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M2AN - Modélisation mathématique et analyse numérique, tome
27, n° 6 (1993), p. 777-799

http://www.numdam.org/item?id=M2AN_1993__27_6_777_0

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INTERACTION OF CONCENTRATED MASSES IN A HARMONICALLY OSCILLATING SPATIAL BODY WITH NEUMANN BOUNDARY CONDITIONS (*)

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Communiqué par E. SANCHEZ-PPALENCIA

Abstract. — The asymptotics of eigenvalues of the Neumann spectral problem with concentrated masses is derived and justified. The resultant limiting problem contains integral terms linking all the equations for the separated inclusions into a coupled system. An approach based on the asymptotic theory of elliptic problems in singularly perturbed domains is used, asymptotically sharp estimates of solutions, and an explicit form for the almost inverse to the initial problem operator is presented.

Résumé. — On déduit et on justifie le comportement asymptotique des valeurs propres du problème spectral de Neumann avec masses concentrées. Le problème limite correspondant contient des termes intégraux qui couplent les équations pour les inclusions séparées, qui forment alors un système. On utilise une méthode basée sur la théorie asymptotique des problèmes elliptiques dans des domaines soumis à des perturbations singulières. On donne des estimations asymptotiques fines des solutions ainsi qu'une forme explicite du pseudo-inverse du problème initial.

1. INTRODUCTION

Let Ω and ω_j be three-dimensional domains with smooth closed boundaries $\partial\Omega$ and $\partial\omega_j$, respectively; $j = 1, \dots, J$. Fixing some points $P_1, \dots, P_J$ in Ω we introduce the sets

$$\omega_j(\varepsilon) = \{x \in \mathbb{R}^3 : \varepsilon^{-1}(x - P_j) \in \omega_j\}, \quad j = 1, \dots, J, \quad (1)$$

$$\Omega(\varepsilon) = \Omega \setminus [\omega_1(\varepsilon) \cup \dots \cup \omega_J(\varepsilon)] \quad (2)$$

(*) Manuscript received October 6, 1992.

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depending on a positive parameter ε which is supposed to be so small as $\overline{\omega_j(\varepsilon)} \subset \Omega$. We consider the Neumann spectral problem

$$\Delta_x u(\varepsilon, x) + \lambda(\varepsilon) \gamma(\varepsilon, x) u(\varepsilon, x) = 0, \quad x \in \Omega, \quad (3)$$

$$\partial_n u(\varepsilon, x) = 0, \quad x \in \partial\Omega, \quad (4)$$

where Δ_x is Laplacian, $\partial_n = \partial/\partial n$, n is the outward normal to $\partial\Omega$,

$$\gamma(\varepsilon, x) = \begin{cases} \gamma_0 & , \quad w \in \Omega(\varepsilon), \\ \varepsilon^{-3} \gamma_j(\varepsilon^{-1}(x - P_j)), & x \in \omega_j(\varepsilon), \end{cases} \quad (5)$$

the functions γ_j being positive and smooth in $\bar{\omega}_j$, $\gamma_0 > 0$. The eigenvalues of problem (3), (4) form the sequence

$$0 = \lambda_0 < \lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \lambda_3(\varepsilon) \leq \dots \rightarrow +\infty \quad (6)$$

while an eigenfunction u_0 corresponding to λ_0 is constant. In this paper we will show that for each of the positive eigenvalues the ratio $\lambda_m(\varepsilon)/\varepsilon$ has a limit (as $\varepsilon \rightarrow +0$) coinciding with the corresponding term μ_m of the eigenvalue sequence

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots \rightarrow +\infty \quad (7)$$

of the system

$$\Delta_\xi w^j(\xi) + \mu \gamma_j^0(\xi) \left[w^j(\xi) - \Gamma^{-1} \sum_{k=1}^J \int_{\omega_k} \gamma_k(\eta) w^k(\eta) d\eta \right] = 0, \quad \xi \in \mathbb{R}^3, j = 1, \dots, J. \quad (8)$$

Here the functions w^j are assumed to vanish at infinity,

$$\gamma_j^0(\xi) = \begin{cases} \gamma_j(\xi), & \xi \in \omega_j, \\ 0, & \xi \in \mathbb{R}^3 \setminus \omega_j, \end{cases}$$

$$\Gamma = \Gamma_0 + \Gamma_1 + \dots + \Gamma_J, \quad \Gamma_0 = \gamma_0 \text{mes}_3 \Omega, \quad \Gamma_j = \int_{\omega_j} \gamma_j(\xi) d\xi. \quad (9)$$

Formula (5) means that the « inclusions » (1) have masses Γ_j of the same order (with respect of ε) as in the « matrix » (2). In other words, we deal with a concentrated mass problem.

The paper [1] considered this type of spectral problems for the first time, the Dirichlet problem being investigated. We would also mention the papers [2-4] and the monographies [5, 6] related the same subject and similar ones. Let us recall the results of [1] with respect to equation (3) with Dirichlet condition

$$u(\varepsilon, x) = 0, \quad x \in \partial\Omega. \quad (10)$$

Problem (3), (10) having the positive eigenvalues $\lambda_1(\varepsilon)$, $\lambda_2(\varepsilon)$, ..., the set

$$\left\{ \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \lambda_m(\varepsilon) : m = 1, 2, \dots \right\} \quad (11)$$

coincides with the union of the spectra of the equations

$$\Delta_\xi w^j(\xi) + \mu \gamma_j^0(\xi) w^j(\xi) = 0, \quad \xi \in \mathbb{R}^3, \quad (12)$$

where $j = 1, \dots, J$ and $w^j(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$.

There is an important difference between the limiting problems (8) and (12): the subtrahend in the square braces links all equations (8) into a coupled system, whereas each equation in (12) may be solved separately. From a physical point of view such an interaction of inclusions in (8) has a spontaneous explanation. Naturally, in the case of free surface $\partial\Omega$ (Neumann conditions) the inertial forces in the oscillating body Ω compounded of (1) and (2) must be self-balancing, the above mentioned subtrahend being responsible for the equilibrium while for Diriclet conditions the same being done by exterior forces applied at the clamped surface $\partial\Omega$. To outline coupling of oscillating inclusions we rewrite equation (3) in a special manner.

PROPOSITION 1 : *Any eigenfunction u of problem (3), (4) is such that the function*

$$x \mapsto u'(\varepsilon, x) = u(\varepsilon, x) + \Gamma_0^{-1} \sum_{j=1}^J \int_{\omega_j(\varepsilon)} \gamma_j(\varepsilon, y) u(\varepsilon, y) dy \quad (13)$$

satisfies boundary conditions (4) and the equation

$$\Delta_x u'(\varepsilon, x) + \lambda(\varepsilon) \gamma(\varepsilon, x) \left\{ u'(\varepsilon, x) - \Gamma^{-1} \sum_{j=1}^J \int_{\omega_j(\varepsilon)} \gamma_j(\varepsilon, y) u'(\varepsilon, y) dy \right\} = 0, \quad x \in \Omega. \quad (14)$$

The eigenfunction u is restored by the formula

$$u(\varepsilon, x) = u'(\varepsilon, x) - \Gamma^{-1} \sum_{j=1}^J \int_{\omega_j(\varepsilon)} \gamma_j(\varepsilon, y) u'(\varepsilon, y) dy. \quad (15)$$

Proof : The only point to be verified is the relation

$$\sum_{j=1}^J \int_{\omega_j(\varepsilon)} \gamma_j(\varepsilon, x) u'(\varepsilon, x) dx = \Gamma \Gamma_0^{-1} \sum_{j=1}^J \int_{\omega_j(\varepsilon)} \gamma_j(\varepsilon, x) u(\varepsilon, x) dx$$

which follows from (13) and (9), (5). ■

A derivation of the (resultant) limiting system (8) is given in section 2. To justify formal asymptotic representations we use an approach which differs from the approach in [5, 6] and is based on the results [7, 8] related to general elliptic problems in domains with singularly perturbed boundaries. The main elements of our approach are a reduction of the initial problem (14), (4) to a vector equation containing a slightly perturbed matrix of limiting problem operators and an inverse reduction of a vector equation (with another perturbation !) to problem (14), (4) (see Sect. 4 and Sect. 5, resp. ; the first reduction was applied in [9-11, 8] to other spectral problems). Although both reductions are non-equivalent and the vector equations contain different realizations of the limiting problem operators, the coincidence of the set (11) with the spectrum of system (8) follows (Sect. 7).

We should emphasize that norms of the perturbation operators turn out to be small only while they are defined on special functional spaces with weighted norms. In section 3 we recall well-known facts touching upon solvability of the limiting problems in these spaces being regarded as an application of the theory of elliptic problems in domains with conical points (*cf.* [12, 13], etc.).

Finally, we note that for the Neumann problem with unique small heavy inclusion (i.e. $J = 1$ in (1), (2)) the presence of an integral term in a resultant limiting equation was mentioned in [14] but without complete proofs.

2. FORMAL ASYMPTOTICS

Let us perform the usual change of spectral parameter,

$$\mu(\varepsilon) = \varepsilon^{-1} \lambda(\varepsilon), \quad (16)$$

and let ε tend to $+0$. Equations (3), (4) become the Neumann problem for the Laplace operator in Ω . A solution of this homogeneous problem is constant and is intended to be the principal term of the eigenfunction asymptotics far from points $P_1, \dots, P_J$. In the vicinity of P_j we define the « rapid » coordinates

$$\xi^j = \varepsilon^{-1}(x - P_j) \quad (17)$$

to derive other limiting problem containing the operator $\Delta_{\xi^j} + \mu \gamma_j^0(\xi^j)$; $j = 1, \dots, J$. Thus, we seek formal asymptotics of solutions to spectral problem (3), (4) as follows :

$$\lambda(\varepsilon) = \varepsilon \mu + \dots, \quad (18)$$

$$u(\varepsilon, x) = v^0 + \sum_{j=1}^J w^j(\varepsilon^{-1}(x - P_j)) + \varepsilon v^1(x) + \dots; \quad (19)$$

we denote by dots asymptotic terms that are inessential for our procedure.

In accordance with (5) and (17) we have

$$\begin{aligned} \Delta_x[v^0 + w^j(\varepsilon^{-1}(x - P_j))] + \varepsilon \mu \gamma(\varepsilon, x)[v^0 + w^j(\varepsilon^{-1}(x - P_j))] = \\ = \varepsilon^{-2} \{ \Delta_\xi w^j(\xi^j) + \mu \gamma_j^0(\xi^j)[w^j(\xi^j) + v^0] \} + \dots \end{aligned}$$

in a small neighbourhood of P_j . Hence, the leading terms of representations (18), (19) must be linked by

$$\Delta_\xi w^j(\xi^j) + \mu \gamma_j^0(\xi^j)[w^j(\xi^j) + v^0] = 0, \quad \xi^j \in \mathbb{R}^3. \quad (20)$$

Suppose that for $v^0 \neq 0$ there exist a solution of equation (20) vanishing at infinity. Being a harmonic function in $\mathbb{R}^3 \setminus \bar{\omega}_j$, this solution admits the decomposition

$$w^j(\xi^j) = b_j(4\pi|\xi^j|)^{-1} + O(|\xi^j|^{-2}), \quad |\xi^j| \rightarrow \infty. \quad (21)$$

The Green formula in the ball $B_R = \{\xi : |\xi| < R\}$ leads us to the relation

$$\begin{aligned} \mu \int_{\omega_j} \gamma_j(\xi)(w^j(\xi) + v^0) d\xi = - \lim_{R \rightarrow \infty} \int_{B_R} \Delta_\xi w^j(\xi) d\xi = \\ = - \lim_{R \rightarrow \infty} \int_{\partial B_R} \frac{\partial}{\partial |\xi|} w^j(\xi) ds_\xi = b_j \lim_{R \rightarrow \infty} \int_{\partial B_R} \frac{ds_\xi}{4\pi|\xi|^2} = b_j. \quad (22) \end{aligned}$$

To determine the boundary conditions for v^1 we substitute (18), (19) into (4), take into account (21), (17) and equate to zero the coefficient at ε^1 . As a result, we obtain

$$\partial_n v^1(x) = - \sum_{j=1}^J b_j \partial_n(4\pi r_j)^{-1}, \quad x \in \partial\Omega, \quad (23)$$

where $r_j = |x - P_j|$. The same arguments with respect to (3) give us the following equation

$$\Delta_x v^1(x) = - \mu \gamma_0 v^0, \quad x \in \Omega. \quad (24)$$

The compatibility condition for problem (24), (23) takes the form

$$- \mu v^0 \Gamma_0 = - \sum_{j=1}^J b_j \int_{\partial\Omega} \partial_n(4\pi r_j)^{-1} ds_x = \sum_{j=1}^J b_j. \quad (25)$$

If $\mu \neq 0$ then by using notations (9) and

$$\langle w \rangle_j = \int_{\omega_j} \gamma_j(\xi) w(\xi) d\xi \quad (26)$$

we unite (25) and (22) into the relation

$$v^0 = -\Gamma^{-1} \sum_{j=1}^J \langle w^j \rangle_j. \quad (27)$$

To this end, in equations (20), $j = 1, \dots, J$, we replace v^0 by expression (27) and obtain system (8) which is called the resultant limiting problem.

3. LIMITING PROBLEMS IN WEIGHTED SPACES

There is a lot of approaches to investigate the problems in question. We choose one of them related to the theory of elliptic problems in domains with piecewise smooth boundaries (the points $P_1, \dots, P_J$ and the infinitely remote point can be regarded as conical ones). This choice is prescribed by the estimate technique we use in the following sections.

Denote by $V_{\beta, \sigma}^{\ell}(\Omega)$ and $V_{\beta}^{\ell}(\mathbb{R}^3)$ the Hilbert spaces of functions defined on Ω and $\mathbb{R}^3$ with the weighted norms

$$\left\{ \begin{aligned} \|v; V_{\beta, \sigma}^{\ell}(\Omega)\| &= \left(\sum_{s=0}^{\ell} \|(r_{\min} + \sigma)^{\beta - \ell + s} \nabla_x^s v; L_2(\Omega)\|^2 \right)^{1/2}, \\ \|w; V_{\beta}^{\ell}(\mathbb{R}^3)\| &= \left(\sum_{s=0}^{\ell} \|(|\xi| + 1)^{\beta - \ell + s} \nabla_{\xi}^s w; L_2(\mathbb{R}^3)\|^2 \right)^{1/2}. \end{aligned} \right. \quad (28)$$

Here $\beta \in \mathbb{R}$, $\sigma \geq 0$, $\ell = 0, 1, \dots$ and $r_{\min} = \min\{r_1, \dots, r_J\}$, $\nabla_x^s v$ is the collection of order s derivatives of v . We will deal with the cases $\sigma = \varepsilon > 0$ and $\sigma = 0$. Note that for any β and $\sigma > 0$ the space $V_{\beta, \sigma}^{\ell}(\Omega)$ coincides with Sobolev space $H^{\ell}(\Omega)$ but the norm equivalence constants depend on σ .

Let $\chi \in C_0^{\infty}(\mathbb{R})$ be a cut-off function, $\chi(r) = 1$ ($= 0$) while $r < 1$ (> 2). For $\alpha \in (0, 1)$ we introduce new cut-off functions with supports depending on ε ; namely,

$$\left\{ \begin{aligned} \chi_j^{\alpha}(\varepsilon, x) &= \chi(\varepsilon^{-\alpha} r_j), \quad X_j^{\alpha}(\varepsilon, \xi) = \chi(\varepsilon^{1-\alpha} |\xi|), \\ X_0^{\alpha}(\varepsilon, x) &= 1 - \sum_{j=1}^J \chi_j^{\alpha}(\varepsilon, x). \end{aligned} \right. \quad (29)$$

LEMMA 2 : 1° If $y \in V_{\beta, \varepsilon}^{\ell}(\Omega)$ and

$$y^0 = X_0^{\alpha} v, \quad y^j(\varepsilon, \xi) = X_j^{\alpha}(\varepsilon, \xi) y(P_j + \varepsilon \xi)$$

then $y^0 \in V_{\beta, 0}^{\ell}(\Omega)$, $y^j \in V_{\beta}^{\ell}(\mathbb{R}^3)$ and

$$\|y^0; V_{\beta, 0}^{\ell}(\Omega)\| + \varepsilon^{\beta - \ell + 3/2} \|y^j; V_{\beta}^{\ell}(\mathbb{R}^3)\| \leq C_1 \|y; V_{\beta, \varepsilon}^{\ell}(\Omega)\|.$$

2° If $z \in V_\beta^\ell(\mathbb{R}^3)$ and $z^0(\varepsilon, x) = \chi_j^\alpha(\varepsilon, x) z(\varepsilon, \varepsilon^{-1}(x - P_j))$, $z' = X_j^\alpha z$ then $z^0 \in V_{\beta, \varepsilon}^\ell(\Omega)$, $z' \in V_\beta^\ell(\mathbb{R}^3)$ and

$$\varepsilon^{\ell - \beta - 3/2} \|z^0; V_{\beta, \varepsilon}^\ell(\Omega)\| + \|z'; V_\beta^\ell(\mathbb{R}^3)\| \leq C_2 \|z; V_\beta^\ell(\mathbb{R}^3)\|.$$

The constants C_1 and C_2 depend neither on $\varepsilon \in (0, \varepsilon_0]$ nor on y and z .

Proof: The desired estimates follow immediately from the norm definitions (28). Therefore we shall mention two facts only. First, for a homogeneous operator $Q(\nabla_x)$ with constant coefficients the inequalities $\varepsilon^\alpha < r_j < 2\varepsilon^\alpha$ hold on the supports of the coefficients of the commutator $[Q, \chi_j^\alpha] = Q\chi_j^\alpha - \chi_j^\alpha Q$ and that is why the estimate of y^0 (and z') is valid. Second, due to (17) and (28) we have

$$\|Z; V_\beta^\ell(\mathbb{R}^3)\| = \varepsilon^{\ell - \beta - 3/2} \|x \mapsto Z(\varepsilon^{-1}(x - P_j)); V_{\beta, \varepsilon}^\ell(\Omega)\|$$

while $\text{supp } Z \subset \{\xi : |\xi| \leq R\varepsilon^{-\alpha}\}$ and ε is sufficiently small, the multiplier $\varepsilon^{\ell - \beta - 3/2}$ leading to the same result in the estimate of Z^0 (and y'). ■

We consider two continuous mappings

$$N_\beta \equiv \{\Delta_x, \partial_n\} : V_{\beta, 0}^2(\Omega) \rightarrow V_{\beta, 0}^0(\Omega) \times H^{1/2}(\partial\Omega), \quad (30)$$

$$L_\beta \equiv \Delta_\xi : V_\beta^2(\mathbb{R}^3) \rightarrow V_\beta^0(\mathbb{R}^3) \quad (31)$$

which are the operators of the Neumann problem in Ω and of the Laplace equation in $\mathbb{R}^3$ respectively.

PROPOSITION 3 : 1° If $\beta \in (1/2, 3/2)$ then N_β is a Fredholm operator and $\text{Ind } N_\beta \equiv \dim \ker N_\beta - \dim \text{coker } N_\beta = 0$ and $\ker N_\beta$ consists of constant functions. Under the condition $1/2 < \beta$, $\gamma < 3/2$ a solution $v \in V_{\gamma, 0}^2(\Omega)$ of the problem

$$\{\Delta_x v, \partial_n v\} = \{f, g\} \in V_{\beta, 0}^0(\Omega) \times H^{1/2}(\partial\Omega) \quad (32)$$

belongs to $V_{\beta, 0}^2(\Omega)$.

2° Operator (31) is an isomorphism provided $\beta \in (1/2, 3/2)$.

Proof: Both assertions are well-known facts (see, for example, chapter 2 [13] and chapter 1 [7]) and we outline only some features of reasoning for the first one. Operator $N_{2-\beta}$ is adjoint to N_β with respect to the Green formula

$$\int_\Omega v \Delta_x u \, dx - \int_{\partial\Omega} v \partial_n u \, ds_x = \int_\Omega u \Delta_x v \, dx - \int_{\partial\Omega} u \partial_n v \, ds_x.$$

Hence, problem (32) has the solution $v \in V_{\beta, 0}^2(\Omega)$ if and only if

$$\int_\Omega fV \, dx - \int_{\partial\Omega} gV \, ds_x = 0 \quad \forall V \in \ker N_{2-\beta}.$$

Besides, the subspaces $\ker N_\beta$ and $\text{coker } N_\beta$ do not depend on $\beta \in (1/2, 3/2)$ (due to the last assertion in 1° being derived with the help of the asymptotic representations of a solution near the points $P_1, \dots, P_J$; cf. Proposition 1.6.2 [13]). To this end, by virtue of Hardy inequality the space $V_0^1(\Omega)$ which contains $V_1^2(\Omega)$ coincides with $H^1(\Omega)$ and therefore $\ker N_\beta = \{\text{const.}\}$. ■

We put

$$If = f - [\text{mes}_3 \Omega]^{-1} \int_{\Omega} f \, dx, \quad \pi_h v = \int_{\Omega} v h \, dx \quad (33)$$

where $h \in C^\infty(\bar{\Omega})$ and $h_\Omega \equiv \int_{\Omega} h \, dx \neq 0$. In the following sections we shall consider the operator

$$\begin{aligned} \mathbb{N}_\beta(\mu) &\equiv \{\Delta_x, \mu \pi_h\} : \tilde{V}_\beta^2(\Omega) \equiv \\ &\equiv \{v \in V_{\beta,0}^2(\Omega) : \partial_n v = 0 \text{ on } \partial\Omega\} \rightarrow IIV_{\beta,0}^0(\Omega) \times \mathbb{C} \end{aligned} \quad (34)$$

corresponding to the Neumann problem which is supplemented by integral conditions avoiding its kernel and cokernel. Properties of (34) follow immediately from Proposition 3 (1°).

COROLLARY 4 : *Let $\beta \in (1/2, 3/2)$. Operator $\mathbb{N}_\beta(\mu)$ is an isomorphism for any $\mu \neq 0$ and $\mu = 0$ is simple eigenvalue, $\ker \mathbb{N}_\beta(0) = \{\text{const.}\}$. If $\delta^{-1} > |\mu| > \delta > 0$ then the norms of $\mathbb{N}_\beta(\mu)$ and $\mathbb{N}_\beta(\mu)^{-1}$ are majorized by a constant depending only on β, δ, Ω and h_Ω .*

Let us introduce the vector-function $w = (w^1, \dots, w^J)$. Rewriting the left-hand side of system (8) in a vector form we connect with the resultant limiting problem the continuous linear bundle

$$\mathbb{C} \ni \mu \mapsto \mathbb{L}_\beta(\mu) \equiv \mathbb{L}_\beta^0 + \mu \mathbb{L}^1 : V_\beta^2(\mathbb{R}^3)^J \rightarrow V_\beta^0(\mathbb{R}^3)^J \quad (35)$$

where $\mathbb{L}_\beta^0 = L_\beta \times \dots \times L_\beta$.

THEOREM 5 : *Let $\beta \in (1/2, 3/2)$. The eigenvalues of the bundle $\mathbb{L}_\beta(\mu)$ form sequence (7). The corresponding eigenvectors $w_h = (w_h^1, \dots, w_h^J)$ can be normalized by the conditions*

$$\begin{aligned} \langle w_h, w_q \rangle &\equiv \sum_{j=1}^J \left\{ \langle w_h^j, \bar{w}_q^j \rangle_j - \right. \\ &\quad \left. - \Gamma^{-1} \langle w_h^j \rangle_j \sum_{k=1}^J \langle \bar{w}_q^k \rangle_k \right\} = \delta_{h,q}, \quad h, q = 1, 2, \dots, \end{aligned} \quad (36)$$

where a bar means complex conjugate, $\langle w^j \rangle_j$ is integral (26) and $\delta_{h,q}$ is Kronecker symbol. There is no vector associated with an eigenvector. The eigenvalues and the eigenvectors do not depend on $\beta \in (1/2, 3/2)$.

Proof: Since $\mathbb{L}_\beta^0$ is an isomorphism (Proposition 3 (2°)) and the mapping $\mathbb{L}_\beta^1 : V_\beta^2(\mathbb{R}^3)^J \rightarrow V_\beta^0(\mathbb{R}^3)^J$ is compact (compare (35) with (8)) the resolvent $\mathbb{L}_\beta(\mu)^{-1}$ exists for all $\mu \in \mathbb{C}$ with exception of points $\mu_1, \mu_2, \dots$ which are finite multiplicity eigenvalues, the only possible accumulation point being at infinity (cf. [15, 5]).

It is clear that $\langle w, y \rangle = \overline{\langle y, w \rangle}$ for $w, y \in V_\beta^2(\mathbb{R}^3)^J$. By virtue of Hölder inequalities we get

$$\Gamma^{-1} \left| \sum_{j=1}^J \langle w^j \rangle_j \right|^2 \leq \Gamma^{-1} \left(\sum_{j=1}^J \Gamma_j^{1/2} \langle w^j \overline{w^j} \rangle_j^{1/2} \right)^2 \leq \Gamma^{-1} \sum_{k=1}^J \Gamma_k \sum_{j=1}^J \langle w^j \overline{w^j} \rangle_j$$

and therefore according to (9) we obtain

$$\langle w, w \rangle \geq \Gamma_0 \Gamma^{-1} \sum_{j=1}^J \langle w^j \overline{w^j} \rangle_j. \quad (37)$$

Moreover, for eigenvectors w_h^j, w_q^j , equations (8) and Green formula provide

$$\mu_h \langle w_h, w_q \rangle = \sum_{j=1}^J \int_{\mathbb{R}^3} \nabla_\xi w_h^j \cdot \nabla_\xi w_q^j d\xi; \quad (38)$$

the integrals converge because w_h^j, w_q^j are harmonic functions in $\mathbb{R}^3 \setminus \bar{\omega}_j$ with representations (21). Relations (37) and (38) state lack of associated vectors, possibility of normalization (36) and eigenvalues to be positive. Besides, one can verify the minimization principle

$$\mu_{h+1} = \inf \left\{ \sum_{j=1}^J \int_{\mathbb{R}^3} |\nabla_\xi w^j|^2 d\xi : w = \{w^1, \dots, w^J\} \in V_\beta^2(\mathbb{R}^3)^J, \right. \\ \left. \Delta_\xi w^j = 0 \text{ in } \mathbb{R}^3 \setminus \bar{\omega}_j, \langle w, w \rangle = 1, \langle w, w_q \rangle = 0, q = 1, \dots, h \right\}$$

to show sequence (7) to be unlimited (we will not use this property itself). To complete the proof we note that an eigenvector $w_h \in V_\beta^2(\mathbb{R}^3)^J$ admits decompositions (21), $j = 1, \dots, J$, and, hence, belongs to $V_\gamma^2(\mathbb{R}^3)^J$ with arbitrary $\gamma \in (1/2, 3/2)$. ■

4. RÉDUCTION OF THE INITIAL PROBLEM TO A VECTOR EQUATION

The goals of this section is to establish that to each solution $\{\lambda(\varepsilon), u'(\varepsilon, x)\}$ of the spectral problem (14), (4) there corresponds the vector

$$U = \{v, w^1, \dots, w^J\} \quad (39)$$

so that equality

$$\mathbb{T}U - \mathbb{S}U + \mu(\varepsilon)(\mathbb{P} - \mathbb{Q})U = 0 \quad (40)$$

is valid. Here $\mu(\varepsilon)$ is the rescaled spectral parameter (16), $\mathbb{S}$ and $\mathbb{Q}$ are operators with small norms, $\mathbb{T} + \mu\mathbb{P}$ is the $(J+1) \times (J+1)$ -matrix formed by the limiting problem operators (34) and (35), $\mathbb{T}_{0j} = \mathbb{T}_{j0} = 0$ and $\mathbb{P}_{0j} = \mathbb{P}_{j0} = 0$ for $j = 1, \dots, J$.

We determine the entries of vector (39) by the formulae

$$\begin{cases} v(\varepsilon, x) = X_0^\nu(\varepsilon, x) u'(\varepsilon, x), \\ w^j(\varepsilon, \xi^j) = \varepsilon^{\beta-1/2} X_j^\sigma(\varepsilon, \xi^j) u'(\varepsilon, P_j + \varepsilon \xi^j), j = 1, \dots, J, \end{cases} \quad (41)$$

(cf. Lemma 2 with $\beta - 1/2 = \beta - \ell + 3/2$ at $\ell = 2$) where X_h^α is the cut-off function in (29),

$$0 < \sigma < 1/2 < \nu < 1. \quad (42)$$

In order to pick out the suitable operators $\mathbb{S}_{0h}$ and $\mathbb{Q}_{0h}$ for the first line

$$(\mathbb{T}_{00} + \mu(\varepsilon)\mathbb{P}_{00})v = (\mathbb{S}_{00} + \mu(\varepsilon)\mathbb{Q}_{00})v + \sum_{j=1}^J (\mathbb{S}_{0j} + \mu(\varepsilon)\mathbb{Q}_{0j})w^j \quad (43)$$

of system (40) we multiply equation (14) by X_0^ν :

$$\Delta_x(X_0^\nu u') - [\Delta_x, X_0^\nu]u' + \varepsilon\mu(\varepsilon)\gamma X_0^\nu \left\{ u' - \Gamma^{-1} \sum_{j=1}^J \int_{\omega_j(\varepsilon)} \gamma u' dy \right\} = 0.$$

Here $[A, B] = AB - BA$. Using (41) and (29) we can rewrite this equation in the form

$$\begin{aligned} \Delta_x v(\varepsilon, x) = & -\varepsilon\mu(\varepsilon)\gamma_0 v(\varepsilon, x) - \\ & -\varepsilon^{1/2-\beta} \sum_{j=1}^J ([\Delta_x, \chi_j^\nu(\varepsilon, x)] w^j(\varepsilon, \varepsilon^{-1}(x - P_j)) \\ & - \varepsilon\mu(\varepsilon)\gamma_0 X_0^\nu(\varepsilon, x) \Gamma^{-1} \langle w^j \rangle_j), x \in \Omega. \end{aligned}$$

Evidently, boundary conditions (4) are fulfilled and similarly to (34) we will omit them in notations of operators. It may be possible now to determine the entries of the matrices $\mathbb{S}$ and $\mathbb{Q}$ by comparing of (43) and (44). However we must pay attention to non-invertibility of operator N_β in (30) (Proposition 3). That is why we prefer to deal with operator (34) but we will indicate the necessary integral condition after finding operators in the other lines of system (40)

$$\begin{aligned} \mathbb{T}_{jj} w^j + \mu(\varepsilon) \sum_{k=1}^J \mathbb{P}_{jk} w^k = & (\mathbb{S}_{j0} + \mu(\varepsilon)\mathbb{Q}_{j0})v + \\ & + \sum_{k=1}^J (\mathbb{S}_{jk} + \mu(\varepsilon)\mathbb{Q}_{jk})w^k, j = 1, \dots, J. \end{aligned} \quad (45)$$

We multiply equation (14) by $\varepsilon^{\beta+3/2} \chi_j^\sigma$, go over to rapid coordinates (17) and obtain

$$\varepsilon^{\beta-1/2} \Delta_\xi (X_j^\sigma u') - \varepsilon^{\beta-1/2} [\Delta_\xi, X_j^\sigma] u' + \\ + \varepsilon^{\beta+5/2} \mu(\varepsilon) \gamma X_j^\sigma \left\{ u' - \Gamma^{-1} \sum_{k=1}^J \int_{\omega_k(\varepsilon)} \gamma u' dy \right\} = 0,$$

where the functions u' and γ depend on $P_j + \varepsilon \xi^j$. Applying (41), (29) and (5) we conclude that

$$\Delta_\xi w^j(\varepsilon, \xi^j) + \mu(\varepsilon) \gamma_j^0(\xi^j) \left\{ w^j(\varepsilon, \xi^j) - \Gamma^{-1} \sum_{k=1}^J \langle w^k \rangle_k \right\} = \\ = \varepsilon^{\beta-1/2} [\Delta_\xi, X_j^\sigma(\varepsilon, \xi^j)] v(\varepsilon, P_j + \varepsilon \xi^j) - \\ - \varepsilon^3 \mu(\varepsilon) \gamma_{0j}'(\xi^j) \left\{ \varepsilon^{\beta-1/2} X_j^\sigma(\varepsilon, \xi^j) (1 - X_j^{1/2}(\varepsilon, \xi^j)) v(\varepsilon, P_j + \varepsilon \xi^j) \right. \\ \left. + X_j^{1/2}(\varepsilon, \xi^j) w_j(\varepsilon, \xi^j) - X_j^\sigma(\varepsilon, \xi^j) \Gamma^{-1} \sum_{k=1}^J \langle w^k \rangle_k \right\}, \quad \xi^j \in \mathbb{R}^3, \quad (46)$$

where $\gamma_{0j}'(\xi) = 0$ for $\xi \in \omega_j$ and $\gamma_{0j}'(\xi) = \gamma_0$ for $\xi \in \mathbb{R}^3 \setminus \bar{\omega}_j$. Equalities (46) and (45) lead to the definition of $\mathbb{S}_{jh}$ and $\mathbb{Q}_{jh}$. We should emphasize that it would be a mistake to rewrite the term $\varepsilon^{\beta+5/2} \mu \gamma_{0j}' X_j^\sigma u'$ in the form $\varepsilon^3 \mu \gamma_{0j}' w^j$ because of lack of the inclusion $V_\beta^2(\mathbb{R}^3) \subset V_\beta^0(\mathbb{R}^3)$ and lossing of continuity of operator $\mathbb{S}_{jj}$.

We are going now to calculate $\pi_1 v$ where π_1 is the functional in (33) with $h = 1$ (recall that it is necessary to determine $\mathbb{S}_{0h}$ and $\mathbb{Q}_{0h}$). Let us rewrite the right-hand side F of (44) in the form

$$F = \mu(\varepsilon) Q_{00} v + \sum_{j=1}^J (S_{0j} w^j + \mu(\varepsilon) Q_{0j} w^j).$$

Note that because of v being a solution of the problem (32) with the right-hand side $\{F, 0\}$ the condition

$$\int_{\Omega} F(\varepsilon, x) dx = 0 \quad (47)$$

is valid. Thus, by virtue of (33) we conclude that

$$F = \mu(\varepsilon) \Pi Q_{00} v + \sum_{j=1}^J (\Pi S_{0j} w^j + \mu(\varepsilon) \Pi Q_{0j} w^j). \quad (48)$$

Besides, in accordance with (44) equality (47) transforms into the following one :

$$\varepsilon \mu(\varepsilon) \gamma_0 \pi_1 v = \varepsilon^{1/2-\beta} \sum_{j=1}^J (\varepsilon \mu(\varepsilon) \gamma_0 \Gamma^{-1} \langle w^j \rangle_j \pi_1 X_0^\nu - \int_{\Omega} [\Delta_\lambda, X_j^\nu(\varepsilon, x)] w^j(\varepsilon, \varepsilon^{-1}(\lambda - P_j)) dx) . \quad (49)$$

Let us denote the last integral by I_j . On account of (46), (41), (42) and going over to coordinate (17), we obtain

$$\begin{aligned} \varepsilon^{-1} I_j &= \int_{\mathbb{R}^3} [\Delta_\xi, X_j^\nu] w^j d\xi^j = - \int_{\mathbb{R}^3} X_j^\nu \Delta_\xi w^j d\xi^j = \\ &= \mu(\varepsilon) \left\{ \langle w^j \rangle_j - \Gamma^{-1} \left(\Gamma_j + \varepsilon^3 \gamma \int_{\mathbb{R}^3 \setminus \omega_j} X_j^\nu d\xi^j \right) \sum_{k=1}^J \langle w^k \rangle_k \right\} + \\ &\quad + \varepsilon^3 \mu(\varepsilon) \gamma_0 \int_{\mathbb{R}^3 \setminus \omega_j} X_j^\nu w^j d\xi^j . \end{aligned}$$

Substituting this expression into (49) we find that the total coefficient of $\sum \langle w^j \rangle_j$ equals

$$\begin{aligned} - \mu(\varepsilon) \varepsilon^{7/2-\beta} \Gamma^{-1} \left\{ \Gamma - \sum_{k=1}^J \Gamma_k - \gamma_0 \int_{\Omega} X_0^\nu dx - \right. \\ \left. - \gamma_0 \sum_{k=1}^J \int_{\mathbb{R}^3 \setminus \omega_k(\varepsilon)} X_k^\nu dx \right\} = - \mu(\varepsilon) \varepsilon^{7/2-\beta} \Gamma^{-1} \gamma_0 \sum_{k=1}^J \text{mes}_3 \omega_k \end{aligned}$$

(see (9) and (29)). Therefore, we can rewrite (49) in the form

$$\begin{aligned} \mu(\varepsilon) \pi_1 v &= \mu(\varepsilon) \sum_{j=1}^J q_j w^j \equiv \\ &\equiv - \mu(\varepsilon) \varepsilon^{7/2-\beta} \sum_{j=1}^J \left\{ \int_{\mathbb{R}^3 \setminus \omega_j} X_j^\nu w^j d\xi^j + \Gamma^{-1} \langle w^j \rangle_j \sum_{k=1}^J \text{mes}_3 \omega_k \right\} . \quad (50) \end{aligned}$$

Finally we introduce the operators in the right-hand side of (43) by the formulae

$$\mathbb{S}_{00} = 0, \mathbb{S}_{0j} = \{HS_{0j}, O\}, \mathbb{Q}_{00} = \{PQ_{00}, 0\}, \mathbb{Q}_{0j} = \{PQ_{0j}, q_j\} \quad (51)$$

while taking S_{0j} , Q_{0h} and q_j out from (48), (44) and (50), the other entries of matrices $\mathbb{S}$ and $\mathbb{Q}$ being defined according to representation (46) compared with (45).

Functions v and w_j in (41) vanish in a neighbourhood of P_j and outside a ball respectively. Hence, the inclusion

$$U \in \mathbb{D}_{\beta, \delta} \equiv \tilde{V}_{\beta + \delta, 0}^2(\Omega) \times \prod_{j=1}^J V_{\beta - \delta}^2(\mathbb{R}^3)$$

holds for arbitrary β and δ . Fixing positive numbers β and δ so that $\beta \pm \delta \in (1/2, 3/2)$, we put

$$\mathbb{T} + \mu \mathbb{P} = \left\| \begin{array}{cc} \mathbb{N}_{\beta + \delta}(\mu); & 0 \\ 0 & ; \quad \mathbb{L}_{\beta - \delta}(\mu) \end{array} \right\| \quad (52)$$

and assume all the operators in (38) to act from $\mathbb{D}_{\beta, \delta}$ into

$$\mathbb{R}_{\beta, \delta} \equiv IIV_{\beta + \delta, 0}^0(\Omega) \times \mathbb{C} \times \prod_{j=1}^J V_{\beta - \delta}^0(\mathbb{R}^3).$$

LEMMA 6: *An eigenvalue $\lambda(\varepsilon)$ and a corresponding (non-trivial) eigenfunction $u(\varepsilon, \cdot)$ of problem (14), (4) give rise to a (rescaled) eigenvalue $\mu(\varepsilon)$ and a (non-trivial) eigenvector U of equation (40). The main term in (40) being given by (52) and perturbation operators $\mathbb{S}$, $\mathbb{Q}$ admit the estimate*

$$\|\mathbb{S}; \mathbb{D}_{\beta, \delta} \rightarrow \mathbb{R}_{\beta, \delta}\| + \|\mathbb{Q}; \mathbb{D}_{\beta, \delta} \rightarrow \mathbb{R}_{\beta, \delta}\| \leq c\varepsilon^\varkappa \quad (53)$$

where $\varkappa > 0$ and the constant c does not depend on $\varepsilon \in (0, \varepsilon_0)$ while $\beta \in (1/2, 3/2)$ and δ is small positive.

Proof: We have to verify only estimate (53). Recalling the definition of Q_{0h} in (44) the inequalities

$$\begin{aligned} \|Q_{00} v; V_{\beta + \delta, 0}^0(\Omega)\| &\leq c\varepsilon \|v; V_{\beta + \delta, 0}^2(\Omega)\|, \\ \|Q_{0j} w^j; V_{\beta + \delta, 0}^0(\Omega)\| &\leq c\varepsilon^{3/2 - \beta} \|w^j; V_{\beta - \delta}^2(\mathbb{R}^3)\| \end{aligned}$$

become evident. Taking into account the positions of the supports of commutator $[A_x, \chi_j^\nu]$ coefficients (cf. Lemma 2) we have

$$\begin{aligned} \|S_{0j} w^j; V_{\beta + \delta, 0}^0(\Omega)\| &\leq \\ &\leq c\varepsilon^{(1/2 - \beta) + (\beta + \delta - 1/2)} \|w^j; V_{\beta + \delta}^2(\{\xi^j \in \mathbb{R}^3: \varepsilon^{\nu-1} < |\xi^j| < 2\varepsilon^{\nu-1}\})\| \\ &\leq c\varepsilon^{(2\nu-1)\delta} \|w^j; V_{\beta - \delta}^2(\mathbb{R}^3)\|. \end{aligned}$$

The same estimates hold for ΠQ_{0h} and ΠS_{0j} because of Π being the projector in $V_{\beta+\delta,0}^0(\Omega)$. Besides, in virtue of (50), (41) and (42) the relations

$$\begin{aligned} |q_j w^j| &\leq c \varepsilon^{7/2-\beta} \int_{\{\xi \mid |\xi| < 2 \varepsilon^{\nu-1}\}} |w^j(\xi)| d\xi \leq \\ &\leq c \varepsilon^{7/2-\beta} \left(\int_0^{2 \varepsilon^{\nu-1}} (1+\rho)^{2(2+\delta-\beta)} \rho^2 d\rho \right)^{1/2} \|w^j; V_{\beta-\delta}^2(\mathbb{R}^3)\| \\ &\leq c \varepsilon \|w^j; V_{\beta-\delta}^2(\mathbb{R}^3)\| \end{aligned}$$

are valid.

The desired estimates for the norms of operators $\mathbb{S}_{j0}$ and $\mathbb{Q}_{jh}$ in (45) (or in (46) which is the same) are derived in a similar manner, $S_{i\ell}$ being equal to zero. In view of the support of the cut-off function we obtain

$$\begin{aligned} \|\mathbb{S}_{j0} v; V_{\beta-\delta}^0(\mathbb{R}^3)\| &\leq c \varepsilon^\delta \|v; V_{\beta-\delta,0}^2(\{x: \varepsilon^\sigma < r_j < 2 \varepsilon^\sigma\})\| \leq \\ &\leq c \varepsilon^{(1-2\sigma)\delta} \|v; V_{\beta-\delta,0}^2(\Omega)\|, \\ \|\mathbb{Q}_{j0} v; V_{\beta-\delta}^0(\mathbb{R}^3)\| &\leq c \varepsilon^{1+\delta} \|v; V_{\beta-\delta,0}^0(\Omega)\| \leq c \varepsilon^{1+\delta} \|v; V_{\beta-\delta,0}^2(\Omega)\|, \\ \|\mathbb{Q}_{jk} w^k; V_{\beta-\delta}^0(\mathbb{R}^3)\| &\leq c \varepsilon^3 \|w^k; V_{\beta-\delta}^0(\{\xi^k: |\xi^k| \leq 2 \varepsilon^{-1/2}\})\| \leq \\ &\leq c \varepsilon^2 \|w^k; V_{\beta-\delta}^2(\mathbb{R}^3)\|. \end{aligned}$$

We finish the proof by mentioning that all the constants in the out-coming inequalities depend neither on v , w^j nor on $\varepsilon \in (0, \varepsilon_0)$ and due to (42) all the exponents of ε are positive. ■

5. INVERSE REDUCTION

In the last section we reduced the initial problem to a vector equation. We are going now to deal with the direct opposite, namely, to find such vector equation (40) that for each solution $\{\mu(\varepsilon), U\}$ of it there exist the eigenvalue $\lambda(\varepsilon) = \varepsilon \mu(\varepsilon)$ and the eigenfunction

$$u'(\varepsilon, x) = X_0^\nu(\varepsilon, x) v(\varepsilon, x) + \varepsilon^{1/2-\beta} \sum_{j=1}^J \chi_j^\sigma(\varepsilon, x) w^j(\varepsilon, \varepsilon^{-1}(x - P_j)) \quad (54)$$

of problem (14), (4), v and $w^1, \dots, w^J$ being the entries of U . We should emphasize that the perturbation operators $\mathbb{S}$ and $\mathbb{Q}$ differ from the ones introduced above but we will use the same symbols for them, misunderstanding being excepted.

Let functions v and $w^1, \dots, w^J$ satisfy the equations

$$\begin{aligned} \Delta_x v(\varepsilon, x) = & -\varepsilon \mu(\varepsilon) \gamma_0 v(\varepsilon, x) - \\ & -\varepsilon^{1/2-\beta} \sum_{j=1}^J \left\{ [\Delta_x, \chi_j^\sigma(x)] w^j(\varepsilon, \varepsilon^{-1}(x - P_j)) \right. \\ & \left. + \varepsilon \mu(\varepsilon) \gamma_0 X_0^{1/2}(\varepsilon, x) (\chi_j^\sigma(\varepsilon, x) w^j(\varepsilon, \varepsilon^{-1}(x - P_j)) - \Gamma^{-1} \langle w^j \rangle_j) \right\}, x \in \Omega; \end{aligned} \quad (55)$$

$$\begin{aligned} \Delta_\xi w^j(\varepsilon, \xi^j) + \mu(\varepsilon) \gamma_j^0(\xi^j) \left\{ w^j(\varepsilon, \xi^j) - \Gamma^{-1} \sum_{k=1}^J \langle w^k \rangle_k \right\} = \\ = \varepsilon^{\beta-1/2} [\Delta_\xi, X_j^\nu(\varepsilon, \xi^j)] v(\varepsilon, P_j + \varepsilon \xi^j) - \\ - \varepsilon^3 \mu(\varepsilon) \gamma_{0j}'(\xi^j) X_j^{1/2}(\varepsilon, \xi^j) \left\{ w^j(\varepsilon, \xi^j) - \Gamma^{-1} \sum_{k=1}^J \langle w^k \rangle_k \right\}, \xi^j \in \mathbb{R}^3. \end{aligned} \quad (56)$$

Let us multiply (55) and (56) by $X_0^\nu(\varepsilon, x)$ and $\varepsilon^{-\beta-3/2} X_j^\sigma(\varepsilon, \xi^j)$ respectively, go over to coordinates x in formulae (56), $j = 1, \dots, J$, and sum them up. To this end, we add the result to the modified equality (55) and obtain

$$\begin{aligned} \Delta_x (X_0^\nu v) - [\Delta_x, X_0^\nu] v + \varepsilon^{1/2-\beta} \sum_{j=1}^J \left\{ \Delta_x (\chi_j^\sigma w^j) - [\Delta_x, \chi_j^\sigma] w^j \right\} + \\ + \mu(\varepsilon) \varepsilon^{1/2-\beta} \gamma_j^0 \left(w^j - \Gamma^{-1} \sum_{k=1}^J \langle w^k \rangle_k \right) = \\ = -\varepsilon \mu(\varepsilon) \gamma_0 X_0^\nu v - \varepsilon^{1/2-\beta} \sum_{j=1}^J \left\{ [\Delta_x, \chi_j^\sigma] w^j - \right. \\ \left. - \varepsilon^{3/2-\beta} \mu(\varepsilon) \gamma_0 X_0^{1/2} (\chi_j^\sigma w^j + \Gamma^{-1} \langle w^j \rangle_j) \right\} + \\ + \sum_{j=1}^J \left\{ [\Delta_x, \chi_j^\nu] v - \varepsilon^{3/2-\beta} \mu(\varepsilon) \gamma_{0j}' X_j^{1/2} \left(w^j + \Gamma^{-1} \sum_{k=1}^J \langle w^k \rangle_k \right) \right\}; \end{aligned}$$

the functions w^j and γ_{0j}', γ_j^0 having $\varepsilon^{-1}(x - P_j)$ as argument. In virtue of (54) this equality is transformed to (14) by mutual elimination of commutators and by gathering similar terms. Thus, (55) and (56) are the equations we need and rewriting them in forms (43) and (45) respectively we introduce the operators Q_{0h}, S_{0h} and Q_{jh}, S_{jh} (functionals q_j will be defined below). New estimates of the operator norms take the forms

$$\begin{aligned} \|Q_{00} v; V_{\beta+\delta, 0}^0(\Omega)\| & \leq c\varepsilon \|v; V_{\beta+\delta, 0}^0(\Omega)\|, \\ \|S_{0j} w^j; V_{\beta+\delta, 0}^0(\Omega)\| & \leq c\varepsilon^{(2\sigma-1)\delta} \|w^j; V_{\beta-\delta}^2(\mathbb{R}^3)\|, \\ \|Q_{0j} w^j; V_{\beta+\delta, 0}^0(\Omega)\| & \leq c\varepsilon^{3/2-\beta} \|w^j; V_{\beta-\delta}^2(\mathbb{R}^3)\|, \\ \|S_{j0} v; V_{\beta-\delta}^0(\mathbb{R}^3)\| & \leq c\varepsilon^{(1-2\nu)\delta} \|v; V_{\beta+\delta, 0}^0(\Omega)\|, \\ \|Q_{jk} w^k; V_{\beta-\delta}^0(\mathbb{R}^3)\| & \leq c\varepsilon^{3/2} \|w^k; V_{\beta-\delta}^2(\mathbb{R}^3)\|. \end{aligned}$$

The procedure we used to derive these estimates was the same as in section 4 but a change of positions of the support of commutator coefficient (compare (55), (56) with (44), (46)) forced us to demand δ to be small and negative, inequalities (42) and the inclusions $\beta \pm \delta \in (1/2, 3/2)$ having been preserved. After that all the exponents of ε in the last estimates become positive. It should be outlined particularly that according to assertions 3-5 the weight index modification performed above does not effect the properties of the limiting problem operators forming the matrix $\mathbb{T} + \mu \mathbb{P}$ in (52).

We, of course, subject the function v to conditions (4). Besides, we ought to find out a supplementary integral condition in order to treat the invertible mapping (34). Let F be the right-hand side of (55). The solvability condition for the Neumann problem (47) can be transformed into the relation

$$\varepsilon \mu(\varepsilon) \gamma_0 \int_{\Omega} v \, dx = -\varepsilon^{3/2-\beta} \sum_{j=1}^J \left\{ \mu(\varepsilon) \gamma_0 \int_{\Omega} X_0^{1/2} \chi_j^{\sigma} w^j \, dx - \mu(\varepsilon) \gamma_0 \langle w^j \rangle_j \int_{\Omega} X_0^{1/2} \, dx + \varepsilon^{-1} I_j \right\}$$

while due to (56) we have

$$\begin{aligned} \varepsilon^{-1} I_j &\equiv \int_{\mathbb{R}^3} [\Delta_{\xi}, X_j^{\sigma}] w^j \, d\xi^j = - \int_{\mathbb{R}^3} X_j^{\sigma} \Delta_{\xi} w^j \, d\xi^j = \\ &= -\varepsilon^{\beta-1/2} \int_{\mathbb{R}^3} [\Delta_{\xi}, X_j^{\nu}] v \, d\xi^j + \mu(\varepsilon) \left\{ \langle w^j \rangle_j - \Gamma^{-1} \Gamma_j \sum_{k=1}^J \langle w^k \rangle_k \right\} + \\ &\quad + \varepsilon^3 \mu(\varepsilon) \gamma_0 \int_{\mathbb{R}^3 \setminus \omega_j} X_j^{1/2} \left(w^j - \Gamma^{-1} \sum_{k=1}^J \langle w^k \rangle_k \right) d\xi^j. \end{aligned}$$

Only the first term in the right-hand side of (55) does not vanish on $\text{supp } \chi_j^{\nu}$ because of the last inequalities in (42). Therefore

$$\varepsilon \int_{\mathbb{R}^3} [\Delta_{\xi}, X_j^{\nu}] v \, d\xi^j = - \int_{\Omega} \chi_j^{\nu} \Delta v \, dx = \varepsilon \mu(\varepsilon) \gamma_0 \int_{\Omega} \chi_j^{\nu} v \, dx.$$

Picking the three obtained formulae up we conclude that condition (47) is equivalent with the relation

$$\begin{aligned} \mu(\varepsilon) \int_{\Omega} X_0^{\nu} v \, dx &\equiv \mu(\varepsilon) \pi_X v = \mu(\varepsilon) \sum_{j=1}^J q_j w^j \equiv \\ &\equiv -\mu(\varepsilon) \varepsilon^{7/2-\beta} \sum_{j=1}^J \left\{ \int_{\mathbb{R}^3 \setminus \omega_j} X_j^{\sigma} w^j \, d\xi^j + \Gamma^{-1} \langle w^j \rangle_j \sum_{k=1}^J \text{mes}_3 \omega_k \right\} \quad (57) \end{aligned}$$

which looks like (50), the estimate

$$|q_j w^j| \leq c \varepsilon^{-\delta+2\sigma} \|w^j; V_{\beta-\delta}^2(\mathbb{R}^3)\|$$

with the positive exponent of ε holding true.

Introducing the main operator $\mathbb{T} + \mu \mathbb{D}$ in (40) by (52) with the upper left-hand corner element defined according to (54) where π_h equals π_X in (57), we outline that $\pi_X 1 > \text{const} > 0$ as $\varepsilon \rightarrow 0$ and recall the last assertion of Corollary 4. To complete the construction of the perturbation operators $\mathbb{S}$, $\mathbb{Q}$ we refer to the « old » formula (51) with « new » S_{0h} , Q_{0h} and q_j .

LEMMA 7 : *A solution $\{\mu(\varepsilon), U\}$ of equation (40) gives rise to a solution $\{\lambda(\varepsilon), u'\}$ of problem (14), (4). Operators $\mathbb{S}$ and $\mathbb{Q}$ are subject to (53) where $\beta \in (1/2, 3/2)$, $\kappa > 0$ and δ is small negative. For any $\rho > 0$ one can find positive ε_ρ such as in the case $\varepsilon \in (0, \varepsilon_\rho)$ and $|\mu(\varepsilon)| < \rho$ function (54) is non-trivial provided $U \neq 0$.*

Proof : It is only the last fact which has not been verified. By virtue of (54) $u' = 0$ in Ω if and only if

$$\text{supp } v \subset \bigcup_{j=1}^J \{x : r_j < 2\varepsilon^\nu\}, \quad \text{supp } w^j \subset \{\xi^j : |\xi^j| > \varepsilon^{\sigma-1}\}.$$

In this case the Neumann conditions for v can be replaced by the Dirichlet ones and in (56) the term containing γ_j^0 vanishes. That is why the functions $v \in V_{\beta+\delta,0}^2(\Omega)$ and $w^j \in V_{\beta-\delta}^2(\mathbb{R}^3)$ are solutions of the Dirichlet problem in Ω and the Laplace equation in $\mathbb{R}^3$ (with slight perturbations of operators since $|\mu(\varepsilon)| < \rho$). Therefore $v = 0$ and $w^j = 0$ because of the uniqueness of the solution of these problems for $\beta \pm \delta \in (1/2, 3/2)$ (cf. Proposition 3). ■

Remark 8 : Looking through the transformations made up in this section one can conclude that if for an eigenvector $U^0 (\neq 0)$ the equation

$$(\mathbb{T} + \mathbb{S}) U^1 + \mu(\varepsilon)(\mathbb{P} + \mathbb{Q}) U^1 = -(\mathbb{P} + \mathbb{Q}) U^0$$

has a solution U^1 , then the formula (54) gives non-trivial Jordan chain $\{u^{0'}, u^{1'}\}$ corresponding to the eigenvalue $\lambda(\varepsilon) = \varepsilon \mu(\varepsilon)$ of problem (14), (4). This is impossible and, hence, there are no associated vector of equation (40).

6. SOLVABILITY OF THE INITIAL PROBLEM

Here we show, in particular, that if $\Lambda = \varepsilon^{-1} \lambda$ is not an eigenvalue of the bundle (52) then the operator

$$\mathcal{N}_\beta(\lambda) \equiv \{\Delta_x + \lambda \gamma, \partial_n\} : V_{\beta,\varepsilon}^2(\Omega) \rightarrow V_{\beta,\varepsilon}^0(\Omega) \times H^{1/2}(\partial\Omega) \quad (58)$$

of problem (3), (4) is an isomorphism. We present also asymptotically sharp estimates of solutions in the weighted spaces and an almost inverse to $\mathcal{N}_\beta(\lambda)$ is an explicit form. Namely, we shall construct the operator

$$\mathcal{R}_\beta(\lambda) : V_{\beta, \varepsilon}^0(\Omega) \times H^{1/2}(\partial\Omega) \rightarrow V_{\beta, \varepsilon}^2(\Omega) \quad (59)$$

such that

$$\|\mathcal{N}_\beta(\lambda) \mathcal{R}_\beta(\lambda) - 1; V_{\beta, \varepsilon}^0(\Omega) \times H^{1/2}(\partial\Omega)\| \leq c\varepsilon^\kappa \quad (60)$$

with $\kappa > 0$. Thus, $\mathcal{N}_\beta(\lambda)^{-1} = \mathcal{R}_\beta(\lambda)[\mathcal{N}_\beta(\lambda) \mathcal{R}_\beta(\beta)]^{-1}$ as $\varepsilon \in (0, \varepsilon_A)$ with some positive ε_A depending on Λ .

LEMMA: *If Λ is not an eigenvalue of system (8) then the equation $L_\beta(\Lambda)W = -\Lambda \{\gamma_1^0, \dots, \gamma_J^0\}$ has the solution W_0 , an entry W_0^l of which admits the representation of type (21) with the coefficient $b_j \equiv b_j^0$,*

$$b_j^0 = \Lambda(\langle W_0^l \rangle_j + \Gamma_j - \Gamma^{-1} \Xi \Gamma_j), \quad \Xi = \sum_{k=1}^J \langle W_0^k \rangle_k. \quad (61)$$

The vector-function $W = W_0 + 1 - \Gamma^{-1} \Xi$ satisfies

$$\Delta_\xi W^j(\xi^j) + \Lambda \gamma_j^0(\xi^j) W^j(\xi^j) = 0, \quad \xi^j \in \mathbb{R}^3, \quad (62)$$

$$\tilde{W}^j(\xi) \equiv W^j(\xi) - 1 + \Gamma^{-1} \Xi - b_j^0 (4\pi |\xi|)^{-1} = O(|\xi|^{-2}), \quad |\xi| \rightarrow \infty.$$

Proof: The calculation of b_j^0 follows formulae (22) with evident modifications, properties of W being a consequence of the definition of W_0 . ■

Let $f \in V_{\beta, \varepsilon}^0(\Omega)$ with $\beta \in (1/2, 3/2)$. We put

$$f^0 = X_0^{1/2} f, \quad f^j(\varepsilon, \xi^j) = \varepsilon^{\beta+3/2} X_j^{1/2}(\varepsilon, \xi^j) f(\varepsilon, P_j + \varepsilon \xi^j). \quad (63)$$

It is clear that

$$f(\varepsilon, x) = f^0(\varepsilon, x) + \varepsilon^{-\beta-3/2} \sum_{j=1}^J f^j(\varepsilon, \varepsilon^{-1}(x - P_j)).$$

Besides, due to Lemma 2, 1° taking account of the positions of $\text{supp } f^0$ and $\text{supp } f^j$ we have

$$\|f^0; V_{\beta-\alpha, 0}^0(\Omega)\| + \|f^j; V_{\beta+\alpha}^0(\mathbb{R}^3)\| \leq c\varepsilon^{-\alpha/2} \|f; V_{\beta, \varepsilon}^0(\Omega)\| \quad (64)$$

with a small negative α . Let us consider the problem

$$\begin{cases} \Delta_x u(\varepsilon, x) + \varepsilon \Lambda \gamma(\varepsilon, x) u(\varepsilon, x) = f(\varepsilon, x), & x \in \Omega, \\ \partial_n u(\varepsilon, x) = g(\varepsilon, x), & x \in \partial\Omega, \end{cases} \quad (65)$$

and search for the approximate solution $\hat{u}$ in the form

$$\begin{aligned} \hat{u}(\varepsilon, x) = \varepsilon^{-1} \mathbb{C} \sum_{j=1}^J W^j(\varepsilon^{-1}(x - P_j)) + X_0^\nu(\varepsilon, x) \hat{v}(\varepsilon, x) + \\ + \varepsilon^{-\beta+1/2} \sum_{j=1}^J (\chi_j^\sigma(\varepsilon, x) \hat{w}^j(\varepsilon, \varepsilon^{-1}(x - P_j)) + \langle w^j \rangle_j) \end{aligned} \quad (66)$$

where $\hat{v} \in V_{\beta-\alpha, 0}^2(\Omega) \subset V_{\beta, 0}^2(\Omega)$ and

$$\hat{w} = \{\hat{w}^1, \dots, \hat{w}^J\} \in V_{\beta+\alpha}^2(\mathbb{R}^3)^J \subset V_{\beta}^2(\mathbb{R}^3)$$

are solutions of limiting problems (32) and (8) with the right-hand sides $\{\hat{f}, \hat{g}\}$ and $\{f^1, \dots, f^J\}$ respectively,

$$\hat{f} = f_0 - \Lambda \mathbb{C} \gamma_0 (1 - \Gamma^{-1} \Xi), \quad \hat{g} = g - \mathbb{C} \sum_{j=1}^J b_j^0 \partial_n (4 \pi r_j)^{-1}. \quad (67)$$

Assuming $\beta \pm \alpha \in (1/2, 3/2)$ we get a solution $\hat{w}$ and transform Neumann problem compatibility condition into

$$\mathbb{C} \Lambda (1 - \Gamma^{-1} \Xi) \Gamma_0 + \mathbb{C} \sum_{j=1}^J b_j^0 \int_{\partial \Omega} \partial_n (4 \pi r_j)^{-1} ds_x = \int_{\Omega} f^0 dx - \int_{\partial \Omega} g ds_x.$$

We apply (61), (25) and reduce the last relation to

$$\mathbb{C} = \Lambda^{-1} \Gamma^{-1} \left(\int_{\Omega} f^0 dx - \int_{\partial \Omega} g ds_x \right). \quad (68)$$

Note that under (67) Neumann problem in question has a unique solution $\hat{v}$ subjected to $\int \hat{v} ds_x = 0$. By virtue of Proposition 3 the inclusions $\hat{v} \in V_{\beta-\alpha, 0}^2(\Omega)$, $\hat{w} \in V_{\beta+\alpha}^2(\mathbb{R}^3)^J$ hold for arbitrary $\alpha \in [0, \alpha_0)$ where α_0 is small positive. Moreover, by means of (67), (63) and (64) the inequality

$$\begin{aligned} |\mathbb{C}| + \varepsilon^{\alpha/2} \|\hat{v}; V_{\beta-\alpha, 0}^2(\Omega)\| + \varepsilon^{\alpha/2} \|\hat{w}; V_{\beta+\alpha}^2(\mathbb{R}^3)^J\| \leq \\ \leq c (\|f; V_{\beta, \varepsilon}^2(\Omega)\| + \|g; H^{1/2}(\partial \Omega)\|) \equiv c \|f, g\| \end{aligned} \quad (69)$$

is obtained where c is a constant depending neither on f, g nor on $\varepsilon \in (0, \varepsilon_0)$.

We ought now to estimate discrepancies produced in (63) by approximate solution (66),

$$\Delta_\varepsilon \hat{u} + \varepsilon \Lambda \gamma \hat{u} = f + \Psi_c + \Psi_v + \Psi_w \text{ in } \Omega, \quad \partial_n \hat{u} = g + \psi_c \text{ on } \partial \Omega.$$

Recalling (66) and (62), (63), (67) we obtain for the discrepancy terms the following expressions

$$\begin{aligned}\psi_c(\varepsilon, x) &= \varepsilon^{-1} \mathbb{C} \sum_{j=1}^J \partial_n \tilde{W}^j(\varepsilon^{-1}(x - P_j)), \\ \Psi_c(\varepsilon, x) &= \mathbb{C} \Lambda \gamma_0^0 \sum_{j=1}^J W_0^j(\varepsilon^{-1}(x - P_j)), \\ \Psi_v(\varepsilon, x) &= \varepsilon \Lambda \gamma_0 X_0''(\varepsilon, x) \hat{v}(\varepsilon, x) + [\Delta_x, X_0''(\varepsilon, x)] \hat{v}(\varepsilon, x), \\ \Psi_w(\varepsilon, x) &= \varepsilon^{\beta-1/2} \sum_{j=1}^J \left\{ [\Delta_x, \chi_j^\sigma(\varepsilon, x)] \hat{w}^j(\varepsilon^{-1}(x - P_j)) + \right. \\ &\quad \left. + \varepsilon \Lambda \gamma_0^0 (\chi_j^\sigma(\varepsilon, x) \hat{w}^j(\varepsilon^{-1}(x - P_j)) + \langle \hat{w}^j \rangle_j) \right\}.\end{aligned}$$

Appealing to (62), (17) and (69) we find

$$\|\psi_c; H^{1/2}(\partial\Omega)\| \leq c\varepsilon \|f, g\|.$$

In order to estimate the other terms we use the same approach as in the previous sections. For example,

$$\begin{aligned}\|\Psi_v; V_{\beta, \varepsilon}^0(\Omega)\| &\leq c \left\{ \varepsilon \|\hat{v}; V_{\beta+2, \delta}^2(\Omega)\| + \right. \\ &\quad \left. + \sum_{j=1}^T \|\hat{v}; V_{\beta, \varepsilon}^2(\{x: \varepsilon^\nu < r_j < 2\varepsilon\})\| \right\} \\ &\leq c\varepsilon^{\nu\alpha} \|\hat{v}; V_{\beta-\alpha, 0}^2(\Omega)\| \leq c\varepsilon^{\nu\alpha} (\|f^0; V_{\beta-\alpha, 0}^0(\Omega)\| \\ &\quad + \|g; H^{1/2}(\partial\Omega)\|) \leq c\varepsilon^{(\nu-1/2)\alpha} \|f, g\|.\end{aligned}$$

Finally,

$$\|\Psi_c; V_{\beta, \varepsilon}^0(\Omega)\| \leq c\varepsilon \|f, g\|, \quad \|\Psi_w; V_{\beta, \varepsilon}^0(\Omega)\| \leq c\varepsilon^{\alpha(1/2-\sigma)} \|f, g\|.$$

Thus, collecting these estimates and taking (42) into account we obtain the desired inequality (60) by denoting $\mathcal{R}_\beta(\varepsilon\Lambda) \{f, g\} \equiv \hat{u}$.

From Lemma 2 and formulae (66), (69) with $\alpha = 0$ it follows that the operator (59) norm does not exceed $c\varepsilon^{-1}$. Since the norm of $\mathcal{N}_\beta(\varepsilon\Lambda) - \mathcal{N}_\beta(\varepsilon\mu)$ (see (58)) is majorized by $c\varepsilon |\Lambda - \mu|$, we have proved the following assertion.

THEOREM 10: *If $\Lambda \neq 0$ is not an eigenvalue of system (8) then there exist positive $\tilde{\varepsilon}, \tilde{\rho}$ such that under $\varepsilon \in (0, \tilde{\varepsilon})$ and $|\lambda - \varepsilon\Lambda| < \varepsilon\tilde{\rho}$ operator (58) is an isomorphism. The norms of $\mathcal{N}_\beta(\lambda)$ and $\mathcal{N}_\beta(\lambda)^{-1}$ do not exceed c and $c\varepsilon^{-1}$ respectively, the constant c not depending on ε and λ under the above-mentioned restrictions.*

Remark 11 : The norm $\mathcal{N}_\beta(\lambda)^{-1}$ estimate is asymptotically sharp in the $V_{\beta, \varepsilon}^\ell$ -space scale. It is the first term in the right-hand side of (66) which gives the norm increase as $\varepsilon \rightarrow 0$. One can modify the estimate by handling this term separately.

7. JUSTIFICATION OF THE ASYMPTOTICS

First of all, we recall the equivalence of problems (3), (4) and (14), (4) (the transition from one to the other is realized by (13) or (15)). Besides, zero is an eigenvalue of both problems, the same holding true for the vector equations (40).

The spectrum $\{\mu_0 = 0, \mu_1, \mu_2, \dots\}$ of bundle (52) does not depend on a small δ . Let μ_ℓ be an multiplicity ℓ eigenvalue. Since there are no associated vectors, the estimate

$$\|(\mathbb{T} + \mu \mathbb{P})^{-1} : \mathbb{R}_{\beta, \delta} \rightarrow \mathbb{D}_{\beta, \delta}\| \leq c |\mu - \mu_\ell|^{-1}$$

holds while $0 < |\mu - \mu_*| < \rho_*$ with a positive ρ_* . Thus, in virtue of the theorem on the total algebraic multiplicity stability (see [15, 5]) we conclude, on account of Lemmas 6 and 7, that in both the cases (Sect. 4 and 5) there exist positive numbers ε_* , d_* and t_* such that for $\varepsilon \in (0, \varepsilon_*)$ the eigenvalues of equation (40) laying on the disc

$$B(\mu_*; d) = \{\mu \in \mathbb{C} : |\mu - \mu_*| < d\} \quad (70)$$

belong to the disk $B(\mu_*; t_* \varepsilon^\kappa)$ where $\kappa > 0$ and have total algebraic multiplicity equal to ℓ . The first reduction performed in Section 4 shows that the quantity of numbers $\varepsilon^{-1} \lambda_k(\varepsilon)$ placed in (70) does not exceed ℓ and each of them hits into $B(\mu_*; t_* \varepsilon^\kappa)$. The second reduction together with Remark 8 shows that the same quantity is not smaller than ℓ . Thus, it is equal to ℓ .

The obtained result and Theorem 10 state that for any $\tilde{t} > 0$ there exists a positive $\tilde{\varepsilon}$ such as for $\varepsilon \in (0, \tilde{\varepsilon})$ the total multiplicity of problem (14), (4) eigenvalues in $B(0; \varepsilon \tilde{t})$ coincides with the multiplicity of bundle (52) eigenvalues in $B(0, \tilde{t})$. Therefore we have proved

THEOREM 12 : *The set (11) corresponding to the eigenvalue sequence (6) of problem (3), (4) (or (14), (4)) coincide with the spectrum of the bundle (35). The estimate*

$$|\lambda_k(\varepsilon) - \varepsilon \mu_k| \leq c_k \varepsilon^{1+\kappa}$$

(with positive κ and the constant c_k depending on k) holds.

Remark 13 :

1) To shorten the paper we avoid touching upon convergence of eigenfunctions.

2) When indicating the properties of operators (30), (31) (Proposition 3) we did not use selfadjointness of problems with exception of the end of Theorem 5 and the last assertion in Remark 8 : lack of both the facts having no influence over reasoning in general. Thus, the approach under consideration can be applied for non-symmetric operators since the information on corresponding mappings of type (30), (31) is available (*cf.* [9-11]).

ACKNOWLEDGEMENTS

I thank Professors J. Sanchez-Hubert and E. Sanchez-Palencia for my visit to France and especially for useful and fruitful discussions that prompted me to finish this paper-work.

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