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A SCHWARZ ADDITIVE METHOD WITH HIGH ORDER INTERFACE CONDITIONS AND NONOVERLAPPING SUBDOMAINS (*)

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Abstract — *We prove the convergence of a Schwarz additive method for a nonoverlapping decomposition into rectangles with interface conditions of order two in the tangential direction* © *Elsevier, Paris*

Résumé — *Nous prouvons la convergence d'une méthode de Schwarz additive pour une décomposition sans recouvrement en rectangles avec des conditions d'interface d'ordre 2 dans la direction tangente à l'interface* © *Elsevier, Paris*

1. INTRODUCTION

The rate of convergence of Schwarz type algorithms is very sensitive to the choice of the interface conditions. The original Schwarz method is based on the use of Dirichlet boundary conditions. In order to increase the efficiency of the algorithm, it has been proposed to replace the Dirichlet boundary conditions by more general boundary conditions, see [7] (or in a different context [4]). Choosing artificial boundary conditions as interface conditions is a good choice. In [9], it is shown that using exact artificial boundary conditions leads in some situations to the convergence of the Schwarz method in a number of steps equals to the number of subdomains. The use of such interface conditions is then optimal. Unfortunately, the exact artificial boundary conditions are non local in space and they have to be approximated at various orders by partial differential operators using techniques developed for artificial boundaries, see e.g. [2].

In this paper, we consider a low wave number approximation of the exact artificial boundary conditions involving second order tangential derivatives, Ventcell boundary conditions (see e.g. [2]). We prove convergence for a decomposition of the domain into rectangles. The main motivation for considering such interface conditions is that they lead to a much faster convergence than Fourier-Robin boundary conditions (see [8], [10] for numerical results). To our knowledge, our result of convergence is the first one of this kind. Indeed, in previous works, either the geometry is simpler (decomposition into strips) or the interface boundary conditions are of Fourier-Robin type. In [8], Ventcell boundary conditions are used for a domain decomposition method for the convection-diffusion equation. Convergence is proved only for a decomposition into strips. On the other hand, in [7] and [1], convergence is proved for an arbitrary decomposition of the domain. Fourier-Robin boundary conditions are considered and not Ventcell interface conditions.

The paper is organized as follows: in § 2 the algorithm is defined and notations are given. In § 3, the algorithm is proved to be well-posed. In § 4, convergence is proved by an energy method.

2. THE ALGORITHM

We consider the equation

$$\mathscr{L}(u) \equiv \frac{u}{\epsilon^2} - \Delta u = f \text{ in } \Omega_d,\ u = 0 \text{ on } \partial\Omega_d \tag{1}$$

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where $\Omega_d =]0, L_X[\times]0, H_Y[$, $\epsilon > 0$. This equation arises from an implicit semi-discretization in time of the heat equation with a time step $\Delta t = \epsilon^{-2}$. We want to solve (1) by a nonoverlapping additive Schwarz method, well suited to parallel computers, with interface conditions of order 2 with respect to the tangential direction

$$\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}$$

which is the local approximation of order 2 with respect to ϵ of the exact artificial boundary condition (see below). Such a choice of interface conditions is interesting. Indeed, consider the simplified case of the equation set on $\mathbf{R}^2$ decomposed into two half-planes, $\mathbf{R}^2_-$ and $\mathbf{R}^2_+$. The additive Schwarz method reads:

$$\mathscr{L}(u_1^{n+1}) = f \quad \text{in } \mathbf{R}^2_-, \quad \mathscr{B}_{12}(u_1^{n+1}) = \mathscr{B}_{12}(u_2^n) \text{ at } x = 0$$
$$\mathscr{L}(u_2^{n+1}) = f \quad \text{in } \mathbf{R}^2_+, \quad \mathscr{B}_{21}(u_2^{n+1}) = \mathscr{B}_{21}(u_1^n) \text{ at } x = 0$$

where $\mathscr{B}_{12} = \partial_x - \Lambda_{12}$ and $\mathscr{B}_{21} = \partial_x - \Lambda_{21}$. The operators Λ_{12} and Λ_{21} are convolution operators acting only in the y variable which are to be chosen. Let $\lambda_{12}(k)$ (resp. $\lambda_{21}(k)$) be the symbol of $\Lambda_{12}(k)$ (resp. Λ_{21}) in the Fourier space (k is the dual variable of y for the Fourier transform in the y direction). By performing a Fourier transform in the y direction, the convergence rate in the Fourier space can easily be computed:

$$\rho(k) = \left(\frac{\lambda_{12}(k) - \lambda(k)}{\lambda_{12}(k) + \lambda(k)}\right)\left(\frac{\lambda_{21}(k) - \lambda(k)}{\lambda_{21}(k) + \lambda(k)}\right),$$

where $\lambda(k) = \sqrt{k^2 + 1/\epsilon^2}$.

Let Λ be the operator of symbol $\lambda(k)$. Thus, taking $\Lambda_{12} = \Lambda_{21} = \Lambda$ leads to an optimal convergence rate $\rho(k) = 0$. This amounts to using as interface conditions, the exact artificial boundary conditions (for more details see e.g. [10]). Since $\lambda(k)$ is not a polynomial in k, this leads to using as interface conditions pseudodifferential operators. As for artificial boundary conditions, in order to avoid the complexity and the cost of using Fourier transform in a code, we shall use approximations of Λ by partial differential operators (see [2]). They are obtained by approximating the symbol $\lambda(k)$ by its Taylor expansion in the vicinity of $k = 0$: $\lambda_{12}(k) = \lambda_{21}(k) = 1/\epsilon + \epsilon k^2/2$. In the physical space, this means we take Ventcell boundary conditions as interface conditions:

$$\mathscr{B}_{12} = \partial_x - \left(\frac{1}{\epsilon} - \frac{\epsilon}{2}\partial_{yy}\right) \quad \text{and} \quad \mathscr{B}_{21} = \partial_x - \left(\frac{1}{\epsilon} - \frac{\epsilon}{2}\partial_{yy}\right).$$

The choice of this Taylor approximation is natural since:

- $\lambda(k)$ has a polynomial behaviour in the vicinity of $k = 0$.
- there is a truncation in frequency due to the discretization in space of the equation which is necessary when the equation is solved on a computer.

For a decomposition of the domain into strips, the convergence of the additive Schwarz method with Ventcell boundary conditions as interface conditions has been proved in [10]. The goal of this paper is to extend this result to a decomposition into rectangles. The domain Ω_d is decomposed into rectangles:

$$\overline{\Omega_d} = \overline{\cup_{i,j}]l_i, L_i[\times]h_j, H_j[} = \overline{\cup_{i,j} \Omega_{i,j}}.$$

In order to define the additive Schwarz method, boundary conditions at the corners of the rectangles have to be used. Our proof of convergence led us to consider the jump of the tangential derivative at the corners. Before defining the algorithm, we need some notations.

Notations In dealing with boundary value problems on rectangles with mixed boundary conditions, we shall make a constant use of some notations (see [3]).

Let Ω be the rectangle $]l, L[\times]h, H[$. We denote

$$\Gamma_1 =]l, H[\times\{h\}, \Gamma_2 = \{L\}\times]h, H[, \Gamma_3 =]l, L[\times\{H\}, \Gamma_4 = \{l\}\times]h, H[$$

and $\Gamma = \cup_i \Gamma_i$. The segments are thus numbered in such a way that $\Gamma_{i+1}(\Gamma_5 = \Gamma_1)$ follows Γ_i according to the positive orientation.

We denote by S_i the vertex which is the endpoint of Γ_i:

$$S_1 = (L, h), \quad S_2 = (L, H), \quad S_3 = (l, H) \quad \text{and} \quad S_4 = (l, h).$$

Furthermore n_i (resp. τ_i) is the unit outward normal (resp. tangent) vector on Γ_i so that (n_i, τ_i) is positively oriented.

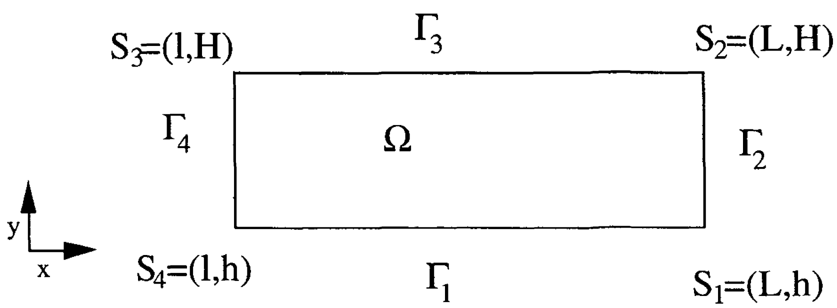

Figure 1. — Notation.

We denote by $(x_i(\sigma), y_i(\sigma))$ the point of Γ which, for small enough $|\sigma|$ is at distance σ (counted algebraically) of S_i along $\partial\Omega$. Consequently $(x_i(\sigma), y_i(\sigma)) \in \Gamma_i$ when $\sigma < 0$ and $(x_i(\sigma), y_i(\sigma)) \in \Gamma_{i+1}$ when $\sigma > 0$. We say that two functions ϕ_j and ϕ_{j+1} defined on Γ_i and Γ_{i+1} respectively are equivalent at S_i if

$$\int_0^{\delta_i} |\phi_i(x_i(-\sigma), y_i(-\sigma)) - \phi_{i+1}(x_i(\sigma), y_i(\sigma))|^2 / \sigma \, d\sigma < \infty .$$

for some $\delta_i > 0$. We shall then write

$$\phi_i \equiv \phi_{i+1} \text{ at } S_i .$$

In considering mixed boundary conditions, it will be convenient to fix a partition of $\{1, 2, 3, 4\}$ in two subsets $\mathscr{D}$ and $\mathscr{A}$. The union of the Γ_i with $i \in \mathscr{D}$ (resp. $\mathscr{A}$) is going to be the boundary where we consider a Dirichlet (resp. artificial) boundary conditions. We have either $u = 0$ on Γ_i if $i \in \mathscr{D}$ or, if $i \in \mathscr{A}$

$$\frac{\partial u}{\partial n_i} + \frac{u}{\epsilon} - \frac{\epsilon}{2} \frac{\partial^2 u}{\partial \tau_i^2} = g_i$$

for some $g_i \in L^2(\Gamma_i)$. Accordingly and concerning corners, we fix, $\mathscr{Ac}$, a subset of $\{1, 2, 3, 4\}$ so that corner conditions are written on S_i, $i \in \mathscr{Ac}$. The set $\mathscr{Ac}$ is such that $\cup_{i \in \mathscr{Ac}} \{S_i\}$ is the set of vertices which do not touch an edge Γ_i with $i \in \mathscr{D}$. We define for $m = 1$ or 2

$$\mathscr{H}^m(\Omega) = \{u \in H^m(\Omega) / u_{|\Gamma_i} \in H^m(\Gamma_i) \text{ for } i \in \mathscr{A} \text{ and } u_{|\Gamma_i} = 0 \text{ for } i \in \mathscr{D}\}$$

which, endowed with its natural norm $\|u\|_{\mathscr{H}^m} = \sqrt{\|u\|^2_{H^m} + \sum_{i \in \mathscr{A}} \|u_{|\Gamma_i}\|^2_{H^m(\Gamma_i)}}$ and the associated scalar product, is a Hilbert space.

With these notations, we can define the algorithm:

DEFINITION 1: *Let $u_{i,j}^n$ be an approximation to u at step n in the interior subdomain $\Omega_{i,j}$, $u_{i,j}^{n+1}$ is defined by:*

$$\mathcal{L}(u_{i,j}^{n+1}) = f \text{ in } \Omega_{i,j}$$

$$\left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u_{i,j}^{n+1}) = \left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u_{i+1,j}^n) \text{ on } \Gamma_{i,j,2}$$

$$\left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u_{i,j}^{n+1}) = \left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u_{i-1,j}^n) \text{ on } \Gamma_{i,j,4}$$

$$\left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u_{i,j}^{n+1}) = \left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u_{i,j+1}^n) \text{ on } \Gamma_{i,j,3}$$

$$\left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u_{i,j}^{n+1}) = \left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u_{i,j-1}^n) \text{ on } \Gamma_{i,j,1}$$

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u_{i,j}^{n+1}) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u_{i+1,j}^n) \text{ at } (x,y) = (L_i, h_j)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(u_{i,j}^{n+1}) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u_{i+1,j}^n) \text{ at } (x,y) = (L_i, H_j)$$

$$\left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(u_{i,j}^{n+1}) = \left(-\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(u_{i-1,j}^n) \text{ at } (x,y) = (l_i, H_j)$$

$$\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u_{i,j}^{n+1}) = \left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u_{i-1,j}^n) \text{ at } (x,y) = (l_i, h_j)\,.$$

For the other subdomains, the definition is similar except on $\partial\Omega_d \cap \partial\Omega_{i,j}$ where $u_{i,j}^{n+1} = 0$.

3. WELL-POSEDNESS OF THE ALGORITHM

The Schwarz algorithm has been defined above in Definition 1. The following theorem shows that it is well posed in $\prod_{i,j} \mathcal{H}^2(\Omega_{i,j})$.

THEOREM 2: *Let $l < L$, $h < H \in \mathbf{R}$, $\Omega =]l, L[\times]h, H[$, $f \in L^2(\Omega)$, $g_i \in L^2(\Gamma_i)$ for $i \in \mathcal{A}$ There exists a unique $u \in \mathcal{H}^2(\Omega)$ satisfying:*

$$\mathcal{L}(u) = f \text{ in } \Omega$$

$$\left(\frac{\partial}{\partial n} + \frac{1}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2}{\partial \tau^2}\right)(u) = g_i \text{ on } \Gamma_i, i \in \mathcal{A}$$

$$\left(\frac{\partial}{\tau_i} - \frac{\partial}{\tau_{i+1}}\right)(u) = h_i \text{ at } S_i, i \in \mathcal{A}c\,.$$

Proof: We first consider the variational formulation in $\mathcal{H}^1(\Omega)$ of the above boundary value problem: Find $u \in \mathcal{H}^1(\Omega)$ such that:

$$\iint_\Omega \frac{uv}{\epsilon^2} + \nabla u \,.\, \nabla v + \sum_{i \in \mathcal{A}} \int_{\Gamma_i} \frac{uv}{\epsilon} + \frac{\epsilon}{2}\frac{\partial u}{\partial \tau_i}\frac{\partial v}{\partial \tau_i} = \iint fv + \sum_{i \in \mathcal{A}} \int_{\Gamma_i} g_i\, v + \frac{\epsilon}{2}\sum_{i \in \mathcal{A}c} h_i\, v(S_i),\ \forall v \in \mathcal{H}^1(\Omega)\,. \tag{2}$$

The term $v(S_i)$ makes sense. Indeed, v is a continuous function on each edge since $v_{|\Gamma_i} \in H^1(\Gamma_i)$, $i = 1, \dots, 4$. Moreover, since $v \in H^1(\Omega)$ we have near S_i $(i = 1, \dots, 4)$ that

$$\int_0^\delta |v_i(x_i(-\sigma), y_i(-\sigma)) - v_{i+1}(x_i(\sigma), y_i(-\sigma))|^2 / \sigma \, d\sigma < \infty$$

for some $\delta > 0$ (see e.g. [3]). Thus, v as a function of the boundary of Ω is continuous at S_i and (2) is well defined.

LEMMA 3: *Problem (2) is well posed.*

Proof: The result follows from an easy application of the Lax-Milgram theorem in the Hilbert space $\mathscr{H}^1(\Omega)$. □

It remains to prove the $\mathscr{H}^2(\Omega)$-regularity. Our proof follows that of [5] where the case $h_i = 0$ was considered. We use interpolation results of [6] and regularity results for elliptic problems on nonsmooth domains of [3]. We will proceed in three steps.

Step 1. *Let u denote the solution to problem (2). On each edge Γ_i, $i = 1, \dots, 4$, $u_{|\Gamma_i} \in H^{3/2}(\Gamma_i)$.*

Proof: For $i \in \mathscr{D}$, the statement is obvious since $u_{|\Gamma_i} = 0$.

Otherwise, in the sense of distributions, we have

$$\frac{u}{\epsilon^2} - \Delta u = f \text{ in } \Omega .$$

Since $\Delta u \in L^2(\Omega)$ and $u \in H^1(\Omega)$, we have (see [3]) that $\frac{\partial u}{\partial n} \in \tilde{H}^{-1/2}(\Gamma_i)$, $i = 1, \dots, 4$ where $\tilde{H}^{-1/2}(\,]s, t[\,)$ is the dual of

$$\tilde{H}^{1/2}(\,]s, t[\,) = \left\{ u \in H_0^{1/2}(\,]s, t[\,) \text{ s.t. } u(y)/\sqrt{(t-y)(y-s)} \in L^2(\,]s, t[\,) \right\} .$$

Hence in the sense of distributions

$$\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2} \frac{\partial^2 u}{\partial \tau_i^2} = g_i \text{ on } \Gamma_i,\ i = 1 \in \mathscr{A} .$$

and $\partial^2 u / \partial \tau_i^2 \in \tilde{H}^{-1/2}(\Gamma_i)$, $i \in \mathscr{A}$. Let Pr^2 denote a right inverse to $\frac{\partial^2 u}{\partial \tau_i^2}$. The operator Pr^2 is continuous from $H^{-1}(\Gamma_i)$ into $H^1(\Gamma_i)$ and from $L^2(\Gamma_i)$ into $H^2(\Gamma_i)$. Thus, by interpolation, Pr^2 is continuous from $\tilde{H}^{-1/2}(\Gamma_i)$ into $H^{3/2}(\Gamma_i)$. Since Pr^2 is unique up to an affine function, $u_{|\Gamma_i} \in H^{3/2}(\Gamma_i)$, $i \in \mathscr{A}$.

Step 2. *Let u denote the solution to problem (2). Then, $u \in H^2(\Omega)$.*

Proof: It follows from the fact that $u \in H^{3/2}(\Gamma_i)$, u as a function of the boundary is continuous at the vertices S_i and regularity results for boundary value problems on polygon (see [3], p. 58).

Step 3. *Let u denote the solution to problem (2). $u_{|\Gamma_i} \in H^2(\Gamma_i)$, $i = 1, \dots, 4$.*

Proof: From $u \in H^2(\Omega)$, it follows that for $i = 1, \dots, 4$, $\partial u / \partial n \in H^{1/2}(\Gamma_i)$. Thus, $\partial^2 u / \partial \tau_i^2 \in L^2(\Gamma_i)$. From standard regularity results, we have $u_{|\Gamma_i} \in H^2(\Gamma_i)$.

Then, it is easy to check that u is also the solution to the problem stated in Theorem 2. □

4. CONVERGENCE PROOF

The proof lies on the energy estimate of Lemma 6. In order to prove it, we shall need two results.

THEOREM 4: *$H^m(\,]l, L[\times]h, H[\,) \cap \mathscr{H}^2(\,]l, L[\times]h, H[\,)$ is dense into $\mathscr{H}^2(\,]l, L[\times]h, H[\,)$ for $m \geqslant 4$.*

Proof: The proof is given in the Annex.

LEMMA 5: *For all* $v \in \mathscr{H}^2(\,]l, L[\, \times \,]h, H[\,)$, *we have*

$$\iint_{]l,L[\times]h,H[} \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} = \iint_{]l,L[\times]h,H[} \left(\frac{\partial^2 v}{\partial x\,\partial y}\right)^2 - \int_{\Gamma_1 \cup \Gamma_3} \frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x\,\partial n} + \int_{\Gamma_2 \cup \Gamma_4} \frac{\partial v}{\partial n}\frac{\partial^2 v}{\partial y^2}$$

Proof: By Theorem 4, it suffices to prove the equality for $v \in H^4 \cap \mathscr{H}^2$. The equality is obtained by integrating by parts first in the x direction and then in the y direction.

We can now prove

LEMMA 6: *Let* $u \in \mathscr{H}^2(\,]l, L[\, \times \,]h, H[\,)$ *such that*

$$\frac{u}{\epsilon^2} - \Delta u = 0\,.$$

Then, we have the following energy estimate:

$$\begin{aligned}
&\iint \frac{3\,u^2}{\epsilon^3} + 4\,\frac{|\nabla u|^2}{\epsilon} + \epsilon\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2\right)\\
&\quad - \int_\Gamma \left(\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau^2}\right)^2 - \left(-\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau^2}\right)^2\\
&\quad + \frac{\epsilon}{2}\left(\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2(L, h) - \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2(L, h)\right)\\
&\quad + \left(\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2(L, H) - \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2(L, h)\right)\\
&\quad + \left(\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2(l, H) - \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2(l, H)\right)\\
&\quad + \left(\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2(l, h) - \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2(l, h)\right) = 0\,.
\end{aligned}$$

Proof: Equation (1) is multiplied by $\frac{3\,u}{\epsilon} - \epsilon\,\Delta u$ and is integrated over $]l, L[\, \times \,]h, H[$:

$$\iint \frac{3\,u^2}{\epsilon^3} + 4\,\frac{|\nabla u|^2}{\epsilon} + \epsilon(\Delta u)^2 - \int \frac{4}{\epsilon}\,u\,\frac{\partial u}{\partial n} = 0$$

Lemma 5 applied to the integral of the term $\epsilon\,\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2}$ yields:

$$\begin{aligned}
&\iint \frac{3\,u^2}{\epsilon^3} + 4\,\frac{|\nabla u|^2}{\epsilon} + \epsilon\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2\right)\\
&\qquad - 2\,\epsilon\int_{\Gamma_1 \cup \Gamma_3} \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\,\partial n} + \int_{\Gamma_2 \cup \Gamma_4} 2\,\epsilon\,\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial y^2} - \int_\Gamma \frac{4}{\epsilon}\,u\,\frac{\partial u}{\partial n} = 0
\end{aligned}$$

By integrating by parts over $\Gamma_1 \cup \Gamma_3$, we obtain

$$\begin{aligned}
&\iint \frac{3\,u^2}{\epsilon^3} + 4\,\frac{|\nabla u|^2}{\epsilon} + \epsilon\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2\right) + \int_\Gamma 2\,\epsilon\,\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial \tau^2} - \frac{4}{\epsilon}\,u\,\frac{\partial u}{\partial n}\\
&\quad + 2\,\epsilon\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(L, h) - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(L, H) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(l, H) - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(l, h)\right) = 0
\end{aligned}$$

The boundary and corner terms can be written as differences of squares:

$$\iint \frac{3\,u^2}{\epsilon^3} + 4\,\frac{|\nabla u|^2}{\epsilon} + \epsilon\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2\right)$$

$$- \int_{\Gamma} \left(\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau^2}\right)^2 - \left(-\frac{\partial u}{\partial n} + \frac{u}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u}{\partial \tau^2}\right)^2$$

$$+ \frac{\epsilon}{2}\left(\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2 (L, h) - \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 (L, h)\right) + \left(\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 (L, h) - \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 (L, H)\right)$$

$$+ \left(\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2 (l, H) - \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 (l, H)\right) - \left(\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2 (l, h) - \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 (l, h)\right) = 0\,.$$

□

We can now prove the:

THEOREM 7: *Assume* $u^0_{i,j} \in \mathscr{H}^2(\Omega_{i,j})$.

Then, the additive Schwarz method (Definition 1) converges in $\mathscr{H}^2$.

Proof: We proceed as in [1]. Equation (1) and the additive Schwarz method are linear so that it suffices to take $f = 0$ and to prove the convergence to zero of $u^n_{i,j}$ as n tends to infinity. Let

$$E^n = \sum_{i,j} \iint \frac{3\,u^{n2}_{i,j}}{\epsilon^3} + 4\frac{|\nabla u^n_{i,j}|}{\epsilon} + \epsilon\left(\left(\frac{\partial^2 u^n_{i,j}}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u^n_{i,j}}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 u^n_{i,j}}{\partial x\,\partial y}\right)^2\right),$$

$$B^n = \sum_{i,j} \int_{\Gamma_{i,j}\backslash\partial\Omega_d} \left(-\frac{\partial u^n_{i,j}}{\partial n} + \frac{u^n_{i,j}}{\epsilon} - \frac{\epsilon}{2}\frac{\partial^2 u^n_{i,j}}{\partial \tau^2}\right)^2$$

and

$$C^n = \frac{\epsilon}{2} \sum_{i,j,\,l_i \neq 0,\, L_i \neq L_X,\, h_i \neq 0,\, H_i \neq H_Y} \left(\frac{\partial u^n_{i,j}}{\partial x} + \frac{\partial u^n_{i,j}}{\partial y}\right)^2 (L_i, h_j) + \left(\frac{\partial u^n_{i,j}}{\partial x} - \frac{\partial u^n_{i,j}}{\partial y}\right)^2 (L_i, H_j)$$

$$+ \left(\frac{\partial u^n_{i,j}}{\partial x} + \frac{\partial u^n_{i,j}}{\partial y}\right)^2 (l_i, H_j) + \left(\frac{\partial u^n_{i,j}}{\partial x} - \frac{\partial u^n_{i,j}}{\partial y}\right)^2 (l_i, h_j)\,.$$

The estimate of Lemma 6 and the definition of the algorithm show that we have

$$E^{n+1} + B^{n+1} + C^{n+1} = B^n + C^n\,.$$

Hence, after summation over n

$$\sum_n E^n \leqslant B^0 + C^0\,,$$

and $\lim_{n \to \infty} E_n = 0$. □

ANNEX

The goal of the annex is to prove

THEOREM A1: $H^m(\,]l, L[\, \times\,]h, H[\,) \cap \mathscr{H}^2(\,]l, L[\, \times\,]h, H[\,)$ *is dense into* $\mathscr{H}^2(\,]l, L[\, \times\,]h, H[\,)$ *for* $m \geqslant 4$.

Proof: The proof is adapted from [3]. We first define

$$\gamma : \mathscr{H}^2 \to \prod_{i=1}^{4} H^2(\Gamma_i) \times H^{1/2}(\Gamma_i)$$

$$u \mapsto \left(\gamma_i(u), \gamma_i\left(\frac{\partial u}{\partial n_i}\right)\right)_{1 \leqslant i \leqslant 4}$$

where γ_i is the trace operator on Γ_i. We know that $Ker(\gamma) = H_0^2(]l, L[\times]h, H[)$. Let

$$Z^2(\Gamma) = \left\{ (g_i, h_i)_{1 \leqslant i \leqslant 4} \in \prod_{i=1}^{4} H^2(\Gamma_i) \times H^{1/2}(\Gamma_i) / g_i(S_i) = g_{i+1}(S_i), \frac{\partial g_i}{\partial \tau_i} \equiv h_{i+1} \text{ at } S_i, \right.$$

$$\left. -\frac{\partial g_{i+1}}{\partial \tau_{i+1}} \equiv h_i \text{ at } S_i, i = 1, \ldots, 4 \text{ and } g_i = 0 \text{ for } i \in \mathscr{D} \right\}$$

We know (see e.g. [3]) that $Im(\gamma) \subset Z^2(\Gamma)$. Conversely, let $(g, h) = (g_i, h_i)_{1 \leqslant i \leqslant 4} \in Z^2(\Gamma)$, there exists $u \in \{u \in H^2(]l, L[\times]h, H[)/\gamma_i(u) = 0 \text{ on } \Gamma_i,\ i \in \mathscr{D}\}$ such that $\gamma(u) = (g, h)$. Since $\gamma_i(u) = g_i$, we have that $u \in \mathscr{H}^2$. Finally, $Im(\gamma) = Z^2(\Gamma)$. The vector space $Z^2(\Gamma)$ is endowed with the norm

$$\|a\|_\gamma = \inf_{u \in \mathscr{H}^2 / \gamma(u) = a} \|u\|_{\mathscr{H}^2}.$$

Since $Ker(\gamma) = H_0^2$ is a closed subspace of the Hilbert space $\mathscr{H}^2$, for each $a \in Z^2(\Gamma)$ there exists a unique $u \in \mathscr{H}^2$ such that $\|a\|_\gamma = \|u\|_{\mathscr{H}^2}$. Let ρ be a right inverse to γ defined as follows

$$\rho : Z^2 \to \mathscr{H}^2$$

$$a \mapsto u \text{ s.t. } \|a\|_\gamma = \|u\|_{\mathscr{H}^2}$$

The operator ρ is by definition a linear continuous operator. It is easy to check that $(Z^2(\Gamma), \|\ \|_\gamma)$ is a Hilbert space.

The vector space $Z^2(\Gamma)$, endowed with the norm

$$\|(g, h)\|_{Z^2}^2 = \sum_{i=1}^{4} \|g_i\|_{H^2(\Gamma_i)}^2 + \|h_i\|_{H^{1/2}(\Gamma_i)}^2 + |g_i(S_i) - g_{i+1}(S_i)|^2$$

$$+ \int_0^{\delta_i} \left| \frac{\partial g_i}{\partial \tau_i}(x_i(-\sigma), y_i(-\sigma)) - h_{i+1}(x_i(\sigma), y_i(\sigma)) \right|^2 / \sigma \, d\sigma$$

$$+ \int_0^{\delta_i} \left| \frac{\partial g_{i+1}}{\partial \tau_{i+1}}(x_i(\sigma), y_i(\sigma)) - h_i(x_i(-\sigma), y_i(-\sigma)) \right|^2 / \sigma \, d\sigma,$$

is also a Hilbert space. We show now that the norms $\|\ \|_\gamma$ and $\|\ \|_{Z^2}$ are equivalent. We know (see e.g. [3]) that there exists $K > 0$ such that $\forall u \in \mathscr{H}^2$, $\|\gamma(u)\|_{Z^2} \leqslant K\|u\|_{\mathscr{H}^2}$. Hence, $\forall a \in Z^2$, $\|a\|_{Z^2} \leqslant K\|a\|_\gamma$. Since $Z^2(\Gamma)$ is a Hilbert space, there exists $c > 0$ such that

$$c\|a\|_\gamma \leqslant \|a\|_{Z^2} \leqslant K\|a\|_\gamma.$$

Thus, $\mathscr{H}^2$ can be written as a direct sum

$$\mathscr{H}^2 = H_0^2 \otimes \rho(Z^2),$$

and any continuous linear form l on $\mathscr{H}^2$ can be represented as

$$\langle l, u\rangle = \langle l_1, u - \rho(\gamma(u))\rangle + \langle l_2, \gamma(u)\rangle$$

where $l_1 \in H^{-2}$ and $l_2 \in Z^{2'}$.

Let l be a linear form on $\mathscr{H}^2$ that vanishes on $H^m \cap \mathscr{H}^2$, $m \geqslant 4$. We show that l vanishes also on $\mathscr{H}^2$ and thus the dense inclusion of $H^m \cap \mathscr{H}^2$ in $\mathscr{H}^2$. The linear form l is decomposed as above into l_1 and l_2. The form l vanishes on $\mathscr{D}(\,]l, L[\, \times\,]h, H[\,) \subset H^m \cap \mathscr{H}^2$ and therefore we have $l_1 = 0$. In other words, $\langle l, u\rangle$ depends only on $\gamma(u)$.

In order to prove that the linear form l vanishes everywhere, it suffices to prove that $\gamma(H^m \cap \mathscr{H}^2)$ is dense into Z^2.

We first study $\gamma(H^m \cap \mathscr{H}^2)$. We know that

$$\gamma(H^m \cap \mathscr{H}^2) = \left\{ (g_i, h_i)_{1 \leqslant i \leqslant 4} \in \prod_{i=1}^{4} H^{m-1/2}(\Gamma_i) \times H^{m-3/2}(\Gamma_i) / g_i(S_i) = g_{i+1}(S_i), \right.$$

$$\left. \frac{\partial g_i}{\partial \tau_i} = h_{i+1} \text{ at } S_i,\ -\frac{\partial g_{i+1}}{\partial \tau_{i+1}} = h_i \text{ at } S_i,\ \frac{\partial h_i}{\partial \tau_i} + \frac{\partial h_{i+1}}{\partial \tau_{i+1}} = 0 \text{ for } i = 1, \dots, 4 \text{ and } g_i = 0 \text{ for } i \in \mathscr{D} \right\}.$$

In order to prove the density, we only have to look at things locally near each corner S_i depending on the kind of the corner. Let $(g_i, h_i, g_{i+1}, h_{i+1}) \in Z^2$ near S_i.

If we assume i and $i+1$ belong to $\mathscr{A}$, the functions $\sigma \mapsto \frac{\partial g_i}{\partial \tau_i}(x_i(-\sigma), y_i(-\sigma)) - h_{i+1}(x_i(\sigma), y_i(\sigma))$ and $\sigma \mapsto \frac{\partial g_{i+1}}{\partial \tau_{i+1}}(x_i(\sigma), y_i(\sigma)) + h_i(x_i(-\sigma), y_i(-\sigma))$ belong to $\tilde{H}^{1/2}(\mathbf{R}_+)$ near zero. There exist two sequences $(\alpha_n)_{n \in \mathbf{N}}$ and $(\beta_n)_{n \in \mathbf{N}}$ in $\mathscr{D}(\mathbf{R}_+)$ which converge to $\frac{\partial g_i}{\partial \tau_i} - h_{i+1}$ and $\frac{\partial g_{i+1}}{\partial \tau_{i+1}} + h_i$ respectively.

The function $g_i + g_{i+1}$ belongs to $H^2(\mathbf{R}_+)$ near zero. Let $(\delta_n)_{n \in \mathbf{N}} \in \mathscr{D}(\overline{\mathbf{R}}_+)$ converge to $g_i + g_{i+1}$ in H^2. The function $g_i - g_{i+1}$ belongs to $H^2 \cap H_0^1(\mathbf{R}_+)$ near zero. We use the

LEMMA A2: *The space*

$$\mathscr{D}_c = \{\lambda \in \mathscr{D}(\overline{\mathbf{R}}_+) / \lambda(0) = 0 \text{ and } \lambda''(0) = 0\}$$

is dense in $H^2 \cap H_0^1(\mathbf{R}_+)$.

Proof: Let $\eta \in \mathscr{D}_c$ s.t. $\eta'(0) = 1$. Let $u \in H^2(\mathbf{R}_+)$. The function $u - u'(0)\,\eta \in H_0^2(\mathbf{R}_+)$. Let $(\phi_n)_{n \in \mathbf{N}} \in \mathscr{D}(\mathbf{R}_+)$ be a sequence that converges to $u - u'(0)\,\eta$ in $H_0^2(\mathbf{R}_+)$. The sequence $(u'(0)\,\eta + \phi_n)_{n \in \mathbf{N}} \in \mathscr{D}_c$ converges to u in $H^2(\mathbf{R}_+)$.

Let $(\lambda_n)_{n \in \mathbf{N}} \in \mathscr{D}_c(\overline{\mathbf{R}}_+)$ converge to $g_i - g_{i+1}$ in $H^2 \cap H_0^1(\mathbf{R}_+)$.

We now define an approximating sequence $(g_i^n, h_i^n, g_{i+1}^n, h_{i+1}^n)$ of $(g_i, h_i, g_{i+1}, h_{i+1})$ as follows:

$$g_i^n = (\lambda_n + \delta_n)/2$$

$$g_{i+1}^n = (-\lambda_n + \delta_n)/2$$

$$h_i^n = \beta^n - (-\lambda_n' + \delta_n')/2$$

$$h_{i+1}^n = -\alpha^n + (\lambda_n' + \delta_n')/2$$

Let us first check that the sequence belongs to $\gamma(H^m \cap \mathscr{H}^2)$ locally near S_i. The regularity of the functions is clear. Moreover, at the corner S_i we have:

$$g_i^n(S_i) - g_{i+1}^n(S_i) = \lambda_n(0) = 0$$

$$\frac{\partial g_i}{\partial \tau_i}(S_i) - h_{i+1}(S_i) = (\lambda_n' + \delta_n')/2 + \alpha_n - (\lambda_n' + \delta_n')/2 = \alpha_n(0) = 0$$

$$\frac{\partial g_{i+1}}{\partial \tau_{i+1}}(S_i) + h_i(S_i) = -(-\lambda_n' + \delta_n')/2 + \beta_n + (-\lambda_n' + \delta_n')/2 = \beta_n(0) = 0$$

$$\frac{\partial h_i}{\partial \tau_i}(S_i) + \frac{\partial h_{i+1}}{\partial \tau_{i+1}}(S_i) = \beta_n' - (-\lambda_n'' + \delta_n'')/2 - \alpha_n' + (\lambda_n'' + \delta_n'')/2 = \lambda_n''(0) = 0 .$$

The convergence of $(g_i^n, h_i^n, g_{i+1}^n, h_{i+1}^n)$ to $(g_i, h_i, g_{i+1}, h_{i+1})$ can easily be checked.

If we assume $i \in \mathscr{D}$ and $i + 1 \in \mathscr{A}$, the proof is very similar. It suffices to take $\delta = -\lambda$.

If we assume i and $i + 1$ belong to $\mathscr{D}$, the proof can be found in [3]. □

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