M2AN - Modélisation mathématique et analyse numérique

Hervé Le Meur

# Non-uniqueness and linear stability of the one-dimensional flow of multiple viscoelastic fluids

*M2AN - Modélisation mathématique et analyse numérique*, tome 31, nº 2 (1997), p. 185-211

<http://www.numdam.org/item?id=M2AN_1997__31_2_185_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODELISATION MATHEMATIQUE ET ANALYSE NUMERIQUE

(Vol 31 n° 2 1997 p 185 a 212)

# NON-UNIQUENESS AND LINEAR STABILITY OF THE ONE-DIMENSIONAL FLOW OF MULTIPLE VISCOELASTIC FLUIDS (*)

by Hervé LE MEUR (1)

Abstract — *In the present paper we study the Couette and Poiseuille flows of multiple viscoelastic fluids for various constitutive laws We first discuss the range of parameters that ensure uniqueness for such flows Then we study the linear stability of the Couette flow for the Phan Thien Tanner and modified Phan Thien Tanner models*

Resume — *Dans cet article nous etudions les ecoulements de Poiseuille et de Couette de plusieurs fluides viscoelastiques pour differentes lois constitutives Nous donnons des limites aux conditions d unicite pour le modele de Johnson Segalman (plan et axisymetrique) ceux de Phan Thien Tanner (PTT) et Phan Thien Tanner Modifie (MPTT) Enfin nous etudions la stabilite lineaire de l ecoulement de Couette d un fluide de type PTT ou MPTT*

## 1 INTRODUCTION

The study of flows of multiple viscoelastic fluids is of a great industrial importance Let us, for instance, recall the problem of coextrusion of two or more materials in order to produce bicomponent fibers with specific properties, or the one of transportation of oils, gums, using the lubricating effect

This paper is concerned with one dimensional motions of multiple ViscoElastic Fluids (VEF) obeying a Johnson Segalman, Phan-Thien Tanner (PTT [11]) or Modified Phan Thien Tanner (MPTT [12]) model We shall assume that the total stress in each layer $i$ can be decomposed into

$$\underline{\underline{\sigma}}^i_{tot} = - p^i \underline{\underline{I}} + 2 \eta^i_{sol} \underline{\underline{D}}[\underline{u}^i] + \underline{\underline{\tau}}^i ,$$

where $p^i$ is the pressure, $\eta^i_{sol}$ the solvent Newtonian viscosity, $\underline{\underline{D}}[\underline{u}^i]$ the rate of strain tensor and $\underline{\underline{\tau}}^i$ the extrastress tensor In each layer $i$, the constitutive law satisfied by $\underline{\underline{\tau}}^i$ is assumed to be of the form

$$g_\epsilon(\underline{\underline{\tau}}^i)\, \underline{\underline{\tau}}^i + \lambda^i \frac{\mathcal{D}_a \underline{\underline{\tau}}^i}{\mathcal{D}t} = 2 \eta^i_{pol} \underline{\underline{D}}[\underline{u}^i] ,$$

(*) Manuscript received October 30 1995 revised version received May 15 1996
(1) CNRS and Universite Paris XI Bât 425 91405 Orsay Cedex Herve LeMeur@maths u psud fr

where $\eta'_{pol}$ is the elastic viscosity (polymer contribution), $\lambda'$ the relaxation time, $a$ a mathematical parameter in $[-1, 1]$ and $\mathscr{D}_a(\,.\,)/\mathscr{D}t$ the interpolated Oldroyd derivative between the Upper Convected Maxwell ($a = 1$) and the Lower Convected Maxwell ($a = -1$) derivatives :

$$\frac{\mathscr{D}_a \underline{\underline{\tau}}}{\mathscr{D}t} = \frac{\partial \underline{\underline{\tau}}}{\partial t} + (\underline{u}\,.\,\nabla)\,\underline{\underline{\tau}} - \frac{(a-1)}{2}\,(\underline{\underline{\tau}}\,\underline{\underline{\nabla u}} + \underline{\underline{\nabla u}}^T\,\underline{\underline{\tau}}) - \frac{(a+1)}{2}\,(\underline{\underline{\tau}}\,\underline{\underline{\nabla u}}^T + \underline{\underline{\nabla u}}\,\underline{\underline{\tau}})\,.$$

The scalar function $g_{\epsilon'}(\underline{\underline{\tau}}')$ will be either 1 for the Johnson Segalman model, $\exp(\epsilon'\lambda/(\eta_{sol} + \eta_{pol})\,\mathrm{tr}\,\underline{\underline{\tau}})$ for the PTT one, or $1 + \epsilon'\lambda/(\eta_{sol} + \eta_{pol})\,\mathrm{tr}\,\underline{\underline{\tau}}$ for the MPTT one, where $\epsilon'$ is a nonnegative parameter.

Among the numerous popular models, the strictly interpolated Johnson Segalman ones ($a \in\, ]-1, 1[$) have an asset to the pure Maxwell one ($a = 1$) since they have a bounded extra stress, whatever the shear stress $\dot{\gamma}$ may be. Moreover, they have a non-zero second normal stress difference, which is seen ([7] for example) as more physical.

Let us first recall some mathematical results about these models and one dimensional flows. The existence of the one-fluid Poiseuille/Couette flow for an interpolated Johnson-Segalman fluid has been extensively described by Guillopé and Saut [1]. Similarly to [2] they prove existence and uniqueness, for any pressure gradient (Poiseuille flow) or upper plate velocity (Couette flow), if the dimensionless polymer viscosity $\varepsilon = \eta_{pol}/(\eta_{sol} + \eta_{pol})$ is less than 8/9. Beyond 8/9, uniqueness holds only for a limited range of parameters, out of which, they find a continuum of solutions which are continuous but not $C^1$. This modeling is drastically different from the case $a = 1$ (UCM) and $a = -1$ (LCM) where existence and uniqueness is always true. This non-uniqueness remains for the PTT/MPTT models, as was already noticed in [13].

In the same article [1], they study the one dimensional Lyapunov stability of Couette flow and prove under some assumptions the unconditional $L^2$ stability of Couette flow and the conditional $H^2$ stability. These results are based on projected equations and one dimensional perturbations, using standard *a priori* estimates methods. Their linear stability study of Couette flow proves linear stability if either $\varepsilon$ is less than 8/9, or $\varepsilon$ is greater than 8/9 with some other restrictions on the flow parameters.

Furthermore, Guillopé and Saut show in [3] that, for a general 2D flow, if $\varepsilon$ and the external force are small enough, then, there exists a unique asymptotically $L^2$ stable solution.

In Section 2 of the present article, we intend to prove existence and uniqueness (if $\varepsilon$ is less than 8/9) for the plane Poiseuille/Couette flows of multiple VEF obeying a Johnson Segalman model. We also prove the boundedness of one dimensional perturbations for these flows, using a non-common formulation. Section 3 is devoted to the same problem in axisymmetric geometry. We also prove that the solution has a zero azimuthal velocity, and provide explicit formulae. Then, in Section 4, we investigate the problem of existence and uniqueness for the plane Poiseuille/Couette flow of $n$ VEF obeying a PTT [11] or an MPTT [12] model. In particular, we give necessary and sufficient conditions for the MPTT fluids and sufficient conditions for the PTT ones to ensure uniqueness. Moreover, we stress some drawbacks of these models and propose a modification that eliminates them. Finally, in Section 5, we prove some results on the spectrum of the linearized operator of the Couette flow of a PTT or an MPTT fluid. Last, we give sufficient conditions for the spectrum to be on the right side of the imaginary axis and conclude to linear stability, because of the analyticity of the underlying semi-group.

## 2. POISEUILLE/COUETTE FLOW IN A PLANE GEOMETRY

### 2.1. Modeling

In most experimental dies, a piston pushes a fluid to the exit. This surfassic strength can be transformed into a volumic pressure gradient that forces the fluid to flow. This flow will be assumed invariant in $x$ (we neglect the entrance and the exit) so that the velocity depends only on $y \in [0, y_{n+1}]$. In a pure Poiseuille flow, there is zero boundary conditions, but there is a non-zero pressure gradient, the so-called pressure drop $\partial p/\partial x$. On the other hand, in a pure shearing flow (so-called Couette flow) there is no pressure gradient, but the upper plate has a non zero velocity. As in the Poiseuille case, the geometry of a Couette flow is assumed to be invariant under $x$ translations (*see fig. 1*). Invariance under $x$ translations leads to an extrastress depending only on $y$. For both flows, the incompressibility and the adhesion on the fixed plate finally gives the representation $(u(y), 0)$ for the velocity.

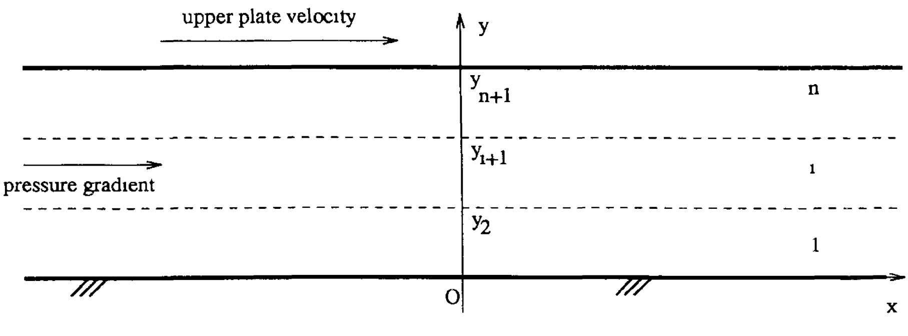


**Figure 1. — Overall geometry.**

One may also consider flows of mixed type ([5] for example), where the pressure gradient and the upper plate velocity are arbitrary. We will call such flows Poiseuille/Couette flows.

The external forces other than the pressure one will be neglected

The dimensioned equations satisfied by the velocity $\underline{u}(y) = (u(y), 0)$, pressure $p$ and extrastress $\underline{\underline{\tau}}$ are, in $\Omega_i$ :

$$\begin{cases} \rho^i\left(\dfrac{\partial \underline{u}^i}{\partial t} + \underline{u}^i \cdot \nabla \underline{u}^i\right) - \eta^i_{sol}\,\Delta \underline{u}^i + \nabla p^i = \operatorname{div} \underline{\underline{\tau}}^i\,, \\ \operatorname{div} \underline{u}^i = 0\,, \\ g_{\epsilon^i}(\underline{\underline{\tau}}^i)\,\underline{\underline{\tau}}^i + \lambda^i \dfrac{\mathcal{D}_a \underline{\underline{\tau}}^i}{\mathcal{D}t} = 2\,\eta^i_{pol}\,\underline{\underline{D}}[\underline{u}^i]\,, \end{cases} \tag{1}$$

where $\rho^i$ is the density of the fluid $i$.

The interface conditions are of two types. The first one is provided by physics ; the non-miscibility of the fluids gives the continuity of normal velocities : $\underline{u}^i \cdot \underline{n} = \underline{u}^i_{\text{interface}} \cdot \underline{n} = \underline{u}^{i+1} \cdot \underline{n}$, where $\underline{n}$ is a unit normal from fluid $i$ to fluid $i+1$. Furthermore, in presence of a viscosity, we can assume that the tangential velocities are continuous. Finally, both components of the velocity will be assumed to be continuous :

$$\underline{u}^i\big|_{\text{interface}} = \underline{u}^{i+1}\big|_{\text{interface}}\,. \tag{2}$$

The second type is provided by the variational formulations of the differential equations :

$$[\![\rho(\underline{u} - \underline{u}_{\text{interface}})]\!] \cdot \underline{n} = 0\,, \tag{3}$$

$$[\![-p\underline{\underline{I}} + 2\,\eta_{sol}\,\underline{\underline{D}}[\underline{u}] + \underline{\underline{\tau}}]\!] \cdot \underline{n} = -\,2\,HS\underline{n}\,, \tag{4}$$

where $2\,H$ is the curvature of the interface, $S$ is a constant of surface tension and $[\![\,\cdot\,]\!] = (\,\cdot\,)_i - (\,\cdot\,)_{i+1}$ denotes the jump from fluid $i$ to fluid $i+1$ The equation (3) reduces to $0 = 0$, with the physical hypothesis (2) exposed above and all the interface conditions are (2) and (4).

Last, the boundary conditions are :

$$\underline{u}(0) = 0\,; \quad \underline{u}(y_{n+1}) = \underline{u}_{wall}\,. \tag{5}$$

In order to make the above equations dimensionless, we will use a characteristic velocity $U_{dim}$ and the following scales :

$$L_{dim} = y_{n+1}\,; \quad \eta = \sum_i (\eta^i_{sol} + \eta^i_{pol})\,; \quad t_{dim} = \frac{L_{dim}}{U_{dim}}\,; \quad \mathcal{T}_{dim} = \eta\,\frac{U_{dim}}{L_{dim}}, \tag{6}$$

for the space, viscosity, time and stress variables respectively. In the sequel, we will use the same symbols for dimensionless variables and fields as we used for dimensional ones. Moreover, we introduce some dimensionless parameters : the Reynolds numbers $\mathrm{Re}^i = \rho^i\, U_{dim}\, L_{dim}/\eta$, the Weissenberg numbers $\mathrm{We}^i = \lambda^i\, U_{dim}/L_{dim}$, the polymeric viscosity $\varepsilon^i = \eta^i_{pol}/\eta^i \in [0, 1[$, the ratios of viscosities $m^i = \eta^i/\eta = (\eta^i_{sol} + \eta^i_{pol})/\eta$ and the coefficient of superficial tension $T$.

With these notations, the dimensionless velocity, pressure and extrastress $\underline{u}, p, \underline{\underline{\tau}}$ verify :

$$\begin{cases} \mathrm{Re}^i\left(\dfrac{\partial \underline{u}^i}{\partial t} + \underline{u}^i \cdot \nabla \underline{u}^i\right) - m^i(1-\varepsilon^i)\,\Delta \underline{u}^i + \nabla p^i = \operatorname{div} \underline{\underline{\tau}}^i\,, \\ \operatorname{div} \underline{u}^i = 0\,, \\ g_{\varepsilon^i}(\underline{\underline{\tau}}^i)\,\underline{\underline{\tau}}^i + \mathrm{We}^i\,\dfrac{\mathscr{D}_a \underline{\underline{\tau}}^i}{\mathscr{D}t} = 2\,m^i\,\varepsilon^i\,\underline{\underline{D}}[\underline{u}^i]\,, \\ [\![\underline{u}]\!] = 0\,, \\ [\![- p\underline{\underline{I}} + 2\,m(1-\varepsilon)\,\underline{\underline{D}}[\underline{u}] + \underline{\underline{\tau}}]\!] \cdot \underline{n} = -\,2\,HT\underline{n}\,, \\ \underline{u}(0) = 0\,, \\ \underline{u}(1) = \underline{u}_{wall}\,. \end{cases} \tag{7}$$

*Remark 2.1 :* In [8], we used local scales in order to have the same volumic equations as [1]. The major drawback of this method is that the jump conditions take the ratios of local scales into account. Moreover, one may expect those equations, on non-continuous fields, to be more difficult to solve numerically than the one used in the present article.

### 2.2. Existence and uniqueness

As a consequence of what precedes, we are interested in the stationary solutions of

$$\begin{cases} -\,m(1-\varepsilon)\,\Delta \underline{u} + \underline{\nabla} p = \operatorname{div} \underline{\underline{\tau}}\,, & (8) \\ \underline{u} = (u(y), 0)\,, & (9) \\ \underline{\underline{\tau}} - \dfrac{\mathrm{We}}{2}\,[(a-1)\,(\underline{\underline{\tau}}\,\underline{\underline{\nabla u}} + \underline{\underline{\nabla u}}^T\,\underline{\underline{\tau}}) + (a+1)\,(\underline{\underline{\tau}}\,\underline{\underline{\nabla u}}^T + \underline{\underline{\nabla u}}\,\underline{\underline{\tau}})] = 2\,m\varepsilon \underline{\underline{D}}[\underline{u}]\,, & (10) \end{cases}$$

in each sub domain $i$, where we drop the superscript $i$. We complete these equations with the interface conditions

$$[\![- p\underline{\underline{I}} + 2\,m(1-\varepsilon)\,\underline{\underline{D}}[\underline{u}] + \underline{\underline{\tau}}]\!]\,\underline{n} = -\,2\,HT\underline{n}\,, \tag{11}$$

$$[\![\underline{u}]\!] = 0\,, \tag{12}$$

where $2H$ is the sum of the principal curvatures (null in the plane case) of the $i$th interface, $T$ the surface tension coefficient and the boundary conditions

$$\underline{u}^1(0) = 0 \quad \text{and} \quad \underline{u}^{n+1}(1) = \underline{u}_{wall}\,. \tag{13}$$

The following theorem states the existence and discusses the uniqueness of a plane Poiseuille/Couette flow of $n$ VEF obeying an interpolated Johnson Segalman law with an upper plate velocity $\underline{u}_{wall}$ and a pressure loss $f^i = -\partial p^i/\partial x \geqslant 0$.

THEOREM 2.1 : *Let the velocity $\underline{u}_{wall}$, the pressure drops $f^i = -\partial p^i/\partial x \geqslant 0$, the viscosities $0 \leqslant \varepsilon^i < 1$, and $a^i \in [-1, 1]$ be given. The equations (8, 9, 10) in every subdomain i closed by the boundary condition (13) and the interface condition (11) at every interface admit a solution $(\underline{u}, p, \underline{\underline{\tau}})$, with continuous velocity, such that :*

$$\begin{cases} p(x,y) = \tau_{22}^i(y) - f^i x + P_0^i & \forall y \in [y_i, y_{i+1}]\,, \\ u(y) = u^i(y_i) + \displaystyle\int_{y_i}^{y} \Phi^i(Z_{\alpha^i}(y'))\,dy' & \forall y \in [y_i, y_{i+1}]\,, \\ \tau_{11}^i, \tau_{12}^i, \ \text{and}\ \tau_{22}^i \ \text{are given by (14)}\,, \end{cases}$$

*where $P_0^i = P_0$, $f^i = f$ and $\alpha^i = \alpha$ are independent on i. The $u^i(y_i)$ are all determined by equations (21, 13), $\alpha$ is a solution of $u^n(1)(\alpha^1) = u_{wall}$ and the function $\Phi^i$ is a solution to (16).*

*If $\varepsilon^i < 8/9$ and $a^i \neq \pm 1$* or *$a^i = \pm 1$, the solution $(\underline{u}^i, p^i, \underline{\underline{\tau}}^i)$ is unique for any f and $\underline{u}_{wall}$. If $\varepsilon^i > 8/9$ and $a^i \neq \pm 1$, then, there is non-uniqueness of solutions for a certain range of parameters.*

*Proof :* First, we solve the volumic equations in the generic layer $i$ and, so, will omit the superscript $i$. From (10), one can easily find the extrastress :

$$\begin{cases} \tau_{11} = \dfrac{(1+a)\, m\varepsilon\, \mathrm{We}\, u'^2(y)}{1 + \mathrm{We}^2(1-a^2)\, u'^2(y)}\,, \\ \tau_{12} = \dfrac{m\varepsilon u'(y)}{1 + \mathrm{We}^2(1-a^2)\, u'^2(y)}\,, \\ \tau_{22} = \dfrac{(a-1)\, m\varepsilon\, \mathrm{We}\, u'^2(y)}{1 + \mathrm{We}^2(1-a^2)\, u'^2(y)}\,. \end{cases} \tag{14}$$

We report these formulae in (8) and easily get the pressure :

$$p(x,y) = -fx + P_0 + \tau_{22}(y)\,. \tag{15}$$

We readily obtain the following equation for $\Phi = u'(y)$, where $k^2 = \mathrm{We}^2(1 - a^2)$ and $Z_\alpha(y) = fy - \alpha$

$$k^2 m(1 - \varepsilon)\, \Phi^3 + k^2 Z_\alpha(y)\, \Phi^2 + m\Phi + Z_\alpha(y) = 0 . \tag{16}$$

In the case where (16) is not a cubic equation ($a = \pm 1$), the solution $\Phi$ is unique. For all the other cases, there is clearly a solution, the discriminant of this equation being negative for all external force if $\varepsilon < 8/9$. So, if $\varepsilon < 8/9$ there will be a unique $u'$ for any given $k, Z, \alpha$ and, if $\varepsilon > 8/9$, there is a range for the parameters in which there will be multiple solutions $u'$. If $\Phi(Z_\alpha(y))$ is the solution to (16), we have, in each layer $i$:

$$u^i(y) = u^i(y_i) + \int_{y_i}^{y} \Phi^i(Z_{\alpha^i}(y'))\, dy' \quad \forall y \in [y_i, y_{i+1}] . \tag{17}$$

Till now, the only difference of our proof with Guillopé and Saut's one is that they prove that the solution $\Phi$ exists as the inverse of a given function. In view of future explicit computations, we prefer to have explicit formulae, even though they are not easy to handle, instead of the implicit solution to the same equation.

So far, the velocity in each sub domain is determined up to two constants per domain $u^i(y_i)$ and $\alpha^i$. We shall now determine these constants through the interface and boundary conditions.

The equation (11) at the $i$th interface gives, after projection :

$$\begin{cases} f^i = f^{i+1} , & (18) \\ P_0^i = P_0^{i+1} , & (19) \\ \alpha^i = \alpha^{i+1} . & (20) \end{cases}$$

Then we perform an induction on the layer $i$ to prove that, for all $i \geqslant 1$, $u^{i+1}(y_{i+1})$ is strictly increasing in $\alpha^1$.

In domain 1, the velocity $u^1(y_1)$ is zero due to the boundary condition (adhesion) and $\alpha^1$ is unknown. For each $i \geqslant 1$, the continuity of velocity is written :

$$u^{i+1}(y_{i+1}) = u^i(y_{i+1}) = u^i(y_i) + \int_{y_i}^{y_{i+1}} \Phi^i(Z_{\alpha^i}(y'))\, dy' . \tag{21}$$

The quantity $u^i(y_i)$ depends in an increasing manner on $\alpha^1$ by the induction hypothesis (and is null if $i = 1$). Moreover, the integral is strictly increasing in $\alpha^1$, thanks to Lemma 2.1 proved in [8]. This completes the induction.

LEMMA 2 1 *Let $\varepsilon$ in $]0, \frac{8}{9}[$, $(y, z)$ in $\mathbb{R}^2$ such that $y < z$ and $f \in \mathbb{R}^+$ If $\Phi(x)$ is the (unique) solution of*

$$k^2 m(1 - \varepsilon) \Phi^3(x) + k^2 x\Phi^2(x) + m\Phi(x) + x = 0 , \tag{22}$$

*then, $\Psi$ $\alpha \mapsto \int_y^z \Phi(fy' - \alpha)\, dy'$ is a strictly increasing one-to-one map from $\mathbb{R}$ onto $\mathbb{R}$ If $\varepsilon > 8/9$, $\Psi$ is not increasing, but, still maps $\mathbb{R}$ onto $\mathbb{R}$*

As a consequence, if $n \geqslant 1$, the upper plate velocity $u^n(1) \left( = u^n(y_n) + \int_y^1 \Phi^n(Z_\alpha(y'))\, dy' \right)$ is a strictly increasing function in $\alpha^1 = \alpha^n$ from $\mathbb{R}$ to $\mathbb{R}$ if $\varepsilon < 8/9$ Thus, in a Poiseuille/Couette flow under the hypothesis $\varepsilon < 8/9$, a unique choice of $\alpha^1$ leads to the fulfilling of the upper boundary condition (13)

If $\varepsilon > 8/9$ and lies in the range of multiplicity (see [1]), any choice of $\Phi$, solution to (16) leads to multiple solutions in $\alpha^1$ and so for the velocity $u$ $\square$

*Remark 2 2* We give for computing purpose the formulae for $u'$

$$u'(y) = \left( \left( -q + \sqrt{q^2 + 4p^3/27}/2 \right)^{1/3} + \left( -q - \sqrt{q^2 + 4p^3/27}/2 \right)^{1/3} \right),$$

where

$$4p^3 + 27q^2 = \frac{1}{k^6(1-\varepsilon)^3} \left[ 4 + \frac{k^2 Z^2(y)(27\varepsilon^2 - 36\varepsilon + 8)}{m^2(1-\varepsilon)} + \frac{4k^4 Z^4}{m^4(1-\varepsilon)} \right],$$

and

$$q = \left( \frac{2Z^3 k^2}{27m^2(1-\varepsilon)^2} - \frac{Z(3\varepsilon - 2)}{3(1-\varepsilon)} \right) / (k^2 m(1-\varepsilon)),$$

though there is no explicit formula for $\alpha$

### 2.3. Bounded perturbations

We are interested in the non-stationary one dimensional flow $(\underline{u} = (u(y,t), 0))$ of a single VEF The classical equations (7) are equivalent to the following system closer to mechanical interpretations (see [9], [8])

$$\begin{cases} \operatorname{Re} \dfrac{D\underline{u}}{Dt} - (1-\varepsilon)\, \Delta \underline{u} + \underline{\nabla} p = \operatorname{div} \underline{\underline{\tau}}\,, & (23) \\ \begin{cases} \dfrac{D\underline{\underline{R}}_a}{Dt} = \underline{\underline{R}}_a\, \underline{\underline{m}}_a^T(\underline{u})\,, \\ \underline{\underline{R}}_a(\underline{x}, t, t) = \underline{\underline{I}}\,, \end{cases} & (24) \\ \underline{\underline{m}}_a(\underline{u}) = \underline{\underline{\Omega}}[\underline{u}] - a\underline{\underline{D}}[\underline{u}]\,, & (25) \\ \underline{\underline{W}}_a + \operatorname{We} \dfrac{D\underline{\underline{W}}_a}{Dt} = 2\,\varepsilon \underline{\underline{R}}_a\, \underline{\underline{D}}[\underline{u}]\, \underline{\underline{R}}_a^T\,, & (26) \\ \underline{\underline{W}}_a = \underline{\underline{R}}_a\, \underline{\underline{\tau}}(x)\, \underline{\underline{R}}_a^T & (27) \\ \underline{u}(0) = 0,\ \underline{u}(1) = \underline{u}_{wall} & (28) \end{cases}$$

Let $(\underline{u}_s = (u_s(y), 0), p_s, \underline{\underline{\tau}}_s)$ be a stationary solution We set $\underline{u} = \underline{u}_s + \underline{u}$, $\underline{\underline{\tau}} = \underline{\underline{\tau}}_s + \underline{\underline{\tau}}$, $p = p_s + p$ in (23-28), where $(\underline{u} = (u(y), 0), \underline{\underline{\tau}}, p)$, to have a nonlinear perturbation about the stationary solution In view of this study in the following subsections, we investigate the properties of $\underline{\underline{R}}_a$ and define the matrix $\underline{\underline{m}}_a = \begin{pmatrix} 0 & 1-a \\ -(1+a) & 0 \end{pmatrix}$ Straightforward calculations, not reproduced here, give, in the non-stationary case, the following lemma

LEMMA 2 2 *If $u_s(y,t)$ is in $L^1_{loc}(\mathbb{R}^+, L^2_{loc}(0,1))$, there is a unique solution to (24) given by*

$$\begin{cases} \underline{\underline{R}}_a(\underline{x}, t, s) = \underline{\underline{I}} \cos \omega(y,t,s) + \underline{\underline{m}}_a^T \dfrac{\sin \omega(y,t,s)}{\sqrt{1-a^2}}\,, \\ \omega(y,t,s) = \dfrac{\sqrt{1-a^2}}{2} \displaystyle\int_s^t \frac{\partial u}{\partial y}(t')\, dt' \end{cases} \tag{29}$$

*This function is in $L^\infty(\mathbb{R}^+, L^\infty(0,1))$ Moreover, for any $\underline{\underline{\tau}}$, and $\underline{\underline{W}}_a$ symmetrical matrices ,*

$$\begin{aligned} |\underline{\underline{R}}_a\, \underline{\underline{\tau}}\, \underline{\underline{R}}_a^T| &\leqslant \frac{1+|a|}{\sqrt{1-a^2}}\, |\underline{\underline{\tau}}|\,, \\ |\underline{\underline{R}}_a^{-1}\, \underline{\underline{W}}_a\, \underline{\underline{R}}_a^{-T}| &\leqslant \frac{1+|a|}{\sqrt{1-a^2}}\, |\underline{\underline{W}}_a|\,, \end{aligned} \tag{30}$$

*where $|\,.\,|$ is the Euclidean norm in $\mathbb{R}^4$*

Notice that, even in the case of a stationary velocity $\underline{u}_s$, the stationary $\underline{\underline{R}}_{as}$ will depend on $t$. Thanks to Lemma 2.2, we can study the boundedness of perturbations in two different ways.

### 2.3.1. *First boundedness result*

Let $a \neq 0$. Through a simple translation ;

$$\underline{\underline{W}}'_a = \underline{\underline{W}}_a + \frac{\varepsilon}{\mathrm{We}\, a} \underline{\underline{R}}_a \underline{\underline{R}}_a^T , \tag{31}$$

we can change the second member of the constitutive equation and derive the new one :

$$\underline{\underline{W}}'_a + \mathrm{We} \frac{D\underline{\underline{W}}'_a}{Dt} = \frac{\varepsilon}{a\, \mathrm{We}} \underline{\underline{R}}_a \underline{\underline{R}}_a^T . \tag{32}$$

Since we are interested in nonlinear perturbations, we replace $\underline{u}$, $\underline{\underline{\tau}}$, $p$, $\underline{\underline{R}}_a$ and $\underline{\underline{W}}'_a$ by $\underline{u}_s + \underline{u}$, $\underline{\underline{\tau}}_s + \underline{\underline{\tau}}$, $p_s + p$, $(\underline{\underline{R}}_{as} + \underline{\underline{R}}_a)$ and $(\underline{\underline{W}}'_{as} + \underline{\underline{W}}'_a)$ respectively, in the new nonlinear system (23, 24, 25, 27, 31, 32). Then, we have the following theorem :

THEOREM 2.2 : *Let* $\varepsilon \in [0, 1[$, $a \neq 0$, $f \geqslant 0$ *and* $(\underline{u}_s = (u_s(y), 0), \underline{\underline{\tau}}_s, p_s)$ *be a stationary solution of (7). Then, any one dimensional perturbation* $(\underline{u} = (u(y), 0), \underline{\underline{\tau}}, p)$, *about a stationary solution, is such that*

$$\underline{u} \in L^\infty(\mathbb{R}^+ ; L^2(0,1)), \quad \underline{\underline{\tau}} \in L^\infty(\mathbb{R}^+ ; L^\infty(0,1)),$$

$$\underline{u} \in L^2(0, T ; H^1(0,1)) \quad \forall T \in \mathbb{R}^+ .$$

*Proof :* Lemma 2.2 tells us that $(\underline{\underline{R}}_{as} + \underline{\underline{R}}_a)(\underline{\underline{R}}_{as} + \underline{\underline{R}}_a)^T$ and $\underline{\underline{R}}_{as} \underline{\underline{R}}_a^T$ are in $L^\infty(\mathbb{R}^+, L^\infty(0,1))$. Then it is an easy consequence of (32), which reduces to a pure transport problem, that $\underline{\underline{W}}'_{as} + \underline{\underline{W}}'_a \in L^\infty(\mathbb{R}^+ ; L^\infty(0,1))$. For the same reasons on the stationary fields, one can see that $\underline{\underline{W}}'_{as}$ is also in $L^\infty(\mathbb{R}^+ ; L^\infty(0,1))$ and so that $\underline{\underline{W}}'_a$ is in the same space. From (30), one conclude that $\underline{\underline{\tau}}$ is also in $L^\infty(\mathbb{R}^+ ; L^\infty(0,1))$. Then, it is easy to find the result on the velocity, using (23). □

*Remark 2.3 :* With more tedious computation, the next subsection will extend the same boundedness result to the case $a = 0$, under an extra condition on $\varepsilon$. The present proof is much simpler than the next one, and we believe it could be of some interest in future calculations, for instance because of its mechanical interpretation (see [9]).

2.3.2. *Second boundedness result*

In this subsection, we state a result similar to Theorem 2.2, that can also be applied if $a = 0$. The complete proof can be found in [8] pp. 41-49.

THEOREM 2.3 : *Let* $\varepsilon \in \left[0, \frac{1-a^2}{2(1+|a|)}\right[$, $a \in \left]-1, 1\right[$ *and* $(\underline{u}_s, \underline{\underline{\tau}}_s, p_s, \underline{\underline{R}}_{as})$ *be a stationary solution of (23, 24, 25, 26, 27, 28). Then, every one dimensional perturbation* $(\underline{u} = (u(y), 0), \underline{\underline{\tau}}, p)$ *is such that*

$$\underline{u} \in L^\infty(\mathbb{R}^+ ; L^2(0,1)), \quad \underline{\underline{\tau}} \in L^\infty(\mathbb{R}^+ ; L^\infty(0,1)),$$

$$\underline{u} \in L^2(0, T; H^1(0,1)) \quad \forall T \in \mathbb{R}^+ .$$

*Scheme of the proof :*

The continuity results (30) assessed in Lemma 2.2, and a lengthy but standard energy method applied to (23) and (26) yields

$$\frac{d|\underline{\underline{W}}_a|^2}{dt} + \frac{1}{\mathrm{We}}\left(2 - \frac{2\sqrt{2}\,\varepsilon(1+|a|)\,\delta}{\sqrt{1-a^2}} - \frac{\sqrt{2}\,\varepsilon(1+|a|)\,\gamma}{\sqrt{1-a^2}}\right)|\underline{\underline{W}}_a|^2 \leqslant$$

$$\leqslant f(t) + \frac{\sqrt{2}\,\varepsilon(1+|a|)\,|\underline{\nabla} u|^2}{\sqrt{1-a^2}\;\gamma\,\mathrm{We}}, \tag{33}$$

$$\frac{d|u|^2}{dt} + \frac{1}{\mathrm{Re}}\left(2(1-\varepsilon) - \frac{\beta(1+|a|)}{\sqrt{2(1-a^2)}} - \theta\right)|\underline{\nabla} u|^2 \leqslant g(t) +$$

$$+ \frac{(1+|a|)\,|\underline{\underline{W}}_a|^2}{\sqrt{2(1-a^2)}\,\mathrm{Re}\,\beta}. \tag{34}$$

Here, the $(\beta, \gamma, \delta, \theta)$ arise from Young inequalities and $f(t)$ and $g(t)$ are some functions in $L^\infty$. An optimal choice of $(\beta, \gamma, \delta, \theta)$ and a linear combination of these two inequalities completes the proof of the theorem. □

## 3. POISEUILLE FLOW IN AXISYMMETRIC GEOMETRY

### 3.1. Existence and uniqueness

The next theorem states the existence/uniqueness of an axisymmetric stationary Poiseuille flow of *n* VEF (*see fig.* 2) obeying a Johnson-Segalman model.

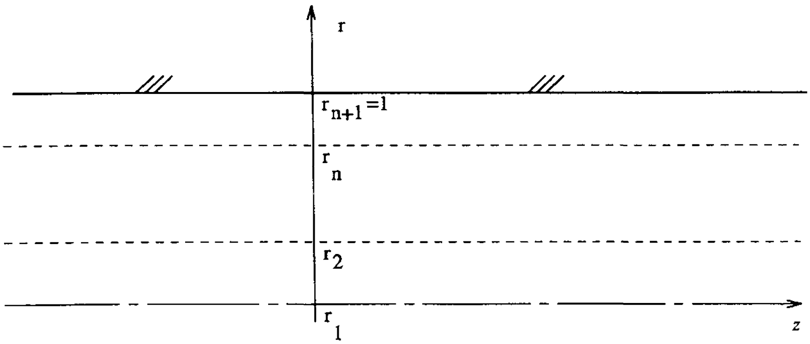


**Figure 2. — Axisymmetric geometry.**

It also shows the mathematical impossibility to have cylinders of fluids turning around the axis of symmetry, since the azimuthal velocity is proved to be null. Basically, this is linked to the second order of the differential equations involved, the boundedness of the velocity on the axis, and the zero boundary condition. In addition to the previous notations, let $T^i \in \mathbb{R}^+$ be the coefficient of surface tension at the $i$th interface.

THEOREM 3.1 : *Let the dimensionless pressure drop $f^i = -\partial p^i/\partial z > 0$, $m^i > 0$ and $a^i \in [-1, 1]$ be given. There exists a solution with continuous velocity $((0, v, w), p, \underline{\tau})$, in cylindrical coordinates, of (8, 37, 11, 12) in n infinite cylinders with a bounded velocity on the axis. These solutions are such that, in each domain i :*

$$\begin{cases} v^i(r) = 0\,, \\ w^i(r) = w^i(r_i) + \displaystyle\int_{r_i}^{r} \Phi^i(f^i\, r'/2)\, dr'\,, \\ p^i(r, z) = -f^i\, z + \xi^i(r)\,, \\ \xi^i(r) = \displaystyle\int_{r_i}^{r} \frac{1}{r'} \frac{\partial(r'\tau^i_{rr})}{\partial r}\, dr' + P^i_0\,, \\ \tau^i_{rr}\; \tau^i_{rz} \quad \text{and} \quad \tau^i_{zz} \text{ are given by } (38)\,, \end{cases} \tag{35}$$

*where $f^i = f$ is independent on i, and $\Phi^i$ is a solution of (40) with $Z(r) = fr/2$.*

*The relations :*

$$\begin{cases} w^n(r_{n+1}) = 0 \\ w^i(r_i) = - \sum_{j=i}^{n} \int_{r_j}^{r_{j+1}} \Phi^j(fr'/2)\, dr' \quad i = 1, n , \\ [\![- \xi + \tau_{rr}]\!] (r_{i+1}) = - T^i / r_{i+1} \quad i = 1, n-1 , \end{cases} \tag{36}$$

*determine the $w^i(r_i)$ and the $P_0^i$ up to a global pressure constant.*

*If $a^i \neq \pm 1$ and $\varepsilon^i < 8/9$ or $a^i = \pm 1$, the solution $\Phi^i$ and $((0, v^i, w^i), p^i, \underline{\underline{\tau}}^i)$ are unique.*

*If $a^i \neq \pm 1$ and $\varepsilon^i < 8/9$, there exists a range of parameters for which non unique solutions exist.*

*Proof :* The scheme of the proof is very similar to the plane case one.

First, we must write the cylindrical version of the equations in the layer $i$ (the superscript will be omitted in the first part of the proof). Let us denote by $(\tau_{i,j})$ the matrix of the extrastress in cylindrical coordinates $(\tau_{11} = \tau_{rr}, \ldots)$, and by $DU$ the rate of strain tensor in cylindrical coordinates. As is proved in [8], the constitutive equation in the cylindrical coordinates is

$$\begin{aligned} (\tau_{i,j}) - \mathrm{We} \Bigg[ & \frac{(a-1)}{2} [(\tau_{i,j}) DU + DU^T (\tau_{i,j})] \\ & + \frac{(a+1)}{2} [(\tau_{i,j}) DU^T + DU(\tau_{i,j})] \\ & + \frac{v}{r} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\tau_{i,j}) + \frac{v}{r} (\tau_{i,j}) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg] \\ = m\varepsilon (DU + DU^T) . \end{aligned} \tag{37}$$

Hereafter, we will assume that $v = 0$. The complete proof that $v = 0$ can be found in [8]. Then, one readily obtains, from (37), the non-zero components of the extrastress in the domain :

$$\begin{cases} \tau_{rr} = \dfrac{m\varepsilon \, \mathrm{We}\,(a-1) B^2}{1 + k^2 B^2} , \\ \tau_{rz} = \dfrac{m\varepsilon B}{1 + k^2 B^2} , \\ \tau_{zz} = \dfrac{m\varepsilon \, \mathrm{We}\,(a+1) B^2}{1 + k^2 B^2} , \\ B = \dfrac{\partial w}{\partial r} . \end{cases} \tag{38}$$

These formulae are reported in the equation of conservation of momentum (23). As a first consequence, we find :

$$\begin{cases} p(r,z) & = -fz + \xi(r)\,, \\ \xi'(r) & = \dfrac{1}{r}\dfrac{\partial(r\tau_{rr})}{\partial r}\,. \end{cases} \tag{39}$$

After some easy calculation, we prove that $w$ satisfies the same cubic equation in $w' = \Phi$ as in the plane case (16) with a new $Z_\alpha(r) = (fr^2/2 - \alpha)/r$ :

$$k^2 m(1-\varepsilon)\,\Phi^3 + k^2 Z_\alpha(r)\,\Phi^2 + m\Phi + Z_\alpha(r) = 0\,. \tag{40}$$

Equation (40) always admits a solution, but, under the hypothesis $\varepsilon < 8/9$ or $a = \pm 1$, it is unique for all parameters. We will denote it by $\Phi(Z_\alpha(r))$. Then, the velocity $w^i$ in the domain $i$ is such that :

$$w^i(r) = w^i(r_i) + \int_{r_i}^{r} \Phi^i(Z_{\alpha_i}(r'))\,dr'\,.$$

So far, the velocity in each sub domain is determined up to two constants per domain : $w^i(r_i)$ and $\alpha^i$.

Equation (11), written in cylindrical coordinates, on the $i$th interface $(r = r_{i+1})$ reads

$$\left[\!\left[ -p(r,z)\begin{vmatrix}1\\0\\0\end{vmatrix} + m(1-\varepsilon)\begin{vmatrix}0\\0\\\dfrac{\partial w}{\partial r}\end{vmatrix} + \begin{vmatrix}\tau_{rr}\\0\\\tau_{rz}\end{vmatrix}\right]\!\right] = -\frac{T^i}{r_{i+1}}\begin{vmatrix}1\\0\\0\end{vmatrix}\,. \tag{41}$$

The first component of (41) gives

$$f^i = f^{i+1} \quad \text{and} \quad [\![-\xi + \tau_{rr}]\!](r_{i+1}) = -T^i/r_{i+1}\,, \tag{42}$$

and then, the third component of (41) can be written :

$$(f^i r_{i+1}/2 - \alpha^i) = (f^{i+1} r_{i+1}/2 - \alpha^{i+1}) \Rightarrow \alpha^i = \alpha^{i+1}\,.$$

Moreover, the velocity at the centerline being bounded, one can easily prove that $\alpha^1 = 0$ and so are the $\alpha^i$. Then, the continuity of the velocity is written :

$$w^{i+1}(r_{i+1}) = w^i(r_{i+1}) = w^i(r_i) + \int_{r_i}^{r_{i+1}} \Phi^i(fr'/2)\,dr'\,, \tag{43}$$

which leads to (36), thanks to the boundary condition $w^n(r_{n+1}) = 0$.

It is clear that if (40) admits multiple solutions, the final solutions will be multiple. So, the proof of the Theorem 3.1 is complete. □

*Remark 3.1 :* The explicit formulae are the same as in the plane case, with $Z_\alpha = fr/2$, except that they are easier to compute since $\alpha^i = 0$.

### 3.2. Boundedness result in the axisymmetric geometry

The next theorem is very similar to Theorem 2.2 and applies to an axisymmetric geometry.

THEOREM 3.2 : *Let* $\varepsilon \in [0, 1[$, $a \neq 0$, $f > 0$ *and* $(\underline{u}_s = (0, 0, w_s(r)), \underline{\underline{\tau}}_s, p_s)$ *be a stationary solution of (7). Then, every perturbation* $(\underline{u}, \underline{\underline{\tau}}, p)$ *with* $\underline{u} = (0, 0, w(r, t))$ *satisfies*

$$\underline{u} \in L^\infty(\mathbb{R}^+ ; L^2(0, 1)), \quad \underline{\underline{\tau}} \in L^\infty(\mathbb{R}^+ ; L^\infty(0, 1)),$$

$$\textit{and} \quad \underline{u} \in L^2(0, T ; H^1(0, 1)) \quad \forall T \in \mathbb{R}^+ .$$

*Proof:* As in the plane case (see Lemma 2.2), we introduce the equivalent system (23-28). We only need to compute the matrix $\underline{\underline{R}}_a$, and find, through tedious but straightforward calculations :

$$\underline{\underline{R}}_a(r, t ; s) =$$

$$\begin{pmatrix} \cos\omega\cos^2\theta & \cos\omega\sin\theta\cos\theta & \dfrac{\sin\omega}{\sqrt{1-a^2}}(1-a)\cos\theta \\ \cos\omega\sin\theta\cos\theta & \cos\omega\sin^2\theta & \dfrac{\sin\omega}{\sqrt{1-a^2}}(1-a)\sin\theta \\ \dfrac{-\sin\omega}{\sqrt{1-a^2}}(1+a)\cos\theta & \dfrac{-\sin\omega}{\sqrt{1-a^2}}(1+a)\sin\theta & \cos\omega \end{pmatrix}, \tag{44}$$

$$\omega(r, t ; s) = \frac{\sqrt{1-a^2}}{2}\int_s^t \frac{\partial w}{\partial r}(r, t')\, dt' . \tag{45}$$

The remaining of the proof is the same as the one of theorem 2.2. It all relies on the fact that $\underline{\underline{R}}_a \in L^\infty(\mathbb{R}^+, L^\infty(0, 1))$. □

## 4. EXISTENCE/UNIQUENESS FOR PTT INTERPOLATED MODELS

The PTT models read :

$$g_{\epsilon}(\underline{\underline{\tau}})\,\underline{\underline{\tau}} + \mathrm{We}\frac{\mathcal{D}_a\underline{\underline{\tau}}}{\mathcal{D}t} = 2\,\varepsilon\underline{\underline{D}}[\underline{u}] , \tag{46}$$

where $g_{\epsilon}(\underline{\underline{\tau}}) = \tilde{g}\,(\varepsilon' \mathrm{We}\,\mathrm{tr}\,\underline{\underline{\tau}}\,)$ is a scalar function and $\epsilon'$ is a positive parameter. For the PTT case $\tilde{g}\,(X) = \exp(X)$ (*cf.* [11]), while $\tilde{g}\,(X) = 1 + X$ for the MPTT one. The other constants $a, \varepsilon, m, f$ and We have the same meaning as in Section 2.1.

In this Section, we prove two theorems. The first one assesses the existence of the Poiseuille/Couette flow of $n$ PTT or MPTT fluids in a plane geometry. The second one gives the limiting parameters for the uniqueness of these solutions.

THEOREM 4.1 (Existence) : *Let* $a^i \in [-1, 1]$, $\varepsilon^i \in [0, 1[$, $m^i$, $\epsilon'^i > 0$ *and* $We^i$, $f^i \geqslant 0$. *The stationary system (8, 9, 46) completed with the boundary condition (13) and interface conditions (11, 12) has at least one solution.*

THEOREM 4.2 (Uniqueness) : *Under the assumptions of Theorem 4.1, we have :*

*For a MPTT fluid ;*

- $1 > a > 0 \quad \epsilon' > \dfrac{1 - a^2}{2\, m\varepsilon(1 - \varepsilon)\,|a|}\left(\varepsilon - \dfrac{8}{9}\right) \Leftrightarrow$ *uniqueness for all f.*
- $-1 < a < 0 \left(0 \leqslant \epsilon' < \dfrac{1 - a^2}{2\, m\varepsilon(1 - \varepsilon)\,|a|}\left(\dfrac{8}{9} - \varepsilon\right)\right.$ *or* $\left.\left(\epsilon' > \dfrac{1 - a^2}{2\,\varepsilon\,|a|}\right)\right) \Leftrightarrow$ *uniqueness for all f.*

*For a PTT fluid ;*

- $1 > a > 0 \quad \epsilon' > (1 - a^2)/(2\, am(1 - \varepsilon)) \Rightarrow$ *uniqueness for all f.*
- $-1 < a < 0$ *Let* $K = (m\varepsilon\epsilon'|a|)/(1 - a^2)$. $\exists K_0 < 1 / K < K_0$ *and* $\varepsilon < 8/9 \Rightarrow$ *uniqueness for all f.*

*Moreover, for a PTT or a MPTT fluid ;*

*If* $a = 0$, *the condition* $\varepsilon < 8/9$ *ensures uniqueness.*

*If* $a = 1$, *there is always uniqueness.*

*If* $a = -1$, *the condition* $\varepsilon < 2/(2 + \exp 3/2)$ *gives rise to multiplicity of solutions for a PTT fluid, while a MPTT fluid will exhibit multiple solutions if* $\varepsilon\epsilon' We^2/m$ *is small enough.*

*If the parameters of every layer ensure uniqueness for the one-fluid flow, then uniqueness holds for the flow of n PTT or MPTT fluids.*

Before stating the proof, let us notice, that, in [13], Keunings and Crochet studied the elongational flow of a PTT or MPTT fluid. In their article, they already exhibited some curves (*cf.* their *fig. 3* and *4*) which proved the multiplicity of solutions. Only one of them had a Newtonian limit as the product of the rate of elongational $\dot{\alpha}$ by the relaxation time $\lambda$ tends to zero.

*Proof* The stationary volumic equations (8, 9, 46) of the Poiseuille/Couette flow of a PTT or MPTT fluid give, after some easy calculation

$$
\begin{aligned}
(a-1)\,\tau_{11} &= (a+1)\,\tau_{22}\,, \\
g(\underline{\underline{\tau}})\,\tau_{11} &= \mathrm{We}\,u'(y)\,(a+1)\,\tau_{12}\,, \\
g(\underline{\underline{\tau}})\,\tau_{12} &= u'(y)\left(m\varepsilon + \mathrm{We}\,\frac{(a+1)}{2}\,\tau_{22} + \mathrm{We}\,\frac{(a-1)}{2}\,\tau_{11}\right), \\
g(\underline{\underline{\tau}})\,\tau_{22} &= \mathrm{We}\,u'(y)\,(a-1)\,\tau_{12}\,, \\
\tau_{12} &= -\,m(1-\varepsilon)\,u'(y) + Z_\alpha(y)\,, \\
p &= -fx + \tau_{22}(y) + P_0\,,
\end{aligned}
\tag{47}
$$

with the same term $Z_\alpha(y) = fy - \alpha$ as in Section 2 and two constants $P_0, \alpha$ We will study separately the cases $a = \pm 1$, $a = 0$ and $a \in\ ]-1, +1[\setminus\{0\}$

If $a = 1$, we eliminate $u'$ and $\tau_{12}$, and have to solve the following equation in $\tau_{11}$

$$
h(\epsilon'\mathrm{We}\,\tau_{11}) = \epsilon'\mathrm{We}\,\tau_{11}(\varepsilon + (1-\varepsilon)\,\tilde{g}\,(\epsilon'\mathrm{We}\,\tau_{11}))^2 = 2\,\varepsilon\epsilon'\mathrm{We}^2\,Z_\alpha^2(y)/m
\tag{48}
$$

Clearly, the solution $\tau_{11}$ is positive Differentiating $h$ for either the PTT or the MPTT model, we easily see that (48) has a unique solution for any $\epsilon' > 0$ Moreover, $\tau_{11} \to +\infty$ as the pressure drop $f$ tends to $+\infty$ Thus $\tau_{11} \in \mathbb{R}^+$ is an optimal estimate for all pressure drop $f$

If $a = -1$, we eliminate $\tau_{12}$ and $u'$ and have to solve in $\tau_{22}$

$$
\begin{aligned}
h(\epsilon'\mathrm{We}\,\tau_{22}) &= \epsilon'\mathrm{We}\,\tau_{22}(\varepsilon + (1-\varepsilon)\,\tilde{g}\,(\epsilon'\mathrm{We}\,\tau_{22}))^2 \\
&= -\,2\,\varepsilon\epsilon'\mathrm{We}^2\,Z_\alpha^2(y)/m
\end{aligned}
\tag{49}
$$

Clearly, $\tau_{22}$ is negative As $X(\varepsilon + (1-\varepsilon)\,\tilde{g}\,(X))^2 \to -\infty$ when $X \to -\infty$, both models have a solution The main difference with the previous case $(a = 1)$ is that $h$ is not always monotonic on $\mathbb{R}^-$ as can be seen from figure 3

For the PTT model, only $\varepsilon > 2/(2 + \exp 3/2) \simeq 0\,3$ assures uniqueness for any right-hand side $\varepsilon\epsilon'\mathrm{We}^2\,Z_\alpha^2/m$ Otherwise, there are multiple solutions $\tau_{22}$ and $u'$ for a certain range of $\varepsilon$, $\epsilon'$, We, $f$, $m$ or $\alpha$

For the MPTT model, figure 3 shows that if $\varepsilon\epsilon'\mathrm{We}^2\,Z_\alpha^2(y)/m$ is small enough (it is the case on the axis of symmetry where $Z_\alpha = 0$, or if $\varepsilon\epsilon'\mathrm{We}^2$ is small enough !), there will be multiple solutions Here again, $\varepsilon$, but also We and $\epsilon'$, have to be not too small to permit uniqueness ! Let us stress that M Renardy and Y Renardy [6] quoted a private communication of U Akbay claiming that there were instabilities for the Couette flow of a LCM

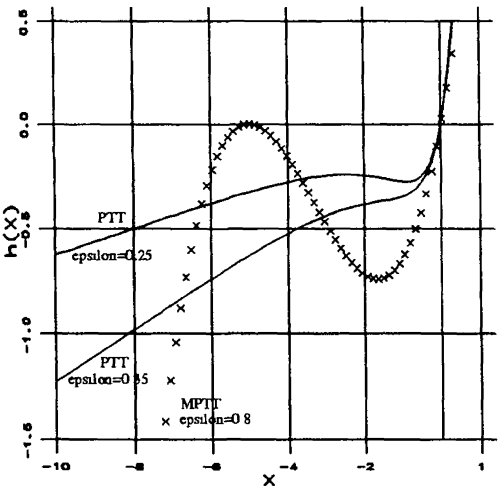


**Figure 3 — Shape of the functions $h$**

fluid $(a = -1)$ The non-uniqueness found here could mean that the linearized system of equations has 0 as an eigenvalue, and so, that the flow lies on a neutral curve, enabling these instabilities to occur

If $a = 0$, the trace of $\underline{\underline{\tau}}$ is zero and so the model gives the same predictions as a Johnson Segalman one Therefore the conditions of uniqueness for one fluid are the same as Guillopé Saut's [1] $\varepsilon < 8/9$

If $a \in \,]-1, 1[\setminus\{0\}$, through various eliminations, (47) exhibits a boundedness requirement

$$\frac{-\varepsilon m}{(a+1)\,\mathrm{We}} < \tau_{22} < 0 < \tau_{11} < \frac{\varepsilon m}{(1-a)\,\mathrm{We}} \tag{50}$$

Then, the trace of $\underline{\underline{\tau}}$ being $2\,a\tau_{22}/(a-1)$, it will be of the sign of $a$ Once we have eliminated $u'$, $\tau_{12}$ and $\tau_{11}$, we want to solve the following equation in $T = \tau_{22}\,\mathrm{We}(a+1)/(\varepsilon m)$

$$\begin{aligned} h(T) &= \frac{T}{1+T}\left(\varepsilon(1+T) + (1-\varepsilon)\,\tilde{g}\left(\frac{2\,\varepsilon\varepsilon' maT}{a^2-1}\right)\right)^2 \\ &= \mathrm{We}^2(a^2-1)\,Z_\alpha^2/m^2\,, \end{aligned} \tag{51}$$

with $T \in \,]-1, 0[$ (see (50)) Obviously, this equation has a solution for both models Let us stress that if the pressure drop is arbitrary in $\mathbb{R}$, then the solution $T$ is arbitrary in $]-1, 0[$ This proves that (50) is optimal

For the MPTT model, the computation of the derivative of $h$ gives a necessary and sufficient condition depending on the sign of $a$

- $1 > a > 0 \quad \epsilon' > \dfrac{1-a^2}{2\,m(1-\varepsilon)\,|a|}\left(\varepsilon - \dfrac{8}{9}\right) \Leftrightarrow$ uniqueness,
- $-1 < a < 0 \quad 0 \leqslant \epsilon' < \dfrac{1-a^2}{2\,m\varepsilon(1-\varepsilon)\,|a|}\left(\dfrac{8}{9} - \varepsilon\right) \Leftrightarrow$ uniqueness.

Let us notice that as $(8/9 - \varepsilon)/(1 - \varepsilon) < 1$, the last condition ensures that $g(\underline{\tau}) \geqslant 0$ for all external conditions. This will appear as crucial in the study of stability in Section 5.

Similar calculations for the PTT model do not give so simple conditions since the equations are not algebraic. We only found sufficient conditions :

- $1 > a > 0 \quad \epsilon' > \dfrac{1-a^2}{2\,a(1-\varepsilon)\,m} \Rightarrow$ uniqueness,
- $-1 < a < 0$ Let $K = m\varepsilon\epsilon'\,|a|/(1-a^2) < 1$. There exists $0 < K_0 < 1$ such that, if $0 \leqslant K < K_0$ and $\varepsilon < 8/9$, then uniqueness holds.

Concerning the existence/uniqueness of multiple VEF Poiseuille/Couette flows, we use the same method as for the Johnson-Segalman model. The interface relations are the same as in the Johnson-Segalman case (18, 19, 20) and give $\alpha^i, f^i$. Assuming that the conditions of existence and uniqueness written above are satisfied for fluid $i$, we write $u'(y)$ from (47) :

$$u^i(y) = u^i(y_i) + \int_{y_i}^{y} \frac{(-fy' + \alpha)}{m(1-\varepsilon) + (m\varepsilon + \mathrm{We}\,(a+1)\,\tau_{22}(y', \alpha))/(\tilde{g}\,(2\,\epsilon'\mathrm{We}\,a\tau_{22}(y', \alpha)/(a-1)))}\,dy', \tag{52}$$

where $\tilde{g}\,(X)$ is either $\exp X$ for the PTT case, or $1 + X$ for the MPTT one, and $\tau_{22}(y', \alpha)$ is the unique solution of (51). Assuming that the conditions required for uniqueness hold, one can prove, in the same way as for Lemma 2.1 that the function

$$\alpha \mapsto \int_{y_i}^{y_{i+1}} \frac{(-fy' + \alpha)\,dy'}{m(1-\varepsilon) + (m\varepsilon + \mathrm{We}\,(a+1)\,\tau_{22}(y', \alpha))/(\tilde{g}\,(2\,\epsilon'\mathrm{We}\,a\tau_{22}(y', \alpha)/(a-1)))}$$

is strictly increasing and that it maps $\mathbb{R}$ onto $\mathbb{R}$. We then prove as in Section 2.2 that $u^n(y_{n+1})$ is strictly increasing with respect to $\alpha^1$. So the proof of the existence and uniqueness for any pressure drop $f$ or any $u_{wall}$ is complete. □

*Remark 4 1* We must notice that, as in the Johnson-Segalman model, the extrastress is unbounded with respect to $\gamma$ if $a = \pm 1$, and bounded if $a \neq \pm 1$ (50) This seems to be an asset to these models in addition to their non-zero second normal stress difference This property remains in the axi-symmetric geometry

*Remark 4 2* Let us stress that the cases $a < 0$ and $a > 0$ give opposite conditions on $\epsilon'$ This is due to the fact that the trace of $\underline{\underline{\tau}}$ is of the sign of $a$ If $a$ is negative, the $g_\epsilon(\underline{\underline{\tau}})$ term of the PTT model can be very small and destabilize the differential equation This disadvantage is more acute for the MPTT model, whose $g_\epsilon(\underline{\underline{\tau}})$ term can even get negative To solve these problems, we propose the following constitutive equation depending on the second invariant of $\underline{\underline{\tau}}$

$$\tilde{g}\left(\epsilon' \mathrm{We} \sqrt{\operatorname{tr}\left(\underline{\underline{\tau}}\,\underline{\underline{\tau}}\right)}\right)\underline{\underline{\tau}} + \mathrm{We}\frac{\mathscr{D}_a \underline{\underline{\tau}}}{\mathscr{D}t} = 2\,\iota \underline{\underline{D}}[\underline{u}]\,,$$

for any $\tilde{g}$ positive and strictly increasing on $\mathbb{R}^+$ Unlike the PTT and MPTT models, this equation may not have a null or even negative damping term Moreover, it also takes shearing stress into account Last, we have proved existence and uniqueness for all $\epsilon' > 0$ and all $a \in [-1, 1]$ of the plane Poiseuille/Couette flow of $n$ such fluids

## 5. LINEAR STABILITY OF THE PTT/MPTT COUETTE FLOW

In Section 4, we have given sufficient conditions to ensure the uniqueness of some stationary solutions of (47), that do always exist As is explained in [1] and [10], in the case of the Navier-Stokes equations in a bounded domain, we know that the nonlinear stability is given by the linear stability, which occurs if and only if the spectrum of the linearized stationary operator is on the right side of the imaginary axis Recent results of M Renardy [14] have brought a new insight on Couette flows of viscoelastic fluids and proved under weak assumptions that the principle of linear stability holds

In this Section, we first prove that the linearized operator of the one dimensional non-stationary Couette flow of a PTT or a MPTT fluid is analytic Then, we look for sufficient conditions on the controlling parameters, to ensure that the spectrum will be on the right side of the imaginary axis Last we conclude to the linear stability under some conditions

To do this, we project the non stationary equations of a Couette flow and linearize them about a stationary solution $((u_s(y), 0), p_s, \underline{\underline{\tau}}_s)$ found in the previous Section, and obtain the system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1-\varepsilon}{\mathrm{Re}} u'' - \dfrac{1}{\mathrm{Re}} \tau' & = 0 , \\ \dfrac{\partial \sigma}{\partial t} - (a+1)\, \tau_s u' + \epsilon' \sigma_s \tilde{g}\,'(\epsilon' \mathrm{We}\, \mathrm{tr}\, \underline{\underline{\tau}}_s)\, \gamma & \\ \quad + \left( \dfrac{g_\epsilon(\underline{\underline{\tau}}_s) + \epsilon' \mathrm{We}\, \sigma_s \tilde{g}\,'(\epsilon' \mathrm{We}\, \mathrm{tr}\, \underline{\underline{\tau}}_s)}{\mathrm{We}} \right) \sigma - (a+1)\, u_s' \tau & = 0 , \\ \dfrac{\partial \tau}{\partial t} - \dfrac{\varepsilon + (a+1)\, \gamma_s \mathrm{We}}{\mathrm{We}} u' + & \\ \quad + \left( \epsilon' \tau_s \tilde{g}\,'(\epsilon' \mathrm{We}\, \mathrm{tr}\, \underline{\underline{\tau}}_s) - \dfrac{(a-1)\, u_s'}{2} \right) \sigma + \dfrac{g_\epsilon(\underline{\underline{\tau}}_s)}{\mathrm{We}} \tau & \\ \quad + (\epsilon' \tau_s \tilde{g}\,'(\epsilon' \mathrm{We}\, \mathrm{tr}\, \underline{\underline{\tau}}_s) - (a+1)\, u_s'/2)\, \gamma & = 0 , \\ \dfrac{\partial \gamma}{\partial t} - (a-1)\, \tau_s u' + (\epsilon' \gamma_s \tilde{g}\,'(\epsilon' \mathrm{We}\, \mathrm{tr}\, \underline{\underline{\tau}}_s))\, \sigma - (a-1)\, u_s' \tau & \\ \quad + (g_\epsilon(\underline{\underline{\tau}}_s)/\mathrm{We} + \epsilon' \gamma_s \tilde{g}\,'(\epsilon' \mathrm{We}\, \mathrm{tr}\, \underline{\underline{\tau}}_s))\, \gamma & = 0 , \\ u(0) = u(1) = 0 , & \end{cases} \tag{53}$$

where $\begin{pmatrix} \sigma & \tau \\ \tau & \gamma \end{pmatrix}$ is the matrix of extrastress either indexed with $s$ for the stationary solution about which we perturb, or non-indexed for the perturbating extrastress, $\tilde{g}\,'$ the derivative of the function $\tilde{g}$ introduced in the previous Section ($e^x$ for the PTT model and $1 + x$ for the MPTT one) Let us stress that this system is the one of a PTT/MPTT Couette flow under one-dimensional perturbations

Denoting the vector $(u, \sigma, \tau, \gamma)^T$ by $U$ and the spatial operator of (53) by $\mathscr{L}$ in $(L^2(0, 1))^4$, we are interested, in a first part, in the location of the spectrum of $\mathscr{L}$ with respect to the imaginary axis To that purpose, we define the domain of the unbounded linear operator $\mathscr{L}$ as the set of $(u, \sigma, \tau, \gamma) \in H_0^1(0, 1) \times (L^2(0, 1))^3$ such that $-\varepsilon u_{yy} - \tau_y \in L^2(0, 1)$ With this definition and using the same method as in [1], one could prove the following theorem

THEOREM 5 1 *If* $a \in [-1, 1]$, $\varepsilon \in [0, 1[$, $\epsilon' \in \mathbb{R}^+$,

$\mathscr{L}$ *is a closed operator in* $(L^2(0, 1))^4$ *with dense domain,*

$\mathscr{L}$ *is m-sectorial with vertex* $-\Lambda$ *for some* $\Lambda > 0$ *and semi-angle* $\frac{\pi}{4}$

A consequence of this Theorem is that $\mathscr{L}$ is analytic and so that the linear stability is governed by the location of the spectrum of $\mathscr{L}$ with respect to the imaginary axis Before stating the next Theorem which brings some insight on the multiplicity of eigenvalues, we introduce the following notations

$$
\begin{aligned}
A &= -\frac{\pi^2}{\mathrm{Re}}(1-\varepsilon)\,, \\
B &= -\left(2\,\frac{g_\epsilon(\underline{\underline{\tau}}_s)}{\mathrm{We}} + \epsilon'(\sigma_s+\gamma_s)\,\tilde{g}\,'(\epsilon'\,\mathrm{We}\,\mathrm{tr}\,\underline{\underline{\tau}}_s)\right), \\
C &= \frac{\pi^2}{\mathrm{Re}}\left(\frac{g_{\epsilon'}(\underline{\underline{\tau}}_s)}{\mathrm{We}}\,\frac{\tau_s}{u'_s} - (1-\varepsilon)\,B\right), \qquad (54) \\
D &= \left(\frac{g_{\epsilon'}(\underline{\underline{\tau}}_s)}{\mathrm{We}}\right)^2 + 4\,a\epsilon' u'_s \tau_s\,\tilde{g}\,'(\epsilon'\,\mathrm{We}\,\mathrm{tr}\,\underline{\underline{\tau}}_s) + u_s'^2(1-a^2)\,, \\
E &= \frac{\pi^2}{\mathrm{Re}}\left[\left(\frac{g_\epsilon(\underline{\underline{\tau}}_s)}{\mathrm{We}}\right)^2\frac{\tau_s}{u'_s} + \frac{g_\epsilon(\underline{\underline{\tau}}_s)}{\mathrm{We}}(a+1)\,\gamma_s + (1-\varepsilon)\,D\right]
\end{aligned}
$$

THEOREM 5 2 *Assume that* $a \in [-1, 1]$, $\varepsilon \in [0, 1[$ $\epsilon' \in \mathbb{R}^+$ *and let A, B, C, D, E be given by (54)*

- *If* $\lambda$ *is one of the three roots of*

$$
\left(\frac{g_\epsilon(\underline{\underline{\tau}}_s)}{\mathrm{We}} - \lambda\right)(\lambda^2 + \lambda B + D)\,, \qquad (55)
$$

*then it is an eigenvalue of countable multiplicity of* $\mathscr{L}$

- *The spectrum contains only eigenvalues Except the three above-mentioned eigenvalues, they are of finite multiplicity, and roots of*

$$
n^2 = \lambda_n \frac{\lambda_n^2 + B\lambda_n + D}{A\lambda_n^2 + C\lambda_n - E}
$$

*The sequence* $\lambda_n$ *satisfies* $\lambda_n - An^2 \to 0$ *and thus,*

$$
\exists \eta > 0 / \forall n \in \mathbb{N},\ |\lambda_{n+1} - \lambda_n| \geqslant \eta > 0 \qquad (56)
$$

- *The spectrum will contain 0, with infinite multiplicity, if and only if, one over the three terms* $g_{\epsilon'}(\underline{\underline{\tau}}_s)/\mathrm{We}$, *D or E is zero In the first two cases, the multiplicity is countable, while it is uncountable in the third one*

*Sketch of the proof* (we refer the reader to [8] for more details)

First, to prove the theorem, we try to solve the subsystem of $(\mathscr{L} - \lambda \underline{\underline{I}})\, \underline{U} = 0$ that comes from the linearization of the constitutive equations to have $\tau(u')$ Using (47), this subproblem can be rewritten in the form

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \begin{vmatrix} \sigma \\ \tau \\ \gamma \end{vmatrix} = \frac{u'}{\mathrm{We}} \frac{g_\epsilon(\underline{\underline{\tau}}_s)}{u'_s} \begin{vmatrix} \sigma_s \\ \tau_s \\ \gamma_s \end{vmatrix}, \tag{57}$$

where $\underline{\underline{A}}$, a $3 \times 3$ matrix, whose characteristic polynomial is

$$\left( \frac{g_\epsilon(\underline{\underline{\tau}}_s)}{\mathrm{We}} - \lambda \right) [\lambda^2 + B\lambda + D] \tag{58}$$

If $\lambda$ is an eigenvalue of $\underline{\underline{A}}$, one can prove that (57) has non-zero solutions such that $\tau = C_1 u' + C_2$ where $C_1$ and $C_2$ are two constants independent on $y$ (thanks to the stationary Couette flow properties) The conservation of momentum restricts to a countable set of admissible $C_1$ which enables only countable multiplicity for these eigenvalues of $\mathscr{L}$ and $\underline{\underline{A}}$

If $\lambda$ is not an eigenvalue of $\underline{\underline{A}}$, the Cramer's formulae give

$$\tau = \frac{-\lambda g_\epsilon(\underline{\underline{\tau}}_s)\, \tau_s / (\mathrm{We}\, u'_s) + \mathrm{Re}\, E/\pi^2 - (1 - \varepsilon) D}{\lambda^2 + B\lambda + D} u' =$$

$$= F(\lambda, \mathrm{We}, u'_s, \epsilon', \varepsilon, a, \sigma_s, \tau_s, \gamma_s)\, u' \tag{59}$$

Using the notations (54), we easily show that the eigenvalues such that $(1 - \varepsilon) + F(\lambda, \quad) \neq 0 \quad (\Rightarrow \lambda \neq 0)$ are solutions of

$$\frac{(1 - \varepsilon) + F(\lambda, \mathrm{We}, \quad)}{\mathrm{Re}\, \lambda} = \frac{1}{n^2 \pi^2} \tag{60}$$

$$\Leftrightarrow \lambda^3 + \lambda^2 (A n^2 + B) + \lambda (C n^2 + D) - n^2 E = 0$$

Let us notice that $\epsilon' = 0$ in (60) gives the equation (5 8) of [1] Thanks to (60), one can easily prove that $\lambda_n - A n^2 \to 0$ when $n \to \infty$, which ensures (56)

If $(1 - \varepsilon) + F(\lambda, \quad) = 0$ (which implies $\lambda = 0$ and so $E = 0$), any $u \in H^2 \cap H_0^1$ and corresponding $\sigma$, $\tau$, $\gamma$ will be convenient and the multiplicity is uncountable

Then, through the same calculations as in [1], one can prove that the spectrum contains only eigenvalues

The last point of Theorem 5 2 is then easy since the spectrum will contain 0 if it is a root of (55) (i e $g_\epsilon(\underline{\underline{\tau}}_s)/\mathrm{We}$ or $D$ is zero), or if 0 is a root of (60) $(E = 0)$, in which case, the multiplicity in uncountable □

Next, we state our main result on the linear stability of Couette PTT or MPTT flows

THEOREM 5 3 *Let* $K = \varepsilon\epsilon'|a|/(1-a^2)$ *where* $a \neq \pm 1$ *and* $\varepsilon > 0$ *There exists* $K_0$ *in* $]0, 2\varepsilon/(3(1-\varepsilon))$ *[and* $K_-$, $K_+$ *in* $]0, 1/3[$ *for the MPTT model, and there exists* $K_0 \in\ ]0, 1[$ *for the PTT model, depending on the flow parameters, such that under the conditions given in the array hereafter, then the Couette flow of a PTT or MPTT fluid is linearly stable under one-dimensional perturbations*

*Moreover, if* $g_\epsilon(\underline{\underline{\tau}}_s) < 0$ *(case MPTT* $a < 0$*), then the flow is unstable*

| | PTT case | MPTT case |
|---|---|---|
| $a = +1$ | no condition | no condition |
| $0 < a < 1$ | $K > \dfrac{2\varepsilon}{3(1-\varepsilon)}$ | $K > 2\varepsilon/(3(1-\varepsilon))$ or<br>$K < 2\varepsilon/(3(1-\varepsilon))$ and<br>if $\varepsilon < 4/5$<br>$k^2 > (2\varepsilon - 1)/(1-\varepsilon)$ or<br>$k^2 < (2\varepsilon - 1)/(1-\varepsilon)$ and $0 < K_0 < K < \dfrac{2\varepsilon}{3(1-\varepsilon)}$<br>if $\varepsilon > 4/5$<br>$k^2 > (2-\varepsilon)^2/(12(1-\varepsilon)^2)$ or<br>$k^2 < (2-\varepsilon)^2/(12(1-\varepsilon)^2)$ and<br>$0 < K_0 < K < \dfrac{2\varepsilon}{3(1-\varepsilon)}$ |
| $a = 0$ | $\varepsilon < 8/9$ or $\varepsilon \geqslant 8/9$ and $k^2 \in\ ]0, \kappa_-[\ \cup\ ]\kappa_+, +\infty[$ cf [1] | |
| $-1 < a < 0$ | $k^2 > \dfrac{2\varepsilon - 1}{1-\varepsilon}$<br>and<br>$0 < K < K_0$ | $1/3 < k^2$ and<br>if $(2-\varepsilon)^2/(12(1-\varepsilon)^2) < k^2$ $\forall K < 1/3$<br>if $\dfrac{2\varepsilon - 1}{1-\varepsilon} < k^2 < \dfrac{(2-\varepsilon)^2}{12(1-\varepsilon)^2}$<br>$K \in\ ]0, K_-[\ \cap\ ]0, 1/3[$<br>if $\dfrac{2\varepsilon}{3(1-\varepsilon)} < k^2 < \dfrac{(2-\varepsilon)^2}{12(1-\varepsilon)^2}$<br>$K \in\ ]K_+, 1/3[\ \cap\ ]0\ 1/3[$ |
| $a = -1$ | $\epsilon' = 0$ or We $= 0$ | |

We only give the sketch of the proof The interested reader can find a more complete one in [8], where explicit values for $K_-$ and $K_+$ can be found

*Sketch of the proof* In the first part, we seek sufficient conditions for the eigenvalues to have positive real parts To do so, we first study the eigenvalues of the $3 \times 3$ matrix $\underline{\underline{A}}$ introduced in the proof of theorem 5 2, splitting the cases $a \geqslant 0$, $-1 < a < 0$ and $a = -1$ Then, we look for the eigenvalues roots of (60) for which we use the Routh-Hurwitz criterion (see [16] p 490) that gives necessary and sufficient conditions ensuring that the roots of the cubic polynomial (60) have positive real parts

$$\begin{cases} An^2 + B < 0 \quad \forall n \geqslant 1 \,, \\ E > 0 \,, \\ - \; (An^2 + B)\,(Cn^2 + D) - n^2\,E > 0 \quad \forall n \geqslant 1 \end{cases} \tag{61}$$

The study of these conditions requires to split cases on $a$ and on the model and leads to the sufficient conditions summarized in the above array

In a second part, we use Theorem 5 1 to ensure that the location of the spectrum on the right of the imaginary axis gives the linear stability This last argument is basically due to the properties of the one dimensional flows

One might also conclude thanks to a theorem of M Renardy [15], and the second point of Theorem 5 2 ensures that the spectrum on the right of the imaginary axis gives stability (principle of linear stability) □

*Remark 5 1* An oversimplified, but sufficient in some experiments, version of theorem 5 3 is that if $a > 0$, large $\epsilon'$ give stability, while, if $a < 0$, only small $\epsilon'$ give it

## 6 CONCLUSION

In this article, we proved the existence of solutions for the Poiseuille/Couette flow of $n$ fluids, obeying interpolated Johnson Segalman models in either plane or axisymmetric geometries, extending some results of [1] We gave some limiting parameters for uniqueness to occur We also proved that these flows, submitted to 1-D perturbations, remain bounded, even in the range where there is no uniqueness Only the case $a = 0$ and $\epsilon > 1/2$ remains uninvestigated To obtain these results, we used a non-common formulation which might be of some interest for future theoretical studies because of its physical meaning

We also proved the existence of solutions for the plane Poiseuille/Couette flow of $n$ PTT or MPTT fluids for all flow parameters Under certain conditions, uniqueness or non-uniqueness can be guaranteed. The case $a = -1$ gave unnatural results which should prevent numerical analysts from using these equations. Then, we proposed a modification of these PTT/MPTT models which leads to existence and uniqueness, takes shearing stress into account and removes some drawbacks of the PTT and MPTT models.

Last, we gave sufficient conditions for the linear one-dimensional stability of a plane Couette PTT/MPTT flow.

The author wishes to thank Professor J. C. Saut for directing this work.

## REFERENCES

[1] C. GUILLOPÉ & J. C. SAUT, 1990, "Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type", M²AN Vol. 24, n° 3, 369-401.

[2] R. W. KOLKKA, G. R. IERLEY, M. G. HANSEN & R. A. WORTHING, 1987, "On the stability of viscoelastic parallel shear flows", Technical Report, F.R.O.G. Michigan Technological University.

[3] C. GUILLOPÉ & J. C. SAUT, 1990, "Existence results for the flow of viscoelastic fluids with a differential constitutive law". *Nonlinear Analysis Theory, Methods & Applications,* Vol. 15, No 9, 849-869.

[4] R. J. GORDON & SCHOWALTER, 1972, *Trans. Soc. Rheol.,* **16,** 79.

[5] L. A. DÁVALOS-OROZCO, 1992, "Capillary instability due to a shear stress on the free surface of a viscoelastic fluid layer", *J. Non-Newtonian Fluid Mech.,* **45,** 171-186.

[6] M. RENARDY & Y. RENARDY, 1986, "Linear stability of plane Couette flow of an Upper Convected Maxwell fluid", *J. Non-Newtonian Fluid Mech.,* **22,** 23-33.

[7] G. M. WILSON & B. KHOMAMI, 1992, "An experimental investigation of interfacial instabilities in multilayer flow of viscoelastic fluids. I. Incompatible polymer systems", *J. Non-Newtonian Fluid Mech.,* **45,** 355-384.

[8] H. LE MEUR, 1994, *Existence, unicité et stabilité d'écoulements de fluides viscoélastiques avec interface,* PhD Thesis of University Paris-Sud Orsay.

[9] D. D. JOSEPH, *Fluid dynamics of Visco Elastic liquids,* Applied Mathematical Sciences 84 Springer Verlag.

[10] D. D. JOSEPH, 1976, *Stability of fluid motions,* Vol. I, II Springer.

[11] N. PHAN-THIEN & R. I. TANNER, 1977, "A new constitutive equation derived from network theory", *J. Non-Newtonian Fluid Mech.,* **2,** 353-365.

[12] R. I. TANNER, *Viscoélasticité non linéaire : Rhéologie et modélisation numérique,* Ecoles CEA-EDF-INRIA, 27-30/01/1992.

[13] R. KEUNINGS & M. J. CROCHET, 1984, "Numerical simulation of the flow of a viscoelastic fluid through an abrupt contraction", *J. Non-Newtonian Fluid Mech.*, **14**, 279-299.

[14] M. RENARDY, "On the linear stability of parallel shear flows of viscoelastic fluids of Jeffreys type" to appear.

[15] M. RENARDY, 1993, "On the type of certain $C^0$ Semigroups", *Comm. Part. Diff. Eq.*, 18 (7 & 8), 1299-1307.

[16] P. HENRICI, 1974, *Applied and Computational Complex Analysis*, vol. I, John Wiley, New-York.