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THE MORTAR ELEMENT METHOD FOR THREE DIMENSIONAL FINITE ELEMENTS (*)

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Abstract — *We present in this paper the application of the mortar element method in the case where finite element methods are used in each subdomains The novelty is in the possibility of dealing with tetrahedra for the finite element mesh This paper thus completes the analysis of the mortar element methods that allows for coupling different spectral and/or finite element methods in adjascent subdomains in a flexible and optimal way*

Resume — *Dans cet article nous presentons la methode des elements avec joints appliquee au cas ou des elements finis sont utilises sur chaque sous domaine La nouveaute reside dans la possibilite de considerer en 3D des maillages tetrêdriques Cet article complete donc l analyse de la methode des elements avec joints qui permet un couplage flexible et optimal de differentes discretisations spectrale elements finis sur des sous domaines adjacents*

1. INTRODUCTION

When domain decomposition is used for the approximation of the solution of some partial differential equation, a large problem is splitted up into a set of smaller ones that can, for example, fit easily on each processor of a parallel machine This ability can be used further on by tuning the approximation technique to the proper caracteristic of each smaller problem In this direction, it comes naturally into mind that one could use, locally to each subdomain, the proper discretization parameter, and maybe even, the proper discretization, adapted to the local behaviour of the solution The mortar element method was invented originally (in 1987) to provide an optimal tool in this framework

We refer to [9] and [10] for a general presentation of the mortar element method, to [12] and [4] for PhD thesis on the coupling of finite element method with spectral element methods in 2D and 3D respectively, to [16] and [2] (see also [3]) for the coupling in the pure spectral element context and finally to [1] and [14] where applications of this idea are used in the context of finite element simulations

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Applications of the mortar element method are also present in situations where domain decomposition is not (only) involved for parallel purpose but has been used to *mesh* the global domain Indeed, a complex geometry can often be decomposed into nonoverlapping subgeometries that are more easily meshed independantly The framework of *conforming* approximation is then very stringent to allow for the more flexible use of this concept Indeed it forces the interface of the subdomains to have coincident meshes This prevents in particular, the use of tetrahedra in one subdomain with hexahedra in an adjacent one Our goal in this paper is to present and analyse the mortar element method in this particular context

2 DEFINITION OF THE METHOD

2.1. Presentation of the discrete space

The method we propose here is adapted to the discretization of three dimensional, second order problems that are written under a variational formulation in a domain Ω of $\mathbb{R}^3$ The main concern is then to provide an approximation of the space $H^1(\Omega)$, hence, there is not much restriction in focussing on the problem of the Laplace equation with homogeneous boundary conditions Find $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla u \, \nabla v \, dx = \langle f, v \rangle , \tag{2 1}$$

(the symbol $\langle \cdot, \cdot \rangle$ is the duality pairing between $H_0^1(\Omega)$ and $(H^{-1}(\Omega))$ The starting point of this method is a decomposition of the domain Ω where the partial differential equation is to be solved We consider a partition of Ω

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}^k \quad \text{with} \quad \Omega^k \cap \Omega^\ell = \emptyset , \quad \text{if } k \neq \ell$$

With each subdomain Ω^k we then associate a regular triangulation made of elements that are either hexahedra or tedrahedra In order to avoid the techniques required for the treatment of the curved boundaries, we assume, again for the sake of simplicity, that each subdomain Ω^k (and thus Ω) is a polyhedron and also that each face of Ω^k that meets the boundary $\partial\Omega$ is entirely inside $\partial\Omega$ We denote by $\Gamma^{k,i}$, $1 \leq i \leq F(k)$ the faces of Ω^k Each such face inherits a triangulation made of either triangular or quadrilateral elements (that are all entire faces of an element of the triangulation of Ω^k) We shall assume, in what follows that these (2D) triangulations are uniformely regular. Note that, since the triangulations on two adjacent subdomains are independant, the interface $\gamma^{k,\ell} = \overline{\Omega}^k \cap \overline{\Omega}^\ell$ is provided with two different and independant (2D) meshes.

The triangulation $\mathcal{T}_h(\Omega^k)$ being chosen over each Ω^k, then comes the definition of the finite element functions. We choose locally the finite element method that is best suited to the local properties of the solution. Let us assume that we work with the simple generic case of linear finite elements. We first define the finite element functions locally and introduce the space

$$X_k(\Omega^k) = \{v_h^k \in \mathscr{C}^0(\Omega^k), v_h^k|_{\partial\Omega} = 0, \quad \forall \mathbf{t} \in \mathcal{T}_h(\Omega^k), v_h^k|_{\mathbf{t}} \in \mathscr{P}^1(\mathbf{t})\}$$

where $\mathscr{P}^1(\mathbf{t})$ is the set of all linear function over $\mathbf{t}$. The global finite element approximation will consist of functions whose restriction over each Ω^k belongs to $X_h(\Omega^k)$. Since the interface is provided with two independant meshes, the contraint of continuity of the global function over Ω is not compatible with good approximation properties of the discrete space. In the general case, such a continuity requirement would "block" all degrees of freedom over $\gamma_{k,\ell}$. Inversely, imposing no condition on the jump across the interface is also known to be a bad choice. In what follows we express the matching that is sufficient to ensure the optimality of the global approximation.

The mortar element method first deals with the skeleton of the decomposition, i.e. the union of all interfaces

$$\mathscr{S} = \bigcup_{k,\ell} \gamma^{k,\ell} = \bigcup_{k=1}^{K} \partial\Omega^k$$

and consists in chosing one of the decompositions of $\mathscr{S}$. This decomposition is made of *mortars*, noted γ_m, $1 \leqslant m \leqslant M$ that are disjoint i.e.

$$\mathscr{S} = \bigcup_{m=1}^{M} \bar{\gamma}_m, \quad \gamma_m \cap \gamma_n = \Phi, \quad \text{if } m \neq n.$$

and, in addition, satisfy the fundamental hypothesis that each mortar coincides with an entire face of one of the subdomains, i.e.

$$\forall m, 1 \leqslant m \leqslant M, \quad \exists k, 1 \leqslant k \leqslant K, \exists i, 1 \leqslant i \leqslant F(k), \quad \gamma_m = \Gamma^{k,i},$$

if a mortar coincides with both (entire) faces of connected subdomains, we choose one of the two so that there exists an application from the set $\{1, ..., M\}$ into the set $\{1, ..., K\}$ that associates to each mortar index m the corresponding subdomain index $k(m)$. Note that each mortar is consequently meshed with (2D) elements that are all entire faces of (3D) elements of Ω^k.

It is well known that the trace of the solution u of our problem over $\mathscr{S}$ is of prime importance in the domain decomposition framework Indeed, would it be known, then the solution u could be computed locally within each subdomain by solving K independant Dirichlet problems over each element Ω^k This is at the basis of the Schur complement method We thus introduce a space $W_h(\mathscr{S})$ of discretization for this trace (called the *mortar space* hereafter)

$$W_h(\mathscr{S}) = \{\varphi = L^2(\mathscr{S}), \forall k, 1 \leqslant k \leqslant K, \exists w_h^k \in X_h(\Omega^k),$$
$$\forall m, 1 \leqslant m \leqslant M, \quad \varphi|_{\gamma} = (w_h^{k(m)})|_{\gamma} \}$$

This set being defined, we introduce the space of approximation over Ω

$$X_h = \Big\{ v_h \in L^2(\Omega), v_h^k = v_{h|\Omega^k} \in X_h(\Omega^k), \exists \varphi \in W_h(\mathscr{S}),$$

$$\forall k, 1 \leqslant k \leqslant K, \quad \forall \iota, 1 \leqslant \iota \leqslant F(k),$$

$$\text{if } \Gamma^{k,\iota} \text{ is a mortar}, \quad (v_h^k)|_{\Gamma^{k,\iota}} = \varphi|_{\Gamma^{k,\iota}}$$

$$\text{otherwise}, \forall \psi \in \tilde{w}_h^{k,\iota}, \quad \int_{\Gamma^{k,\iota}} [(v_h^k)|_{\Gamma^{k,\iota}} - \varphi|_{\Gamma^{k,\iota}}]\, \psi = 0 \Big\} \tag{2 2}$$

where the finite element set $\tilde{w}_h^{k,\iota}$ is defined locally on each $\Gamma^{k,\iota}$ (actually, only on those faces that are not mortars) Let $w_h^{k,\iota}$ be the space of all restrictions to $\Gamma^{k,\iota}$ of the elements of X_h In our applications, $\tilde{w}_h^{k,\iota}$ is an appropriate subspace of $w_h^{k,\iota}$, with same dimension as $w_h^{k,\iota} \cap H_0^1(\Gamma^{k,\iota})$, that will be defined in the next subsection

Note that this definition of the finite element space of approximation leads to a non unique definition of the values of the discrete functions on the edges of the skeleton even if the global mesh would allow for a standard conforming definition of this space In the context of the parallel implementation, this leads to a major improvement of this formulation of the mortar element method over the previous ones as is explained in [5] and [6] Indeed, it allows for reducing the amount of communications between different subdomains (see also [13] for similar ideas)

The last point that needs to be adressed now is the definition of the space $\tilde{w}_h^{k,\iota}$ In this paper, it will be solely given in the case where the face $\Gamma^{k,\iota}$ is meshed with triangular elements. The case of quadrilateral elements is explained in [4] and [7] and the general case where both quadrilateral and triangular elements are involved is easily deduced from these two papers.

2.2. Definition of the space $\tilde{w}_h^{k,i}$

We are in the case where the face $\Gamma^{k,i}$ is meshed with triangular elements. We denote by $\mathbf{a}_p$, $1 \leqslant p \leqslant P(k,i)$, the set of all vertices of the triangles and distinguish the internal nodes that belong to $\Gamma^{k,i}$ (numbered from 1 to $P_0(k,i)$) from those that belong to the boundary of $\Gamma^{k,i}$ (numbered from $P_0(k,i)+1$ to $P(k,i)$). With all these nodes are associated the shape functions h_p so that any element χ of $w_h^{k,i}$ can be written as

$$\chi = \sum_{p=1}^{P(k,i)} \chi(\mathbf{a}_p)\, h_p$$

and those elements that belong to $H_0^1(\Gamma^{k,i})$ can be written as

$$\chi = \sum_{p=1}^{P_0(k,i)} \chi(\mathbf{a}_p)\, h_p\,.$$

The vertices $\mathbf{a}_p$, $P_0(k,i)+1 \leqslant p \leqslant P(k,i)$ belong to the same triangles as internal nodes within $\Gamma^{k,i}$. We denote by $\mathbf{a}_p^q$, $1 \leqslant q \leqslant Q(p)$ those vertices inside $\Gamma^{k,i}$ that belong to a side of a triangle with end point $\mathbf{a}_p$. For each such p, $P_0(k,i)+1 \leqslant p \leqslant P(k,i)$, we choose $Q(p)$ positive real numbers λ_q^p with

$$\forall p,\, P_0(k,i)+1 \leqslant p \leqslant P(k,i), \quad \sum_{q=1}^{Q(p)} \lambda_q^p = 1\,.$$

The definition of the space $\tilde{w}_h^{k,i}$ is then

$$\tilde{w}_h^{k,i} = \left\{ \chi \in w_h^{k,i}, \quad \forall p,\, P_0(k,i)+1 \leqslant p \leqslant P(k,i), \quad \chi(\mathbf{a}_p) = \sum_{q=1}^{Q(p)} \lambda_q^p\, \chi(\mathbf{a}_p^q) \right\}$$

it can also be written as

$$\tilde{w}_h^{k\,\ell} = \left\{ \chi \in w_h^{k\,\ell}, \quad \chi = \sum_{p=1}^{P_0(k\,\ell)} \chi(\mathbf{a}_p)\, h_p + \sum_{p=P_0(k\,\ell)+1}^{P(k\,\ell)} \left[\sum_{q=1}^{Q(p)} \lambda_q^p \chi(\mathbf{a}_p^q) \right] h_p \right\}$$

2.3. Presentation of the discrete problem

From the variational formulation of the problem (2 1) together with the definition of the discrete space X_h, it is an easy matter to define a discrete problem corresponding to a Galerkin approximation It consists in finding a solution $u_h \in X_h$ such that

$$\forall v_h \in X_h, \quad \int_\Omega \nabla u_h \nabla v_h \, dx = \int_\Omega f v_h \, dx$$

The well posedness of this problem is easy to prove It is a standard consequence of the Poincaré inequality in the case where each subdomain Ω^k has an edge on $\partial\Omega$, the general case is treated in [9] Thus there exists a unique solution to this problem

Another formulation of the method can also be given by making use of a Lagrangian and expressing the gluing process as an external constraint and not as a part of the definition of the discrete space This is the route followed by [14], [15] for example, see also [5]

The unknowns of the problem are then

(a) all the values of u_h at the internal nodes within each subdomain,

(b) the mortar values within each mortar,

(c) the values of u_h^k on each node of the edges of Ω^k

(remind that the discrete functions are multivalued on each edge) Note that the nodal values of u_h^k at the interior of each face that is not a mortar are derived from the nodal values of the mortar function together with the nodal values on each edge of the corresponding face For more details on the matrix form of the method, we refer to [6]

3. NUMERICAL ANALYSIS OF THE METHOD

This analysis follows the same lines as the original proofs in [8] and [9] The second Strang lemma in the nonconforming situation where $X_h \not\subset H_0^1(\Omega)$ gives the error estimate

$$\sum_{k=1}^{K} \| u - u_h \|_{H^1(\Omega_k)} \leqslant$$

$$c \left(\inf_{v_h \in X_h} \sum_{k=1}^{K} \| u - v_h \|_{H^1(\Omega_k)} + \sup_{v_h \in X_h} \frac{\int_{\mathcal{S}} \frac{\partial u}{\partial \mathbf{n}} [v_h]}{\sum_{k=1}^{K} \| v_h \|_{H^1(\Omega_k)}} \right), \quad (3.1)$$

where $[v_h]$ denotes the jump of v_h across the interfaces of $\mathcal{S}$ and $\frac{\partial u}{\partial \mathbf{n}}$ denotes the normal derivative of u. The first error term in (3.1) is known as the approximation error, the second term is the consistency error and is a consequence of the discontinuity of the elements of X_h through the interface.

3.1. Analysis of the consistency error

A bound for the consistency error is derived as follows

$$\left| \int_{\mathcal{S}} \frac{\partial u}{\partial \mathbf{n}} [v_h] \right| = \left| \sum_{1 \leqslant k < \ell \leqslant K} \int_{\gamma^{k,\ell}} \frac{\partial u}{\partial \mathbf{n}} (v^k - \phi + \phi - v^\ell) \right|$$

where we remind that $\gamma_{k,\ell} = \Omega^k \cap \Omega^\ell$ and ϕ is the mortar function associated with v_h. Since ϕ coincides with either v^k or v^ℓ over $\gamma^{k,\ell}$ — say here with v^ℓ — we have

$$\left| \int_{\mathcal{S}} \frac{\partial u}{\partial \mathbf{n}} [v_h] \right| = \left| \sum_{\Gamma^{k,\prime} \text{ not a mortar}} \int_{\Gamma^{k,\prime}} \frac{\partial u}{\partial \mathbf{n}} (v^k - \phi) \right| .$$

Using now (2.2), we deduce that

$$\forall \psi \in \tilde{W}_h^{k,\prime}, \quad \left| \int_{\mathcal{S}} \frac{\partial u}{\partial \mathbf{n}} [v_h] \right| = \left| \sum_{\Gamma^{k,\prime} \text{ not a mortar}} \int_{\Gamma^{k,\prime}} \left(\frac{\partial u}{\partial \mathbf{n}} - \psi \right) (v^k - \phi) \right| ,$$

so that

$$\left| \int_{\mathcal{S}} \frac{\partial u}{\partial n} [v_h] \right| \leqslant \inf_{\psi \in \tilde{W}_h^{k,\prime}} \left| \sum_{\Gamma^{k,\prime} \text{ not a mortar}} \int_{\Gamma^{k,\prime}} \left(\frac{\partial u}{\partial \mathbf{n}} - \psi \right) (v^k - \phi) \right|$$

$$\leqslant \sum_{\Gamma^{k,\prime} \text{ not a mortar}} \inf_{\psi \in \tilde{W}_h^{k,\prime}} \left\| \frac{\partial u}{\partial \mathbf{n}} - \psi \right\|_{[H^{1/2}(\Gamma^{k,\prime})]'} \| v^k - \phi \|_{H^{1/2}(\Gamma^{k,\prime})}$$

$$\leqslant \sum_{\Gamma^{k,\prime} \text{ not a mortar}} \inf_{\psi \in \tilde{W}_h^{k,\prime}} \left\| \frac{\partial u}{\partial \mathbf{n}} - \psi \right\|_{[H^{1/2}(\Gamma^{k,\prime})]'}$$

$$[\| v^k \|_{H^{1/2}(\Gamma^{k,\prime})} + \| \phi \|_{H^{1/2}(\Gamma^{k,\prime})}] .$$

Recalling once more that ϕ coincides with one of the traces of v_h over $\mathcal{S}$, we deduce from the standard trace theorem that

$$\left| \int_{\mathcal{S}} \frac{\partial u}{\partial \mathbf{n}} [v_h] \right| \leqslant \sum_{\Gamma^{k,\ell} \text{ not a mortar}} \inf_{\psi \in \tilde{w}_h^{k,\ell}} \left\| \frac{\partial u}{\partial \mathbf{n}} - \psi \right\|_{[H^{1/2}(\Gamma^{k,\ell})]'} \sum_{k=1}^{K} \| v_h \|_{H^1(\Omega^k)} .$$

It is well known that, for any $g \in H^1(\Gamma^{k,\ell})$

$$\inf_{\psi \in \tilde{w}_h^{k,\ell}} \| g - \psi \|_{L^2(\Gamma^{k,\ell})} \leqslant ch \; \| g \|_{H^1(\Gamma^{k,\ell})}$$

so that, denoting by p_h the projection operator from $L^2(\Gamma^{k,\ell})$ onto $\tilde{w}_h^{k,\ell}$, and using a interpolation argument, we have

$$\| g - p_h(g) \|_{L^2(\Gamma^{k,\ell})} \leqslant ch^{1/2} \; \| g \|_{H^{\frac{1}{2}}(\Gamma^{k,\ell})} .$$

Besides a standard Aubin Nitsche argument leads to

$$\| g - p_h(g) \|_{[H^{1/2}(\Gamma^{k,\ell})]'} \leqslant ch^{1/2} \; \| g - p_h(g) \|_{L^2(\Gamma^{k,\ell})} .$$

Combining these two results yields

$$\inf_{\psi \in \tilde{w}_h^{k,\ell}} \left\| \frac{\partial u}{\partial \mathbf{n}} - \psi \right\|_{[H^{1/2}(\Gamma^{k,\ell})]'} \leqslant ch \; \left\| \frac{\partial u}{\partial \mathbf{n}} \right\|_{H^{1/2}(\Gamma^{k,\ell})} ,$$

the trace theorem gives then

$$\left| \int_{\mathcal{S}} \frac{\partial u}{\partial \mathbf{n}} [v_h] \right| \leqslant ch \| u \|_{H^2(\Omega)} \sum_{k=1}^{K} \| v_h \|_{H^1(\Omega^k)} .$$

From the definition of the matching, the consistency error is then of optimal order.

3.2. Analysis of the best approximation error

Let us turn now to the best approximation error. The key point of its analysis is given by the stability property of the following operator defined from $L^2(\Gamma^{k,\ell})$ into $w_h^{k,\ell} \cap H_0^1(\Gamma^{k,\ell})$ as follows : $\forall \chi \in L^2(\Gamma^{k,\ell})$,

$$\forall \psi \in \tilde{w}_h^{k,\ell}, \quad \int_{\Gamma^{k,\ell}} (\chi - \pi_h^{k,\ell} \chi) \, \psi = 0 . \tag{3.2}$$

LEMMA 3.1 : *There exists a constant $c > 0$ such that :* $\forall \chi \in L^2(\Gamma^{k,\ell})$,

$$\| \pi_h^{k,\ell} \chi \|_{L^2(\Gamma^{k,\ell})} \leqslant c \| \chi \|_{L^2(\Gamma^{k,\ell})} . \tag{3.3}$$

Before proving this result, we give, as a corollary, our main result concerning the approximation error.

THEOREM 3.2 : *For any real numbers σ_k, $1 \leqslant \sigma_k \leqslant 2$, there exists a constant c, independant of h such that, for any function $v \in H_0^1(\Omega)$ with $v|_{\Omega^k} \in H^{\sigma_k}(\Omega^k)$,*

$$\inf_{v_h \in X_h} \left(\sum_{k=1}^{K} \| v|_{\Omega^k} - v_h|_{\Omega^k} \|^2_{H^1(\Omega^k)} \right)^{\frac{1}{2}} \leqslant c \sum_{k=1}^{K} h^{\sigma_k - 1} \| v|_{\Omega^k} \|_{H^{\sigma_k}(\Omega^k)} .$$

Proof: It follows by using the same lines as in [9]. With each function fullfilling the previous hypothesis we associate a discrete function, defined locally within each subdomain, that satisfy

$$\| v|_{\Omega^k} - \tilde{v}_{h|\Omega^k} \|_{L^2(\Omega^k)} + h \| v_{|\Omega^k} - \tilde{v}_{h|\Omega^k} \|_{H^1(\Omega^k)} \leqslant C h^{\sigma_k} \| v_{|\Omega^k} \|_{H^{\sigma_k}(\Omega^k)} , \tag{3.4}$$

(take for instance the discrete interpolate of v). Such a function does not satisfy the matching condition across the interfaces. To cope with this, we first define the mortar function that will derive the value of the discrete function in X_h that will approximate v. This mortar function ϕ, over each γ_m will be chosen as the restriction of $\tilde{v}_{h|\Omega^{k(m)}}$. By construction, this element belongs to $W_h(\mathscr{S})$. We shall now modify the values of $\tilde{v}_h$ over each face that is not a mortar. Let $\Gamma^{\ell,\iota}$ be such a face and define the element of $w_h^{\ell,\iota} \cap H_0^1(\Gamma^{\ell,\iota})$ by $\pi_h^{\ell,\iota}[(\tilde{v}_{h|\Omega^\ell} - \phi)_{|\Gamma^{\ell,\iota}}]$. This element can be extended into an element of $X_h(\Omega^\ell)$, noted $r_h^{\ell,\iota} = \mathscr{R}_h^{\ell,\iota}(\pi_h^{\ell,\iota}[(\tilde{v}_{h|\Omega^\ell} - \phi)_{|\Gamma^{\ell,\iota}}])$ that vanishes over each face of Ω^ℓ except $\Gamma^{\ell,\iota}$. Such a lifting operator exists as is proven in e.g. [11]. In addition, this operator can be chosen to satisfy the stability property

$$\| r_h^{\ell,\iota} \|_{H^1(\Omega^\ell)} \leqslant c \| \pi_h^{\ell,\iota}[(\tilde{v}_{h|\Omega^\ell})_{|\Gamma^{\ell,\iota}} - \phi_{|\Gamma^{\ell,\iota}}] \|_{H_{00}^{1/2}(\Gamma^{\ell,\iota})} .$$

The new approximation of v that belongs to X_h by construction, is thus

$$v_{h|\Omega^\ell} = \tilde{v}_{h|\Omega^\ell} + \sum_{\Gamma^{k,\iota} \text{ not a mortar}} r_h^{\ell,\iota} .$$

Making use of an inverse inequality over $w_h^{\ell,\iota}$ (remind the uniform assumption done on the triangulation of each face), we derive, in as standard manner that

$$\|(v - v_h)_{|\Omega^\ell}\|_{H^1(\Omega^\ell)}$$
$$\leqslant \|(v - \tilde{v}_h)_{|\Omega^\ell}\|_{H^1(\Omega^\ell)} + \|r_h^{\ell,\iota}\|_{H^1(\Omega^\ell)}$$
$$\leqslant \|(v - \tilde{v}_h)_{|\Omega^\ell}\|_{H^1(\Omega^\ell)} + \|\pi_h^{\ell,\iota}[(\tilde{v}_{h|\Omega^\ell})_{|\Gamma^{\ell,\iota}} - \phi_{|\Gamma^{\ell,\iota}}]\|_{H_{00}^{1/2}(\Gamma^{\ell,\iota})}$$
$$\leqslant \|(v - \tilde{v}_h)_{|\Omega^\ell}\|_{H^1(\Omega^\ell)} + h^{-1/2}\|\pi_h^{\ell,\iota}[(\tilde{v}_{h|\Omega^\ell})_{|\Gamma^{\ell,\iota}} - \phi_{|\Gamma^{\ell,\iota}}]\|_{L^2(\Gamma^{\ell,\iota})}$$
$$\leqslant \|(v - \tilde{v}_h)_{|\Omega^\ell}\|_{H^1(\Omega^\ell)} + h^{-1/2}\|(\tilde{v}_{h|\Omega^\ell})_{|\Gamma^{\ell,\iota}} - \phi_{|\Gamma^{\ell,\iota}}\|_{L^2(\Gamma^{\ell,\iota})}$$
$$\leqslant \|(v - \tilde{v}_h)_{|\Omega^\ell}\|_{H^1(\Omega^\ell)} + h^{-1/2}(\|(\tilde{v}_{h|\Omega^\ell})_{|\Gamma^{\ell,\iota}} - v_{|\Gamma^{\ell,\iota}}\|_{L^2(\Gamma^{\ell,\iota})})$$
$$+ \|((v_{|\Omega^\ell})_{|\Gamma^{\ell,\iota}} - \phi)_{|\Gamma^{\ell,\iota}}\|_{L^2(\Gamma^{\ell,\iota})}),$$

recalling that ϕ is the trace over the mortars of $\tilde{v}_h$, it is straightforward that

$$\sum_{k=1}^{K} \|(v - v_h)_{|\Omega^k}\|_{H^1(\Omega^k)}$$
$$\leqslant c\left(\sum_{k=1}^{K} \|(v - v_h)_{|\Omega^k}\|_{H^1(\Omega^k)} + \sum_{k=1}^{K} h^{-1/2}\|(v - v_h)_{|\mathcal{S}}\|_{L^2(\mathcal{S})}\right),$$

from (3.4) together with a trace bound, we derive

$$\sum_{k=1}^{K} \|(v - v_h)_{|\Omega^k}\|_{H^1(\Omega^k)} \leqslant ch \sum_{k=1}^{K} \|v\|_{H^2(\Omega^k)}.$$

We are left now with the proof of the stability property (3.3)

3.3. Stability of the operator $\pi_h^{k,\iota}$

The proof of Lemma 3.1 is performed in two steps. For the sake of simplification, we shall skip any reference to the exponent ${}^{k,\iota}$ in the different notations, assuming that we are working on a reference domain e.g. $\Gamma =]-1, 1[^2$.

The first step involves the $L^2(\Gamma)$-projection operator p_h over $\tilde{w}_h$ defined by : $\forall \chi \in L^2(\Gamma)$, $p_h \chi \in \tilde{w}_h$ is such that :

$$\forall \psi \in \tilde{w}_h, \quad \int_\Gamma (\chi - p_h \chi)\, \psi = 0 . \tag{3.5}$$

The point that is adressed in this step consists in the proof of

LEMMA 3.3 : *There exists a constant* α, $0 < \alpha < 1$ *such that :* $\forall \chi \in L^2(\Gamma)$,

$$\| \pi_h \chi - p_h \chi \|_{L^2(\Gamma)} \leqslant \alpha \| \pi_h \chi \|_{L^2(\Gamma)} .$$

Proof : Let us first recall the following two formula valid on any triangle $\mathcal{T}$ with vertex A, B, C : $\forall \varphi \in \mathcal{P}_1(\mathcal{T})$,

$$\int_{\mathcal{T}} \varphi^2\, dx = \frac{|\mathcal{T}|}{6} (\varphi^2(A) + \varphi^2(B) + \varphi^2(C) \tag{3.6}$$

$$+ \varphi(A)\, \varphi(B) + \varphi(B)\, \varphi(C) + \varphi(A)\, \varphi(C))$$

$$= \frac{|\mathcal{T}|}{12} ([\varphi(A) + \varphi(B) + \varphi(C)]^2 + \varphi^2(A) + \varphi^2(B) + \varphi^2(C)) \tag{3.7}$$

$$= \frac{|\mathcal{T}|}{12} ([\varphi(A) + \varphi(B)]^2 + [\varphi(B) + \varphi(C)]^2$$

$$+ [\varphi(A) + \varphi(C)]^2) . \tag{3.8}$$

Only for clarity, we shall assume that the mesh over Γ is structured and composed of isosceles rectangle triangles so that any node $\mathbf{a}_p$ on the edges of Γ that is not a vertex satisfies $Q(p) = 2$ as follows.
The proof is based on the following simple equality : $\forall \chi \in L^2(\Gamma)$,

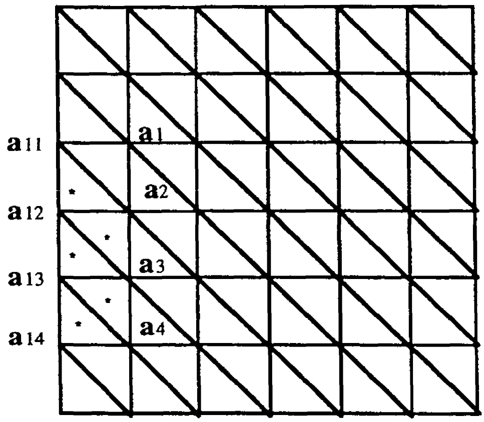
$$\| \pi_h \chi - p_h \chi \|_{L^2(\Gamma)} = \inf_{\chi_h \in \tilde{w}_h} \| \pi_h \chi - \chi_h \|_{L^2(\Gamma)} ,$$

and the construction of a suitable χ_h. Let us define $\chi_h \in \tilde{w}_h$ such that

$$\forall p, 1 \leqslant p \leqslant P_0 \quad \chi_h(\mathbf{a}_p) = \pi_h \chi(\mathbf{a}_p) ,$$

hence

$$\pi_h \chi - \chi_h = \sum_{p = P_0 + 1}^{P} \chi_h(\mathbf{a}_p)\, h_p .$$



Example of the triangulation of a face.

Using (3 6) to compute the L^2 -norms of this expression in the five triangles with a star, we arrive at the

$$\|\pi_h \chi - \chi_h\|_{L^2(\Gamma)} = \frac{|\mathcal{T}|}{6} (\chi_h(\mathbf{a}_{11})\, \chi_h(\mathbf{a}_{12})$$

$$+ \chi_h(\mathbf{a}_{12})\, \chi_h(\mathbf{a}_{13}) + \chi_h(\mathbf{a}_{13})\, \chi_h(\mathbf{a}_{14}))$$

$$+ 3 \frac{|\mathcal{T}|}{6} (\chi_h(\mathbf{a}_{12})^2 + \chi_h(\mathbf{a}_{13})^2) + \text{positive term},$$

where we have written the only expressions where the influence of $\mathbf{a}_3$ will enter in Using the fact that χ_h belongs to $\tilde{w}_h$, we deduce

$$\|\pi_h \chi - \chi_h\|_{L^2(\Gamma)} \leqslant \frac{|\mathcal{T}|}{8} ([\chi_h(\mathbf{a}_2) + \chi_h(\mathbf{a}_3)]^2 +$$

$$+ [\chi_h(\mathbf{a}_3) + \chi_h(\mathbf{a}_4)]^2) + \frac{|\mathcal{T}|}{6} (\chi_h(\mathbf{a}_3)^2)$$

On the other hand, the use of (3 7) or (3.8) on each of the triangles sharing the vertex $\mathbf{a}_3$ allows for proving that

$$\|\pi_h \chi - \chi_h\|_{L^2(\Gamma)} \geqslant \frac{|\mathcal{T}|}{6} ([\chi_h(\mathbf{a}_2) + \chi_h(\mathbf{a}_3)]^2 +$$

$$+ [\chi_h(\mathbf{a}_3) + \chi_h(\mathbf{a}_4)]^2) + \frac{5\, |\mathcal{T}|}{12} (\chi_h(\mathbf{a}_3)^2),$$

which proves that for the contributions where this corner arises the inequality of the Lemma holds with the constant $\alpha = \frac{3}{4}$. For the contributions of the corners of Γ, the same strategy can be used which ends the proof.

The second step is shorter and will end the proof of Lemma 3.1. It is readily checked that

$$\int_\Gamma (\pi_h \chi)^2 = \int_\Gamma (\pi_h \chi - p_h \chi)\, \pi_h \chi + \int_\Gamma p_h \chi \pi_h \chi \,.$$

Using now (3.2) and (3.5), we derive

$$\int_\Gamma (\pi_h \chi)^2 = \int_\Gamma (\pi_h \chi - p_h \chi)^2 + \int_\Gamma p_h \chi \pi_h \chi \,,$$

so that using the fact that p_h is a contraction in $L^2(\Gamma)$ we deduce

$$\int_\Gamma (\pi_h \chi)^2 \leq \int_\Gamma (\pi_h \chi - p_h \chi)^2 + \|\chi\|_{L^2(\Gamma)} \|\pi_h \chi\|_{L^2(\Gamma)} \,,$$

and from lemma 3.2 we then deduc that

$$\|\pi_h \chi\|_{L^2(\Gamma)} \leq \frac{1}{1-\alpha} \|\chi\|_{L^2(\Gamma)} \,.$$

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