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APPROXIMATION PROPERTIES OF PERIODIC INTERPOLATION BY TRANSLATES OF ONE FUNCTION (*)

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Communicated by P. J. LAURENT

To the memory of Professor Dr. Lothar COLLATZ

Abstract. — *In the paper [5] Golomb has derived a Hilbert space approach to periodic splines of odd degree on uniform meshes which were studied systematically for the first time by Quade and Collatz [9]. Golomb's approach has been extended to more general methods of periodic interpolation by translates of a given periodic function g [1, 2, 3, 8]. It is the objective of this paper to investigate approximation properties of these interpolation methods in spaces of periodic functions which are closely related to g and extend the results of [4]. As an application approximation properties of periodic splines of even degree are obtained.*

Résumé. — *Dans l'article [3] Golomb a présenté une construction hilbertienne des fonctions-spline périodiques de degré impair dans un réseau uniforme, fonctions étudiées systématiquement pour la première fois par Quade et Collatz [7]. La construction de Golomb a été généralisée pour les méthodes d'interpolation par les translates d'une fonction périodique g . Le but de cet article est d'étudier des propriétés d'approximation des méthodes d'interpolation par les translates de g dans des espaces des fonctions périodiques qui sont associées à la fonction g . En application, nous déduisons les propriétés d'approximation des fonctions-spline périodiques de degré pair.*

1. THE INTERPOLATION METHOD

Let g be a fixed real valued periodic function from the Wiener algebra $\mathcal{A}_{2\pi}$ of those continuous periodic functions from $\mathcal{C}_{2\pi}$ which possess an absolutely convergent Fourier series. The inner product of $f, g \in \mathcal{C}_{2\pi}$ is defined by

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(t) g(t)^* dt.$$

The exponential functions are denoted by $e_k(t) = \exp(ikt)$, $k \in \mathbb{Z}$. If

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$g \in \mathcal{A}_{2\pi}$, then

$$g(t) = \sum_{k=-\infty}^{\infty} (g, e_k) e_k(t), \quad \sum_{k=-\infty}^{\infty} |(g, e_k)| < \infty.$$

Let $t_j = 2\pi j/n$, $j \in \mathbb{Z}$, be a uniform mesh depending on $n \in \mathbb{N}$. The discrete inner product is given by

$$\langle f, g \rangle = \frac{1}{n} \sum_{k=0}^{n-1} f(t_k) g(t_k)^*.$$

The discrete Fourier transform and the finite Fourier transform are related by *aliasing* :

$$\langle f, e_j \rangle = \sum_s \langle f, e_{j+sn} \rangle \quad (1)$$

where $j \in \mathbb{Z}$ and $f \in \mathcal{A}_{2\pi}$.

The *basis functions* b_j , $0 \leq j < n$, are obtained via the discrete Fourier transform from the generating function g :

$$b_j(t) = \langle g(t - \cdot), e_{-j} \rangle = \sum_s \langle g, e_{j+sn} \rangle e_{j+sn}(t). \quad (2)$$

We have the fundamental relations

$$b_j(t_k) = e_j(t_k) b_j(0) \quad (j, k \in \mathbb{Z}). \quad (3)$$

We assume that

$$b_j(0) \neq 0 \quad (0 \leq j < n) \quad (4)$$

and define

$$Q_n(f)(t) = \langle f, e_0 \rangle + \sum_{k=1}^{n-1} \langle f, e_k \rangle b_k(t)/b_k(0) \quad (5)$$

for any $f \in \mathcal{C}_{2\pi}$.

Discrete Fourier transform yields the interpolation properties

$$Q_n(f)(t_j) = f(t_j) \quad (0 \leq j < n). \quad (6)$$

The space of interpolants is given by (see [2, 3])

$$V_n(g) = \langle 1, g(\cdot - t_1) - g, \dots, g(\cdot - t_{n-1}) - g \rangle. \quad (7)$$

We assume that the finite Fourier transform of the generating function

g possesses certain properties such that the *existence conditions* $b_j(0) \neq 0$ are valid and certain approximation properties can be established.

Let d_k , $k \in \mathbb{Z}$, be a sequence of real numbers satisfying the following relations

$$d_{-k} = d_k, \quad d_k > d_{k+1} > 0 \quad (k \in \mathbb{N}).$$

Moreover we assume that

$$d_{rm} \leq \alpha_r d_m (r, m \in \mathbb{N}), \quad \alpha := \sum_{r=1}^{\infty} \alpha_r < \infty.$$

We consider mainly two cases of generating functions g :

$$(I) \quad g(t) = \sum_k d_k e_k(t).$$

$$(II) \quad g(t) = \sum_k -i \cdot \operatorname{sgn}(k) d_k e_k(-\pi/n) e_k(t).$$

The generating function g defining the interpolation process is independent of the number n of interpolation points in case I while it is dependent in case II.

Trigonometric interpolation is characterized by the choice

$$(III) \quad g(t) = \sum_{k=-m}^m e_k(t) (m = [n/2]).$$

In this case g depends also on n . We refer to [3] for the proof of the validity of the existence condition.

Two examples are given for possible choices of the sequence (d_k) .

Example a:

$$d_k = k^{-q} \quad (k \in \mathbb{N}).$$

Here q is a real number greater than 1. If q is *even*, case I yields periodic odd degree spline interpolation on uniform mesh as studied by Golomb [3]. The generating function is the well known Bernoulli function $P_q(t)$ up to a factor

$$g(t) = \sum_{k=1}^{\infty} k^{-2r} \cos(kt) = \frac{(-1)^r}{2} P_{2r}(t).$$

The function g is a spline of degree $2r$ with deficiency 1, while $g(\cdot - t_k) - g$ is a spline of degree $2r - 1$.

If q is *odd*, then case II yields periodic even degree midpoint spline interpolation on a uniform mesh [3]. The generating function is the *shifted*

Bernoulli function $P_q(t - \pi/n)$ up to a factor

$$g(t) = \sum_{k=1}^{\infty} k^{-2r-1} \sin(k(t - \pi/n)) = \frac{(-1)^r}{2} P_{2r+1}(t - \pi/n).$$

Example b :

$$d_k = e^{-bk} \quad (k \in \mathbb{N}).$$

Here b is a positive real number. Both cases I and II yield rational trigonometric interpolation processes (see [2, 3]).

In case I the generating function is related to the Poisson kernel

$$P_b(t) = \frac{\sinh(b)}{2(\cosh(b) - \cos(t))},$$

$$g(t) = \sum_{k=1}^{\infty} e^{-kb} \cos(kt) = P_b(t) - \frac{1}{2}.$$

In case II the generating function is related to the *shifted conjugate* Poisson kernel

$$Q_b(t) = \frac{\sin(t)}{2(\cosh(b) - \cos(t))},$$

$$g(t) = \sum_{k=1}^{\infty} e^{-kb} \sin(k(t - \pi/n)) = Q_b(t - \pi/n).$$

2. UNIFORM BOUNDEDNESS OF THE INTERPOLATION PROJECTORS

The Wiener algebra $\mathcal{A}_{2\pi}$ is a dense subalgebra of $\mathcal{C}_{2\pi}$. $\mathcal{A}_{2\pi}$ is a Banach space of continuous periodic functions with respect to the norm (see [6])

$$\|f\|_a = \sum_{k=-\infty}^{\infty} |(g, e_k)|.$$

Moreover, we have

$$\|f\|_{\infty} \leq \|f\|_a \quad (f \in \mathcal{A}_{2\pi})$$

where

$$\|f\|_{\infty} = \sup \{ |f(t)| : t \in \mathbb{R} \}.$$

We will first investigate approximation properties of the Fourier partial sum projector which is in some sense a universal approximation operator. The

Fourier partial sum projector F_n is defined by

$$F_n(f) = \sum_{k=-m}^m (f, e_k) e_k \quad (m = [n/2]).$$

Clearly, F_n is a bounded linear operator on $\mathcal{C}_{2\pi}$. On the other hand, it is well known that the norms $\|F_n\|$, $n \in \mathbb{N}$, are not uniformly bounded. According to the Banach Steinhaus principle [7], $F_n(f)$, $n \in \mathbb{N}$, is not convergent for every continuous periodic function. Therefore it is natural to consider F_n as a bounded linear operator from $\mathcal{A}_{2\pi}$ into $\mathcal{C}_{2\pi}$. It follows from the definition of the norms that

$$\|F_n(f)\|_{\infty} \leq \|f\|_a \quad (f \in \mathcal{A}_{2\pi}).$$

Thus, the norms $\|F_n\|$, $n \in \mathbb{N}$, are uniformly bounded. Since

$$F_n(e_k) = e_k(|k| < n/2),$$

an application of the Banach Steinhaus principle yields uniform convergence

$$\lim_{n \rightarrow \infty} \|f - F_n(f)\|_{\infty} = 0 \quad (f \in \mathcal{A}_{2\pi}).$$

Replacing $\mathcal{C}_{2\pi}$ by $\mathcal{A}_{2\pi}$ is not a severe restriction from the practical point of view since $\mathcal{A}_{2\pi}$ contains all Lipschitz-continuous periodic functions [6] and in particular those functions from $\mathcal{C}_{2\pi}$ which are piecewise $\mathcal{C}^1$ [10].

Next we will establish an analogous result for the interpolation projectors Q_n , $n \in \mathbb{N}$. The interpolation projector Q_n is a bounded linear projector from $\mathcal{A}_{2\pi}$ into $\mathcal{C}_{2\pi}$ with $Q_n(f) \in \mathcal{A}_{2\pi}$ for $f \in \mathcal{A}_{2\pi}$.

THEOREM 1: *There is a positive constant c independent of n such that*

$$\|Q_n(f)\|_{\infty} \leq c \|f\|_a \quad (f \in \mathcal{A}_{2\pi}),$$

i.e., the interpolation projectors Q_n are uniformly bounded as projectors from $\mathcal{A}_{2\pi}$ into $\mathcal{C}_{2\pi}$.

Proof: We start with case I:

$$g(t) = \sum_k d_k e_k(t).$$

Recall that

$$b_j(t) = \sum_s (g, e_{j+sn}) e_{j+sn}(t) = \sum_s d_{j+sn} e_{j+sn}(t)$$

where $0 < j < n$. Since $d_{j+sn} > 0$ we obtain

$$|b_j(t)| \leq b_j(0), \quad 0 < j < n.$$

Next remember that

$$Q_n(f)(t) = \langle f, e_0 \rangle + \sum_{k=1}^{n-1} \langle f, e_k \rangle b_k(t)/b_k(0).$$

This implies

$$|Q_n(f)(t)| \leq \sum_{k=0}^{n-1} |\langle f, e_k \rangle|.$$

Since $\langle f, e_j \rangle = \sum_s \langle f, e_{j+sn} \rangle$, we can conclude

$$|Q_n(f)(t)| \leq \sum_{k=0}^{n-1} \left| \sum_s \langle f, e_{k+sn} \rangle \right| \leq \sum_{k=0}^{n-1} \sum_s |\langle f, e_{k+sn} \rangle| = \|f\|_a.$$

Thus the assertion holds for case I.

Next we consider case II. Recall that

$$b_j(t) = \sum_s \langle g, e_{j+sn} \rangle e_{j+sn}(t).$$

Since

$$g(t) = \sum_k -i \cdot \operatorname{sgn}(k) d_k e_k(\pi/n) e_k(t),$$

we obtain

$$|b_j(t)| = \left| \sum_{s=0}^{\infty} d_{j+sn} (-1)^s e_{sn}(t) - \sum_{s=1}^{\infty} d_{-j+sn} (-1)^s e_{-sn}(t) \right|,$$

$$|b_j(0)| = \sum_{s=0}^{\infty} (d_{j+sn} + d_{n-j+sn}) (-1)^s$$

where $0 < j < n$.

Due to the properties of the sequence $d = (d_k)$ we obtain

$$|b_j(0)| \geq d_j - d_n \geq d_j(1 - \alpha_2),$$

$$|b_j(t)| \leq d_j + \sum_{s=1}^{\infty} d_{j+sn} + \sum_{s=1}^{\infty} d_{-j+sn} \leq d_j(1 + \alpha)$$

where $0 < j < n$. This implies

$$|b_j(t)/b_j(0)| \leq (1 + \alpha)/(1 - \alpha_2)$$

where $t \in \mathbb{R}$ and $0 < t < n$. Proceeding as in the proof for case I we can

conclude

$$|Q_n(f)(t)| \leq (1 + \alpha)/(1 - \alpha_2) \|f\|_u.$$

Thus the assertion holds also for case II.

For completeness we consider also case III (trigonometric interpolation). Recall that

$$g(t) = \sum_{k=-m}^m e_k(t) \quad (m = [n/2]),$$

$$b_j(t) = \sum_s (g, e_{j+sn}) e_{j+sn}(t).$$

This implies

$$b_j(t) = e_j(t), \quad 0 < j < n/2, \quad b_j(t) = e_{-n+j}(t), \quad n/2 < j < n$$

for odd n and

$$b_j(t) = e_j(t), \quad 0 < j < n/2, \quad b_j(t) = e_{-n+j}(t), \quad n/2 < j < n,$$

$$b_{n/2}(t) = e_{n/2}(t) + e_{-n/2}(t)$$

for even n . Again we have

$$|b_j(t)/b_j(0)| \leq 1, \quad 0 < j < n.$$

Thus the assertion holds also for case III.

To establish uniform convergence of $Q_n(f)$ for $f \in \mathcal{A}_{2\pi}$ via the Banach-Steinhaus principle [7] we have to verify uniform convergence of $Q_n(f)$ on a dense subset of $\mathcal{A}_{2\pi}$. This will be done in the following section.

3. APPROXIMATION OF THE EXPONENTIALS

The approximation properties of Q_n will first be investigated for the exponential functions e_j .

THEOREM 2: *For any $k \in \mathbb{Z}$ the asymptotic error relation*

$$\|e_k - Q_n(e_k)\|_\infty = \mathcal{O}(d_{[n/2]}) \quad (n \rightarrow \infty)$$

holds.

Proof: We consider case I. Then we have

$$g(t) = \sum_k d_k e_k(t) = d_0 + \sum_{k=1}^{\infty} 2 d_k \cos(kt),$$

i.e., $g(t)$ is real valued. This implies $Q_n(\bar{f}) = \overline{Q_n(f)}$.

Since $\bar{e}_k = e_{-k}$ and $Q_n(e_0) = e_0$ we may assume $0 < k \leq n/2$. Recall that

$$b_k(t) = \sum_s d_{k+sn} e_{k+sn}(t) = e_k(t) \sum_s d_{k+sn} e_{sn}(t).$$

Then we obtain

$$\begin{aligned} |e_k(t) - Q_n(e_k)(t)| &= |e_k(t) - b_k(t)/b_k(0)| = \\ &= |b_k(0) - b_k(t) e_{-k}(t)| / |b_k(0)| \\ &\leq \sum_{s \neq 0} 2 d_{k+sn} / d_k \leq \sum_{s=1}^{\infty} 2 d_{s[n/2]} / d_k \leq (2 \alpha / d_k) d_{[n/2]}. \end{aligned}$$

Thus, the assertion holds for case I.

We consider now case II. Then we have

$$g(t) = \sum_k -i \cdot \text{sgn}(k) d_k e_k(-\pi/n) e_k(t) = \sum_{k=1}^{\infty} 2 d_k \sin(k(t - \pi/n)),$$

i.e., $g(t)$ is real valued which implies $Q_n(\bar{f}) = \overline{Q_n(f)}$. Since $\bar{e}_k = e_{-k}$ and $Q_n(e_0) = e_0$, again we may assume $0 < k \leq n/2$. Recall that

$$b_k(t) = -i \cdot e_k(t) \left(\sum_{s=0}^{\infty} d_{k+sn} (-1)^s e_{sn}(t) - \sum_{s=1}^{\infty} d_{-k+sn} (-1)^s e_{-sn}(t) \right).$$

Then we obtain

$$\begin{aligned} |e_k(t) - Q_n(e_k)(t)| &= |e_k(t) - b_k(t)/b_k(0)| = \\ &= |b_k(0) - b_k(t) e_{-k}(t)| / |b_k(0)| \\ &\leq \left| \sum_{s=1}^{\infty} d_{k+sn} (-1)^s (1 - e_{sn}(t)) - \sum_{s=1}^{\infty} d_{-k+sn} (-1)^s (1 - e_{-sn}(t)) \right| / |b_k(0)| \\ &\leq (2 \alpha / ((1 - \alpha_2) d_k)) d_{[n/2]}. \end{aligned}$$

Thus, the assertion holds also for case II.

For the sake of completeness we note that Theorem 2 is trivially true for case III (*trigonometric interpolation*) since in this case we have

$$e_k = Q_n(e_k) \quad (|k| < [n/2]).$$

THEOREM 3 : For any $f \in \mathcal{A}_{2\pi}$ $Q_n(f)$ converges uniformly to f as n tends to infinity :

$$\|f - Q_n(f)\|_{\infty} \rightarrow 0 \quad (n \rightarrow \infty).$$

Proof: Note that the exponentials form a dense subalgebra of $\mathcal{A}_{2\pi}$. In view of Theorem 1 and Theorem 2 the Banach Steinhaus principle is applicable and yields a proof of Theorem 3.

4. QUANTITATIVE ERROR BOUNDS

We use the sequence $d = (d_k)$ to introduce a *smooth* subspace $\mathcal{A}_{2\pi}^d$ of the Wiener algebra $\mathcal{A}_{2\pi}$:

$$\mathcal{A}_{2\pi}^d := \left\{ f \in \mathcal{A}_{2\pi} : \sum_{k \neq 0} |(f, e_k)|/d_k < \infty \right\}.$$

Then we can introduce a linear operator $D : \mathcal{A}_{2\pi}^d \rightarrow \mathcal{A}_{2\pi}$ being defined by

$$D(f) = \sum_{k \neq 0} d_k^{-1} (f, e_k) e_k.$$

The space $\mathcal{A}_{2\pi}^d$ and the operator D are related to the generating function g and the interpolation projector Q_n . It is useful to investigate first the approximation properties in $\mathcal{A}_{2\pi}^d$ of the Fourier partial sum projector F_n .

THEOREM 4: If $f \in \mathcal{A}_{2\pi}^d$ then

$$\|f - F_n(f)\|_{\infty} = o(d_{[n/2]}) \quad (n \rightarrow \infty).$$

Proof: Note first that

$$F_n(D(f)) = D(F_n(f)) \quad (f \in \mathcal{A}_{2\pi}^d).$$

Then we have

$$\begin{aligned} |f(t) - F_n(f)(t)| &\leq \sum_{|k| > [n/2]} |(f, e_k)| = \|f - F_n(f)\|_a = \\ &= \sum_{|k| > [n/2]} |(D(f)f, e_k)| d_k \leq \left(\sum_{|k| > [n/2]} |(D(f)f, e_k)| \right) d_{[n/2]} \\ &= \|D(f) - F_n(D(f))\|_a \cdot d_{[n/2]}, \end{aligned}$$

i.e., we have

$$\|f - F_n(f)\|_{\infty} \leq \|f - F_n(f)\|_a \leq \|D(f) - F_n(D(f))\|_a \cdot d_{[n/2]}.$$

Since $D(f) \in \mathcal{A}_{2\pi}$, $\|D(f) - F_n(D(f))\|_a = o(1)(n \rightarrow \infty)$, and the proof of Theorem 4 is complete.

THEOREM 5 : If $f \in \mathcal{A}_{2\pi}^d$ then

$$\|f - Q_n(f)\|_\infty = \mathcal{O}(d_{[n/2]}) \quad (n \rightarrow \infty).$$

Proof : We have

$$\begin{aligned} |f(t) - Q_n(f)(t)| &\leq |f(t) - F_n(f)(t)| + |F_n(f)(t) - Q_n(F_n(f))(t)| \\ &\quad + |Q_n(F_n(f))(t) - Q_n(f)(t)|. \end{aligned}$$

In view of Theorem 1 we have

$$|Q_n(F_n(f))(t) - Q_n(f)(t)| \leq c \|f - F_n(f)\|_a.$$

This implies

$$\begin{aligned} |f(t) - Q_n(f)(t)| &\leq C \|f - F_n(f)\|_a + |F_n(f)(t) - Q_n(F_n(f))(t)| \\ &\leq C \|f - F_n(f)\|_a + \sum_{|k| \leq [n/2]} |(f, e_k)| \cdot \|e_k - Q_n(e_k)\|_\infty \end{aligned}$$

where $C > 0$ is a constant.

In view of Theorem 2 we can conclude

$$\begin{aligned} |f(t) - Q_n(f)(t)| &\leq \\ &\leq C \|f - F_n(f)\|_a + \left(\sum_{0 < |k| \leq [n/2]} |(f, e_k)| \right) \cdot (2 \alpha / ((1 - \alpha_2) d_k)) d_{[n/2]} \\ &\leq C \|f - F_n(f)\|_a + (2 \alpha / (1 - \alpha_2)) d_{[n/2]} \|D(f)\|_a. \end{aligned}$$

Now an application of Theorem 4 completes the proof of Theorem 5.

The approximation properties of the trigonometric interpolation projector are more closely related to those of the Fourier partial sum projector.

THEOREM 6 : Let Q_n be the projector of trigonometric interpolation. Then for any $f \in \mathcal{A}_{2\pi}^d$ the asymptotic error relation

$$\|f - Q_n(f)\|_\infty = o(d_{[n/2]}) \quad (n \rightarrow \infty)$$

holds.

Proof : As in the proof of Theorem 5 we have

$$\begin{aligned} |f(t) - Q_n(f)(t)| &\leq |f(t) - F_n(f)(t)| + |F_n(f)(t) - Q_n(F_n(f))(t)| \\ &\quad + |Q_n(F_n(f))(t) - Q_n(f)(t)|. \end{aligned}$$

Since Q_n is the trigonometric interpolation projector, we have

$$F_n(f) = Q_n(F_n(f)).$$

Taking into account Theorem 1 we obtain

$$|f(t) - Q_n(f)(t)| \leq C \|f - F_n(f)\|_a$$

where $C > 0$ is a constant. Now an application of Theorem 4 completes the proof.

As an application we consider

$$d_k = k^{-q} \quad (k \in \mathbb{N}, q \in \mathbb{N}, q > 1).$$

Then we have

$$\mathcal{A}_{2\pi}^d := \left\{ f \in \mathcal{A}_{2\pi} : \sum_{k \neq 0} |k|^q |(f, e_k)| < \infty \right\},$$

i.e.,

$$\mathcal{A}_{2\pi}^d = \{f \in \mathcal{C}_{2\pi}^q : D^q f \in \mathcal{A}_{2\pi}\}$$

where $D^q f$ denotes the q -th derivative of f .

THEOREM 7: For any $f \in \mathcal{C}_{2\pi}^q$ with $D^q f \in \mathcal{A}_{2\pi}$ the asymptotic error relation

$$\|f - Q_n(f)\|_\infty = \mathcal{O}(n^{-q}) \quad (n \rightarrow \infty)$$

holds.

If $q = 2r$, $r \in \mathbb{N}$, case I yields the error estimate of Golomb [5] for odd degree periodic splines.

If $q = 2r + 1$, $r \in \mathbb{N}$, case II extends the error estimate of Golomb to even degree periodic midpoint splines.

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