M2AN - MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

C. CARASSO
G. PASA

An optimal viscosity profile in the secondary oil recovery

M2AN - Modélisation mathématique et analyse numérique, tome 32, nº 2 (1998), p. 211-221
<http://www.numdam.org/item?id=M2AN_1998__32_2_211_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODELISATION MATHEMATIQUE ET ANALYSE NUMERIQUE

(Vol 32, n° 2, 1998, p 211 à 221)

AN OPTIMAL VISCOSITY PROFILE IN THE SECONDARY OIL RECOVERY

C. CARASSO (*) et G. PASA (**)

Abstract — *The secondary recovery process is a manner to produce oil from a porous medium, by displacing it with a second fluid (usually water) If the second fluid is less viscous, then the well-known Saffman-Taylor instability appears, producing the "fingering" phenomenon An intermediate region with variable viscosity μ, containing a polymer mixed with water can be considered between water and oil A Sturm-Liouville problem is obtained from the study of the linear stability of the straight initial interfaces between mixture and oil The eigenvalues of this system are the growth constants σ (in time) of the perturbations and may be controlled by μ The surface tension due to the mixture water-polymer gives us a maximum value of σ in terms of the wave numbers of the perturbations In this paper, the maximum value of σ is minimized, by giving an explicit upper bound for a "minimizing" viscosity of the mixture water-polymer and also a lower bound for the "optimal" length of the intermediate region (injection length) This bounds are obtained in terms of the jump of the viscosity on the interface with the oil In this way, an improvement of the stability is obtained © Elsevier, Paris*

Résumé — *La « récupération assistée » est une technique permettant d'obtenir le pétrole contenu dans un milieu poreux Elle s'applique en déplaçant le pétrole a l'aide d'un autre fluide (usuellement de l'eau) Si le deuxième fluide a une viscosité plus petite que celle du pétrole, on a alors le phénomène bien connu de digitation mis en évidence par Saffman et Taylor On considère, entre l'eau et pétrole, une région intermédiaire avec une viscosité variable μ, qui contient un mélange eau-polymère La viscosité μ de ce mélange est croissante de l'eau vers le pétrole On étudie la stabilité de l'interface ainsi obtenue entre le mélange et le pétrole Les valeurs propres du problème de Sturm-Liouville ainsi obtenu sont les constantes d'augmentation σ (en temps) des perturbations , il est possible de contrôler ces valeurs propres à l'aide de la viscosité μ La tension superficielle produite par le mélange nous donne une valeur maximale de σ, comme fonction du nombre d'onde des perturbations Dans ce travail, on obtient la viscosité μ du mélange qui minimise la valeur maximale de σ On a ainsi la viscosité qui minimise le phénomène de digitation et améliore très nettement la technique de « récupération assistée » On obtient aussi l'expression exacte de la borne supérieure de la viscosité « minimisante » et une borne inférieure pour la longueur « optimale » de la région occupée par le mélange (longueur d'injection) Ces bornes sont fonctions du saut de la viscosité sur l'interface avec le pétrole On a ainsi une amélioration de la stabilité © Elsevier, Paris*

*

This work wad done while the second author was visiting l'Equipe d'Analyse Numérique, University of Saint-Etienne (France), with the support of the Région Rhône Alpes (program TEMPRA)

1. INTRODUCTION

We study the "secondary recovery" process: the oil contained in a porous medium is obtained by displacing it with a second fluid (usually water).

The water viscosity μ_1 is less than the oil viscosity μ_2 ; then the instability of the interfaces between water and oil appears, which first was studied in the well-known paper of Saffman and Taylor [1], 1958. Because of this phenomenon, it is possible to obtain water rather than oil. A surface tension on the interface between water and oil may decrease this instability.

Gorell and Homsy [3], 1983, consider an intermediate region (i.r.), between water and oil, of length l, containing a given quantity of polymer mixed with water. The viscosity of this mixture, denoted by $\mu(x)$, increases from the water to the oil viscosity. We have to find the viscosity $\mu(x)$ which minimises the "fingering" phenomenon.

We consider a steady two-dimensional system in the plane (x, y). The part $-\infty < x < -l$ of the porous medium is filled with water, the region $-l < x < 0$ is filled with the mixture polymer-water and the region $0 < x < \infty$ is filled with the oil.

(*) Manuscript received February 27, 1996, Revised January 3, 1997
(*) UMR 5585 An Numérique Lyon St-Etienne, 23 rue Dr Paul Michelon 42023 Sr-Etienne cedex 2, France e-mail carasso@anumsun1 univ-st-etienne fr
(**) Institute of Mathematics, Roumanian Academy, PO Box 1-764, RO-70700, Bucharest, Romania e-mail gpasa@stoilow imar ro

We suppose a steady flow of water with a velocity U at $-\infty$. The unknown viscosity $\mu(x)$ of the mixture polymer-water is supposed invertible as a function of the solute concentration. The flow in the bidimensional porous medium is given by the Darcy law.

The obtained system, together with the continuity equation for the velocity, give us a Sturm-Liouville problem. We have to study, for this system, the growth constants σ of the sinusoidal perturbations u' of the straight initial interfaces between the mixture and oil:

$$u'(x, y, t) = f(x) \cdot \exp(iky + \sigma t)$$

where k is the perturbation wave number in the vertical direction. The growth constants σ are the eigenvalues of the considered system.

The numerical results of [3] give us the maximal value of σ for the wave numbers k in a finite region. The "optimal" viscosity μ, obtained numerically, has an exponential form.

The same model is studied in [4], in the case of a small quantity of polymer, by using an asymptotic development; the "minimizing" viscosity is constant.

An existence theorem for the eigenvalues of the above Sturm-Liouville problem is given in [5], by using the method of successive approximations.

Some numerical results concerning σ are given in [6]; but the largest value of σ is obtained for $k \to \infty$, in discordance with the results of [3].

An existence theorem for the "minimizing" viscosity is given in [7], by using the Rayleigh's quotient.

In this paper, we obtain an upper bound for the optimal viscosity of the mixture (i.r.) and a lower bound for the "injection lengtht" l, in terms of $\mu(0)$ — the value of the viscosity on the interface between (i.r.) and oil. We improve the value of σ, obtained without (i.r.), by choosing an appropriate value for $\mu(0)$. The considered Sturm-Liouville system is discretized by using a finite-difference method, and the eigenvalues are localized with the Gerschgorin's theorem.

The obtained upper bound for the "minimizing" viscosity in the (i.r.) is of exponential type (with respect to the horizontal variable x), in agreement with the numerical results of [3].

We give an explanation for the lack of agreement between the numerical results of [3] and [6]. If the ratio $\mu(x)'/\mu(x)$ for $x \in$ (i.r.) is bounded by a certain expression, then the largest value of σ is obtained for k in a finite region. The considered viscosities of [6] were not satisfying this condition.

In the last part, a strategy is given to decrease the "fingering" phenomenon and to improve the stability of the interfaces.

2. BASIC EQUATIONS; STATEMENT OF THE PROBLEM

We consider a bidimensional porous medium $Ox_1 y$. The velocity of the fluid is denoted by w, with the components (u, v). The flow in the porous medium is given by the Darcy law and the continuity equation for the velocity:

$$\operatorname{div}(w) = 0 \tag{1}$$

$$\operatorname{grad}(P) = -\mu \cdot w \tag{2}$$

where μ is the viscosity multiplied by the inverse of the permeability and P is the pressure.

The viscosity in the intermediate region (i.r.) is supposed invertible with respect to the concentration of the polymer-solute; therefore, we consider a "continuity" equation for the viscosity:

$$D\mu/Dt = 0 . \tag{3}$$

The system (1)-(2)-(3) has a steady basic solution in variables (x_1, y) :

$$u = U, v = 0, \mu = \mu_0(x_1 - Ut), P = -U \int_{x_0}^{x_1} \mu(s - Ut)\, ds = P_0(x_1, t)\,. \tag{4}$$

We denote by g_x the derivative with respect to x. By a change of variable, we consider the moving referential $x = x_1 - Ut$.

We have to study the stability of the steady solution (4). If this solution is perturbed, we get the system:

$$u'_x + v'_y = 0 \tag{5}$$

$$P'_x = -\mu' U - \mu_0 u'$$

$$P'_y = -\mu_0 v'$$

$$\mu'_t + u' . (\mu_0)_x = 0$$

where u', v', P', μ' are the perturbations of the velocity, pressure and viscosity. The above equations are linear; therefore the perturbations are decomposable in Fourier series and each component may be studied separately.

We start with the following form of the horizontal component of the velocity:

$$u'(x, y, t) = f(x) . \exp(iky + \sigma t)\,. \tag{6}$$

In this formula, σ is the growth constant (in time) and $k > 0$ is the wave number in the vertical direction y. By using the system (5) and the perturbation (6), we obtain the following Sturm-Liouville system:

$$-(\mu_0 . f_x)_x + k^2 \mu_0 f = k^2 \frac{U}{\sigma} (\mu_0)_x . f \tag{7}$$

The boundary conditions are obtained with the Laplace law; u' is continuous on the interface and the difference of the pressure is equal with the surface tension multiplied by the curvature of the perturbed interface.

Let $x = \eta_0$ be the straight interface of the steady solution (4); then the interface of the perturbed solution is $x = \eta_0 + \eta'$ where $\eta' \ll \eta_0$, and we get:

$$\eta_t = u' \quad \text{for} \quad x = \eta_0 + \eta'\,.$$

The relation (6) gives us the expression of η'; we obtain:

$$P(\eta_0 + \eta') = P_0(\eta_0) - \mu_0(\eta_0) \{f_x(\eta_0)/k^2 + f(\eta_0)\, U/\sigma\} \exp(iky + \sigma t) \tag{8}$$

as in [3].

The system (7) with the boundary conditions (8) was used in [3] to obtain the well known solution of Saffman and Taylor [1] (with two constant viscosity profiles, without intermediate region). If we consider

$$\mu_0 = \begin{cases} \mu_2 & \text{if } x > \eta_0 \\ \mu_1 & \text{if } x < \eta_0\,, \end{cases}$$

then for the eigenfunction f we get:

$$f(x) = \begin{cases} f(\eta_0) \exp(-k(x - \eta_0)) & \text{if } x > \eta_0 \\ f(\eta_0) \exp(k(x - \eta_0)) & \text{if } x < \eta_0 \end{cases}$$

and the growth constant is given by

$$\sigma = \frac{(\mu_2 - \mu_1)\, Uk - Tk^3}{\mu_2 + \mu_1}. \tag{9}$$

In the above formula, T is the surface tension on $x = \eta_0$ The last relation may be used to obtain a maximum value σ_m of σ, in terms of k (the wave number):

$$\sigma_m = \frac{2(\mu_2 - \mu_1)\, U}{3(\mu_2 + \mu_1)\sqrt{3}} \cdot \sqrt{\frac{(\mu_2 - \mu_1)\, U}{T}}. \tag{10}$$

This maximum value is obtained for the wave number k_m:

$$k_m = \frac{1}{\sqrt{3}} \sqrt{\frac{(\mu_2 - \mu_1)\, U}{T}}. \tag{11}$$

It is possible to see that the surface tension T may decrease the value of σ_m. We have to prove that the presence of the intermediate region (i.r.) gives us a smaller maximum value of σ. An "optimal" viscosity in the (i.r.) is used to minimize the "fingering" phenomenon.

The boundary conditions which must be added for the equation (7) are obtained by using the relation (8) on the interfaces $x = -l$ and $x = 0$:

$$\mu_0(0)\, f_x(0) = \{-k\mu_2 + Uk^2(\mu_2 - \mu_0(0))/\sigma - Tk^2/\sigma\} f(0), \tag{12}$$

$$\mu_0(0)\, f_x(-l) = \{k\mu_1 + Uk^2(\mu_1 - \mu_0(-l))/\sigma\} f(-l). \tag{13}$$

We consider the surface tension T only on the interface $x = 0$ (with the oil), as in [3]. The equation (7) and the conditions (12), (13) give us a Sturm-Liouville problem. The particularity of this problem consists in the presence of the eigenvalues $1/\sigma$ in the boundary conditions.

We introduce the following adimensional quantities

$$\sigma^* = \frac{2\sigma}{3\sqrt{3}\,\sigma_m}, \quad \mu^*(x) = \mu_0(x)/\mu_1, \quad k^* = k/\left(k_m\sqrt{3}\right), \quad \alpha = \mu_2/\mu_1,$$

$$x^* = k_m x\sqrt{3}, \quad L = k_m l\sqrt{3}, \quad f^*(x) = f(x)/U, \quad \lambda = 1/\sigma. \tag{14}$$

The surface tension T is considered only at $x = 0$; therefore $\mu(x)$ is discontinuous only for $x = 0$; at $x = -l$ we impose the continuity of μ, $\mu(-L) = 1$. Therefore the relation (13) has a simpler form. By using (14) (we omit the * for simplicity), the system (7)-(12)-(13) may be written as:

$$-(\mu f_x)_x + k^2 \mu f = \lambda k^2 \beta \mu_x . f, \quad x \in\,]-L, 0[\tag{15}$$

$$f_x(0) = (\lambda a + b) f(0)$$

$$f_x(-L) = k . f(-L)$$

with the following expressions for a, b, β:

$$a = k^2 \beta\{\alpha - \mu(o) - k^2(\alpha - 1)\}/\mu(0), \tag{16}$$

$$b = -k . \alpha/\mu(o), \quad \beta = \frac{\alpha + 1}{\alpha - 1}.$$

We consider a given quantity of polymer contained in (i.r.); as μ is invertible with respect to the concentration, we have the following restriction:

$$\int_{-L}^{0} \mu(x) = C \tag{17}$$

Let $g(L, \mu_L) = m(L, \mu_L)^{-1}$, where $m(L, \mu_L)$ is the smallest eigenvalue of the Sturm-Liouville problem (15), then:

$$\sigma \leqslant g(L, \mu_L) .$$

Therefore we have to find the injection length $\bar{L}$ and the viscosity $\bar{\mu}$ such that:

$$(P) \quad g(\bar{L}, \bar{\mu}) = Min\,[g(L, \mu)\,; \quad L \in \mathbf{R}_+, \quad \mu \in V] .$$

Here V is the space of the increasing functions of $C^1[-L, 0)$ which verify the conditions $\mu(0) \leqslant \alpha$, $\mu(-L) = 1$.

Moreover, we impose for μ the condition

$$[Ln(\mu)]_x \leqslant \frac{2\,.\,(\alpha - \mu(0))^{3/2}}{\alpha\sqrt{27(\alpha - 1)}} . \tag{17'}$$

3. LOCALIZATION OF THE THE EIGENVALUES OF THE DISCRETIZED PROBLEM

The problem (15) with the restrictions (17) is solved by using the finite-difference method. We consider the following discretization of the interval $]-L, 0[$ by the points x_i:

$$x_i = -i\,.\,h, \quad h = \Delta x = \frac{L}{M}, \quad i = 0, 1, 2, \ldots, M .$$

The derivative of a function $g(x)$ is approximated by:

$$\frac{dg}{dx}(x_i) \approx \frac{g(x_i + h/2) - g(x_i - h/2)}{h} .$$

In the sequel, we denote by $d/dx = {}'$ the derivative with respect to x.

The condition at $x = 0$ for the problem (15) is:

$$\frac{f(x_0) - f(x_1)}{h} \approx (\lambda\,.\,a + b) f(x_0)$$

or (denoting by f_i the approximated value of $f(x_i)$, $i = 0, 1, 2, \ldots, M$)

$$\frac{1 - bh}{ha} f_0 - \frac{1}{ha} f_1 = \lambda f_0 \tag{18}$$

where

$$\frac{1 - bh}{h^2}(\mu_{1/2}) f_0 - \frac{1}{h^2}(\mu_{1/2}) f_1 = \lambda \frac{a\mu_{1/2}}{h} f_0 , \tag{18'}$$

with the following notations:

$$x_{i+1/2} = -(i + 1/2)\, h = -ih - h/2 = x_i - h/2$$

$$x_{i-1/2} = -(i - 1/2)\, h = -ih + h/2 = x_i + h/2$$

$$\mu_{i+1/2} = \mu(x_i + 1/2), \quad \mu_{i-1/2} = \mu(x_i - 1/2)\,.$$

We have, for $x_i \in \,]-L, 0[$:

$$-(\mu f')'(x_i) \approx \frac{\mu_{i-1/2}(f_{i-1} - f_i) - \mu_{i+1/2}(f_i - f_{i+1})}{h^2}$$

and we obtain the discretization of (15):

$$-\frac{1}{h^2}(\mu_{i-1/2}) f_{i-1} + \frac{1}{h^2}(\mu_{i-1/2} + \mu_{i+1/2}) f_i - \frac{1}{h^2}(\mu_{i+1/2}) f_{i+1}$$

$$+ k^2 \mu_i f_i = \lambda k^2 \beta \mu_i' f_i,\; i = 1, 2, \ldots, (M-2)\,. \tag{19}$$

The last equation, for $x_{M-1,}$ is of a particular form; we use the condition at $x = -L$ of the problem (15) as follows:

$$\frac{f(x_{M-1}) - f(x_M)}{h} \approx k \,.\, f(x_M)$$

or

$$f_{M-1} - (1 + kh) f_M = 0\,,$$

where

$$-\frac{1}{h^2}(\mu_{M-1/2}) f_{M-1} + \frac{1 + kh}{h^2}(\mu_{M-1/2}) f_M = 0\,. \tag{20}$$

The above relation and relation (19) (for $i = M - 1$) are used to obtain the equation for the "last" point:

$$-\frac{1}{h^2}(\mu_{M-3/2}) f_{M-2} + \frac{1}{h^2}\left[\mu_{M-3/2} + \mu_{M-1/2}\left(1 - \frac{1}{1+kh}\right)\right] f_{M-1}$$

$$+ k^2(\mu_{M-1}) f_{M-1} = \lambda k^2 \beta(\mu'_{M-1}) f_{M-1}\,. \tag{21}$$

Finally, we get the system:

$$\begin{pmatrix} \delta & -u_1 & 0 & 0 & \cdot & 0 & 0 & 0 \\ -u_1 & \gamma_1 & -u_2 & 0 & \cdot & 0 & 0 & 0 \\ 0 & -u_2 & \gamma_2 & -u_3 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & -u_{M-2} & \gamma_{M-2} & -u_{M-1} \\ 0 & 0 & 0 & 0 & \cdot & 0 & -u_{M-1} & \Gamma \end{pmatrix} \cdot \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_{M-2} \\ f_{M-1} \end{pmatrix}$$

$$= \lambda \cdot \begin{pmatrix} s & 0 & 0 & \cdot & 0 & 0 \\ 0 & w_1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & w_2 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & 0 & w_{M-1} \end{pmatrix} \cdot \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \cdot \\ \cdot \\ f_{M-2} \\ f_{M-1} \end{pmatrix}$$

with the notations:

$$\delta = \frac{(1 - hb)\,\mu_{1/2}}{h^2}, \quad s = \frac{a\mu_{1/2}}{h}, \quad u_i = \frac{\mu_{i-1/2}}{h^2}, \quad i = 1, 2, \ldots, (M-1)$$

$$\gamma_i = \frac{\mu_{i-1/2} + \mu_{i+1/2}}{h^2} + k^2 \mu_i, \quad i = 1, 2, \ldots, (M-2)$$

$$\Gamma = \frac{\mu_{M-3/2} + \left(1 - \frac{1}{1 + kh}\right) \mu_{M-1/2}}{h^2} + k^2 \mu_{M-1},$$

$$w_i = k^2 \beta \mu_i', \quad i = 1, 2, \ldots, (M-1)$$

The above discretized system may be written as:

$$Af = \lambda Df \tag{22}$$

where D is a diagonal matrix and f is the vector with components f_i, $i = 0, 1, 2, \ldots, M-1$.

The matrices A and D are symmetric, therefore the eigenvalues λ are real.

We have the conditions: $\mu > 0$ and $\mu' > 0$, then D is positive for $a > 0$. Therefore, an upper bound is obtained for the wave numbers which give us the positive eigenvalues. By using (16), D is positive definite iff:

$$k^2 \leqslant \frac{\alpha - \mu(0)}{\alpha - 1} = k_{cr}^2 . \tag{23}$$

The matrix A is "diagonal dominant", hence it is positive definite (we emphasize that $b < 0$).

It is possible to write the system (22) as follows

$$Mf = \lambda f \quad \text{with} \quad M = D^{-1} A . \tag{24}$$

The Gerschgorin's theorem gives us the localization of the eigenvalues λ_i, $i = 1, 2, \dots, M$ of the system (24). These eigenvalues are contained in the union of the intervals:

$$\left|\lambda - \frac{\delta}{s}\right| \leqslant \frac{u_1}{s} \tag{25}$$

$$\left|\lambda - \frac{\gamma_i}{w_i}\right| \leqslant \frac{u_i + u_{i+1}}{w_i}, \quad i = 1, 2, \dots, (M-2)$$

$$\left|\lambda - \frac{\Gamma}{w_{M-1}}\right| \leqslant \frac{u_{M-1}}{w_{M-1}}.$$

We can obtain an "explicit" form of (25) in terms of a and μ:

$$\left|\lambda - \frac{1 - bh}{ha}\right| \leqslant \frac{1}{ah}$$

$$\left|\lambda - \frac{h^2 k^2 \mu_i + \mu_{i-1/2} + \mu_{i+1/2}}{\theta\mu'_i}\right| \leqslant \frac{\mu_{i-1/2} + \mu_{i+1/2}}{\theta\mu'_i}, i = 1, 2, \dots, (M-2)$$

$$\left|\lambda - \frac{\mu_{M-3/2} + \left(1 - \frac{1}{1 + kh}\right)\mu_{M-1/2} + h^2 k^2 \mu_{M-1}}{\theta\mu'_{M-1}}\right| \leqslant \frac{\mu_{M-3/2}}{\theta\mu'_{M-1}}$$

where $\theta = h^2 k^2 \beta$.

We can also obtain the following form for the system (25):

$$\left|\lambda - \frac{1 - bh}{ha}\right| \leqslant \frac{1}{ha} \tag{26}$$

$$\left|\lambda - \frac{\mu_{i-1/2} + \mu_{i+1/2}}{\theta\mu'_i} - \frac{\mu_i}{\beta\mu'_i}\right| \leqslant \frac{\mu_{i-1/2} + \mu_{i+1/2}}{\theta\mu'_i}, \quad i = 1, 2, \dots, (M-2)$$

$$\left|\lambda - \frac{\mu_{M-3/2} + \mu_{M-1/2}}{\theta\mu'_{M-1}} - \frac{\mu_{M-1}}{\beta\mu'_{M-1}} + \frac{\mu_{M-1/2}}{\theta\mu'_{M-1}(1 + kh)}\right| \leqslant \frac{\mu_{M-3/2}}{\theta\mu'_{M-1}}.$$

4. THE OPTIMAL VISCOSITY PROFILES

The growth constant of the perturbed problem is given by $\sigma = 1/\lambda$. By using the relations (26) we obtain the following estimates:

$$\sigma \leqslant Max\left\{\frac{k\beta}{\alpha}(\alpha - \mu(0) - k^2(\alpha - 1)); \frac{\beta\mu'_i}{\mu_i}, i = 1, \dots, (M-2); \left[\frac{\mu_{M-1}}{\beta\mu'_{M-1}} + \frac{\mu_{M-1/2}}{\theta\mu'_{M-1}} \cdot \frac{kh}{1 + kh}\right]^{-1}\right\} \tag{27}$$

or

$$\sigma \leqslant Max\left\{\frac{k\beta}{\alpha}(\alpha - \mu(0) - k^2(\alpha - 1)); \frac{\beta\mu'_i}{\mu_i}, i = 1, \dots, M-1\right\} = g(L, \mu_L). \tag{27'}$$

The maximum of the function

$$k \to H(k) = \frac{k\beta}{\alpha}\{\alpha - \mu(0) - k^2(\alpha - 1)\}$$

is obtained for k_0 such that:

$$k_0^2 = \frac{\alpha - \mu(0)}{3(\alpha - 1)}, \tag{28}$$

and is equal to

$$F(\mu(0)) = \frac{2}{3}\frac{\beta}{\alpha} \cdot \frac{(\alpha - \mu(0))^{3/2}}{\sqrt{3(\alpha - 1)}}. \tag{29}$$

We emphasize that the condition (23) is verified. Therefore:

$$\sigma \leqslant Max\left\{F(\mu(0))\,; \frac{\beta\mu_i'}{\mu_i}, i = 1, \dots, (M-1)\right\}$$

We use now the condition (17') and get:

$$\beta\frac{\mu_i'}{\mu_i} \leqslant F(\mu(0))\,; i = 1, 2, \dots, (M-1), \tag{30}$$

therefore

$$\sigma \leqslant g(L, \mu_L) \leqslant F(\mu(0)) = \frac{2}{3}\frac{\beta}{\alpha} \cdot \frac{(\alpha - \mu(0))^{3/2}}{\sqrt{3(\alpha - 1)}}.$$

It is possible to write the restriction (17') as follows:

$$(Ln\mu)' \leqslant F(\mu(0))/\beta, \tag{30'}$$

and integrating on $[-L, x]$ we obtain

$$\int_{-L}^{x}(Ln\mu)'\,ds \leqslant \int_{-L}^{x}\frac{F(\mu(0))}{\beta}\,ds, x \in\ ]-L, 0[\ .$$

By using the condition $\mu(-L) = 1$ we get:

$$\mu(x) \leqslant \exp\left\{(x + L) \cdot \frac{F(\mu(0))}{\beta}\right\}. \tag{31}$$

We can obtain the following form of the condition (17):

$$\int_{-L}^{0}\exp\left[(x + L)\frac{F(\mu(0))}{\beta}\right]dx \geqslant C$$

and therefore we get

$$L \geqslant \frac{\beta}{F(\mu(0))} \cdot Ln\left\{\frac{CF(\mu(0))}{\beta} + 1\right\} = \bar{L}. \tag{32}$$

The relation (31) implies

$$\mu(0) \leqslant \exp\left\{\frac{L}{\beta} \cdot F(\mu(0))\right\},$$

therefore we obtain

$$1 \leqslant \mu(0) \leqslant \mu_c \leqslant \alpha .$$

Here μ_c is the solution of the equation

$$\mu_c = \exp\left\{\frac{\bar{L}}{\beta} \cdot F(\mu_c)\right\},$$

or, in an equivalent form,

$$\mu_c = \frac{CF(\mu_c)}{\beta} + 1 = C\frac{2}{3\,\alpha}\frac{(\alpha - \mu_c)^{3/2}}{\sqrt{3(\alpha - 1)}} + 1 . \tag{33}$$

Therefore, the optimal viscosity profiles are defined in the interval $[-\bar{L}, 0]$ such that

$$\bar{L} = \frac{\beta}{F(\mu_c)} \quad Ln\left[\frac{CF(\mu_c)}{\beta} + 1\right]$$

and verify the relation

$$\mu(x) \leqslant \exp\left\{(x + L)\frac{F(\mu_c)}{\beta}\right\} . \tag{31'}$$

We recall that μ increases, and that μ_c is the solution of (33).

5. CONCLUSIONS

The utilization of the (i.r.) give us a smaller value for σ than those obtained by Saffman and Taylor (without intermediate region).

To obtain an improvement, we can use the first of relations (14) and the relation (30). We emphasize that the above results were obtained with adimensional quantities, therefore we get:

$$\sigma = \frac{3\,\sigma_m\,\sigma^*\sqrt{3}}{2} \leqslant \sigma_m \Leftrightarrow \sigma^* \leqslant \frac{2}{3\sqrt{3}} . \tag{34}$$

It is possible to see that an improvement is obtained if the following condition is verified

$$F(\mu(0)) \leqslant \frac{2}{3\sqrt{3}} . \tag{35}$$

The above relation may be given in terms of $\mu(0)$, and the improvement condition is

$$\alpha - (\alpha - 1)\left(\frac{\alpha}{\alpha + 1}\right)^{2/3} \leqslant \mu(0) . \tag{36}$$

In conclusion, for $\mu(0) = \mu_c$, the relation (31') gives us an upper bound for the "minimizing" viscosity and (32) gives us a lower bound for the injection length of the (i.r.).

The "dangereous" interval with respect to the wave number k is contained in the finite region defined by (23), if the conditions (30) are verified: in this region the eigenvalues are positive.

It is possible to explain the lack of agreement between the numerical results of [6] and [3], when the restrictions (30) are not verified. Indeed, in this situation, one of the quantities $v_i = (\beta\mu_i')/(\mu_i)$ may be larger than $F(\mu(0))$; then the largest value of σ is equal to v_i and is obtained for large values of k. We have proved that if we impose the "restrictions" (30), then $F(\mu(0))$ (which doesn't depend on k) is an upper bound for the growth constants. In this way, the maximum value for σ is obtained for the finite value k_0.

REFERENCES

[1] P. G. SAFFMAN, G. TAYLOR (1958), The penetration of a fluid in a porous medium or Hele-Shaw cell containing a more viscous liquid, *Proc. Roy. Soc.*, **A245,** pp. 312-329.

[2] R. L. CHOUKE, P. Van der MEURS, C. Van der POEL (1959), The stability of a slow, immiscible, viscous liquid-liquid displacements in a permeable media, *Trans AIME,* **216,** pp. 188-194.

[3] S. B. GORELL and G. M. HOMSY (1983), A theory of the optimal policy of oil recovery by the secondary displacement process, *SIAM J. Appl. Math.,* **43,** pp. 79-98.

[4] G. I. PASA and D. A. POLIŠEVSKI (1992), Instability of interfaces in oil recovery, *Int. J. Engng. Sci.,* **30,** pp. 161-167.

[5] G. I. PASA (1994), An existence theorem for a problem in oil recovery, *Rev. Roum. Mec. Appl.,* **39,** pp. 72-81.

[6] G. I. PASA (1994), A numerical characterization of the growth constant in oil recovery, Preprint IMAR No. 43.

[7] G. I. PASA (1996), An existence theorem for a control problem in oil recovery, *Numer. Funct. Analysis and Optimiz.*, **17,** pp. 921-933.