 = 0$ and $a_k = 1$ for $k > 1$.

The matching relation for the corrector terms (see [15]) implies the following decay when $\xi \to +\infty$

$$\lim_{\xi \to +\infty} \mathbf{b}_i^{bl}(s, \xi, \tau, t) = 0\,. \tag{3.16}$$

At the interface $\partial\Omega$, we have

$$\begin{cases} \mathbf{b}_i^{out} \,.\, \mathbf{n} = \mathbf{b}_i^{s} \,.\, \mathbf{n} + \psi_i \\ \mathbf{b}_i^{out} \,.\, \mathbf{s} = \chi_\Omega^{-1}(\mathbf{b}_i^{s} \,.\, \mathbf{s} + \phi_i) \end{cases} \tag{3.17}$$

where $\mathbf{b}_i^{bl} = \phi_i\, \mathbf{s} + \psi_i\, \mathbf{n}$, $\mathbf{n}$ is the outer normal of Ω and $\mathbf{s}$ is the tangential vector.

3.2.3. *Expression of the first order term*

PROPOSITION 1: *The first order term of the induction perturbation is given by*

$$\mathbf{b}_0 = \begin{cases} \mathbf{b}_0^{out} & \text{in } \Omega_{out}, \\ \mathbf{b}_0^{bl} & \text{in } \Omega\,, \end{cases}$$

where $\mathbf{b}_0^{out}$ *is solution outside* Ω *of the following exterior problem*

$$\begin{cases} \operatorname{div} \mathbf{b}_0^{out} = 0 & \text{in } \Omega_{out}, \\ \operatorname{rot} \chi^{-1}\, \mathbf{b}_0^{out} = \mathbf{j}_0(\mathbf{x}, \tau, t) & \text{in } \Omega_{out}, \\ \mathbf{b}_0^{out} \,.\, \mathbf{n} = 0 & \text{on } \partial\Omega\,, \\ \lim_{\|\mathbf{x}\| \to +\infty} \|\mathbf{b}_0^{out}(\mathbf{x}, \tau, t)\| = 0\,, \end{cases}$$

with $\mathbf{j}_0$ *is* 2π*-periodic in* t.

Inside the conductor Ω, *we have* $\mathbf{b}_0^{s} = 0$ *and the corrector term is only tangential and it is given by* $\mathbf{b}_0^{bl} = \phi_0\, \mathbf{s}$ *where*

$$\phi_0(s, \xi, \tau, t) = Real\left(\sum_{k \in \mathbb{Z}_*} b_k(s, \xi, \tau) \exp(\imath k t)\right)$$

and

$$b_k = a_k(s, \tau) \exp\left(- \sqrt{|k|\, \chi/2}\, (1 + sign\,(k)\, \imath)\, \xi \right),$$

where a_k *are the Fourier's coefficients of the tangential trace on* $\partial\Omega$ *of the outside induction* $\mathbf{b}_0^{out}(\mathbf{P}(s), \tau, t) \,.\, \mathbf{s}$, *i.e.*

$$a_k(s, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \chi_\Omega\, \mathbf{b}_0^{out}(\mathbf{P}(s), \tau, t) \,.\, \mathbf{s} e^{-\imath kt}\, dt \,.$$

Inside the conductor the main quantity for applications using magnetic induction, like induction heating or stirring effect, is the induced current. The induced current generates Joule effect and Lorentz Forces. The following corollary gives us the first term of its expansion.

COROLLARY 1: *The expansion of the induced current* $\mathbf{j}_{ind}$ *begins as follows*

$$\mathbf{j}_{ind} = - \epsilon^{-1/2}\, \mathrm{Real} \left(\sum_{k \in \mathbb{Z}_*} \sqrt{\frac{|k|\, \chi}{2}}\, (1 + sign(k)\, \imath)\, b_k(s, \xi, \tau) \exp(\imath kt) \right) \mathbf{e}_y + o(1)\,.$$

Remark 3: The above proposition shows that in the conductor the first order term of the oscillating induction field is concentrated in the boundary layer near the surface. Moreover the induction is only tangential in the conductor.

Proof of the proposition: For the first order, we have the following set of equations

$$\begin{cases} \operatorname{div} \mathbf{b}_0^{out} = 0 & \text{in } \Omega_{out}, \\ \operatorname{rot} \chi^{-1}\, \mathbf{b}_0^{out} = \mathbf{j}_0(\mathbf{x}, \tau, t) & \text{in } \Omega_{out}, \\ \lim\limits_{\|\mathbf{x}\| \to +\infty} \|\mathbf{b}_0^{out}(\mathbf{x}, \tau, t)\| = 0\,. & \end{cases}$$

$$\begin{cases} \dfrac{\partial \mathbf{b}_0^s}{\partial t} = 0 & \text{in } \Omega\,, \\ \operatorname{div} \mathbf{b}_0^s = 0, & \text{in } \Omega\,. \end{cases} \tag{3.18}$$

$$\begin{cases} \dfrac{\partial \phi_0}{\partial t} - \chi^{-1} \dfrac{\partial^2 \phi_0}{\partial \xi^2} = 0 & \text{for } \xi > 0\,, \\ \dfrac{\partial \psi_0}{\partial t} = \dfrac{\partial \psi_0}{\partial \xi} = 0 & \text{for } \xi > 0\,, \end{cases} \tag{3.19}$$

with the interface condition

$$\mathbf{b}_0^{out} \,.\, \mathbf{n} = \mathbf{b}_0^s \,.\, \mathbf{n} + \psi_0, \tag{3.20}$$

$$\phi_0 = \chi_\Omega\, \mathbf{b}_0^{out} \,.\, \mathbf{s} - \mathbf{b}_0^s \,.\, \mathbf{s}\,. \tag{3.21}$$

Because $\mathbf{b}_0^s$ and ψ_0 are 2π-periodic and their mean values are equal to zero, we obtain

$$\mathbf{b}_0^s = 0 \quad \text{and} \quad \psi_0 = 0\,.$$

Then by (3.20) we know $\mathbf{b}_0^{out} \cdot \mathbf{n}$ on $\partial\Omega$ and we solve (3.2.3) with this boundary value. Now by (3.21), (3.19) and (3.16) the first order term of the tangential part of the corrector satisfies

$$\begin{cases} \dfrac{\partial \phi_0}{\partial t} - \chi^{-1} \dfrac{\partial^2 \phi_0}{\partial \xi^2} = 0 & \text{for } \xi > 0 , \\ \phi_0(s, 0, \tau, t) = \chi_\Omega \mathbf{b}_0^{out}(\mathbf{P}(s), \tau, t) \cdot \mathbf{s} & \text{for } \xi = 0 , \\ \lim_{\xi \to +\infty} \phi_0(s, \xi, \tau, t) = 0 , \\ \phi_0 \text{ is } 2\pi \text{ periodic in } t . \end{cases} \tag{3.22}$$

We introduce the Fourier expansion in t of $\mathbf{b}_0^{out}(\mathbf{P}(s), \tau, t) \cdot \mathbf{s}$ i.e.

$$\phi_0(s, 0, \tau, t) = \chi_\Omega \, \mathbf{b}_0^{out}(\mathbf{P}(s), \tau, t) \cdot \mathbf{s} = \mathrm{Real}\left(\sum_{k \in \mathbb{Z}} a_k(s, \tau) \exp(\imath k t) \right).$$

By using the Fourier transformation in t on the system (3.22) we obtain

$$\begin{cases} \imath k b_k - \chi^{-1} \dfrac{\partial^2 b_k}{\partial \xi^2} = 0 & \text{for } \xi > 0 , \\ b_k(s, 0, \tau, t) = a_k(s, \tau) & \text{on } \xi = 0 , \\ \lim_{\xi \to +\infty} b_k(s, \xi, \tau, t) = 0 , \end{cases}$$

where b_k are the Fourier's coefficients of ϕ_0.

It is easy to see that the solution is $b_k = a_k \exp\left(-\sqrt{\chi |k| /2}(1 + sign(k)\, \imath)\, \xi \right)$. This finishes the proof.

Proof of the corollary: The induced current is given by

$$\mathbf{j}_{ind} := \mathrm{rot}\, \mathbf{B} .$$

Due to the fact that the induction field is only concentrated in the boundary layer we obtain in terms of the local variables of the curve the following expression

$$\mathbf{j}_{ind} = \frac{-1}{1 + \rho(s)\, r} \left(\frac{\partial}{\partial r} ((1 + \rho(s)\, r)\, \mathbf{B} \cdot \mathbf{s}) + \frac{\partial \mathbf{B} \cdot \mathbf{n}}{\partial s} \right) \mathbf{e}_y ,$$

where $\mathbf{e}_y = \mathbf{n} \wedge \mathbf{s}$, see [3] for more details. Then the first order term writes

$$\mathbf{j}_{ind}^0 = -\frac{\partial}{\partial r} \phi_0 \, \mathbf{e}_y .$$

By plugging ϕ_0 defined in proposition 1 we obtain the expression of the induced current.

Remark 4: One important pont is to see that the expansion of the induced current begins by $\epsilon^{-1/2}$. This means that the modulus of $\mathbf{j}_{ind}$ is large in the boundary layer.

3.2.4. *Higher order terms*

By this approach we have constructed the first order term of the induction field. Now by using the following iterative scheme, we can build an approximation of the induction at any order.

Let $\mathbf{b}^s_{i-1}$, $\mathbf{b}^{out}_{i-1}$, $\mathbf{b}^{bl}_{i-1} = \phi_i \, \mathbf{s} + \psi_i \, \mathbf{n}$ be known; then

Step 1

Solve independently $\mathbf{b}^s_i$ in Ω and the normal part of the corrector ψ_i solution of (3.13) and (3.15).

Step 2

Construct the induction field outside Ω with the condition $\mathbf{b}^{out}_i \, . \, \mathbf{n} = \psi_i$ solution of (3.12).

Step 3

Construct the tangential part of the corrector ϕ_i with the condition $\phi_i = \chi_\Omega \, \mathbf{b}^{out}_i \, . \, \mathbf{s}$ solution of (3.15).

Since $\mathbf{b}$ is periodic in t, by (3.15) it is easy to show that inside the conductor the regular part of $\mathbf{b}$ satisfies the following proposition.

PROPOSITION 2: *The regular part of the induction field in the conductor vanishes*

$$\mathbf{E}^s_N \, \mathbf{b} \equiv 0 \quad \forall N .$$

Proof: Inside the conductor the regular part of the induction field is solution of

$$\begin{cases} \imath \mathbf{b}^s_0 = 0 \\ \imath \mathbf{b}^s_{2k+1} = 0 \\ \imath \mathbf{b}^s_{2k} = \operatorname{rot} \operatorname{rot} \mathbf{b}^s_{2k-2} , \end{cases} \tag{3.23}$$

then obviously we have $\mathbf{b}^s_{2k} = 0$ for all k.

This proposition means that inside the conductor the oscillating term of the induction field is only concentrated in the boundary layer of thickness $\sqrt{\epsilon}$. So all effects outside this layer are due to the mean value of the induction.

In next section we apply the above scheme for an harmonic current and for $N = 1$.

4. A PARTICULAR CASE: A HARMONIC CURRENT

We are interested in this section in constructing the first and second order terms when the domain is a cross section of a vertical column of metal. Now we consider a given harmonic current density (i.e. it is only an oscillating term) in Ω_{out} like

$$\mathbf{J}(\mathbf{x}, t) = \text{Real} \left(\mathbf{j}_d(\mathbf{x}) \, e^{\imath t} \right) = \mathbf{j}_d(\mathbf{x}) \cos t . \tag{4.1}$$

4.1. General cross section of a column

PROPOSITION 3: *The approximation of order ϵ of the induction perturbation field is given by*

$$\mathbf{b} = \text{Real} \left(\begin{Bmatrix} \mathbf{b}^{out}_0(\mathbf{x}) + \sqrt{\epsilon} \, \dfrac{\bar{\alpha}}{\chi_\Omega} \, \mathbf{b}^{out}_1(x) \\ \mathbf{b}^{bl}_1(\mathbf{x}) \end{Bmatrix} e^{\imath t} \right) + o(\sqrt{\epsilon})$$

where $\mathbf{b}_0^{out}$, *and* $\mathbf{b}_1^{out}$ *are defined by*

$$\begin{cases} \operatorname{div} \mathbf{b}_0^{out} = 0 & in\ \Omega_{out}, \\ \operatorname{rot} \chi^{-1} \mathbf{b}_0^{out} = \mathbf{j}_d(\mathbf{x}) & in\ \Omega_{out}, \\ \mathbf{b}_0^{out} \cdot \mathbf{n} = 0 & on\ \partial\Omega, \\ \lim\limits_{\|\mathbf{x}\| \to +\infty} \|\mathbf{b}_0^{out}(\mathbf{x})\| = 0, \end{cases} \tag{4.2}$$

$$\begin{cases} \operatorname{div} \mathbf{b}_1^{out} = 0 & in\ \Omega_{out}, \\ \operatorname{rot} \chi^{-1} \mathbf{b}_1^{out} = 0 & in\ \Omega_{out}, \\ \mathbf{b}_1^{out} \cdot \mathbf{n} = -\frac{\partial}{\partial s} g_0(s) & on\ \partial\Omega, \\ \lim\limits_{\|\mathbf{x}\| \to +\infty} \|\mathbf{b}_1^{out}(\mathbf{x})\| = 0, \end{cases} \tag{4.3}$$

and $\mathbf{b}_1^{bl}$ *by*

$$\mathbf{b}_1^{bl}\left(s, \frac{r}{\sqrt{\epsilon}}\right) = \left(g_0(s)\,\mathbf{s} + \sqrt{\epsilon}\left(\left(\frac{\bar{\alpha}}{\chi} g_1(s) - \frac{\rho(s)}{2} \frac{r}{\sqrt{\epsilon}} g_0(s)\right)\mathbf{s} - \frac{\bar{\alpha}}{\chi} \frac{\partial g_0}{\partial s}\mathbf{n}\right)\right) e^{-\alpha \frac{r}{\sqrt{\epsilon}}}, \tag{4.4}$$

where $\alpha = \beta(1+i)$, $\bar{\alpha}$ *its conjugate value,* $\beta = \sqrt{\chi/2}$, $g_i(s) = \chi_\Omega\, \mathbf{b}_i^{out}(\mathbf{P}(s)) \cdot \mathbf{s}$ *and* $\rho(s)$ *is the curvature of* $\partial\Omega$.

Proof: Due to the fact that (3.12) is linear and $\mathbf{J}$ is given by (4.1) we look for the following form of the induction

$$\mathbf{B} = \mathrm{Real}\,(\mathbf{B}(\mathbf{x})\, e^{it}).$$

Proposition 1 easily gives that the first order term is solution of (1) outside Ω and inside the corrector term $\mathbf{b}_0^{bl} = \phi_0\,\mathbf{s} + \psi_0\,\mathbf{n}$ is given by

$$\begin{cases} \psi_0(s, \xi) = 0, \\ \phi_0(s, \xi) = g_0(s) \exp(-\beta(1+\imath)\,\xi). \end{cases}$$

where $g_0(s) = \chi_\Omega\, \mathbf{b}_0^{out}(\mathbf{x}) \cdot \mathbf{s}$ for $\mathbf{x} \in \partial\Omega$.

Now by (3.12)-(3.15) the second order term of the regular part is solution of

$$\begin{cases} \imath \mathbf{b}_1^s = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{b}_1^{out} = 0 & \text{in } \Omega_{out}, \\ \operatorname{rot} \chi^{-1} \mathbf{b}_1^{out} = 0 & \text{in } \Omega_{out}, \\ \lim\limits_{\|\mathbf{x}\| \to +\infty} \|\mathbf{b}_1^{out}(\mathbf{x}, \tau, t)\| = 0. \end{cases}$$

and the corrector term is solution for $\xi > 0$

$$\begin{cases} \dfrac{\partial \psi_1}{\partial t} = \chi^{-1} \dfrac{\partial^2 \phi_0}{\partial s\, \partial \xi} \\ \dfrac{\partial \psi_1}{\partial \xi} = -\rho(s)\, \psi_0 + \dfrac{\partial \phi_0}{\partial s} - \rho(s)\, \xi \dfrac{\partial \psi_0}{\partial \xi} \\ \dfrac{\partial \phi_1}{\partial t} - \chi^{-1} \dfrac{\partial^2 \phi_1}{\partial \xi^2} = \chi^{-1}\left(\dfrac{\partial^2 \psi_0}{\partial s\, \partial \xi} + \rho(s) \dfrac{\partial \phi_0}{\partial \xi}\right). \end{cases}$$

Because $\mathbf{b}_1^s = 0$ the transmission conditions on $\partial\Omega$ give

$$\mathbf{b}_1^{out} . \mathbf{n} = \psi_1 \quad \text{and} \quad \phi_1 = \chi_\Omega \, \mathbf{b}_1^{out} . \mathbf{s} \, .$$

The construction of ψ_1 needs the following compatibility condition

$$\chi^{-1} \frac{\partial^3 \phi_0}{\partial s \, \partial \xi^2} = \frac{\partial^2 \phi_0}{\partial t \, \partial s} \, .$$

We can write it as follows

$$\frac{\partial}{\partial s} \left[\frac{\partial \phi_0}{\partial t} - \chi^{-1} \frac{\partial^2 \phi_0}{\partial \xi^2} \right] = 0 \, .$$

By the construction of ϕ_0 this condition is satisfied. Then by solving this set of equation we find the expressions given in the proposition. If we come back to the real value, the second order approximation of the induction is given by outside the conductor i.e. in Ω_{out}

$$\mathbf{b} = \mathbf{b}_0^{out} (\mathbf{x}) \cos t + \frac{\beta}{\chi} \sqrt{\epsilon} \, \mathbf{b}_1^{out} (x) \, (\cos t + \sin t) + O(\epsilon)$$

and inside the conductor

$$\mathbf{b} = \begin{pmatrix} \left(g_0(s) + \sqrt{\epsilon} \left(\frac{\beta}{\chi} g_1(s) - \frac{\rho(s)}{2} g_0(s) \frac{r}{\sqrt{\epsilon}} \right) \right) \cos \left(t - \beta \frac{r}{\sqrt{\epsilon}} \right) + \sqrt{\epsilon} \frac{\beta}{\chi} g_1(s) \sin \left(t - \beta \frac{r}{\sqrt{\epsilon}} \right) \\ - \sqrt{\epsilon} \frac{\bar{\alpha}}{\chi} \frac{\partial g_0}{\partial s} \left(\cos \left(t - \beta \frac{r}{\sqrt{\epsilon}} \right) + \sin \left(t - \beta \frac{r}{\sqrt{\epsilon}} \right) \right) \end{pmatrix} e^{-\beta \frac{r}{\sqrt{\epsilon}}} + O(\epsilon)$$

with $\mathbf{b}_0^{out}$ resp. $\mathbf{b}_1^{out}$ solution of (4.2) resp. (4.3).

PROPOSITION 4: *The behaviour in time of the first order term of the induction* $\mathbf{B}_0$ *is given by*

$$\lim_{\tau \to +\infty} \left[\mathbf{B}_0 (\mathbf{x}, \tau, t) - \sqrt{2} \, \mathbf{b}_0 (\mathbf{x}) \, e^{it} \right] = 0$$

where $\mathbf{b}_0$ *is given by*

$$\begin{cases} \operatorname{rot} \mathbf{b}_0 = \mathbf{j}_d & \text{in } \Omega_{out}, \\ \operatorname{div} \mathbf{b}_0 = 0 & \text{in } \Omega_{out}, \\ \mathbf{b}_0 . \mathbf{n} = 0 & \partial\Omega \, , \\ \mathbf{b}_0 (\mathbf{x}) = \sqrt{2} \, \mathbf{b}_0 (\mathbf{P}(s)) . \mathbf{s} e^{-\beta \frac{d(x)}{\sqrt{\epsilon}}} \mathbf{s} & \text{in } \Omega \, , \end{cases}$$

where $\mathbf{P}(s)$ *is the orthogonal projection of* $\mathbf{x}$ *on* $\partial\Omega$, $d(\mathbf{x})$ *is the distance to* $\partial\Omega$.

Proof: The stationary solution, $\mathbf{B}_{ST}$, $\mathbf{E}_{ST}$ of (3.8) is given by

$$(P_{ST})\begin{cases} \operatorname{div} \mathbf{B}_{ST} = 0 & \text{in } \mathbb{R}^d, \\ \operatorname{rot} \chi^{-1} \mathbf{B}_{ST} = 0 & \text{in } \Omega_{out}, \\ \operatorname{rot} \mathbf{E}_{ST} = 0 & \text{in } \mathbb{R}^d, \\ \operatorname{rot} \operatorname{rot} \chi^{-1} \mathbf{B}_{ST} = 0 & \text{in } \Omega\,, \\ [\mathbf{B}_{ST} \cdot \mathbf{n}] = 0 & \text{on } \partial\Omega\,, \\ [\mathbf{B}_{ST} \wedge \mathbf{n}] = 0 & \text{on } \partial\Omega\,, \\ [\mathbf{E}_{ST} \wedge \mathbf{n}] = 0 & \text{on } \partial\Omega\,, \\ \lim_{\|\mathbf{x}\| \to +\infty} \|\mathbf{B}_{ST}\| = 0\,. \end{cases}$$

It is easy to show that the solution of (P_{ST}) is $\mathbf{B}_{ST} = \mathbf{E}_{ST} = 0$. Then the behaviour of the induction is given only by the perturbation term. The first order term is given by proposition 3, this finishes the proof.

This proposition means that the behaviour of the induction field for large time is given by the oscillating field. The mean part is here only to relax the initial condition. This could be viewed as a justification to the fact that when we have (4.1), we can search the induction with the harmonic approximation.

In this case expansion of the induced current inside Ω begins by

$$\mathbf{j}_{ind} = \frac{\beta}{\sqrt{\epsilon}}\, g_0(s) \left[\cos\left(t - \beta \frac{r}{\sqrt{\epsilon}}\right) - \sin\left(t - \beta\frac{r}{\sqrt{\epsilon}}\right)\right] e^{-\beta \frac{r}{\sqrt{\epsilon}}}\, \mathbf{e}_y + O(1)$$

with $r = dist\,(M, \partial\Omega)$ when M is inside Ω and $\beta = \sqrt{\chi/2}$.

4.2. Numerical results for cylinder column

In this section we illustrate the results given in proposition 3. We consider that Ω is the unit disk, and the given current is defined by

$$\mathbf{j}(\mathbf{x}, t) := \cos(t) \sum_{k=0}^{4} I_k\, \delta_{\mathbf{x}_k}(\mathbf{x})\, \mathbf{e}_y$$

where $\mathbf{x}_k = 2.0 \exp(\imath\theta_k)$, $\theta_k = k\pi/2$, $I_k = (-1)^k$ and $\delta_{\mathbf{x}_k}$ is the Dirac function at point $\mathbf{x}_k$. This kind of current density is classical used in casting or guiding jet, see ([2]). We solve the problem satisfied by the first and second order term with the help of the potential. These exterior problems (4.2) and (4.3) can be easy rewriting in potential form like

$$\begin{cases} \Delta\phi = 0 & \text{for } |\mathbf{x}| > 1\,, \\ \phi(\mathbf{x}, t) = g(\mathbf{x}, t) & \text{for } |\mathbf{x}| = 1\,, \end{cases}$$

Then, we have to solve two harmonic problems with different Dirichlet conditions. To solve the above system we introduce the Fourier expansion of g defined as follows

$$g(r, \theta, t) = \sum_{k \in \mathbb{Z}} a_k(t) \exp(\imath k\theta)$$

then the solution of the above system writes

$$\phi(r, \theta, t) = \sum_{k \in \mathbb{Z}} a_k(t)\, r^{-|k|} \exp(ik\theta)$$

see [5] for more details.

In this part we show the behaviour in space of the first and second order terms of the induction and we compare the two first terms with respect to ϵ.

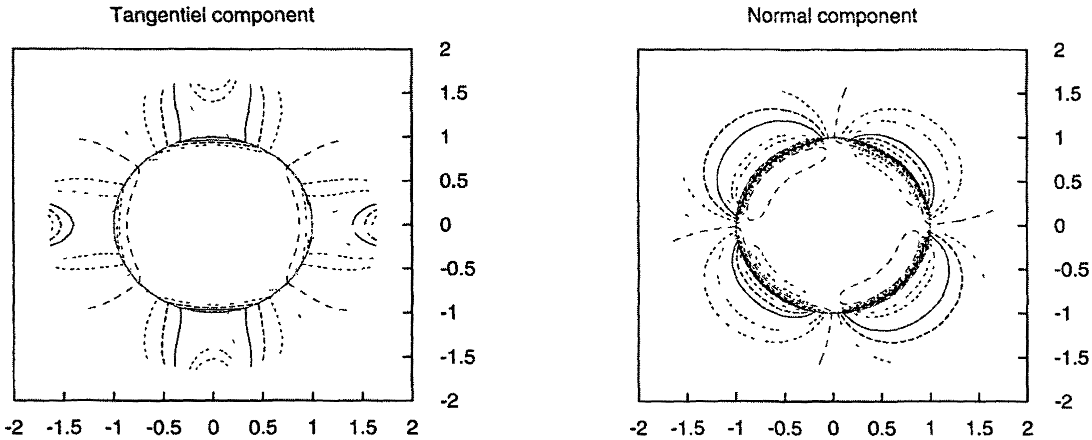

Figure 4.1. — Second order induction field for $\epsilon = 0.005$

Figure 4.1 shows the isovalues of the approximation of the induction field at the second order for $\epsilon = 5.10^{-3}$ and at time $t = 1$. The maximum of the tangential field is 0.5 while the maximum of the normal field is 0.04. In the next figures the horizontal axis represents the intensity of the function while in vertical axis we put the value of $|\mathbf{x}|$.

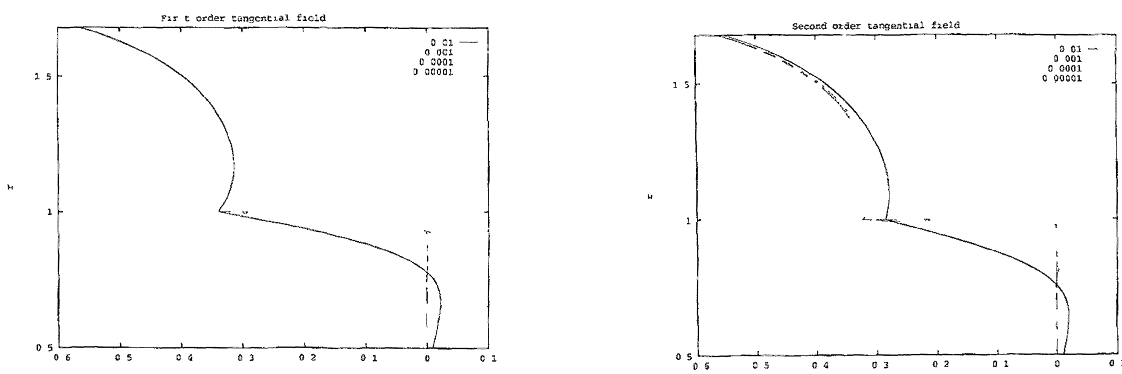

Figure 4.2. — Different order of approximation for the tangential magnetic field.

Now *figure 4.2* shows for $\theta = 0$ and $t = 1$, the first and the second order approximation of the tangential magnetic field for different values of $\epsilon = (0.01, 0.001, 0.0001, 0.00001)$. At the first order we see that the frequency has no influence on the magnetic field outside the conductor (i.e. for $r > 1$).

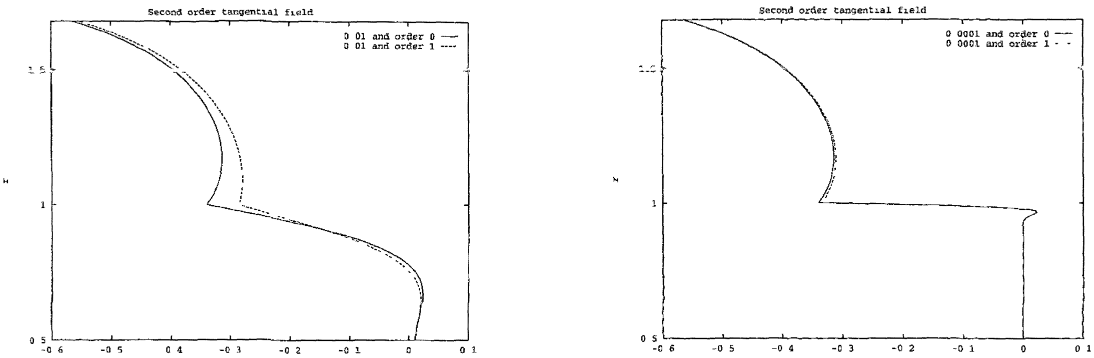

Figure 4.3. — Comparison for different values of ϵ at $\theta = 0$ between the first and the second order of the tangential magnetic field.

Figure 4.3 compares for $\theta = 0$ and $t = 1$, the first order approximation with the second order approximation of the tangential magnetic field for different values of $\epsilon = (0.01, 0.0001)$. Here, we can see the influence of the second order term on the exterior problem even for the smaller value of ϵ.

5. CONCLUSION

We have shown that in the case of solid conductors the magnetic induction can be well described for high frequencies as a superposition of a mean value field and an oscillating field. Moreover, only the oscillating term has a boundary layer behaviour and it does not propagate in the depth of the conductor. The induction is only concentrated near the surface of the conductor also, the diffusion of the magnetic field in the conductor is only due to the mean value of the imposed density current. By our decomposition we can solve the induction problem without the harmonic approximation and then solves transient problems. Finally, we have an explicit expression for the first order term of the induced current in a conductor and then it is easy to construct the Joule force for thermal heating or Lorentz force for stirring, levitation, ...

In the particular case when the mean value of the current vanishes we have shown that for the first order term the behaviour of the induction for large time is given only by the oscillating term. This point could be view as a justification of the harmonic approximation.

6. APPENDIX

We introduce the new local variables defined in *figure 6.4*. We assume the surface γ has a normal parametrisation g, then the tangent is given by $\mathbf{s} = g'(s)$. Let the point $\mathbf{M}(x, y)$ in Ω we denote the distance $\mathbf{MP}$ by r. The pair (r, s) will be the new coordinate of the point $\mathbf{M}$. If the δ-neighbourhood of the surface $\{0 \leqslant s \leqslant s_0\} \times \{0 \leqslant r \leqslant \delta\}$ is small enough there exists a one to one correspondence between the two coordinates (x, y) and (s, r) expressed by

$$\mathbf{M} = (x, y) = \mathbf{P}(s) - r\mathbf{n}.$$

A function Θ is decomposed as

$$\Theta = \phi\mathbf{s} + \psi\mathbf{n}.$$

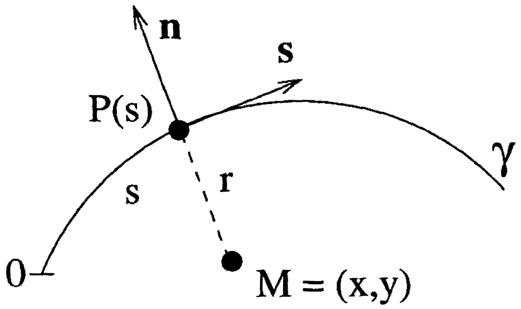

Figure 6.4. — Curvilinear system of coordinate.

In this new coordinate the Maxwell's equations are given by (see [3] for more details)

$$\begin{cases} \frac{1}{h_1}\left[\frac{\partial \phi}{\partial s} - \frac{\partial h_1 \psi}{\partial r}\right] \\ \epsilon^{-1}\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \kappa} - \chi^{-1}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{1}{h_1}\frac{\partial \psi}{\partial s}\right) + \frac{\partial}{\partial r}\left(\frac{\phi}{h_1}\frac{\partial h_1}{\partial r}\right)\right) = \mathbf{F}\,.\,\mathbf{s} \\ \epsilon^{-1}\frac{\partial \psi}{\partial t} + \frac{\partial \phi}{\partial \kappa} - \chi^{-1}\left(\frac{1}{h_1}\frac{\partial^2 \phi}{\partial r\,\partial s} + \frac{1}{h_1}\frac{\partial}{\partial s}\left(\frac{1}{h_1}\frac{\partial \psi}{\partial s}\right) + \frac{1}{h_1}\frac{\partial}{\partial s}\left(\frac{\phi}{h_1}\frac{\partial h_1}{\partial r}\right)\right) = \mathbf{F}.\mathbf{n} \end{cases}$$

with $h_1 = 1 + \rho(s)\, r$, where ρ is the curvature of the surface. It is positive when the corresponding centre of curvature lies on the side of the surface to which the normal **n** points.

If we apply the transformation

$$\begin{cases} s := s \\ \xi := r/\sqrt{\epsilon}\,, \end{cases}$$

then the above system becomes

$$\begin{cases} D_0\,\Theta + \sqrt{\epsilon}\,D_1\,\Theta = 0 \\ \frac{1}{\epsilon}\frac{\partial \Theta}{\partial t} + \frac{\partial \Theta}{\partial \tau} - \chi^{-1}\frac{1}{\epsilon}A_\epsilon\,\Theta = F \end{cases}$$

where the operators are defined by

$$A_\epsilon\,\Theta = \sum_{i=0}^{\infty} \epsilon^{i/2} A_i\,\Theta$$

with

$$D_0\,\Theta = \frac{\partial \psi}{\partial \xi} \quad D_1\,\Theta = \rho(s)\,\psi - \frac{\partial \phi}{\partial s} + \rho(s)\,\xi\,\frac{\partial \psi}{\partial \xi}$$

$$A_0\,\Theta = \begin{pmatrix} \frac{\partial^2}{\partial \xi^2} & 0 \\ 0 & 0 \end{pmatrix} \quad A_1\,\Theta = \begin{pmatrix} \rho\frac{\partial}{\partial \xi} & \frac{\partial^2}{\partial \xi\,\partial s} \\ \frac{\partial^2}{\partial \xi\,\partial s} & 0 \end{pmatrix}$$

$$A_i\,\Theta = (-1)^i \rho^{i-3}\,\xi^{i-2}\bar{A}_i$$

$$\bar{A}_i\,\Theta = \begin{pmatrix} -\rho^3\left[(i-1) + \xi\frac{\partial}{\partial \xi}\right] & -\rho^2\left[(i-1)\frac{\partial}{\partial s} + \xi\frac{\partial^2}{\partial \xi\,\partial s}\right] \\ \rho\left[(i-1)\left(i\rho'/2 + \rho\frac{\partial}{\partial s}\right) - \rho\xi\frac{\partial^2}{\partial \xi\,\partial s}\right] & (i-1)\left(\rho\frac{\partial^2}{\partial s^2} + \frac{(i-2)}{2}\rho'\frac{\partial}{\partial s}\right) \end{pmatrix}.$$

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