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C.-H. BRUNEAU

P. FABRIE

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NEW EFFICIENT BOUNDARY CONDITIONS FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS : A WELL-POSEDNESS RESULT (*)

by C.-H. BRUNEAU (1) and P. FABRIE (1, 2)

Abstract. — *Efficient natural conditions on open boundaries for incompressible flows are derived from a weak formulation of Navier-Stokes equations. Energy estimates in velocity-pressure are established from a mixed formulation and a rigourous proof of existence of solutions is given. As an illustration, the conditions are written down for the flow behind an obstacle in a channel. Moreover, numerical tests have shown the accuracy and robustness of such conditions.*

INTRODUCTION

The aim of this work is to find out boundary conditions that convey properly the vortices through an artificial limit of the domain. These last ten years, many authors have dealt with this problem for various equations, the most famous of which is the wave equation. Following the theory of absorbing boundary conditions, Halpern [8] for the linear advection diffusion equation and Halpern-Schatzman [9] for the linearized Navier-Stokes equations derive artificial conditions that yield a well-posed problem. In [2], Begue-Conca-Murat-Pironneau review a family of boundary conditions on dynamical pressure for stationary Stokes and Navier-Stokes equations, show that these conditions lead to well-posed problems and give some numerical experiments. Their conditions in vorticity are, in some way, natural boundary conditions for a weak formulation in velocity-pressure.

In this paper, we present natural boundary conditions for a weak formulation in velocity-pressure involving the stress tensor

$$\sigma(U,p) = \frac{2}{Re} D(U) - pI\,, \quad \text{with} \quad D(U)_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For Stokes problem, these conditions reduce to

$$\sigma(U,p)\,.\,n = G$$

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(1) Mathématiques Appliquées de Bordeaux, Université Bordeaux I, 33405 Talence, France.
(2) LEPT-ENSAM-URA 873.

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on open boundaries where n is the unit outward normal vector.

For Navier-Stokes equations, we take into account the contribution of convection terms to avoid reflections on artificial boundaries. We establish in the general case, existence of a weak solution for unsteady flows and uniqueness in two dimension. The proof follows classical techniques using Cauchy-Kovaleska regularization. Moreover, our conditions have been used successfully to compute the flow behind a cylinder in a channel [3]. The numerical results at high Reynolds numbers, exhibit accurate solutions without any reflections even when strong vortices cross the artificial limit of the domain.

1. WEAK FORMULATION

The goal of this work is to find out open boundary conditions for incompressible Navier-Stokes equations. Let Ω be a connected bounded domain in $\mathbb{R}^N$ $(N \leqslant 3)$ with smooth boundary $\partial\Omega$; we assume that $\partial\Omega$ has two connected components Γ_0 and $\Gamma_1 = \Gamma_D \cup \Gamma_N$ with meas $(\Gamma_D) \neq 0$ meas $(\Gamma_N) \neq 0$ and $\Gamma_D \cap \Gamma_N = \emptyset$ (see *fig. 1*).

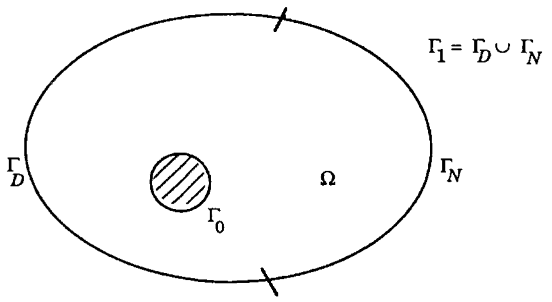

Figure 1. — The domain Ω.

We want to solve the following evolution problem for $t \in (0, T)$

$$(\mathscr{P}) \quad \begin{array}{lll} \partial_t U + (U \cdot \nabla) \cdot U - \operatorname{div} \sigma(U, p) = 0 & \text{in} & Q_T = \Omega \times (0, T) \\ \operatorname{div} U = 0 & \text{in} & Q_T \\ U(x, 0) = U_{in}(x) & \text{in} & \Omega \\ U(\,.\,, t) = 0 & \text{on} & \Gamma_0 \times (0, T) \\ U(\,.\,, t) = G_D & \text{on} & \Gamma_D \times (0, T) \\ \text{and an artificial condition} & \text{on} & \Gamma_N \end{array}$$

and we assume that there exists G_1 an extension of G_D to Γ_1 such that $\int_\Gamma G_1 \, d\gamma = 0.$

We shall precise the functional spaces and the regularity of the data in Section 2.

1.1. Auxiliary Stokes problem

For solving the problem $(\mathscr{P})$, it is convenient to introduce a divergence free function U_0 such that the trace of U_0 on Γ_0 is zero and the trace of U_0 on Γ_1 is G_1. Without any loss of generality we take (U_0, p_0) solution of the following Stokes problem

$$(\mathscr{S}) \quad \begin{array}{lll} \operatorname{div} \sigma(U_0, p_0) = 0 & \text{in} & \Omega \\ \operatorname{div} U_0 = 0 & \text{in} & \Omega \\ U_0 = 0 & \text{in} & \Gamma_0 \\ U_0 = G_1 & \text{on} & \Gamma_1 . \end{array}$$

Under hypothesis of regularity on Ω and G_1, the problem $(\mathscr{S})$ has always a unique regular solution [14]. So, by setting $V = U - U_0$ and $q = p - p_0$, the problem $(\mathscr{P})$ is equivalent to $(\mathscr{P}_{\text{hom}})$

$$(\mathscr{P}_{\text{hom}}) \quad \begin{array}{lll} \partial_t V + ((V + U_0) . \nabla) . (V + U_0) - \operatorname{div} \sigma(V, q) = 0 & \text{in} & \Omega \times (0, T) \\ \operatorname{div} V = 0 & \text{in} & \Omega \times (0, T) \\ V(x, 0) = V_{in}(x) = U_{in}(x) - U_0(x) & \text{in} & \Omega \\ V(. , t) = 0 & \text{on} & \Gamma_0 \cup \Gamma_D \times (0, T) \\ \text{and an artificial condition} & \text{on} & \Gamma_N . \end{array}$$

In this work, we establish a family of natural boundary conditions on Γ_N for $(\mathscr{P}_{\text{hom}})$. These conditions can depend on U_0 and, as U_0 is uniquely determined by G_1, the only arbitrary action is the choice of G_1. In Section 3, we show that the physics of the problem generaly yields a canonical extension.

1.2. Formal open boundary conditions

Let us denote (Ψ, π) a couple of regular test functions such that Ψ vanishes on $\Gamma_0 \cup \Gamma_D$. Assuming the solution of $(\mathscr{P}_{\text{hom}})$ is smooth enough, we can write by splitting the convection term

$$\int_\Omega \partial_t V . \psi \, dx + \int_\Omega (V . \nabla) V . \Psi \, dx + \int_\Omega (U_0 . \nabla) V . \Psi \, dx +$$

$$+ \int_\Omega V . \nabla U_0 . \Psi \, dx + \int_\Omega U_0 . \nabla U_0 . \Psi \, dx - \int_\Omega \operatorname{div} \sigma(V, q) . \Psi \, dx = 0 .$$

Let us take three arbitrary real numbers α_i, $1 \leqslant i \leqslant 3$. Integrating by part, we get

$$\int_\Omega \partial_t V \cdot \Psi \, dx + \frac{1}{2} \int_\Omega (V \cdot \nabla V \cdot \Psi - V \cdot \nabla \Psi \cdot V) \, dx + \frac{1}{2} \int_{\Gamma_N} V \cdot n V \cdot \Psi \, d\gamma +$$

$$+ \int_\Omega (\alpha_1 U_0 \cdot \nabla V \cdot \Psi - (1 - \alpha_1) U_0 \cdot \nabla \Psi \cdot V) \, dx + (1 - \alpha_1) \int_{\Gamma_N} U_0 \cdot n V \cdot \Psi \, d\gamma$$

$$+ \int_\Omega (\alpha_2 V \cdot \nabla U_0 \cdot \Psi - (1 - \alpha_2) V \cdot \nabla \Psi \cdot U_0) \, dx + (1 - \alpha_2) \int_{\Gamma_N} V \cdot n U_0 \cdot \Psi \, d\gamma$$

$$+ \int_\Omega (\alpha_3 U_0 \cdot \nabla U_0 \cdot \Psi - (1 - \alpha_3) U_0 \cdot \nabla \Psi \cdot U_0) \, dx + (1 - \alpha_3)$$

$$\times \int_{\Gamma_N} U_0 \cdot n U_0 \cdot \Psi \, d\gamma + \int_\Omega \sigma(V, q) : \nabla \Psi \, dx - \int_{\Gamma_N} \sigma(V, q) \cdot n \cdot \Psi \, d\gamma = 0 .$$

Indeed, we point out to the reader that by symmetry

$$D(V) : \nabla \Psi = D(V) : D(\Psi)$$

and that

$$qI : \nabla \Psi = q \operatorname{div} \Psi .$$

Moreover, we gather some boundary terms to obtain for instance

$$\frac{1}{2} \int_{\Gamma_N} V \cdot n V \cdot \Psi \, d\gamma + (1 - \alpha_1) \int_{\Gamma_N} U_0 \cdot n V \cdot \Psi \, d\gamma =$$

$$= \int_{\Gamma_N} \left(\frac{1}{2} V \cdot n + (1 - \alpha_1) U_0 \cdot n \right) V \cdot \Psi \, d\gamma = \int_{\Gamma_N} h(V) \, V \cdot \Psi \, d\gamma .$$

If $h(V)$ is a non negative term, we can derive an *a priori* estimate for the velocity and the pressure as it is shown below. Otherwise, we must vanishe at least the negative part. Indeed, if we remark that we can write

$$h(V) = h(V)^+ - h(V)^- = 2 \, h(V)^+ - | h(V) |$$

it is possible to keep in the weak formulation either $h(V)^+$ or $2 \, h(V)^+$.

Further, in the general case, an external force F can be applied on Γ_N.

So, the weak formulation reads

$$\int_{\Omega} \partial_t V \cdot \Psi + \frac{1}{2} (V \cdot \nabla V \cdot \Psi - V \cdot \nabla \Psi \cdot V) \, dx$$

$$+ \int_{\Omega} (\alpha_1 U_0 \cdot \nabla V \cdot \Psi - (1 - \alpha_1) U_0 \cdot \nabla \Psi \cdot V) \, dx$$

$$+ \int_{\Omega} (\alpha_2 V \cdot \nabla U_0 \cdot \Psi - (1 - \alpha_2) V \cdot \nabla \Psi \cdot U_0) \, dx$$

$$(\mathcal{F}) + \int_{\Omega} (\alpha_3 U_0 \cdot \nabla U_0 \cdot \Psi - (1 - \alpha_3) U_0 \cdot \nabla \Psi \cdot U_0) \, dx$$

$$+ \frac{2}{Re} \int_{\Omega} D(V) : D(\Psi) \, dx$$

$$- \int_{\Omega} q \operatorname{div} \Psi \, dx + \beta \int_{\Gamma_N} \left(\frac{1}{2} V \cdot n + (1 - \alpha_1) U_0 \cdot n \right)^+ V \cdot \Psi \, d\gamma$$

$$+ \beta \int_{\Gamma_N} ((1 - \alpha_2) V \cdot n + (1 - \alpha_3) U_0 \cdot n)^+ U_0 \cdot \Psi \, d\gamma = \int_{\Gamma_N} F \cdot \Psi \, d\gamma ,$$

$$\int_{\Omega} \pi \operatorname{div} V \, dx = 0 ,$$

$$V(\cdot , 0) = V_{in}$$

where β is a non negative real number.

Under some regularity assumptions on (V, q), the weak formulation yields

$$(\mathcal{P}_{\text{hom}}) \quad \begin{array}{ll} \partial_t V + ((V + U_0) \cdot \nabla) \cdot (V + U_0) - \operatorname{div} \sigma(V, q) = 0 & \text{in} \quad \Omega \times (0, T) \\ \operatorname{div} V = 0 & \text{in} \quad \Omega \times (0, T) \\ V(x, 0) = V_{in}(x) & \text{in} \quad \Omega \\ V(\cdot , t) = 0 & \text{on} \quad \Gamma_0 \cup \Gamma_D \times (0, T) \\ \sigma(V, q) \cdot n + \Big(\beta \Big(\frac{1}{2} V \cdot n + (1 - \alpha_1) U_0 \cdot n \Big)^+ - & \\ \quad - \Big(\frac{1}{2} V \cdot n + (1 - \alpha_1) U_0 \cdot n \Big) \Big) V + & \\ (\beta ((1 - \alpha_2) V \cdot n + (1 - \alpha_3) U_0 \cdot n)^+ - & \\ \quad - ((1 - \alpha_2) V \cdot n + (1 - \alpha_3) U_0 \cdot n)) U_0 = F & \text{on} \quad \Gamma_N \times (0, T) . \end{array}$$

To well understand the boundary condition, we point out to the reader that the nonlinear term $V \cdot \nabla V \cdot \Psi$ in $(\mathscr{F})$ must be symmetrized to obtain an energy estimate [10], [13], [14].

On the contrary, for the other contributions of the convection term, we have the choice to symmetrize them or not. That corresponds to $\alpha_i = 1/2$ or $\alpha_i = 1$. On the other hand, β taking the values 0, 1 or 2 leads to vanishe respectively $h(V)$, $-h(V)^-$ or $-|h(V)|$in the above expression. However, from the mathematical point of view, we need only to assume that β is a non negative real number to get a well-posed problem.

As an example, for $\alpha_1 = 1/2$ and $\alpha_2 = \alpha_3 = 1$, the boundary condition reduces to

$$\sigma(V, q) \cdot n + \frac{1}{2}\left[\beta((V + U_0) \cdot n)^+ - (V + U_0) \cdot n\right] V = F \quad \text{on} \quad \Gamma_N \times (0, T)$$

and if we consider only the three values of β 0, 1/2 or 1, it can be written on the form

$$\sigma(V, q) \cdot n - \frac{1}{2}\Theta((V + U_0) \cdot n) V = F \quad \text{on} \quad \Gamma_N \times (0, T)$$

where Θ is one of the following real functions

$$\Theta(a) = a, \quad \Theta(a) = -a^- \quad \text{or} \quad \Theta(a) = -|a|.$$

Finally the initial problem reads

$$(\mathscr{P}) \quad \begin{array}{ll} \partial_t U + (U \cdot \nabla) \cdot U - \operatorname{div} \sigma(U, p) = 0 & \text{in} \quad \Omega \times (0, T) \\ \operatorname{div} U = 0 & \text{in} \quad \Omega \times (0, T) \\ U(x, 0) = U_{in}(x) & \text{in} \quad \Omega \\ U(\cdot, t) = 0 & \text{on} \quad \Gamma_0 \times (0, T) \\ U(\cdot, t) = G_D & \text{on} \quad \Gamma_D \times (0, T) \\ \multicolumn{2}{l}{\sigma(U, p) \cdot n - \frac{1}{2}\Theta(U \cdot n + (1 - 2\alpha_1) U_0 \cdot n)(U - U_0) -} \\ \multicolumn{2}{l}{\Theta((1 - \alpha_2) U \cdot n + (\alpha_2 - \alpha_3) U_0 \cdot n) U_0 = \sigma(U_0, p_0) \cdot n + F} \\ & \text{on} \quad \Gamma_N \times (0, T) \end{array}$$

where F is equal to zero on artificial boundaries and α_i is equal to 1/2 or 1.

Let us note that, for Stokes flow, the artificial boundary conditions reduce to

$$(0) \quad \sigma(U, p) \cdot n = \sigma(U_0, p_0) \cdot n + F \quad \text{on} \quad \Gamma_N \times (0, T)$$

which is the natural condition. Moreover for Navier-Stokes flow, when $\alpha_1 = 1/2$, $\alpha_2 = \alpha_3$ and $U \cdot n \geq 0$ on $\Gamma_N \times (0, T)$ we get again this condition with $\Theta(a) = -a^-$. That means that for an outgoing flow, it is sufficient to impose Stokes boundary conditions.

We point out to the reader that the condition

$$\sigma(U, p) \cdot n = 0 \quad \text{on} \quad \Gamma_N \times (0, T)$$

is not always compatible with a reference flow.

Remark 1 : If instead of $\sigma(U, p)$, we use the pseudo tensor

$$\tilde{\sigma}(U, p) = \frac{1}{Re} \nabla U - pI ,$$

then the boundary conditions read

$$\tilde{\sigma}(U, p) \cdot n - \frac{1}{2} \Theta(U \cdot n + (1 - 2\alpha_1) U_0 \cdot n)(U - U_0) -$$

$$- \Theta((1 - \alpha_2) U \cdot n + (\alpha_2 - \alpha_3) U_0 \cdot n) U_0 = \tilde{\sigma}(U_0, p_0) \cdot n + F \quad \text{on } \Gamma_N ,$$

and in the same way they reduce to

$$\tilde{\sigma}(U, p) = \tilde{\sigma}(U_0, p_0) \cdot n + F \quad \text{on} \quad \Gamma_N \times (0, T)$$

for Stokes flows.

2. EXISTENCE OF WEAK SOLUTIONS

In this section, we prove the existence of weak solutions (V, q) of $(\mathscr{F})$ and uniqueness in two dimension. We first introduce the following notations

$$\mathbb{L}^p(\Omega) = (L^p(\Omega))^N, p \geq 1, \text{provided with the norm } |\cdot|_p \text{ or } |\cdot| \text{ for } p = 2$$

$$\mathbb{H}_0 = \{V \in \mathbb{L}^2(\Omega) \,;\, \text{div } V = 0 \,;\, n = 0 \text{ on } \Gamma_0 \cup \Gamma_D\}$$

$$\mathbb{H}^s(\Omega) = (H^s(\Omega))^N, s \geq 0, \text{provided with the norm } \|\cdot\|_s$$

$$\mathbb{H}_D^1(\Omega) = \{V \in \mathbb{H}^1(\Omega) \,;\, V = 0 \text{ on } \Gamma_0 \cup \Gamma_D\}$$

$$\mathbb{H}_{D,0}^1(\Omega) = \{V \in \mathbb{H}_D^1(\Omega) \,;\, \text{div } V = 0\}$$

$$\mathbb{X} = \{\Psi \in H^1(0, T \,;\, \mathbb{H}_D^1(\Omega)) \,;\, \Psi(T) = 0\}$$

$$\mathbb{X}_0 = \{\Psi \in \mathbb{X} \,;\, \text{div } \Psi = 0\}$$

$$\mathbb{Y} = \{\pi \in H^1(0, T \,;\, L^2(\Omega)) \,;\, \pi(T) = 0\}$$

$$\mathbb{V} = L^\infty(0, T \,;\, \mathbb{L}^2(\Omega)) \cap L^2(0, T \,;\, \mathbb{H}_D^1(\Omega))$$

$$\mathbb{V}_0 = L^\infty(0, T \,;\, \mathbb{L}^2(\Omega)) \cap L^2(0, T \,;\, \mathbb{H}_{D,0}^1(\Omega))$$

$$\mathbb{Q} = \mathbb{Y}' .$$

Then, we write a mixed formulation in order to easily derive the artificial boundary conditions in a formal way. For $U_0 \in \mathbb{H}^2(\Omega)$, $V_{in} \in \mathbb{H}_0$ and $F \in H^{-1/2}(\Gamma_N)$, we seek a couple $(V, q) \in \mathbb{V} \times \mathbb{Q}$ such that we have in $\mathscr{D}'(0, T)$

$$\int_\Omega \partial_t V \cdot \Psi + \frac{1}{2}(V \cdot \nabla V \cdot \Psi - V \cdot \nabla \Psi \cdot V)\, dx$$

$$+ \int_\Omega (\alpha_1 U_0 \cdot \nabla V \cdot \Psi - (1 - \alpha_1) U_0 \cdot \nabla \Psi \cdot V)\, dx$$

$$+ \int_\Omega (\alpha_2 V \cdot \nabla U_0 \cdot \Psi - (1 - \alpha_2) V \cdot \nabla \Psi \cdot U_0)\, dx$$

$$+ \int_\Omega (\alpha_3 U_0 \cdot \nabla U_0 \cdot \Psi - (1 - \alpha_3) U_0 \cdot \nabla \Psi \cdot U_0)\, dx$$

$$+ \frac{2}{Re} \int_\Omega D(V) : D(\Psi)\, dx$$

$$(\mathscr{F}) \quad - \langle q, \operatorname{div} \Psi \rangle_\Omega + \beta \int_{\Gamma_N} \left(\frac{1}{2} V \cdot n + (1 - \alpha_1) U_0 \cdot n\right)^+ V \cdot \Psi\, d\gamma$$

$$+ \beta \int_{\Gamma_N} ((1 - \alpha_2) V \cdot n + (1 - \alpha_3) U_0 \cdot n)^+ U_0 \cdot \Psi\, d\gamma = \langle F \cdot \Psi \rangle_{\Gamma_N},$$

$$\int_\Omega \pi \operatorname{div} V\, dx = 0 \quad \forall (\Psi, \pi) \in \mathbb{X} \times \mathbb{Y},$$

$$V(\,.\,, 0) = V_{in}.$$

Remark 2 : To give sense to the equality $V(\,.\,, 0) = V_{in}$ we first take the test function Ψ in $\mathbb{X}_0$.

On one hand, for the bidimensional problem, the solution is regular enough in the variable V to take Ψ in $\mathbb{V}_0$; so $\partial_t V$ belongs to $L^2(0, T ; \mathbb{H}^1_{D,0}(\Omega)')$ and then V is continuous from $[0, T]$ into $\mathbb{H}_0$. On the other hand, in 3D we can see that V is continuous from $[0, T]$ into

$$[\mathbb{H}^1_{D,0}(\Omega), (\mathbb{H}^1_{D,0}(\Omega) \cap \mathbb{H}^2(\Omega))']_{1/2} \quad ([10]).$$

So it is important to take a free divergence initial condition.

2.1. Regularized problem

We approximate the Navier-Stokes equations by the artificial compressibility method (see [14]).

That is to seek a couple $(V_\varepsilon, q_\varepsilon) \in \mathbb{V} \times L^\infty(0, T; L^2(\Omega))$ such that for a small $\varepsilon > 0$ we have in $\mathscr{D}'(0, T)$

$$\int_\Omega \partial_t V_\varepsilon . \Psi + \frac{1}{2} (V_\varepsilon . \nabla V_\varepsilon . \Psi - V_\varepsilon . \nabla \Psi . V_\varepsilon) \, dx$$

$$+ \int_\Omega (\alpha_1 U_0 . \nabla V_\varepsilon . \Psi - (1 - \alpha_1) U_0 . \nabla \Psi . V_\varepsilon) \, dx$$

$$+ \int_\Omega (\alpha_2 V_\varepsilon . \nabla U_0 . \Psi - (1 - \alpha_2) V_\varepsilon . \nabla \Psi . U_0) \, dx$$

$$+ \int_\Omega (\alpha_3 U_0 . \nabla U_0 . \Psi - (1 - \alpha_3) U_0 . \nabla \Psi . U_0) \, dx$$

$$+ \frac{2}{Re} \int_\Omega D(V_\varepsilon) : D(\Psi) \, dx$$

$$(\mathscr{F}) \quad - \langle q_\varepsilon, \operatorname{div} \Psi \rangle_\Omega + \beta \int_{\Gamma_N} \left(\frac{1}{2} V_\varepsilon . n + (1 - \alpha_1) U_0 . n \right)^+ V_\varepsilon . \Psi \, d\gamma$$

$$+ \beta \int_{\Gamma_N} ((1 - \alpha_2) V_\varepsilon . n + (1 - \alpha_3) U_0 . n)^+ U_0 . \Psi \, d\gamma = \langle F . \Psi \rangle_{\Gamma_N},$$

$$\varepsilon \int_\Omega \partial_t q_\varepsilon \pi \, dx + \int_\Omega \operatorname{div} V_\varepsilon \pi \, dx = 0 \quad \forall (\Psi, \pi) \in \mathbb{X} \times \mathbb{Y},$$

$$V_\varepsilon(. , 0) = V_{in},$$

$$q_\varepsilon(. , 0) = q_{in}.$$

Remark 3 : To give sense to the equalities $V_\varepsilon(\,.\,,0) = V_{in}$ and $q_\varepsilon(\,.\,,0) = q_{in}$ we distinguish the 2D problem from the 3D one.

For the bidimensional problem, when ε is fixed, the solution is regular enough to get $\partial_t V_\varepsilon$ in $L^2(0,T;\mathbb{H}^1_D(\Omega)')$ and then V_ε is continuous from $[0,T]$ into $\mathbb{L}^2(\Omega)$. In the same way, we can easily see that q_ε is also continuous from $[0,T]$ into $L^2(\Omega)$. For the 3D case, we have the same result for the pressure but V is only continuous from $[0,T]$ into $[\mathbb{H}^1_D(\Omega),(\mathbb{H}^1_D(\Omega)\cap\mathbb{H}^2(\Omega))']_{1/2}$ ([10]). We take a free divergence initial datum V_{in} to avoid discontinuity at $t=0$ for the limit problem.

PROPOSITION 1 : *For* $U_0 \in \mathbb{H}^2(\Omega)$, $V_{in} \in \mathbb{H}_0$, $q_{in} \in L^2(\Omega)$ *and* $F \in \mathbb{H}^{-1/2}(\Gamma_N)$, *the problem* $(\mathscr{F}_\varepsilon)$ *admits at least one solution which is unique in the 2D case.*

Sketch of proof. By Galerkin method, we can show easily that there exists at least one couple $(V_\varepsilon, q_\varepsilon)$ solution of $(\mathscr{F}_\varepsilon)$ which is unique in two dimension. We refer to [10], [14] for the idea of the proof.

2.2. *A priori* estimates

In order to pass to the limit when ε goes to zero, it is convenient to establish the following *a priori* estimates independently of ε.

PROPOSITION 2 : *For each* $T>0$, *there exist some constants* c_1, c_2, c_3 *which depend only on* T *such that*

$$\sup_{t\in(0,T)} |V_\varepsilon(t)| \leqslant c_1 \tag{1}$$

$$\int_0^T |\nabla V_\varepsilon(t)|^2\,dt \leqslant c_2 \tag{2}$$

$$\sup_{t\in(0,T)} \sqrt{\varepsilon}\,|q_\varepsilon(t)| \leqslant c_3\,. \tag{3}$$

Proof: We first give the proof in 2D. In this case, the solution is regular enough to take the test functions $(\Psi, \pi) \in \mathbb{V} \times L^{\infty}(0, T; L^2(\Omega))$ in $(\mathcal{F}_{\varepsilon})$. So, taking $(\Psi, \pi) = (V_{\varepsilon}, q_{\varepsilon})$ and summing both equations, we get

$$\frac{1}{2}\frac{d}{dt}|V_{\varepsilon}|^2 + \frac{\varepsilon}{2}\frac{d}{dt}|q_{\varepsilon}|^2 + \frac{2}{Re}\int_{\Omega} D(V_{\varepsilon}) : D(V_{\varepsilon})\,dx +$$

$$+ \beta \int_{\Gamma_N} \left(\frac{1}{2} V_{\varepsilon} \cdot n + (1 - \alpha_1)\, U_0 \cdot n\right)^+ V_{\varepsilon} \cdot V_{\varepsilon}\, d\gamma$$

$$\leqslant |2\,\alpha_1 - 1|\; |U_0|_{\infty}|\nabla V_{\varepsilon}|\; |V_{\varepsilon}| + |\alpha_2|\; |V_{\varepsilon}|_3^2 |\nabla U_0|_3 +$$

$$+ |1 - \alpha_2|\; |U_0|_{\infty}|\nabla V_{\varepsilon}|\; |V_{\varepsilon}|$$

$$+ |\alpha_3|\; |U_0|_{\infty}|\nabla U_0|\; |V_{\varepsilon}| + |\,1 - \alpha_3|\; |U_0|_4^2|\nabla V_{\varepsilon}|$$

$$+ \beta|((1 - \alpha_2)\, V_{\varepsilon} \cdot n + (1 - \alpha_3)\, U_0 \cdot n)^+|_{3,\Gamma_N}|U_0|_{3,\Gamma_N}|V_{\varepsilon}|_{3,\Gamma_N}$$

$$+ \|F\|_{-1/2,\Gamma_N}\|V_n\|_{1/2,\Gamma_N}$$

then, by Korn inequality we have ([5])

$$|D(V_{\varepsilon})|^2 \geqslant \eta|\nabla V_{\varepsilon}|^2$$

and by Sobolev embeddings for $N \leqslant 3$ and interpolation inequality [1]

$$|U_0|_{\infty} \leqslant c(\Omega)\; \|U_0\|_2$$

$$|\nabla U_0|_3 \leqslant c(\Omega)\; \|U_0\|_2$$

$$|V_{\varepsilon}|_3 \leqslant c(\Omega)\; |V_{\varepsilon}|^{1/2}|\nabla V_{\varepsilon}|^{1/2}$$

where $c(\Omega)$ denotes a generic constant. Moreover, by Sobolev embedding and continuity of the trace operator, we have on Γ_N

$$|V_{\varepsilon}|_{3,\Gamma_N} \leqslant c(\Gamma_N)\; \|V_{\varepsilon}\|_{1/3,\Gamma_n} \leqslant c(\Gamma_N, \Omega)\; \|V_{\varepsilon}\|_{5/6}\,.$$

So, by interpolation inequality

$$|V_{\varepsilon}|_{3,\Gamma_N} \leqslant c(\Gamma_N, \Omega)\; |V_{\varepsilon}|^{1/6}|\nabla V_{\varepsilon}|^{5/6}\,.$$

Thus

$$\frac{1}{2}\frac{d}{dt}|V_\varepsilon|^2+\frac{\varepsilon}{2}\frac{d}{dt}|q_\varepsilon|^2+\frac{2\eta}{Re}|\nabla V_\varepsilon|^2 \leqslant$$

$$\leqslant c(\Omega)\left(|2\alpha_1-1|+|1-\alpha_2|\right)\|U_0\|_2|\nabla V_\varepsilon|\,|V_\varepsilon|$$

$$+c(\Omega)\,|\alpha_2|\,\|U_0\|_2|\nabla V_\varepsilon|\,|V_\varepsilon|$$

$$+c(\Omega)\,|\alpha_3|\,\|U_0\|_2^2|V_\varepsilon|+c(\Omega)\,|1-\alpha_3|\,\|U_0\|_2^2|\nabla V_\varepsilon|$$

$$+c(\Gamma_N,\Omega)\,\beta|1-\alpha_2|\,\|U_0\|_2|V_\varepsilon|^{1/3}|\nabla V_\varepsilon|^{5/3}$$

$$+c(\Gamma_N,\Omega)\,\beta|1-\alpha_3|\,\|U_0\|_2^2|V_\varepsilon|^{1/6}|\nabla V_\varepsilon|^{5/6}$$

$$+c(\Gamma_N,\Omega)\,\|F\|_{-1/2,\Gamma_N}|\nabla V_\varepsilon|\,.$$

Then, using Young inequality for any positive real numbers a, b

$$ab\leqslant\frac{1}{p}\left(\frac{a}{\delta}\right)^p+\frac{1}{q}(b\delta)^q$$

where $\frac{1}{p}+\frac{1}{q}=1$ and δ is a positive real number of our choice, we finally have

$$\frac{d}{d}|V_\varepsilon|^2+\varepsilon\frac{d}{dt}|q_\varepsilon|^2+\frac{2\eta}{Re}|\nabla V_\varepsilon|^2\leqslant c_4|V_\varepsilon|^2+c_5$$

where c_4 and c_5 depend only on the data.

Using Gronwall lemma we obtain the inequalities (1), (2) and (3) for the 2D case.

For the 3D case, the proof above is formal. To be rigourous, we need to derive first these estimates on the finite dimension approximation by Galerkin method. Then, as all the interpolation inequalities and Sobolev embeddings used in the 2D proof are valid in 3D, we get the same *a priori* estimates by lower semi-continuity of the norm in a reflexive space.

The estimates given in proposition 2 are not enough to pass to the limit when ε goes to zero. We need, in addition, an estimate on a fractional derivative in time of the velocity and an estimate on the pressure. These two estimates are given in the next propositions.

PROPOSITION 3 : *For each $T>0$ there exists a constant c_6 which depends only on T such that*

$$\int_0^T|D_t^\gamma V_\varepsilon(t)|^2\,dt\leqslant c_6\quad\forall\gamma<1/4\,.$$

Proof : For the sake of simplicity we only give here the proof in 2D using 3D valid Sobolev embeddings. For the 3D case the result comes via Galerkin approximation as it is pointed out in the previous proof.

We follow the ideas developed in [10] and [14]. Let $\tilde{V}$ be the extension of V by zero outside of $(0, T)$, we note $\mathscr{F}(\tilde{V})$ its Fourier transform in time and introduce the space

$$\mathscr{H}^{\gamma}(0, T ; \mathbb{H}^1_D(\Omega) ; \mathbb{L}^2(\Omega)) =$$

$$= \left\{ V \in \mathbb{L}^2(0, T ; \mathbb{H}^1_D(\Omega)) ; D^{\gamma}_t \tilde{V} \in L^2(0, T ; \mathbb{L}^2(\Omega)) \right\}$$

where $D^{\gamma}_t \tilde{V}(t) = \mathscr{F}^{-1}((i\tau)^{\gamma} \mathscr{F}(V)(\tau))(t)$

We take $\Psi \in \mathbb{H}^1_D(\Omega)$ in $(\mathscr{F}_{\varepsilon})$ and remark that

$$\partial_t \tilde{V}_{\varepsilon} = \partial_t V_{\varepsilon} + V_{\varepsilon}(0)\, \delta_0 - V_{\varepsilon}(T)\, \delta_T .$$

Then we apply the Fourier transform in time

$$i\tau \int_{\Omega} \mathscr{F}(\tilde{V}_{\varepsilon}) . \Psi\, dx + \frac{1}{2} \int_{\Omega} (\mathscr{F}(\tilde{V}_{\varepsilon} . \nabla \tilde{V}_{\varepsilon}) . \Psi - \mathscr{F}(\tilde{V}_{\varepsilon} \otimes \tilde{V}_{\varepsilon}) : \nabla \Psi)\, dx$$

$$+ \int_{\Omega} (a_1\, U_0 \otimes \Psi : \mathscr{F}(\nabla \tilde{V}_{\varepsilon}) - (1 - a_1)\, U_0 \otimes \mathscr{F}(\tilde{V}_{\varepsilon}) : \nabla \Psi)\, dx$$

$$+ \int_{\Omega} (a_2\, \mathscr{F}(\tilde{V}_{\varepsilon}) \otimes \Psi : \nabla U_0 - (1 - a_2)\, \mathscr{F}(\tilde{V}_{\varepsilon}) \otimes U_0 : \nabla \Psi)\, dx$$

$$+ \int_{\Omega} (a_3\, U_0 . \nabla U_0 . \Psi - (1 - a_3)\, U_0 . \nabla \Psi . U_0)\, \mathscr{F}(\chi_{[0, T]})\, dx$$

$$+ \frac{2}{Re} \int_{\Omega} D(\mathscr{F}(\tilde{V}_{\varepsilon})) : D(\Psi)\, dx - \int_{\Omega} \mathscr{F}(\tilde{q}_{\varepsilon})\, \mathrm{div}\, \Psi\, dx$$

$$+ \beta \int_{\Gamma_N} \mathscr{F}\Big(\Big(\frac{1}{2} \tilde{V}_{\varepsilon} . n + (1 - a_1)\, \chi_{[0, T]}\, U_0 . n \Big)^+ \tilde{V}_{\varepsilon} \Big) . \Psi\, d\gamma$$

$$+ \beta \int_{\Gamma_N} \mathscr{F}((1 - a_2)\, \tilde{V}_{\varepsilon} . n + (1 - a_3)\, \chi_{[0, T]}\, U_0 . n)^+\ U_0 . \Psi\, d\psi$$

$$= \langle \mathscr{F}(\tilde{F}), \Psi \rangle_{\Gamma_N} + \frac{1}{(2\pi)^{3/2}} \int_{\Omega} V_{\varepsilon}(0) . \Psi - V_{\varepsilon}(T) . \Psi e^{-i\tau T}\, dx ,$$

$$i\tau\varepsilon \int_{\Omega} \mathscr{F}(\tilde{q}_{\varepsilon})\, \pi\, dx + \int_{\Omega} \pi\, \mathrm{div}\, (\mathscr{F}(\tilde{V}_{\varepsilon}))\, dx$$

$$= \frac{\varepsilon}{(2\pi)^{3/2}} \int_{\Omega} q_{\varepsilon}(0) . \pi - q_{\varepsilon}(T)\, \pi\, e^{-i\tau}\, dx$$

where $V \otimes W$ denotes the second order tensor whose components are given by $(V \otimes W)_{ij} = V_i V_j$.

Taking $(\Psi, \pi) = (\overline{\mathscr{F}(\tilde{V}_\varepsilon)})(\tau), (\overline{\mathscr{F}(\tilde{q}_\varepsilon)}(\tau))$, we give an estimate of the cubic terms

$$\left| \int_\Omega (\mathscr{F}(\tilde{V}_\varepsilon . \nabla \tilde{V}_\varepsilon) . \overline{\mathscr{F}}(\tilde{V}_\varepsilon) - \mathscr{F}(\tilde{V}_\varepsilon \otimes \tilde{V}_\varepsilon) : \nabla \overline{\mathscr{F}}(\tilde{V}_\varepsilon))\, dx \right| \leqslant$$

$$\leqslant |\mathscr{F}(\tilde{V}_\varepsilon . \nabla \tilde{V}_\varepsilon)|_{6/5} |\overline{\mathscr{F}}(\tilde{V}_\varepsilon)|_6 + |\mathscr{F}(\tilde{V}_\varepsilon \otimes \tilde{V}_\varepsilon)| \, |\nabla \overline{\mathscr{F}}(\tilde{V}_\varepsilon)| \, .$$

From estimates (1) and (2) and by interpolation, $\tilde{V}_\varepsilon$ is bounded in $L^{2/s}(\mathbb{R}\,; H^s(\Omega))$ for any $0 \leqslant s \leqslant 1$.

So, by Sobolev embeddings, $V_\varepsilon . \nabla \tilde{V}_\varepsilon$ is bounded in $L^{4/3}(\mathbb{R}\,; L^{6/5}(\Omega))$ and $|\mathscr{F}(\tilde{V}_\varepsilon . \nabla \tilde{V}_\varepsilon)|_{6/5}$ is bounded in $L^4(\mathbb{R})$. In the same way $|\overline{\mathscr{F}}(\tilde{V}_\varepsilon)|_6$ is bounded in $L^2(\mathbb{R})$.

For the second term, we take $\tilde{V}_\varepsilon$ bounded in $L^{8/3}(\mathbb{R}, H^{3/4}(\Omega))$ and consequently $|\mathscr{F}(\tilde{V}_\varepsilon \otimes \tilde{V}_\varepsilon)|$ is bounded in $L^4(\mathbb{R})$ and $|\nabla \overline{\mathscr{F}}(\tilde{V}_\varepsilon)|$ is bounded in $L^2(\mathbb{R})$.

Finally, there exists a bounded function g_1 in $L^{4/3}(\mathbb{R})$ such that

$$\frac{1}{2} \left| \int_\Omega \mathscr{F}(\tilde{V}_\varepsilon . \nabla \tilde{V}_\varepsilon) . \overline{\mathscr{F}}(\tilde{V}_\varepsilon) - \mathscr{F}(\tilde{V}_\varepsilon \otimes \tilde{V}_\varepsilon) : \nabla \overline{\mathscr{F}}(\tilde{V}_\varepsilon)\, dx \right| \leqslant g_1(\tau) \quad \textit{for a.e. } \tau \textit{ in } \mathbb{R}\, .$$

Using the same technique and assuming $U_0 \in \mathbb{H}^2(\Omega)$, we can show that there exist $g_2 \in L^1(\mathbb{R})$, $g_3 \in L^1(\mathbb{R})$ and $g_4 \in L^2(\mathbb{R})$ such that

$$\left| \int_\Omega (\alpha_1 U_0 \otimes \mathscr{F}(\tilde{V}_\varepsilon) : \mathscr{F}(\nabla \tilde{V}_\varepsilon) - (1 - \alpha_1) U_0 \otimes \right.$$

$$\left. \otimes \mathscr{F}(\tilde{V}_\varepsilon) : \nabla \mathscr{F}(\tilde{V}_\varepsilon))\, dx \right| \leqslant g_2(\tau) \textit{ for a.e. } \tau \textit{ in } \mathbb{R}$$

$$\left| \int_\Omega (\alpha_2 \mathscr{F}(\tilde{V}_\varepsilon) \otimes \mathscr{F}(\tilde{V}_\varepsilon) : \nabla U_0 - (1 - \alpha_2) \mathscr{F}(\tilde{V}_\varepsilon) \otimes \right.$$

$$\left. \otimes U_0 : \nabla \mathscr{F}(\tilde{V}_\varepsilon))\, dx \right| \leqslant g_3(\tau) \textit{ for a.e. } \tau \textit{ in } \mathbb{R}$$

$$\left| \int_\Omega (\alpha_3 U_0 . \nabla U_0 . \mathscr{F}(\tilde{V}_\varepsilon) - (1 - \alpha_3) U_0 . \right.$$

$$\left. . \nabla \mathscr{F}(\tilde{V}_\varepsilon) . U_0) \mathscr{F}(\chi_{[0,T]})\, dx \right| \leqslant g_4(\tau) \textit{ for a.e. } \tau \textit{ in } \mathbb{R}\, .$$

Now, by continuity of the trace operator and from estimates (1) and (2), we remark that $\left(\frac{1}{2}\tilde{V}_\varepsilon \cdot n + (1-\alpha_1)\chi_{[0,T]} U_0 \cdot n\right)^+ \tilde{V}_\varepsilon$ is bounded in $L^1(\mathbb{R}; L^2(\Gamma_N))$; so there exists a function g_5 in $L^2(\mathbb{R})$ such that

$$\beta\left|\int_{\Gamma_N} \mathscr{F}\left(\left(\frac{1}{2}\tilde{V}_\varepsilon \cdot n + (1-\alpha_1)\chi_{[0,T]} U_0 \cdot n\right)^+ \tilde{V}_\varepsilon\right) \cdot \mathscr{F}(\tilde{V}_\varepsilon)\, d\gamma\right| \leqslant g_5(\tau)$$

for a.e. τ *in* $\mathbb{R}$

and, in the same way, as $((1-\alpha_2)\tilde{V}_\varepsilon \cdot n + (1-a_3)\chi_{[0,T]} U_0 \cdot n)^+$ is bounded in $L^2(\mathbb{R}; L^2(\Gamma_N))$, there exists g_6 in $L^1(\mathbb{R})$ such that

$$\beta\left|\int_{\Gamma_N} \mathscr{F}(((1-\alpha_2)\tilde{V}_\varepsilon \cdot n + (1-\alpha_3)\chi_{[0,T]} U_0 \cdot n)^+)\, U_0 \cdot \mathscr{F}(\tilde{V}_\varepsilon)\, d\gamma\right| \leqslant$$

$$\leqslant g_6(\tau) \text{ for a.e. } \tau \text{ in } \mathbb{R}$$

Moreover, the Dirac terms and the second member are bounded by $g_7 \in L^2(\mathbb{R})$. Thus, summing the equations and taking the imaginary part we obtain

$$(4) \quad |\tau|\, |\mathscr{F}(\tilde{V}_\varepsilon)(\tau)|^2 + \varepsilon|\tau|\, |\mathscr{F}(\tilde{q}_\varepsilon)(\tau)|^2 \leqslant h_1(\tau) + h_2(\tau) + h_3(\tau)$$

for a.e. τ *in* $\mathbb{R}$

where $h_1 \in L^1(\mathbb{R})$, $h_2 \in L^2(\mathbb{R})$ and $h_3 \in L^{4/3}(\mathbb{R})$.

To conclude, we first remark that there exist two constants d_1 and d_2 such taht for every $0 \leqslant \sigma \leqslant 1$

$$|\tau|^{1-\sigma} \leqslant d_1 \frac{|\tau|}{1+|\tau|^\sigma} + d_2$$

so the inequality (4) gives

$$\int_{\mathbb{R}} |\tau|^{1-\sigma}(|\mathscr{F}(\tilde{V}_\varepsilon)(\tau)|^2 + \varepsilon|\mathscr{F}(\tilde{q}_\varepsilon)(\tau)|^2)\, d\tau \leqslant$$

$$\leqslant d_1\left\{\int_{\mathbb{R}} \frac{h_1(\tau)}{1+|\tau|^\sigma}\, d\tau + \int_{\mathbb{R}} \frac{h_2(\tau)}{1+|\tau|^\sigma}\, d\tau + \int_{\mathbb{R}} \frac{h_3(\tau)}{1+|\tau|^\sigma}\, d\tau\right\}$$

$$+ d_2\left\{\int_R |\mathscr{F}(\tilde{V}_\varepsilon)(\tau)|^2 + \varepsilon|\mathscr{F}(\tilde{q}_\varepsilon)(\tau)|^2\, d\tau\right\}$$

which is bounded independently of ε as soon as $1/2 < \sigma \leqslant 1$.

Let us denote B the bounded operator defined by

$$\mathbb{X} = \{ V \in H^1(0, T ; \mathbb{H}^1_D(\Omega)) ; V(T) = 0 \}$$

$$\mathbb{R} = \{ q \in H^1(0, T ; L^2(\Omega)) ; q(T) = 0 \}$$

$$V \in \mathbb{X} \mapsto B(V) = \operatorname{div} V \in \mathbb{Y} .$$

Following [6], we prove that B is an onto operator and so its adjoint ${}^tB$ is of closed range. First we observe that, as the time t is a parameter, we only need to show that B is an onto operator from $\mathbb{H}^1_D(\Omega)$ to $L^2(\Omega)$.

Let f be a given function in $L^2(\Omega)$, we build p in $H^2(\Omega)$ solution of

$$\Delta p = f \quad \text{in} \quad \Omega ,$$

$$p = 0 \quad \text{on} \quad \Gamma_N ,$$

$$\partial_n p = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_D ,$$

and g defined by $g = 0$ on $\Gamma_0 \cup \Gamma_D$, $g = \nabla p$ on Γ_N. Then we set $h = g - \nabla p$ on $\partial\Omega$ that belongs to $H^{1/2}(\partial\Omega)$ by construction and checks

$$\int_{\partial\Omega} h \,.\, n \, d\gamma = 0 .$$

Then we find $W = W_0 + \nabla p$ in $\mathbb{H}^1_D(\Omega)$ where W_0 satisfies [6]

$$\operatorname{div} W_0 = 0 , \quad W_0 = h \text{ on } \partial\Omega .$$

PROPOSITION 4 : *There exists a constant c_7 independent of ε such that*

$$\| {}^tBq_\varepsilon \|_{\mathbb{X}'} \leqslant c_7 .$$

Proof : Let us integrate in time the first equation of $(\mathscr{F}_\varepsilon)$ for Ψ in $\mathbb{X}$

$$
\begin{aligned}
& -\int_0^T \int_\Omega V_\varepsilon \, \partial_t \Psi \, dx\, dt + \frac{1}{2} \int_0^T \int_\Omega (V_\varepsilon . \nabla V_\varepsilon . \Psi - V_\varepsilon . \nabla \Psi . V_\varepsilon)\, dx\, dt + \\
& + \int_0^T \int_\Omega (\alpha_1 U_0 . \nabla V_\varepsilon . \Psi - (1 - \alpha_1) U_0 . \nabla \Psi . V_\varepsilon)\, dx\, dt \\
& + \int_0^T \int_\Omega (\alpha_2 V_\varepsilon . \nabla U_0 . \Psi - (1 - \alpha_2) V_\varepsilon . \nabla \Psi . U_0)\, dx\, dt \\
& + \int_0^T \int_\Omega (\alpha_3 U_0 . \nabla U_0 . \Psi - (1 - \alpha_3) U_0 . \nabla \Psi . U_0)\, dx\, dt \\
& + \frac{2}{Re} \int_0^T \int_\Omega D(V_\varepsilon) : D(\Psi)\, dx\, dt \\
& - \langle {}^t B q_\varepsilon, \Psi \rangle + \beta \int_0^T \int_{\Gamma_N} \left(\frac{1}{2} V_\varepsilon . n + (1 - \alpha_1) U_0 . n \right)^+ V_\varepsilon \, \Psi \, d\gamma\, dt \\
& + \beta \int_0^T \int_{\Gamma_N} ((1 - \alpha_\varepsilon) V_\varepsilon . n + (1 - \alpha_3) U_0 . n)^+ \, \Psi \, d\gamma\, dt \\
= & \int_0^T \langle F, \Psi \rangle_{\Gamma_N} dt + \int_\Omega V_0 \, \Psi(0)\, dx .
\end{aligned}
$$

Using estimates (1) and (2) and Hölder inequality, it comes

$$|\langle {}^t B q_\varepsilon, \Psi \rangle| \leq c_7 \, \| \Psi \|$$

where c_7 is a positive constant independent of ε.

2.3. Convergence

Let us recall that the embedding from the space

$$
\begin{aligned}
& \mathscr{H}^\gamma(0, T ; \mathbb{H}^1_D(\Omega) ; \mathbb{L}^2(\Omega)) \\
& = \{ V \in L^2(0, T ; \mathbb{H}^1_D(\Omega)) ; D^\gamma_t V \in L^2(0, T ; \mathbb{L}^2(\Omega)) \}
\end{aligned}
$$

into $L^2(0, T ; \mathbb{L}^2(\Omega))$ is compact ([10], [14]).

From propositions 2 and 3 we know that V_ε is bounded in

$$\mathcal{H}^\gamma(0, T ; \mathbb{H}^1_D(\Omega) ; \mathbb{L}^2(\Omega))$$

so we can extract a sequence ε_n such that

$$V_{\varepsilon_n} \to V \quad L^\infty(0, T ; \mathbb{L}^2(\Omega)) \text{ weak*} \tag{5}$$

$$V_{\varepsilon_n} \to V \quad L^2(0, T ; \mathbb{H}^1_D(\Omega)) \text{ weak} \tag{6}$$

$$V_{\varepsilon_n} \to V \quad L^2(0, T ; \mathbb{L}^2(\Omega)) \text{ strong and a.e.} \tag{7}$$

On the other hand, by proposition 2

$$\varepsilon_n q_{\varepsilon_n} \to 0 \quad L^\infty(0, T ; L^2(\Omega)) \text{ weak*} \tag{8}$$

and by proposition 4, as ${}^tB$ has a closed range,

$${}^tBq_{\varepsilon_n} \to {}^tBq\mathbb{X}' \text{ weak} . \tag{9}$$

Now, let us remark that by interpolation we deduce from (6) and (7) that

$$V_{\varepsilon_n} \to V \quad L^2(0, T ; \mathbb{H}^s(\Omega)) \text{ strong for any } 0 < s < 1 \tag{10}$$

and for $s = 3/4$ we get

$$V_{\varepsilon_n} \to V \quad L^2(0, T ; \mathbb{L}^4(\Omega)) \text{ strong} ;$$

then

$$\int_0^T \int_\Omega (V_{\varepsilon_n} . \nabla V_{\varepsilon_n} \Psi - V_{\varepsilon_n} . \nabla \Psi . V_{\varepsilon_n}) \, dx \, dt \to$$

$$\to \int_0^T \int_\Omega (V . \nabla V . \Psi - V . \nabla \Psi . V) \, dx \, dt .$$

For $s = 5/6$ in (10) and using the continuity of the trace operator in $H^{1/3}(\Gamma_N)$ we obtain

$$\int_0^T \int_{\Gamma_N} \left(\frac{1}{2} V_\varepsilon . n + (1 - \alpha_1) U_0 . n\right)^+ V_\varepsilon . \Psi \, d\gamma \, dt \to$$

$$\to \int_0^T \int_{\Gamma_N} \left(\frac{1}{2} V . n + (1 - \alpha_1) U_0 . n\right)^+ V . \Psi \, d\gamma \, dt ,$$

in the same way we get convergence for the other boundary term.

Finally, as the other terms are linear, there are no difficulties to pass to the limit.

Now, integrating in time the second equation of $(\mathcal{F}_\varepsilon)$ for π in $\mathbb{Y}$

$$- \varepsilon \int_0^T \int_\Omega q_\varepsilon \frac{\partial \pi}{\partial t} \, dx \, dt + \int_0^T \int_\Omega \pi \operatorname{div} V_\varepsilon \, dx \, dt = \varepsilon \int_\Omega q_0 \, \pi(0) \, dx$$

so according to (3), we get at the limit

$$\int_0^T \int_\Omega \pi \operatorname{div} V \, dx \, dt = 0 \, .$$

In conclusion, we have shown

THEOREM 1 : *Let Ω be a connected bounded domain in $\mathbb{R}^N$ $(N \leqslant 3)$ with smooth boundaries, then for U_0 in $\mathbb{H}^2(\Omega)$, V_{in} in $\mathbb{H}_0$ and F in $\mathbb{H}^{-1/2}(\Gamma_N)$, there exists at least one solution (V, q) in $\mathbb{V} \times \mathbb{Q}$ of $(\mathcal{F})$.*

2.4. Uniqueness in 2D

As it is well-known, the uniqueness result is related to the regularity of $\partial_t V$. For $\Psi \in \mathbb{X}_0$, we get from $(\mathcal{F})$

$$\int_\Omega \partial_t V \, . \, \Psi + \frac{1}{2} (V \, . \, \nabla V \, . \, \Psi - V \, . \, \nabla \Psi \, . \, V) \, dx$$

$$+ \int_\Omega (\alpha_1 \, U_0 \, . \, \nabla V \, . \, \Psi - (1 - \alpha_1) \, U_0 \, . \, \nabla \Psi \, . \, V) \, dx$$

$$+ \int_\Omega (\alpha_2 \, V \, . \, \nabla U_0 \, . \, \Psi - (1 - \alpha_2) \, V \, . \, \nabla \Psi \, . \, U_0) \, dx$$

$$+ \int_\Omega (\alpha_3 \, U_0 \, . \, \nabla U_0 \, . \, \Psi - (1 - \alpha_3) \, U_0 \, . \, \nabla \Psi \, . \, U_0) \, dx$$

$$+ \frac{2}{Re} \int_\Omega D(V) : D(\Psi) \, dx$$

$$+ \beta \int_{\Gamma_N} \left(\frac{1}{2} V \, . \, n + (1 - \alpha_1) \, U_0 \, . \, n \right)^+ V \, . \, \Psi \, d\gamma$$

$$+ \beta \int_{\Gamma_N} ((1 - \alpha_2) \, V \, . \, n + (1 - \alpha_3) \, U_0 \, . \, n)^+ \, U_0 \, . \, \Psi \, d\gamma = \langle F, \Psi \rangle_{\Gamma_N} \, .$$

Then we define the following operators from $H^1_{D,0}(\Omega)$ into $H^1_{D,0}(\Omega)'$ by :

$$\langle \mathcal{A} V, \Psi\rangle = \frac{2}{Re}\int_\Omega D(V) : D(\Psi)\, dx$$

$$+ \int_\Omega (a_1\, U_0 . \nabla V . \Psi - (1 - a_1)\, U_0 . \nabla \Psi . V)\, dx$$

$$+ \int_\Omega (a_2\, V . \nabla U_0 . \Psi - (1 - a_2)\, V . \nabla \Psi . U_0)\, dx ,$$

$$\langle \mathcal{B}(V, V), \Psi\rangle = \frac{1}{2}\int_\Omega (V . \nabla V . \Psi - V . \nabla \Psi . V)\, dx ,$$

$$\langle \mathcal{S}(V), \Psi\rangle = \beta \int_{\Gamma_N} \left(\frac{1}{2} V . n + (1 - a_1)\, U_0 . n\right)^+ V . \Psi\, d\gamma$$

$$+ \beta \int_{\Gamma_N} ((1 - a_2)\, V . n + (1 - a_3)\, U_0 . n)^+\, U_0 . \Psi\, d\gamma ,$$

$$\langle \mathcal{G}(V), \Psi\rangle = \int_\Omega (a_3\, U_0 . \nabla U_0 . \Psi - (1 - a_3)\, U_0 . \nabla \Psi . U_0)\, dx - \langle F, \Psi\rangle_{\Gamma_N} .$$

Then we write :

$$\partial_t V + \mathcal{A} V + \mathcal{B}(V, V) + \mathcal{S}(V) + \mathcal{G}(V) = 0 .$$

From existence theorem and regularity properties on V we deduce that for 2D dimension space, all these terms belong to $L^2(0, T ; \mathbb{H}^1_{D,0}(\Omega)')$, and then $\partial_t V$ belongs to $L^2(0, T ; \mathbb{H}^1_{D,0}(\Omega)')$. Then we can prove the following result :

THEOREM 2 : *Let $(V, q) \in \mathbb{V} \times \mathbb{Q}$ be a solution of $(\mathcal{F})$. If the space dimension is equal to two, then V is unique.*

Proof: Let (V_1, q_1) and (V_2, q_2) be two solutions of $(\mathscr{F})$, we set $V = V_2 - V_1$ and $q = q_2 - q_1$. Then (V, q) satisfies

$$\langle \partial_t V, \Psi \rangle + \frac{1}{2} \int_{\Omega} (V_2 . \nabla V + V . \Psi + V . \nabla V_1 . \Psi - V_2 . \nabla \Psi . V - V . \nabla \Psi - V_1)\, dx$$

$$+ \int_{\Omega} (\alpha_1 U_0 . \nabla V . \Psi - (1 - \alpha_1) U_0 . \nabla \Psi . V)\, dx$$

$$+ \int_{\Omega} (\alpha_2 V . \nabla U_0 . \Psi - (1 - \alpha_2) V . \nabla \Psi . U_0)\, dx$$

$$+ \frac{2}{Re} \int_{\Omega} D(V) : D(\Psi)\, dx - \langle q, \operatorname{div} \Psi \rangle_{\Omega}$$

$$+ \beta \int_{\Gamma_N} \left(\frac{1}{2} V_2 . n + (1 - \alpha_1) U_0 . n \right)^+ V . \Psi\, d\gamma$$

$$+ \beta \int_{\Gamma_N} \left\{ \left(\frac{1}{2} V_2 . n + (1 - \alpha_1) U_0 . n \right)^+ \right.$$

$$\left. - \left(\frac{1}{2} V_1 . n + (1 - \alpha_1) U_0 . n \right)^+ \right\} V_1 . \Psi\, d\gamma$$

$$+ \beta \int_{\Gamma_N} \{ ((1 - \alpha_2) V_2 . n + (1 - \alpha_3) U_0 . n)^+$$

$$- ((1 - \alpha_2) V_1 . n + (1 - \alpha_3) U_0 . n)^+ \} U_0 . \Psi\, d\gamma = 0 .$$

$$\forall \Psi \in \mathbb{H}^1_{D,0}(\Omega) .$$

Then, for $\Psi = V$, we get the following inequality

$$\frac{1}{2} \frac{d}{dt} |V|^2 + \frac{2\eta}{Re} |\nabla V|^2 \leqslant$$

$$\leqslant \frac{1}{2} |V|_4^2 |\nabla V_1| + \frac{1}{2} |V|_4 |V_1|_4 |\nabla V| + (2 - \alpha_2) |U_0|_4 |V|_4 |\nabla V| + \alpha_2 |V|_4^2 |\nabla U_0|$$

$$+ \beta \left| \left(\frac{1}{2} V_2 . n + (1 - \alpha_1) U_0 . n \right) \right|_{3, \Gamma_N} |V|_{3, \Gamma_N}^2 + \frac{\beta}{2} |V . n|_{3, \Gamma_N} |V_1|_{3, \Gamma_N} |V|_{3, \Gamma_N}$$

$$+ \beta (1 - \alpha_2) |V|_{3, \Gamma_N} |U_0|_{3, \Gamma_N} |V|_{3, \Gamma_N}$$

as the function $a \to a^+$ is a one-lipschitz function.

By Sobolev embeddings and interpolation [1] we have

$$|V|_p \leqslant c \| V \|_s \leqslant c' |V|^{1-s} |\nabla V|^s \quad \text{with} \quad s = \frac{p-2}{p}.$$

Then using in addition the continuity of the trace operator we get

$$|V|_{q, \Gamma_N} \leqslant c \| V \|_{\sigma, \Gamma_N} \leqslant c' \| V \|_{\sigma + 1/2} \leqslant c'' |V|^{1/2 - \sigma} |\nabla V|^{\sigma + 1/2} \quad \text{with} \quad \sigma = \frac{q-2}{2q}.$$

So we have the following estimate

$$\frac{1}{2} \frac{d}{dt} |V|^2 + \frac{2\eta}{Re} |\nabla V|^2 \leqslant$$

$$\leqslant c_8 (|\nabla V_1| + |\nabla U_0|) |V| \, |\nabla V| + c_9 (|V_1|_4 + |U_0|_4) |V|^{1/2} |\nabla V|^{3/2}$$

$$+ c_{10} \left(\left| \frac{1}{2} V_2 \cdot n + (1 - \alpha_1) U_0 \cdot n \right|_{3, \Gamma_N} + |V_1|_{3, \Gamma_N} + |U_0|_{3, \Gamma_N} \right) |V|^{2/3} |\nabla V|^{4/3}.$$

Now, as V belongs to $\mathbb{V}_0$ and using Young inequality, we show that there exists a function $h_4(t)$ belonging to $L^1(0, T)$ such that

$$\frac{d}{dt} |V|^2 + \frac{2\eta}{Re} |\nabla V|^2 \leqslant h_4 |V|^2.$$

The proof follows using Gronwall lemma.

3. PRACTICAL EXAMPLE

To show the robustness and accuracy of our boundary conditions, we apply them to compute the flow behind a cylinder in a channel (see *fig.* 2).

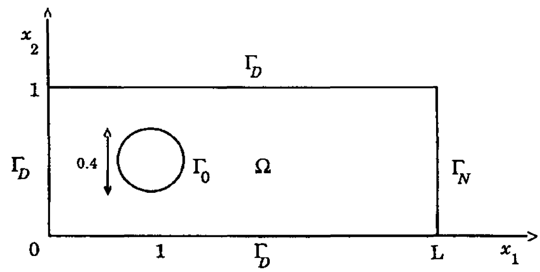

Figure 2. — Domain Ω for the channel.

In this case, the flow is set equal to Poiseuille flow (U_P, p_P) upstream and we can take $(U_0, p_0) = (U_P, p_P)$ and $\sigma(U_0, p_0) = \sigma(U_P, p_P)$ downstream. Indeed, this is true if Γ_N is far enough from the obstacle and numerical tests show that this is still valid when Γ_N is closer.

So the problem $(\mathscr{P})$ reads

$$(\mathscr{P})\quad \begin{array}{lll} \partial_t U + (U \, . \, \nabla) \, . \, U - \operatorname{div} \sigma(U, p) = 0 & \text{in} & \Omega \times (0, T) \\ \operatorname{div} U = 0 & \text{in} & \Omega \times (0, T) \\ U(x, 0) = 0 & \text{in} & \Omega \\ U(\, . \, , t) = 0 & \text{on} & \Gamma_0 \times (0, T) \\ U(\, . \, , t) = U_P & \text{on} & \Gamma_D \times (0, T) \end{array}$$

$$\sigma(U, p) \, . \, n - \frac{1}{2} \Theta(U \, . \, n + (1 - 2\alpha_1) U_P \, . \, n)(U - U_P) -$$

$$- \Theta((1 - \alpha_2) U \, . \, n + (\alpha_2 - \alpha_3) U_P \, . \, n) U_P = \sigma(U_P, P_P) \, . \, n$$

$$\text{on } \Gamma_N \times (0, T) \, .$$

For numerical tests, we set $\alpha_1 = 1/2$, $\alpha_2 = \alpha_3 = 1$, $\theta(a) = -a^-$ and $p_P = 0$ on Γ_N (see [3] for more details). On *figure* 3, we see that the solution obtained on a troncated domain is very closed to the one obtained on a larger one at the same time. Moreover, these conditions are truly robust as we can compute chaotic solutions at high Reynolds numbers (*fig.* 4), which is not the case with the linear condition (0) that produces strong reflections for the same time step discretisation (*fig.* 5).

CONCLUSIONS

We have established a new family of open boundary conditions that lead to a well-posed problem for incompressible Navier-Stokes equations. These conditions are applied to compute the flow behind a cylinder in a channel. Numerical tests show that they are very robust and accurate as they do not induce any reflections downstream even when strong vortices cross the artificial boundary. Finally, these conditions are suitable to simulate the transition to turbulence in an open domain.

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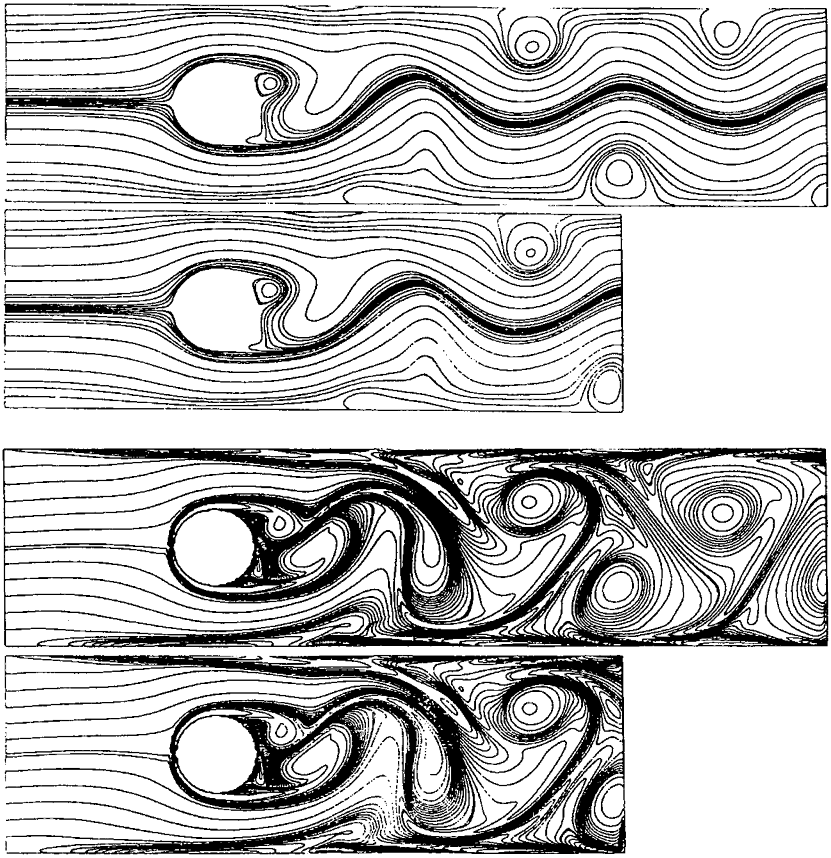

Figure 3. — Stream lines and vorticity lines at Re = 1 000.

Figure 4. — Chaotic solution at Re = 10 000.

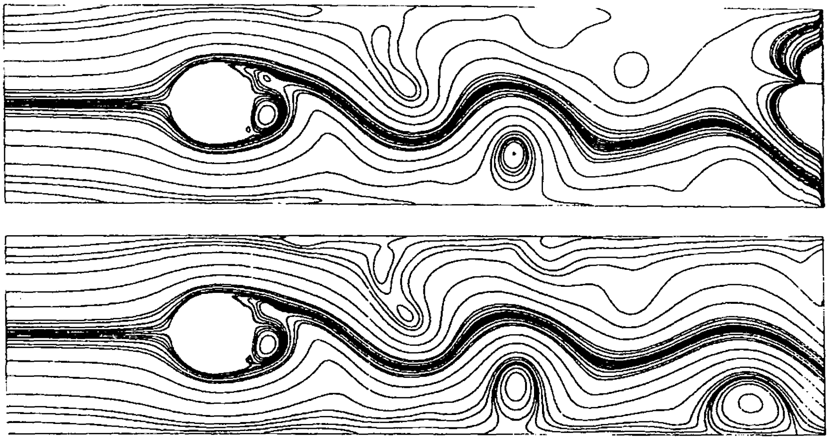

Figure 5. — Comparison of the linear (top) and the full condition (bottom) at Re = 10 000.

REFERENCES

[1] R. A. ADAMS, 1975, *Sobolev spaces,* Academic press, New-York.

[2] C. BÈGUE, C. CONCA, F. MURAT and O. PIRONNEAU, 1987, A nouveau sur les équations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression, *C. R. Acad. Sci. Paris,* **304** série I, pp. 23-28.

[3] Ch.-H. BRUNEAU and P. FABRIE, 1994, Effective downstream boundary conditions for incompressible Navier-Stokes equations. *Int. J. for Num. Methods in Fluids,* **19,** pp. 693-705.

[4] C. CONCA, 1984, Approximation de quelques problèmes de type Stokes par une méthode d'éléments finis mixtes. *Numer. Math.,* **45,** pp. 75-91.

[5] G. DUVAUD, J. L. LIONS, 1972, *Les inéquations en mécanique et en physique,* Dunod.

[6] V. GIRAULT et P. A. RAVIART, 1986, *Finite elements method for Navier-Stokes equations,* Springer Series in Computational Mathematics.

[7] P. M. GRESHO, 1991, Incompressible fluid dynamics : Some fundamental formulation issues. *Annu. Rev. Fluid Mech.,* **23,** pp. 413-453.

[8] L. HALPERN, 1986, Artificial boundary conditions for the linear advection diffusion equation. *Math. Comp.,* **46,** pp. 425-438.

[9] L. HALPERN and M. SCHATZMAN, 1989, Artificial boundary conditions for incompressible viscous flows. *SIAM J. Math. Anal.,* **20,** pp. 308-353.

[10] J. L. LIONS, 1979, *Quelques méthodes de résolution des problèmes aux limites nonlinéaires,* Dunod, Paris.

[11] R. PEYRET and B. REBOURCET, 1982, Développement de jets en fluides stratifiés, *Journal de Mécanique Théorique et Appliquée,* **1,** pp. 467-491.

[12] O. PIRONNEAU, 1986, Conditions aux limites sur la pression pour les équations de Stokes et de Navier-Stokes, *C. R. Acad. Sci. Paris,* **303,** série I, pp. 403-40.

[13] R. TEMAM, 1993, *Navier-Stokes Equations and Nonlinear Functional Analysis,* Regional conference series in applied mathematics.

[14] R. TEMAM, 1979, *Navier-Stokes Equations and numerical Analysis,* 2nd ed. North-Holland, Amsterdam.