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ESTIMATION OF THE CONDUCTIVITY IN THE ONE-PHASE STEFAN PROBLEM : NUMERICAL RESULTS (*)

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Abstract. — *In this paper we develop an iterative algorithm to estimate the time dependent diffusion coefficient from boundary data in a one-phase Stefan problem. It uses a boundary integral representation of the solution of a free boundary value problem. Convergence of this algorithm is established and numerical results are included.*

Résumé. — *Dans ce travail, nous développons une méthode itérative pour l'estimation d'un coefficient dépendant du temps dans un problème Stefan d'une phase utilisant des dates aux frontières. Cette méthode utilise la représentation aux frontières de la solution d'un problème de frontière libre. La convergence de l'algorithme est prouvée et des exemples numériques sont inclus.*

1. INTRODUCTION

We continue our study of the estimation of an unknown time-dependent conductivity coefficient in the Stefan problem :

$$\begin{aligned} &u_t = a(t)\, u_{xx} && 0 < t \leqslant T\,, \quad 0 < x < s(t), \\ &a(t)\, u_x(0, t) = g(t) && 0 < t \leqslant T\,, \\ &u(s(t), t) = 0 && 0 \leqslant t \leqslant T\,, \\ &u(x, 0) = \phi(x) && 0 \leqslant x \leqslant b\,, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\dot{s}(t) = -\, a(t)\, u_x(s(t), t) \quad 0 < t \leqslant T\,, \\ &s(0) = b\,. \end{aligned} \tag{1.2}$$

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Here we concentrate on the numerical solution of the inverse problem, building on the basic results presented in [8]. We assume that the initial value $b > 0$ and the functions g and ϕ are known. We shall suppose throughout that the functions g and ϕ satisfy :

$$g \in C[0, T], \quad \text{with} \quad g(t) \leq 0,\ t \in [0, T],$$
$$\phi \in W^{1,\infty}(0, b), \quad \text{with} \quad \phi(b) = 0,\ \phi(x) > 0,\ x \in [0, b),$$
$$\phi' \text{ continuous at } x = b,$$

and that the coefficient a belongs to the set :

$$\mathscr{A} = \{a \in C[0, T] \mid 0 < \nu \leq a(t) \leq \mu,\ t \in [0, T]\},$$

where ν and μ are given. The forward problem consists of determining, for $a \in \mathscr{A}$, $s(a) \in C[0, T]$ and $u(a) \in C(\Omega_a)$ where $\Omega_a = \{(x, t) \mid 0 < x < s(t; a),\ 0 < t < T\}$, such that the pair s, u is a (classical) solution of (1.1), (1.2). In the inverse, or parameter estimation problem, we suppose that we have observed the system modeled by the above equations to obtain (perhaps only partial) information about s and/or u, and wish to determine the coefficient a.

We treat the parameter estimation problem in the setting of output-least-squares. Let Z be a normed linear space (the observation space), and let $\mathscr{P}_a : C[0, T] \times C(\bar{\Omega}_a) \mapsto Z$ be the observation operator. We assume that we have an observation $z \in Z$, corresponding to the solution of the Stefan problem evaluated at the « true » coefficient a^*. The least-squares functional is given by

$$J(a) = |\mathscr{P}_a(s(a), u(a)) - z|_Z^2,$$

with $(s(a), u(a))$ the solution of the Stefan problem corresponding to the parameter a. The determination of the unknown coefficient is based on the nonlinear least-squares problem :

$$\text{(ID)} \qquad \min J(a) \quad \text{over} \quad a \in \tilde{\mathscr{A}}$$

where $\tilde{\mathscr{A}}$ is a compact (in $C[0, T]$) subset of $\mathscr{A}$. We discuss several examples of the cost functional J below.

As the problem (ID) is infinite dimensional, we are interested in replacing it by a finite dimensional discretized version. Then we may iterate on the unknown parameter, at each step solving the discretized version of the Stefan problem.

In [8] we laid the theoretical groundwork. In order that this paper be self-contained, we summarize the relevant results in section 2, referring the interested reader to [8] for details and proofs.

In section 3 we present an algorithm for the numerical solution of (ID), based on the discretization of an equivalent integral representation of equation (1.1) combined with an iteration on the boundary function. We analyze this algorithm rigorously in section 4. Finally, in section 5, we discuss the implementation and in section 6 we present numerical examples and an alternative method to solve (ID).

2. SUMMARY OF THEORETICAL RESULTS

Here we summarize some facts concerning (1.1), (1.2) that were established in [8], supplementing them with a few results necessary for the analysis of the numerical approximations to be discussed below. We note that while the one dimensional Stefan problem has been well studied (see, e.g., [5] and [6]), earlier work has not focused on the conductivity parameter, and in fact this coefficient is often assumed to be 1 ; we have established a priori estimates for the solution of (1.1) and (1.2) uniformly in a.

2.1. Equivalent integral formulation

We express the solution of system (1.1) in terms of integral equations. First we study equations (1.1) independently of equations (1.2). For this purpose, we define for given $K > 0$ the set

$$\mathcal{S}_{T,K} = \{s \in W^{1,\infty}(0, T) \mid 0 \leq \dot{s}(t) \leq K \text{ for a.a. } t \in [0, T],\ s(0) = b\} ;$$

we shall fix K below. For each $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$ we can solve (1.1) to obtain u. We emphasize that we are now considering u as a solution of (1.1), for fixed but arbitrary $s \in \mathcal{S}_{T,K}$, so that the pair (u, s) is in general not a solution of the Stefan problem. For the reformulation of the problem in terms of integral equations, we define the fundamental solution Γ:

$$\Gamma(x, t ; \xi, \tau) = \frac{1}{2\sqrt{\pi(a(t) - \alpha(\tau))}} \exp\left(\frac{-(x-\xi)^2}{4(\alpha(t) - \alpha(\tau))}\right),$$

with $\alpha(t) := \int_0^t a(\tau)\, d\tau$ and the Green and Neumann functions (depending on a) :

$$\begin{aligned} G(x, t ; \xi, \tau) &= \Gamma(x, t ; \xi, \tau) - \Gamma(-x, t ; \xi, \tau), \\ N(x, t ; \xi, \tau) &= \Gamma(x, t ; \xi, \tau) + \Gamma(-x, t ; \xi, \tau). \end{aligned}$$

It was shown in [8] that for each $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$, solving equation (1.1) for u is equivalent to solving the following integral equation for v

$$\frac{1}{2} v(t\,;a,s) =$$

$$= \int_0^b G(s(t), t\,; \xi, 0)\, \phi'(\xi)\, d\xi - \int_0^t N_x(s(t), t\,; 0, \tau)\, g(\tau)\, d\tau +$$

$$+ \int_0^t N_x(s(t), t\,; s(\tau), \tau)\, a(\tau)\, v(\tau\,; a, s)\, d\tau\,, \tag{2.1}$$

and then evaluating u according to :

$$u(x, t\,; a, s) = \int_0^b N(x, t\,; \xi, 0)\, \phi(\xi)\, d\xi - \int_0^t N(x, t\,; 0, \tau)\, g(\tau)\, d\tau +$$

$$+ \int_0^t N(x, t\,; s(\tau), \tau)\, a(\tau)\, v(\tau)\, d\tau\,. \tag{2.2}$$

Let us write equation (2.1) in the form

$$(\mathfrak{I} - \mathcal{K}(a, s))\, v(a, s) = \mathcal{G}(a, s)\,, \tag{2.3}$$

where for each $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$, we set

$$\mathcal{G}(t\,; a, s) := 2 \int_0^b G(s(t), t\,; \xi, 0)\, \phi'(\xi)\, d\xi -$$

$$- 2 \int_0^t N_x(s(t), t\,; 0, \tau)\, g(\tau)\, d\tau\,,$$

and $\mathcal{K}(a, s) : C[0, T] \to C[0, T]$ is defined by

$$[\mathcal{K}(a, s)\, x]\,(t) = \begin{cases} \int_0^t K(t, \tau\,; a, s)\, x(\tau)\, d\tau\,, & t > 0 \\ 0\,, & t = 0\,, \end{cases}$$

with

$$K(t, \tau\,; a, s) := 2\, a(\tau)\, N_x(s(t), t\,; s(\tau), \tau)\,.$$

PROPOSITION 2.1 : *$\{\mathcal{K}(a, s) \mid (a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}\}$ is collectively compact in $C[0, T]$.*

Proof : It suffices to show that

$$\{\mathcal{K}(a, s)\, f \mid (a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}, f \in C([0, T]), |f|_\infty \leq 1\}$$

is equicontinuous. Without loss of generality, choose $0 < t < \tilde{t} \leq T$ and consider

$$\left| [\mathcal{K}(a,s) f](\tilde{t}) - [\mathcal{K}(a,s) f](t) \right| \leq$$

$$\leq \int_0^t \left| [K(\tilde{t}, \tau; a, s) - K(t, \tau; a, s)] f(\tau) \right| d\tau + \\ + \int_t^{\tilde{t}} \left| K(\tilde{t}, \tau; a, s) f(\tau) \right| d\tau \leq$$

$$\leq 2 \mu |f|_\infty \int_0^t |N_x(s(\tilde{t}), \tilde{t}; s(\tau), \tau) - N_x(s(t), t; s(\tau), \tau)| d\tau + \\ + 4 \mu \tilde{\kappa} |f|_\infty (\tilde{t} - t)^{1/2},$$

where we used (2.15) in [8]. We split the first term

$$\int_0^t \left| N_x(s(\tilde{t}), \tilde{t}; s(\tau), \tau) - N_x(s(t), t; s(\tau), \tau) \right| d\tau \leq$$

$$\leq \frac{1}{4\sqrt{\pi}} \int_0^t \Bigg| (\alpha(\tilde{t}) - \alpha(\tau))^{-3/2} (s(\tilde{t}) - s(\tau)) e^{-\frac{(s(\tilde{t}) - s(\tau))^2}{4(\alpha(\tilde{t}) - \alpha(\tau))}} - (\alpha(t) - \alpha(\tau))^{-3/2} (s(t) - s(\tau)) e^{-\frac{(s(t) - s(\tau))^2}{4(\alpha(t) - \alpha(\tau))}} \Bigg| d\tau$$

$$+ \frac{1}{4\sqrt{\pi}} \int_0^t \Bigg| (\alpha(\tilde{t}) - \alpha(\tau))^{-3/2} (s(\tilde{t}) + s(\tau)) e^{-\frac{(s(\tilde{t}) + s(\tau))^2}{4(\alpha(\tilde{t}) - \alpha(\tau))}} - (\alpha(t) - \alpha(\tau))^{-3/2} (s(t) + s(\tau)) e^{-\frac{(s(t) + s(\tau))^2}{4(\alpha(t) - \alpha(\tau))}} \Bigg| d\tau = I + II.$$

For each $\tau \in [0, T]$, define $g_\tau(t)$, for $t \in [0, T]$ by

$$g_\tau(t) = \begin{cases} 0, & t \leq \tau \\ (\alpha(t) - \alpha(\tau))^{-3/2} (s(t) + s(\tau)) e^{-\frac{(s(t) + s(\tau))^2}{4(\alpha(t) - \alpha(\tau))}}, & t > \tau, \end{cases}$$

and observe that there is a constant C_1, independent of τ, t, $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$ such that

$$\left| \frac{d}{dt} g_\tau(t) \right| \leq C_1 \quad t \in [0, T].$$

Hence we infer

$$II \leqslant \frac{1}{4\sqrt{\pi}} TC_1(\tilde{t} - t) .$$

Term I is dealt with as follows

$$4\sqrt{\pi}\, I \leqslant \int_0^t \left| (\alpha(\tilde{t}) - \alpha(\tau))^{-3/2} - (\alpha(t) - \alpha(\tau))^{-3/2} \right| |s(t) - s(\tau)| \, d\tau +$$

$$+ \int_0^t (\alpha(\tilde{t}) - \alpha(\tau))^{-3/2} |s(t) - s(\tilde{t})| \, d\tau +$$

$$+ \int_0^t (\alpha(\tilde{t}) - \alpha(\tau))^{-3/2} (s(\tilde{t}) - s(\tau)) \left| e^{-\frac{(s(\tilde{t}) - s(\tau))^2}{4(\alpha(\tilde{t}) - \alpha(\tau))}} - e^{-\frac{(s(t) - s(\tau))^2}{4(\alpha(t) - \alpha(\tau))}} \right| d\tau$$

$$= I1 + I2 + I3 .$$

For any $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$ one obtains the bound

$$I2 \leqslant 2\, \nu^{-3/2} K(\tilde{t} - t)^{1/2} .$$

The mean value theorem implies

$$I3 \leqslant K\nu^{-3/2} \frac{1}{4} \int_0^t (\tilde{t} - \tau)^{-1/2} \left| \frac{(s(t) - s(\tau))^2}{\alpha(t) - \alpha(\tau)} - \frac{(s(\tilde{t}) - s(\tau))^2}{\alpha(\tilde{t}) - \alpha(\tau)} \right| d\tau .$$

Using an argument similar to the one used in the discussion of II one concludes the existence of a constant C_3 independent of $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$ such that

$$I3 \leqslant C_3(\tilde{t} - t)^{1/2} .$$

Applying (3.18) in [8] to $I1$ one eventually arrives at

$$I1 \leqslant \frac{2K\mu^3}{\nu^{9/2}} (\tilde{t} - t) \int_0^t \frac{(\tilde{t} - \tau)^2 + (t - \tau)^2}{(t - \tau)^{1/2} (\tilde{t} - \tau)^{3/2} ((t - \tau)^{3/2} + (\tilde{t} - \tau)^{3/2})} d\tau$$

$$\leqslant \frac{2K\mu^3}{\nu^{9/2}} (\tilde{t} - t) [2(\tilde{t} - t)^{-1/2} +$$

$$\left. + \int_0^t \frac{(\tilde{t} - \tau)}{(t - \tau)^{1/2} (\tilde{t} - \tau)^{1/2} ((t - \tau)^{3/2} + (\tilde{t} - \tau)^{3/2})} d\tau \right] .$$

As to the last integral, we argue

$$\int_0^t \frac{(\tilde{t} - \tau)\, d\tau}{(t - \tau)^{1/2}(\tilde{t} - \tau)^{1/2}((t - \tau)^{3/2} + (\tilde{t} - \tau)^{3/2})} \leq$$

$$\leq (\tilde{t} - t)^{-1/2} \int_0^t \frac{d\tau}{(t - \tau)^{1/2}(\tilde{t} - \tau)^{1/2}}$$

$$+ \int_0^t \left(\frac{t - \tau}{\tilde{t} - \tau} \right)^{1/2} \frac{d\tau}{(t - \tau)^{3/2} + (\tilde{t} - \tau)^{3/2}}$$

$$= I11 + I12 .$$

Hölder's Inequality gives ($p < 2$)

$$I11 \leq \left(\frac{2}{2 - p} \right)^{1/p} \left(\frac{2}{q - 2} \right)^{1/q} T^{\frac{1}{p} - \frac{1}{2}} (\tilde{t} - t)^{\frac{1}{q} - 1} .$$

Introducing the new variable of integration $\xi = \dfrac{t - \tau}{\tilde{t} - \tau}$ in $I12$ results in

$$I12 = (\tilde{t} - t)^{-1/2} \int_0^{\frac{t}{\tilde{t}}} \frac{\xi^{1/2}(1 - \xi)^{-1/2}}{1 + \xi^{3/2}}\, d\xi \leq (\tilde{t} - t)^{-1/2} \int_0^1 \xi^{1/2}(1 - \xi)^{-1/2}\, d\xi .$$

Combine the above estimates to obtain

$$I1 \leq C_4 (\tilde{t} - t)^{1/q}$$

for some constant $C_4 > 0$ independent of $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$. □

It was shown in [8] that $\lim_{t \to 0} [\mathscr{G}(a, s)](t) = \phi'(b)$ for any $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$, so that we may continuously extend $\mathscr{G}(a, s)$ to $t = 0$, and hereafter consider $\mathscr{G}(a, s) \in \mathscr{C}[0, T]$ for each $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$. We shall later make use of the following result.

PROPOSITION 2.2 : *Let $\tilde{\mathscr{A}}$ be a compact subset of $\mathscr{A}$. Then the set $\{\mathscr{G}(a, s) \mid (a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T,K}\}$ has compact closure in $C[0, T]$.*

Proof : We consider $\mathscr{G}(a, s) = \mathscr{G}_1(a, s) + \mathscr{G}_2(a, s)$ in two parts, with the following definitions :

$\mathscr{G}_1(a, s) :=$

$$:= \int_0^b \frac{\phi'(\xi)}{\sqrt{\pi \alpha(t)}} \left(\exp\left(\frac{-(s(t) - \xi)^2}{4\alpha(t)} \right) - \exp\left(\frac{-(s(t) + \xi)^2}{4\alpha(t)} \right) \right) d\xi$$

and

$$\mathscr{G}_2(a, s) := \int_0^t \frac{g(\tau)\, s(t)}{\sqrt{\pi}(\alpha(t) - \alpha(\tau))^{3/2}} \exp\left(\frac{-s(t)^2}{4(\alpha(t) - \alpha(\tau))} \right) d\tau .$$

Let us begin with $\mathcal{G}_2$. From estimates in [8] (see proofs of Lemma 3.3, 3.7) it follows that for any $a_1, a_2 \in \tilde{\mathcal{A}}$ and $s_1, s_2 \in \mathcal{S}_{T,K}$ we have

$$|\mathcal{G}_2(a_1, s_1) - \mathcal{G}_2(a_2, s_2)|_\infty \leq C(|a_1 - a_2|_\infty + |s_1 - s_2|_\infty),$$

where C is independent of a_i and s_i. As $\tilde{\mathcal{A}} \times \mathcal{S}_{T,K}$ is a compact subset of $C[0, T] \times C[0, T]$, this gives the desired result for $\mathcal{G}_2$.

Next we argue that the set $\{\mathcal{G}_1(a, s) | (a, s) \in \tilde{\mathcal{A}} \times \mathcal{S}_{T,K}\}$ is uniformly bounded and equicontinuous, so that the Ascoli Theorem will give the result for $\mathcal{G}_1$. The uniform boundedness was shown in [8]. As to the equicontinuity, we fix t_1, and show that $|\mathcal{G}_1(a, s)(t_1) - \mathcal{G}_1(a, s)(t_2)| < \varepsilon$ whenever $|t_1 - t_2| < \delta$, independently of a and s. In [8] (following (2.14)) it was shown that $\lim_{t \to 0} \mathcal{G}_1(a, s)(t) = \phi'(b)$ with this limit uniform in $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$, so that we defined $\mathcal{G}_1(a, s)(0) = \phi'(b)$; this establishes the desired estimate for the case $t_1 = 0$. Now consider $t_1 > 0$. Let us write $\mathcal{G}_1(a, s) := \mathcal{G}_{11}(a, s) + \mathcal{G}_{12}(a, s)$, with the decomposition defined in the obvious way.

We have that

$$\begin{aligned}
&|\mathcal{G}_{11}(a, s)(t_1) - \mathcal{G}_{11}(a, s)(t_2)| = \\
&= \left| \int_0^b \phi'(\xi) \left\{ \frac{1}{\sqrt{\pi \alpha(t_1)}} \exp\left(\frac{-(s(t_1) - \xi)^2}{4\alpha(t_1)} \right) \right.\right. \\
&\quad \left.\left. - \frac{1}{\sqrt{\pi \alpha(t_2)}} \exp\left(\frac{-(s(t_2) - \xi)^2}{4\alpha(t_2)} \right) \right\} d\xi \right| \leq \frac{|\phi'|_\infty}{\sqrt{\pi}} \\
&\quad \times \left(\left| \int_0^b \frac{1}{\sqrt{\alpha(t_1)}} \left\{ \exp\left(\frac{-(s(t_1) - \xi)^2}{4\alpha(t_1)} \right) - \exp\left(\frac{-(s(t_2) - \xi)^2}{4\alpha(t_1)} \right) \right\} d\xi \right| \right. \\
&\quad + \int_0^b \left| \frac{1}{\sqrt{\alpha(t_1)}} \left\{ \exp\left(\frac{-(s(t_2) - \xi)^2}{4\alpha(t_1)} \right) - \exp\left(\frac{-(s(t_2) - \xi)^2}{4\alpha(t_2)} \right) \right\} \right| d\xi \\
&\quad \left. + \int_0^b \left| \left(\frac{1}{\sqrt{\alpha(t_1)}} - \frac{1}{\sqrt{\alpha(t_2)}} \right) \exp\left(\frac{-(s(t_2) - \xi)^2}{4\alpha(t_2)} \right) \right| d\xi \right) \\
&\equiv \frac{|\phi'|_\infty}{\sqrt{\pi}} (I + II + III).
\end{aligned}$$

With a change of variables (in both terms), we obtain

$$I \leqslant 2 \int_{\frac{1}{2}(s(t_1)-b)\alpha(t_1)^{-1/2}}^{\frac{1}{2}s(t_1)\alpha(t_1)^{-1/2}} e^{-\eta^2}\,d\eta - 2\int_{\frac{1}{2}(s(t_2)-b)\alpha(t_1)^{-1/2}}^{\frac{1}{2}s(t_2)\alpha(t_1)^{-1/2}} e^{-\eta^2}\,d\eta$$

$$\leqslant \left\{ 2\int_{\frac{1}{2}(s(t_1)-b)\alpha(t_1)^{-1/2}}^{\frac{1}{2}(s(t_2)-b)\alpha(t_1)^{-1/2}} e^{-\eta^2}\,d\eta - \int_{\frac{1}{2}s(t_1)\alpha(t_1)^{-1/2}}^{\frac{1}{2}s(t_2)\alpha(t_1)^{-1/2}} e^{-\eta^2}\,d\eta \right\}$$

$$\leqslant \frac{2K|t_1-t_2|}{\sqrt{\nu t_1}}\,.$$

Next we see that

$$II \leqslant \frac{1}{\nu^{1/2}t_1^{1/2}}\int_0^b \left(\frac{(s(t_2)-\xi)^2}{4}\left(\frac{1}{\alpha(t_1)}-\frac{1}{\alpha(t_2)}\right)\right)d\xi$$

$$\leqslant \frac{\mu\,|t_2-t_1|}{\nu^{5/2}t_1^{3/2}t_2}\,\frac{s(t_2)^2\,b}{4} \leqslant \frac{\mu b(b+KT)^2|t_2-t_1|}{4\,\nu^{5/2}t_1^{3/2}t_2}\,.$$

We estimate III as

$$III \leqslant b\left|\frac{1}{\sqrt{\alpha(t_1)}}-\frac{1}{\sqrt{\alpha(t_2)}}\right| = b\left|\frac{\alpha(t_2)-\alpha(t_1)}{\sqrt{\alpha(t_1)\,\alpha(t_2)}(\sqrt{\alpha(t_1)}+\sqrt{\alpha(t_2)})}\right|$$

$$\leqslant \frac{b\mu\,|t_2-t_1|}{\nu^{3/2}\sqrt{t_1}\sqrt{t_2}(\sqrt{t_1}+\sqrt{t_2})}\,.$$

As a consequence of the mean value theorem one infers the existence of a constant $C > 0$ independent of $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$ such that

$$|\mathscr{G}_{12}(a,s)(t_1) - \mathscr{G}_{12}(a,s)(t_2)| \leqslant C\,|t_1-t_2|$$

holds. Combining the above estimates it is clear that equicontinuity holds for any $t_1 > 0$, and this completes the proof of the proposition. □

We state the following result from [8].

THEOREM 2.3 : *For each* $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$ *equation* (2.3) *has a unique solution* $v(a, s) \in C[0, T]$ *satisfying*

$$|v(a,s)|_\infty = |(\mathfrak{I} - \mathscr{K}(a,s))^{-1}\,\mathscr{G}(a,s)|_\infty$$

$$\leqslant |(\mathfrak{I} - \mathscr{K}(a,s))^{-1}|\,|\mathscr{G}(a,s)|_\infty$$

$$\leqslant \bar{K}|\mathscr{G}(a,s)|_\infty \leqslant \bar{k}\,,$$

where $\bar{K}$ *and* $\bar{k}$ *are independent of* a *and* s.

2.2. The Stefan Problem and ID

Now we discuss the iteration on the boundary function s. Let $K > 0$ be a given constant, fix $a \in \mathcal{A}$, and define $\mathcal{T}(a, .) : \mathcal{S}_{T,K} \to W^{1,\infty}(0, T)$ by :

$$[\mathcal{T}(a, s)](t) = b - \int_0^t a(\tau)\, v(\tau ; a, s)\, d\tau\,, \tag{2.4}$$

where v represents the solution of equation (2.3). Fir $s \in W^{1,\infty}(0, T)$, let

$$|s|_\beta = |s(0)| + \underset{t \in [0, T]}{\text{ess sup}} \left(|\dot{s}(t)|\, e^{-\beta t} \right),$$

and note that for any $\beta > 0$, this is equivalent to the $W^{1,\infty}(0, T)$ norm. Let $W^{1,\infty}_\beta(0, T)$ represent the space $W^{1,\infty}(0, T)$ equipped with this weighted topology. The following results can be found in [8] (Lemma 3.2, Corollary 3.6, and a modification of Corollary 3.8).

THEOREM 2.4 : *With the definitions above,*

(i) *There exists $K > 0$ (depending only on ν, μ, $|\phi'|_\infty$, $|g|_\infty$, b, T) such that $\mathcal{T}(a, .)$ leaves $\mathcal{S}_{T,K}$ invariant ;*

(ii) $|\mathcal{T}(a, s) - \mathcal{T}(a, \tilde{s})|_\beta \leq \gamma(\beta) |s - \tilde{s}|_\beta$ *for $s, \tilde{s} \in \mathcal{S}_{T,K}$, $a \in \mathcal{A}$, with $\gamma(\beta)$ independent of a, $\gamma(\beta) > 0$, and $\gamma(\beta) \to 0$ as $\beta \to \infty$;*

(iii) $|\mathcal{T}(a, s) - \mathcal{T}(\tilde{a}, s)|_\beta \leq L(\beta) |a - \tilde{a}|_\infty$ *for a, $\tilde{a} \in \mathcal{A}$, $s \in \mathcal{S}_{T,K}$, where L is independent of $s \in \mathcal{S}_{T,K}$.*

Since $\mathcal{S}_{T,K}$ is closed in the $|.|_\beta$ topology (for any $\beta > 0$), this theorem indicates that for appropriately chosen K and β (from now on, we fix K and β at these values), $\mathcal{T}(a, .)$ is a contraction mapping in the β-weighted topology on the set $\mathcal{S}_{T,K}$. Moreover, this contraction map is Lipschitz continuous in the parameter a, uniformly with respect to $s \in \mathcal{S}_{T,K}$. We therefore obtain the following result :

COROLLARY 2.5 : *For each $a \in \mathcal{A}$, $\mathcal{T}(a, .)$ has a unique fixed point $\bar{s}(a)$ in $\mathcal{S}_{T,K}$. The fixed point depends Lipschitz continuously on $a \in \mathcal{A}$. Moreover, let $\bar{v}(a) = v(a, \bar{s}(a))$ be the corresponding solution of* (2.1), *and $\bar{u}(a) = u(a, \bar{s}(a))$ be the corresponding solution of equation* (2.2) ; *then the pair $(\bar{s}(a), \bar{u}(a))$ is the solution of the Stefan problem corresponding to the parameter a.*

Given Theorem 2.4 and Corollary 2.5, we express the solution of the Stefan problem for the given parameter $a \in \mathcal{A}$ as follows ; first solve the coupled equations below for $\bar{s}(a)$, $v(a, \bar{s}(a))$:

$$\begin{aligned} &(\mathcal{I} - \mathcal{K}(a, \bar{s}(a)))\, v(a, \bar{s}(a)) = \mathcal{G}(a, \bar{s}(a))\,, \\ &\bar{s}(a) = \mathcal{T}(a, \bar{s}(a)) \quad \text{with} \quad \mathcal{T}(a, .) \quad \text{as defined in (2.4)}\,. \end{aligned} \tag{2.5}$$

Then obtain $\bar{u}(a)$ as

$$\bar{u}(a) = u(a, \bar{s}(a)) \quad \text{with } u \text{ as defined in (2.2)}. \tag{2.6}$$

Remark : From now on the following notation will be used : for any $(a, s) \in \mathscr{A} \times S_{T,K}$, $v(a, s)$ will designate the solution of (2.1), and $u(a, s)$ will designate the solution of (2.2) obtained with (a, s) and this $v(a, s)$; in general, this is not a Stefan solution. If for a given a, it is the case that s is such that (2.5) also obtains, we designate this Stefan $s(a)$ by $\bar{s}(a)$, and the corresponding Stefan u by $\bar{u}(a) = u(a, \bar{s}(a))$.

The preceeding discussion suggests the following algorithm for the solution of the forward problem, with fixed $a \in \mathscr{A}$. Begin with $s^0 \in \mathscr{S}_{T,K}$ ($s^0 \equiv b$ is a good initial guess in the absence of further information). Iterate on the equation :

$$s^{k+1}(t) = [\mathscr{C}(a, s^k)](t)$$

(for each k we solve (2.3) to obtain $v(a, s^k)$). In the limit as $k \to \infty$, we obtain $\bar{s}(a)$, and the corresponding $\bar{u}(a)$. The discretized Stefan equation is based on this iteration map, combined with a discretized version of the integral equations (2.3).

From Lemma 2.6 and Lemma 3.9 of [8], and using the fact that the β-norm is stronger than the L^∞-norm, we obtain the following continuity result :

THEOREM 2.6 : *Given any* $s_1, s_2 \in \mathscr{S}_{T,K}$, *let*

$$\tilde{\Omega} \equiv \{(x, t) | 0 \leq x \leq \max\ (s_1(t), s_2(t)), 0 \leq t \leq T\} .$$

For any $a_1, a_2 \in \mathscr{A}$, *let* $u(a_i, s_i)$ *represent the solution of* (2.2), *using* $v(a_i, s_i)$ *the solution of* (2.1), *for* $i = 1, 2$. *There exists* $\rho > 0$ *independent of* $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$ *such that*

$$|u(a_1, s_1) - u(a_2, s_2)|_{L^\infty(\tilde{\Omega})} \leq \rho\,(|a_1 - a_2|_\infty + |s_1 - s_2|_\beta) .$$

Finally, we obtain Lipschitz continuity of the Stefan solution on the parameter. The proof of the first statement below follows from Theorem 2.4, and the definition of $\bar{s}(a)$. The second statement is proved in [8] (Theorem 3.5).

COROLLARY 2.7 : *Let* $a, \tilde{a} \in \mathscr{A}$, *and let* $\bar{s}(a)$, $\bar{s}(\tilde{a})$, *and* $\bar{u}(a)$, $\bar{u}(\tilde{a})$ *represent, respectively, the corresponding Stefan boundaries, and solutions of the Stefan problem* (1.1). *Then there exist constants* k, $\tilde{k}$ *independent of* a, $\tilde{a}$ *such that*

$$|\bar{s}(a) - \bar{s}(\tilde{a})|_\beta \leq k\,|a - \tilde{a}|_\infty ,$$

$$|\bar{u}(x, t\,; a) - \bar{u}(x, t\,; \tilde{a})| \leq \tilde{k}\,|a - \tilde{a}|_\infty$$

uniformly in $(x, t) \in \{(x, t) | 0 \leq x \leq \max\ (\bar{s}(t\,; a), \bar{s}(t\,; \tilde{a})), 0 \leq t \leq T\}$.

We now return to the parameter identification problem. In terms of our integral formulation, the parameter estimation problem can be equivalently stated as :

(ID) $$\operatorname*{Min}_{a \in \tilde{\mathcal{A}}} J(a) = \left| \mathcal{P}_a(\bar{s}(a), \bar{u}(a)) - z \right|_Z^2 ,$$

where $(\bar{s}(a), \bar{u}(a))$ is the solution of (2.5), (2.6).

From our continuity results and the compactness of $\tilde{\mathcal{A}}$, we immediately obtain.

THEOREM 2.8 : *Assume $\mathcal{P}_a$ is continuous in a. Then Problem* (ID) *has a solution.*

We shall consider two specific cost functionals. In all cases, we take the compact constraint set to be $\tilde{\mathcal{A}} \equiv \mathcal{A} \cap \left\{ a \in W^{1,\infty}(0, T) \,|\, |\dot{a}|_\infty \leq K_a \right\}$.

As a first example, suppose that we have measurements, denoted $\{z_i\}$, of the temperature u at the fixed end, $x = 0$, at a set of positive times $\{t_i\}$, with $i = 1, 2, \ldots, m$. For this case, $Z = \mathbb{R}^m$, and $\mathcal{P}_a$ (independent of a, and thus trivially continuous) is the operation of selecting the u-component of the Stefan solution pair, and evaluating it at the set of points $(x, t) = (0, t_i)$, for $i = 1, 2, \ldots, m$. We define

$$J1(a) = \sum_{i=1}^{m} \left| z_i - \bar{u}(0, t_i ; a) \right|^2$$

where $\bar{u}(\,.\,,\,.\,; a)$ represents the solution to the system of equations (2.5), (2.6), corresponding to the parameter a. We emphasize that while only $\bar{u}$ appears in the cost functional $J1$, in fact a solution of the Stefan problem for a given $a \in \mathcal{A}$ consists of the pair of functions $\bar{u}$ and $\bar{s}$. The corresponding estimation problem is

(ID1) $$\operatorname*{Min}_{a \in \tilde{\mathcal{A}}} J1(a) .$$

For the second example, suppose the measurements $\{z_i\}$ correspond to observations of the boundary location s at the times $\{t_i\}$. For this case, $Z = \mathbb{R}^m$ and $\mathcal{P}_a$ (again independent of a) represents the operation of selecting the s-component of the Stefan solution pair, and evaluating it at the set of points $t = t_i$, $i = 1, 2, \ldots, m$. Here we define

$$J2(a) = \sum_{i=1}^{m} \left| z_i - \bar{s}(t_i ; a) \right|^2$$

where $\bar{s}(\,.\,; a)$ represents the solution of equation (2.5) corresponding to the parameter a ; notice that for this cost functional there is no need to evaluate equation (2.6). Our estimation problem is

(ID2) $$\operatorname*{Min}_{a \in \tilde{\mathscr{A}}} J2(a).$$

3. NUMERICAL SOLUTION OF THE INVERSE PROBLEM

To approximate the forward problem, we must discretize equation (2.3). It is this discretization of the integral equation on which we would like to focus here, and thus we shall assume that the integrations necessary for the evaluation of $\mathscr{G}(t; a, s)$, $[\mathfrak{C}(a, s)](t)$, and (2.2), can be performed exactly. We replace the integral operator $\mathscr{K}(a, s)$ by an approximation (involving a quadrature rule), to obtain $\mathscr{K}_h(a, s) : C[0, T] \to C[0, T]$ for each $(a, s) \in \mathscr{A} \times \mathscr{S}_{T, K}$. The resulting approximation for equation (2.3) will then be written as

$$(\mathfrak{J} - \mathscr{K}_h(a, s))\, v_h(a, s) = \mathscr{G}(a, s). \tag{3.1}$$

We shall discuss a specific choice for $\mathscr{K}_h$ in section 5. Here we consider any discretization technique which satisfies the following hypothesis :

(H) (i) $\lim_{h \to 0} |\mathscr{K}_h(a, s) f - \mathscr{K}(a, s) f|_\infty = 0$ for each $f \in C[0, T]$, uniformly in $(a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T, K}$, $\tilde{\mathscr{A}}$ a compact subset of $\mathscr{A}$,

(ii) $\mathscr{F} := \{\mathscr{K}_h(a, s) | 0 < h \leq h_0, (a, s) \in \mathscr{A} \times \mathscr{S}_{T, K}\}$ is collectively compact for some $h_0 > 0$,

(iii) $|\mathscr{K}_h(a_1, s_1) - \mathscr{K}_h(a_2, s_2)| \leq C(|a_1 - a_2|_\infty + |s_1 - s_2|_\beta)$, where C is independent of (a_1, s_1), $(a_2, s_2) \in \mathscr{A} \times \mathscr{S}_{T, K}$.

THEOREM 3.1 : *Suppose that $\mathscr{K}_h$ is defined so that* (H) *holds and let $\tilde{\mathscr{A}}$ be a compact subset of $\mathscr{A}$. Then there exists $h_0 > 0$ such that for any $0 < h < h_0$.*

(i) *Equation* (3.1) *admits a unique solution $v_h(a, s)$ for each $(a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T, K}$.*

(ii) *There exists a constant $\bar{v}$ such that $|v_h(a, s)|_\infty \leq \bar{v}$, uniformly in h, $(a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T, K}$.*

(iii) $\lim_{h \to 0} |v_h(a, s) - v(a, s)|_\infty = 0$ *holds uniformly in*

$$(a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T, K}.$$

Proof: Using Lemma 5.1 of [1], we have that the convergence of statement (i) of (H) is uniform in f in compact subsets of $C[0, T]$. We can then use Theorem 5.1 and Corollary 5.2 of [1] to obtain

$$|(\mathscr{K}_h(a, s) - \mathscr{K}(a, s))\, \mathscr{K}_h(a, s)| \to 0 \text{ as } h \to 0,$$
uniformly in $(a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T, K}$.

Since (see Theorem 2.3) we have $|(\mathfrak{I} - \mathcal{K}(a, s))^{-1}| \leq \bar{K}$, for a constant $\bar{K}$ which is independent of $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$, we can argue that, for sufficiently small h,

$$|(\mathcal{K}_h(a, s) - \mathcal{K}(a, s)) \mathcal{K}_h(a, s)| < \frac{1}{|(\mathfrak{I} - \mathcal{K}(a, s))^{-1}|} \quad \text{for all } (a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}.$$

Now we use Theorem 7.1 of [1] to conclude that $(\mathfrak{I} - \mathcal{K}_h(a, s))^{-1}$ exists for all h sufficiently small, and is bounded independently of h, a, s ; this establishes part (i) ; letting $\mathcal{L}$ represent this bound, we immediately obtain statement (i) with $\bar{v} = \mathcal{L}\bar{k}/\bar{K}$ (with the notation of Theorem 2.3). The same theorem also ensures that, for any $f \in C[0, T]$:

$$|(\mathfrak{I} - \mathcal{K}_h(a, s))^{-1} f - (\mathfrak{I} - \mathcal{K}(a, s))^{-1} f|_\infty \leq |(\mathfrak{I} - \mathcal{K}(a, s))^{-1}| \times$$
$$\times \left(\frac{|\mathcal{K}_h(a, s) f - \mathcal{K}(a, s) f|_\infty + |(\mathcal{K}_h(a, s) - \mathcal{K}(a, s)) \mathcal{K}_h(a, s)| \, |(\mathfrak{I} - \mathcal{K}(a, s))^{-1} f|_\infty}{1 - |(\mathfrak{I} - \mathcal{K}(a, s))^{-1}| \, |(\mathcal{K}_h(a, s) - \mathcal{K}(a, s)) \mathcal{K}_h(a, s)|} \right).$$

This shows that $\lim_{h \to 0} |(\mathfrak{I} - \mathcal{K}_h(a, s))^{-1} f - (\mathfrak{I} - \mathcal{K}(a, s))^{-1} f|_\infty = 0$ uniformly in $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$, for any $f \in C[0, T]$.

Now recall that

$$|v_h(a, s) - v(a, s)|_\infty = |(\mathfrak{I} - \mathcal{K}_h(a, s))^{-1} \mathcal{G}(a, s) - (\mathfrak{I} - \mathcal{K}(a, s))^{-1} \mathcal{G}(a, s)|_\infty ,$$

which, together with Proposition 2.2, establishes (iii). □

We now present a computational algorithm for the parameter estimation problem. It essentially consists of iterating on the map $\mathcal{T}(a, .)$, solving (3.1) for $v_h(a, s)$ to use for the evaluation of $\mathcal{T}(a, .)$. In our theoretical analysis, it will be important to know that each s we obtain in the iteration process belongs to $\mathcal{S}_{T,K}$; while Theorem 2.3 guarantees that $\mathcal{S}_{T,K}$ remains invariant under $\mathcal{T}(a, .)$ in case we use v a solution of equation (2.3) in the evaluation of $\mathcal{T}(a, .)$, we have no such guarantee using the approximate integral equation. Thus we must change the definition of the update map for s slightly in the approximation scheme.

Let $f \in C[0, T]$ be given. Define $p[f] \in C[0, T]$ by :

$$p[f] = \begin{cases} 0 & \text{if } f > 0 \\ f & \text{if } -K \leq f \leq 0 \\ -K & \text{if } f \leq -K \end{cases}$$

and note that p satisfies the following properties :

$$\begin{aligned} & p[f] = f \quad \text{if} \quad -K \leqslant f \leqslant 0 , \\ & |p[f_1] - p[f_2]|_\infty \leqslant |f_1 - f_2|_\infty \quad \text{for any} \quad f_1, f_2 \in C[0, T] . \end{aligned} \tag{3.2}$$

Using that map, we now define an approximation to the fixed point iteration on $\mathcal{C}(a, .)$ by :

Fix $a \in \mathcal{A}$, h and N. Set $s_h^0 \equiv b \in \mathcal{S}_{T, K}$.

For $k = 1, 2, \ldots, N$:

Solve the approximate integral equation (3.1)

$$(\mathfrak{I} - K_h(a, s_h^{k-1}))\, v_h(a, s_h^{k-1}) = \mathcal{G}(a, s_h^{k-1}) ; \tag{3.3}$$

Update the boundary :

$$[s_h^k(a)](t) = b - \int_0^t p(a(\tau)\, v_h(\tau ; a, s_h^{k-1}(a)))\, d\tau .$$

It is straightforward to check that the set $\{s_h^k(a)\}$ obtained in this way belongs to $\mathcal{S}_{T, K}$ for any h, k, $a \in \mathcal{A}$. Notice that our approximation has two levels ; we choose h corresponding to the grid size for the integral approximation (the quadrature in the approximation to $\mathcal{K}$), whereas N represents the number of iterations to be performed to approximate the fixed point of the mapping $\mathcal{C}(a, .)$. For a given $a \in \mathcal{A}$, we shall write $s_h^N(a)$, $v_h(a, s_h^N(a))$ to designate the approximate solution of the forward Stefan problem, i.e., the result of iterating on equation (3.3) with the given parameter $a \in \mathcal{A}$. The corresponding approximation to u, designated by $u_h^N(a, s_h^N(a))$, is then obtained by evaluating equation (2.2) using a, $s_h^N(a)$, and $v_h(a, s_h^N(a))$.

We define approximate cost functionals in terms of this approximation to the solution of the Stefan problem. To approximate $J1$, we define

$$J1_h^N(a) = \sum_{i=1}^{m} |z_i - u_h^N(0, t_i ; a, s_h^N(a))|^2$$

where u_h^N is obtained from equation (2.2) with $s_h^N(a)$, $v_h(a, s_h^N(a))$ the result of iteration (3.3).

To approximate $J2$, we define

$$J2_h^N(a) = \sum_{i=1}^{m} |z_i - s_h^N(t_i ; a)|^2$$

where $s_h^N(a)$ is the result of iteration (3.3).

The approximate parameter estimation problems are then :

$(\mathrm{ID1}_h^N)$ $\qquad \min_{a \in \tilde{\mathcal{A}}} J1_h^N(a)$

or

$$(\mathrm{ID}2_h^N) \qquad \operatorname*{Min}_{a \in \tilde{\mathscr{A}}} J2_h^N(a) .$$

In order to implement either of the above problems, we also need to discretize the set $\tilde{\mathscr{A}}$. The necessary modifications of our analysis below are straightforward, but technical, and we shall not pursue this.

4. CONVERGENCE OF THE ALGORITHM

We can prove convergence of the algorithm presented in section 3 by exploiting the fact that each of the cost functionals $J1_h^N$ and $J2_h^N$ involve the solution of an approximate Stefan problem, in which the approximations are defined in such a way as to be convergent uniformly in the parameters and the boundary functions. We present these results here.

PROPOSITION 4.1 : *Assume that* (H) *holds. Then* $J1_h^N$ *and* $J2_h^N$ *are continuous in* $a \in \mathscr{A}$.

Proof : In the second part of the proof we shall verify the continuity of the mappings

$$a \mapsto s_h^N(a) : C\,[0, T] \to W_\beta^{1,\infty}(0, T)$$

and

$$a \mapsto v_h(a, s_h^N(a)) : C\,[0, T] \to C\,[0, T] . \tag{4.1}$$

Once this is shown, continuity of $J2_h^N$ is an immediate consequence. As to the continuity of $J1_h^N$ one first uses techniques of handling singular kernels similar to those in [8] to show that (2.2) and (4.1) imply that $a \mapsto u_h^N(0, t\,; a, s_h^N(a))$ is continuous from $C\,[0, T] \to C\,[0, T]$.

Now we turn to the verification of (4.1). We first prove that the map $(a, s) \to v_h(a, s) : C\,[0, T] \times W_\beta^{1,\infty}(0, T) \to C\,[0, T]$ is continuous ; we argue as in the proof of Theorem 3.1 that

$$\begin{aligned}
&|v_h(a_1, s_1) - v_h(a_2, s_2)|_\infty = \\
&\quad = |(\mathfrak{I} - \mathscr{K}_h(a_1, s_1))^{-1}\,\mathscr{G}(a_1, s_1) - (\mathfrak{I} - \mathscr{K}_h(a_2, s_2))^{-1}\,\mathscr{G}(a_2, s_2)|_\infty \\
&\quad \leq \mathscr{L}\,|\mathscr{G}(a_1, s_1) - \mathscr{G}(a_2, s_2)|_\infty \\
&\qquad + |(\mathfrak{I} - \mathscr{K}_h(a_1, s_1))^{-1}\,\mathscr{G}(a_2, s_2) - (\mathfrak{I} - \mathscr{K}_h(a_2, s_2))^{-1}\,\mathscr{G}(a_2, s_2)|_\infty .
\end{aligned}$$

The estimates derived in [8] (see proofs of Lemma 3.3 and 3.7) can be used to obtain

$$|\mathscr{G}(a_1, s_1) - \mathscr{G}(a_2, s_2)|_\infty \leq \tilde{C}\,(|a_1 - a_2|_\infty + |s_1 - s_2|_\beta) ,$$

for some constant $\tilde{C} > 0$ independent of $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$. As a consequence of (H) (iii) and the identity

$$(I - A)^{-1} - (I - B)^{-1} = (I - A)^{-1}(A - B)(I - B)^{-1}$$

it follows that

$$\left|(\mathfrak{I} - \mathscr{K}_h(a_1, s_1))^{-1} - (\mathfrak{I} - \mathscr{K}_h(a_2, s_2))^{-1}\right| \leq$$
$$\leq \mathscr{L}^2 C\left(|a_1 - a_2|_\infty + |s_1 - s_2|_\beta\right)$$

which implies

$$|v_h(a_1, s_1) - v_h(a_2, s_2)| \leq C_1(|a_1 - a_2|_\infty + |s_1 - s_2|_\beta) \tag{4.2}$$

for some constant C_1 independent of $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$, $0 < h \leq h_0$.

We now consider the mapping $a \mapsto s_h^N(a)$, where $s_h^N(a)$ is constructed by the algorithm described in (3.3). For $k = 1$ one obtains

$$\begin{aligned}|s_h^1(a_1) - s_h^1(a_2)|_\beta &\leq \bar{v}\,|a_1 - a_2|_\infty + \mu\,|v_h(a_1, b) - v_h(a_2, b)|_\infty \\ &\leq (\bar{v} + \mu C_1)|a_1 - a_2|_\infty .\end{aligned}$$

At the second step of the fixed point iteration one has to consider

$$\begin{aligned}|s_h^2(a_1) - s_h^2(a_2)|_\beta &\leq \bar{v}\,|a_1 - a_2|_\infty + \mu\,|v_h(a_1, s_h^1(a_1)) - v_h(a_2, s_h^1(a_2))|_\infty \\ &\leq \bar{v}\,|a_1 - a_2|_\infty + \mu C_1(|a_1 - a_2|_\infty + |s_h^1(a_1) - s_h^1(a_2)|_\beta) \\ &\leq (\bar{v} + \mu C_1)(1 + \mu C_1)|a_1 - a_2|_\infty .\end{aligned}$$

Induction yields

$$|s_h^N(a_1) - s_h^N(a_2)|_\beta \leq (\bar{v} + \mu C_1) \sum_{j=0}^{N-1} (\mu C_1)^j |a_1 - a_2|_\infty .$$

This together with (4.2) completes the proof of the proposition. □

PROPOSITION 4.2 : *Assume that* (H) *holds. suppose* $\lim_{j \to \infty} a^j = a$ *where* $\{a^j\}$ *is an arbitrary sequence in* $\mathscr{A}$. *Then* $|s_h^N(a^j) - \bar{s}(a)|_\beta \to 0$ *and* $|v_h(a^j, s_h^N(a^j)) - v(a, \bar{s}(a))|_\infty \to 0$ *hold as* $N, j \to \infty$, $h \to 0$.

Proof: We first consider the convergence $s_h^N(a^j) \to \bar{s}(a)$:

$$|s_h^N(a^j) - \bar{s}(a)|_\beta \leq |s_h^N(a^j) - s^N(a^j)|_\beta +$$
$$+ |s^N(a^j) - \bar{s}(a^j)|_\beta + |\bar{s}(a^j) - \bar{s}(a)|_\beta .$$

The first term on the right-hand side is estimated as follows :

$$\begin{aligned}
&\left|s_h^N(a^j) - s^N(a^j)\right|_\beta = \\
&= \sup_{t \in [0,T]} \left|e^{-\beta t}(p[a^j v_h(t; a^j, s_h^{N-1}(a^j))] - a^j v(t; a^j, s^{N-1}(a^j)))\right| \\
&\leqslant \left|p[a^j v_h(a^j, s_h^{N-1}(a^j))] - p[a^j v(a^j, s_h^{N-1}(a^j))]\right|_\infty \\
&\quad + \left|p[a^j v(a^j, s_h^{N-1}(a^j))] - a^j v(a^j, s_h^{N-1}(a^j))\right|_\infty \\
&\quad + \sup_{t \in [0,T]} \left|e^{-\beta t}(a^j v(t; a^j, s_h^{N-1}(a^j)) - a^j v(t; a^j, s^{N-1}(a^j)))\right| \\
&\leqslant \mu \left|v_h(a^j, s_h^{N-1}(a^j)) - v(a^j, s_h^{N-1}(a^j))\right|_\infty \\
&\quad + \left|\mathcal{C}_{a^j}(s_h^{N-1}(a^j)) - \mathcal{C}_{a^j}(s^{N-1}(a^j))\right|_\beta \\
&\leqslant \mu \left|v_h(a^j, s_h^{N-1}(a^j)) - v(a^j, s_h^{N-1}(a^j))\right|_\infty + \gamma \left|s_h^{N-1}(a^j) - s^{N-1}(a^j)\right|_\beta
\end{aligned}$$

where we have used (3.2), the fact that each $s_h^N(a^j) \in \mathcal{S}_{T,K}$, and Theorem 2.4. We then obtain

$$\begin{aligned}
&\left|s_h^N(a^j) - \bar{s}(a)\right|_\beta \leqslant \gamma \left|s_h^{N-1}(a^j) - \bar{s}(a)\right|_\beta + \\
&\quad + \mu \left|v_h(a^j, s_h^{N-1}(a^j)) - v(a^j, s_h^{N-1}(a^j))\right|_\infty \\
&\quad + (\gamma + 1)\left|\bar{s}(a^j) - \bar{s}(a)\right|_\beta + \gamma \left|s^{N-1}(a^j) - \bar{s}(a^j)\right|_\beta + \left|s^N(a^j) - \bar{s}(a^j)\right|_\beta .
\end{aligned}$$

This is of the form $e^N(h,j) \leqslant \gamma e^{N-1}(h,j) + R(N,h,j)$, with

$$e^N(h,j) = \left|s_h^N(a^j) - \bar{s}(a)\right|_\beta$$

and

$$\begin{aligned}
&R(N,h,j) = \mu \left|v_h(a^j, s_h^{N-1}(a^j)) - v(a^j, s_h^{N-1}(a^j))\right|_\infty + \\
&\quad + (\gamma + 1)\left|\bar{s}(a^j) - \bar{s}(a)\right|_\beta + \gamma \left|s^{N-1}(a^j) - \bar{s}(a^j)\right|_\beta + \left|s^N(a^j) - \bar{s}(a^j)\right|_\beta .
\end{aligned}$$

By induction, we find

$$e^N(h,j) \leqslant \gamma^N e^0(h,j) + \sum_{k=1}^{N} \gamma^{N-k} R(k,h,j) .$$

Suppose $\varepsilon > 0$ is given. Using part (ii) of Theorem 2.4, Corollary 2.5, and part (iii) of Theorem 3.1, we can make $R(N,h,j) \leqslant \varepsilon^{\frac{1-\gamma}{2}}$ for $N \geqslant N(\varepsilon)$,

$h \leq h(\varepsilon)$ and $j \geq j(\varepsilon)$, for some $N(\varepsilon)$, $h(\varepsilon)$, $j(\varepsilon)$. Then, for all $n \geq 1$, $h \leq h(\varepsilon)$ and $j \geq j(\varepsilon)$, we see that

$$e^{N(\varepsilon)+n}(h, j) \leq \gamma^n e^{N(\varepsilon)}(h, j) + \sum_{k=N(\varepsilon)+1}^{N(\varepsilon)+n} \gamma^{N(\varepsilon)+n-k} R(k, h, j)$$
$$\leq \gamma^n e^{N(\varepsilon)}(h, j) + \frac{\varepsilon}{2}.$$

Finally, $e^N(h, j)$ is bounded uniformly in N, h, and j, (since all the $s_h^N(a^j)$ are in $\mathcal{S}_{T,K}$), and $\gamma < 1$, so we can make $\gamma^n e^{N(\varepsilon)}(h, j) \leq \frac{\varepsilon}{2}$ for n large enough. This establishes the convergence of $s_h^N(a^j) \to \bar{s}(a)$.

Concerning the second statement of the proposition we consider the estimate

$$\left|v_h(a^j, s_h^N(a^j)) - v(a, \bar{s}(a))\right|_\infty \leq$$
$$\leq \left|v_h(a^j, s_h^N(a^j)) - v_h(a, \bar{s}(a))\right|_\infty + \left|v_h(a, \bar{s}(a)) - v(a, \bar{s}(a))\right|_\infty .$$

Due to Theorem 3.1 and (4.2) the right-hand side of the above inequality tends to zero. This completes the proof of the proposition. □

COROLLARY 4.3 : *Assume that* (H) *holds. Suppose* $\lim_{j \to \infty} a^j = a$ *where* $\{a^j\}$ *is an arbitrary sequence in* $\mathcal{A}$. *Then* $u_h^N(0, t; a^j, s_h^N(a^j)) \to \bar{u}(0, t; a)$ *in* $C[0, T]$ *as* $N, j \to \infty$, $h \to 0$.

Proof: Using the definitions of u_h^N and u, the fact that $\int_0^t |N(0, t; s(\tau), \tau)| \, d\tau$ is uniformly bounded in t, a, and s, and Theorem 2.6, we obtain, for a constant C which is independent of j, h, N, and t:

$$\left|u_h^N(0, t; a^j, s_h^N(a^j)) - \bar{u}(0, t; a)\right| =$$
$$= \left|u_h^N(0, t; a^j, s_h^N(a^j)) - u(0, t; a, \bar{s}(a))\right|$$
$$\leq \left|u_h^N(0, t; a^j, s_h^N(a^j)) - u(0, t; a^j, s_h^N(a^j))\right|$$
$$+ \left|u(0, t; a^j, s_h^N(a^j)) - u(0, t; a, \bar{s}(a))\right|$$
$$\leq C\left|v_h(a^j, s_h^N(a^j)) - v(a^j, s_h^N(a^j))\right|_\infty + \rho\left(|a^j - a|_\infty + \left|s_h^N(a^j) - \bar{s}(a)\right|_\beta\right).$$

Now the corollary follows from Theorem 3.1 and Proposition 4.2. □

THEOREM 4.4 : *For any approximation scheme satisfying* (H), *a solution of* $(\mathrm{ID1}_n^N)$ *exists for each N and sufficiently small h, and these solutions* *converge subsequentially in* $C[0, T]$ *to a solution of* (ID1). *The same statement holds for* ($ID2_h^N$) *and* (ID2).

Proof : Since $\tilde{\mathscr{A}}$ is compact in $C[0, T]$, Proposition 4.1 ensures that for each h, N, a solution $\hat{a}_h^N$ of ($ID1_h^N$) (or ($ID2_h^N$)) exists. Moreover, from the compactness of $\tilde{\mathscr{A}}$ we have a convergent subsequence such that $\hat{a}_{h_k}^{N_k} \to \hat{a} \in \tilde{\mathscr{A}}$ in $C[0, T]$ as $h_k \to 0$, $N_k \to \infty$. As in [4] we may use Corollary 4.3 (or Proposition 4.2) to conclude that this limit $\hat{a}$ is a solution of (ID1) (or (ID2)). □

5. NUMERICAL IMPLEMENTATION

We approximate the integral equation using a method presented by K. Atkinson in [2, 3]. The idea is to discretize the integral in $\mathscr{K}$ by Simpson's rule, when this can be done accurately (i.e. where the kernel K is well behaved), and use quadratic interpolation near the singularity of K. In this section, we define the approximations, and then prove that hypothesis (H) of section 3 is satisfied.

As the first step choose $\delta > 0$ large enough so that the $K(t, \cdot\,; a, s)$ is « nice » on $[0, t - \delta]$, $t > \delta$, for any choice of $(a, s) \in \mathscr{A} \times \mathscr{S}_{T,K}$ and as small as possible in order to minimize the effort to carry out the singular integration on $[t - \delta, t)$, $t \in [0, T]$. Subdivide the time interval $[0, T]$ into $2n$ equal subintervals to obtain the quadrature points $t_i = ih$, $i = 0, \dots, 2n$, with $h = \frac{T}{2n}$. Let si denote the number of « singular » intervals contained in $(t - \delta - h, t)$, $t > 0$; i.e. $si = \left[\frac{\delta}{h}\right] + 1$. Without loss of generality we may assume si is odd, i.e. $si = 2j^* + 1$. Hence

$$2j^* h \leq \delta < (2j^* + 1) h . \tag{5.1}$$

Let β_0, β_1, β_2 represent the Lagrange interpolating polynomials and set

$$\Lambda_j f = \sum_{i=0}^{2} \beta_i(\cdot - t_{2j-2}) f(t_{2(j-1)+i}),$$

$$|\Lambda_j f| \leq \ell\, |f|_\infty, \qquad j = 1, \dots, n$$

$f \in C([0, T])$. Furthermore, define

$$\psi_j(t) = \begin{cases} 0 & t < t_{2j} \\ \frac{1}{h}(t - t_{2j}) & t_{2j} \leq t \leq t_{2j+1} \\ 1 & t \geq t_{2j+1}, \end{cases}$$

$$\chi_j(t) = 1 - \psi_j(t), \quad t \in [0, T], \quad j = 1, \dots, n .$$

Finally, we introduce a compact notation for Simpson's approximation ($f \in C([0, T])$)

$$\mathcal{S}_{t_{2j-2}}^{t_{2j}} f = \frac{h}{3} [f(t_{2j-2}) + 4 f(t_{2j-1}) + f(t_{2j})]$$

and recall that

$$\mathcal{S}_{t_{2j-2}}^{t_{2j}} f = \int_{t_{2j-2}}^{t_{2j}} (\Lambda_j f)(\tau)\, d\tau . \tag{5.2}$$

Let $I_k = [t_{2(k-1)}, t_{2k}]$ and for $f \in C([0, T])$, $(a, s) \in \mathcal{A} \times \mathcal{S}_{T, K}$, define the approximate integral operators $\mathcal{K}_h(a, s)$ by

$$\begin{aligned}(\mathcal{K}_h(a, s) f)(t) = \sum_{j=1}^{k} \mathcal{S}_{t_{2j-2}}^{t_{2j}} K(t, \cdot\, ; a, s) f \\ + \psi_{j^*+1+k}(t)\, \mathcal{S}_{t_{2k}}^{t_{2(k+1)}} K(t, \cdot\, ; a, s) f \\ + \chi_{j^*+1+k}(t) \int_{I_{k+1}} K(t, \tau ; a, s)(\Lambda_{k+1} f)(\tau)\, d\tau \\ + \sum_{j=k+2}^{k+j^*+1} \int_{I_j} K(t, \tau ; a, s)(\Lambda_j f)(\tau)\, d\tau \\ + \int_{t_{2(k+j^*+1)}}^{t} K(t, \tau ; a, s)(\Lambda_{k+j^*+2} f)(\tau)\, d\tau ,\end{aligned} \tag{5.3}$$

for $\quad t \in I_{j^*+2+k},\ k = -(j^*+1), \dots, -1, 0, 1, \dots, n - j^* - 2 .$

In the above definition we use the convention that sums and integrals should be set to zero if the upper limit of summation (integration) is less than the lower limit. In the following discussion we shall assume that $h \leqslant \frac{\delta}{6}$, hence, as a consequence of the choice of δ, we conclude

$$t - \tau \geqslant \frac{5\,\delta}{6}, \quad \text{for} \quad t \in I_{j^*+2+k}, \quad \tau \in I_j, \quad j = 1, \dots, k+1, \quad k = 0, \dots, n - j^* - 2 . \tag{5.4}$$

The definition (5.3) ensures $\mathcal{K}_h(a, s) f \in C[0, T]$ and (5.4) implies that

$$|\mathcal{K}_h(a, s)| \leqslant \frac{1}{\sqrt{\delta}} M , \quad h > 0 , \quad (a, s) \in \mathcal{A} \times \mathcal{S}_{T, K} \tag{5.5}$$

holds for some $M > 0$, which is independent of $(a, s) \in \mathcal{A} \times \mathcal{S}_{T, K}$, $0 < h \leqslant \frac{\delta}{6}$.

PROPOSITION 5.1 : *The family of operators*

$$\left\{\mathcal{K}_h(a, s) \mid (a, s) \in \mathcal{A} \times \mathcal{S}_{T, K}, 0 < h \leq \frac{\delta}{6}\right\}$$

is collectively compact in $C([0, T])$.

Proof : Because of (5.5) it suffices to show that the set of functions

$$\left\{\mathcal{K}_h(a, s) f \mid (a, s) \in \mathcal{A} \times \mathcal{S}_{T, K}, 0 < h \leq \frac{\delta}{6}, |f|_\infty \leq 1\right\}$$

is equicontinuous. Choose $t, \tilde{t} \in [0, T]$, and assume without loss of generality that $t < \tilde{t}$. Then there are unique indices $k, \tilde{k} \in \{-(j^* + 1), \ldots, n - j^* - 2\}$ such that

$$t \in I_{j^* + k + 2}, \quad \tilde{t} \in I_{j^* + \tilde{k} + 2} \quad \text{and} \quad k \leq \tilde{k} .$$

In addition we shall also assume $\tilde{t} - t \leq \frac{1}{2} \delta$ which implies

$$\tilde{k} - k < j^* . \tag{5.6}$$

This follows from

$$\begin{aligned} 2(\tilde{k} - k) h &= 2(\tilde{k} + j^* + 1) h - 2(k + j^* + 2) h + 2 h \\ &\leq \tilde{t} - t + 2 h \leq \frac{5}{6} \delta \leq \delta - h < 2 j^* h . \end{aligned}$$

We first consider the case $k < \tilde{k}$. In view of definition (5.3), using (5.2) and our convention concerning void sums, one obtains

$$\begin{aligned} &[\mathcal{K}_h(a, s) f](\tilde{t}) - [\mathcal{K}_h(a, s) f](t) = \\ &= \sum_{j=1}^{k} \int_{t_{2j-2}}^{t_{2j}} A_j([K(\tilde{t}, \cdot ; a, s) - K(t, \cdot ; a, s)] f)(\tau) \, d\tau \\ &\quad + \psi_{j^* + 1 + k}(t) \int_{t_{2k}}^{t_{2(k+1)}} A_{k+1}([K(\tilde{t}, \cdot ; a, s) - K(t, \cdot ; a, s)] f)(\tau) \, d\tau \\ &\quad + \chi_{j^* + 1 + k}(t) \\ &\quad \times \int_{t_{2k}}^{t_{2(k+1)}} [A_{k+1}(K(\tilde{t}, \cdot ; a, s) f)(\tau) - K(t, \tau ; a, s)(A_{k+1} f)(\tau)] \, d\tau \\ &\quad + \sum_{j=k+2}^{\tilde{k}} \int_{t_{2j-2}}^{t_{2j}} [A_j(K(\tilde{t}, \cdot ; a, s) f)(\tau) - K(t, \tau ; a, s)(A_j f)(\tau)] \, d\tau \\ &\quad + \psi_{j^* + 1 + \tilde{k}}(\tilde{t}) \end{aligned}$$

$$\times \int_{t_{2\tilde{k}}}^{t_{2(\tilde{k}+1)}} [\Lambda_{\tilde{k}+1}(K(\tilde{t}, \cdot\,; a, s) f)(\tau) - K(t, \tau\,; a, s)(\Lambda_{\tilde{k}+1} f)(\tau)]\, d\tau$$
$$+ \chi_{j^*+1+\tilde{k}}(\tilde{t}) \int_{t_{2\tilde{k}}}^{t_{2(\tilde{k}+1)}} [K(\tilde{t}, \tau\,; a, s) - K(t, \tau\,; a, s)](\Lambda_{\tilde{k}+1} f)(\tau)\, d\tau$$
$$+ \sum_{j=\tilde{k}+2}^{k+j^*+1} \int_{t_{2(j-1)}}^{t_{2j}} [K(\tilde{t}, \tau\,; a, s) - K(t, \tau\,; a, s)](\Lambda_j f)(\tau)\, d\tau$$
$$+ \int_{t_{2(k+j^*+1)}}^{t} [K(\tilde{t}, \tau\,; a, s) - K(t, \tau\,; a, s)](\Lambda_{k+j^*+2} f)(\tau)\, d\tau \tag{5.7}$$
$$- \int_{t_{2(k+j^*+1)}}^{t} K(\tilde{t}, \tau\,; a, s)(\Lambda_{k+j^*+2} f)(\tau)\, d\tau$$
$$+ \sum_{j=k+j^*+2}^{\tilde{k}+j^*+1} \int_{t_{2(j-1)}}^{t_{2j}} (K(\tilde{t}, \tau\,; a, s)(\Lambda_j f)(\tau))\, d\tau$$
$$+ \int_{t_{2(\tilde{k}+j^*+1)}}^{\tilde{t}} K(\tilde{t}, \tau\,; a, s)(\Lambda_{\tilde{k}+j^*+2} f)(\tau)\, d\tau\,,$$

which may be estimated as follows

$$\left|[\mathscr{K}_h(a, s) f](\tilde{t}) - [\mathscr{K}_h(a, s) f](t)\right|$$
$$\leqslant \sum_{j=1}^{k+1} \int_{t_{2j-2}}^{t_{2j}} \left|\Lambda_j([K(\tilde{t}, \cdot\,; a, s) - K(t, \cdot\,; a\,; s)] f)(\tau)\right| d\tau$$
$$+ \left|\chi_{j^*+1+k}(t)\right.$$
$$\left.\times \int_{t_{2k}}^{t_{2(k+1)}} [\Lambda_{k+1}(K(\tilde{t}, \cdot\,; a, s) f)(\tau) - K(t, \tau\,; a, s)(\Lambda_{k+1} f)(\tau)]\, d\tau\right|$$
$$+ \sum_{j=k+2}^{\tilde{k}} \int_{t_{2j-2}}^{t_{2j}} \left|\Lambda_j(K(\tilde{t}, \cdot\,; a, s) f)(\tau) - K(t, \tau\,; a, s)(\Lambda_j f)(\tau)\right| d\tau$$
$$+ \left|\psi_{j^*+1+\tilde{k}}(\tilde{t})\right.$$
$$\left.\times \int_{t_{2\tilde{k}}}^{t_{2(\tilde{k}+1)}} [\Lambda_{\tilde{k}+1}(K(\tilde{t}, \cdot\,; a, s) f)(\tau) - K(t, \tau\,; a, s)(\Lambda_{\tilde{k}+1} f)(\tau)]\, d\tau\right|$$
$$+ \sum_{j=\tilde{k}+1}^{k+j^*+1} \int_{t_{2(j-1)}}^{t_{2j}} \left|[K(\tilde{t}, \tau\,; a, s) - K(t, \tau\,; a, s)](\Lambda_j f)(\tau)\right| d\tau$$
$$+ \int_{t_{2(k+j^*+1)}}^{t} \left|[K(\tilde{t}, \tau\,; a, s) - K(t, \tau\,; a, s)](\Lambda_{k+j^*+2} f)(\tau)\right| d\tau$$

$$+ \int_{t}^{t_{2(k+j^*+2)}} \left|K(\tilde{t}, \tau ; a, s)(\Lambda_{k+j^*+2} f)(\tau)\right| d\tau$$
$$+ \sum_{j=k+j^*+3}^{\tilde{k}+j^*+1} \int_{t_{2j-2}}^{t_{2j}} \left|K(\tilde{t}, \tau ; a, s)(\Lambda_j f)(\tau)\right| d\tau$$
$$+ \int_{t_{2(\tilde{k}+j^*+1)}}^{\tilde{t}} \left|K(\tilde{t}, \tau ; a, s) \Lambda_{\tilde{k}+j^*+2} f(\tau)\right| d\tau .$$

This is then further estimated by

$$\left|[\mathcal{K}_h(a, s) f](\tilde{t}) - [\mathcal{K}_h(a, s) f](t)\right| \leq$$
$$\leq \sum_{j=1}^{k+1} \int_{t_{2j-2}}^{t_{2j}} \left|\Lambda_j([K(\tilde{t}, . ; a, s) - K(t, . ; a, s)] f)(\tau)\right| d\tau$$
$$+ \left|\chi_{j^*+1+k}(t)\right.$$
$$\left.\times \int_{t_{2k}}^{t_{2(k+1)}} [\Lambda_{k+1}(K(\tilde{t}, . ; a, s) f)(\tau) - K(t, \tau ; a, s)(\Lambda_{k+1} f)(\tau)] \, d\tau\right|$$
$$+ \sum_{j=k+2}^{\tilde{k}} \int_{t_{2j-2}}^{t_{2j}} \left|\Lambda_j(K(\tilde{t}, . ; a, s) f)(\tau) - K(t, \tau ; a, s)(\Lambda_j f)(\tau)\right| d\tau$$
$$+ \left|\psi_{j^*+1+\tilde{k}}(\tilde{t})\right. \tag{5.8}$$
$$\left.\times \int_{t_{2\tilde{k}}}^{t_{2(\tilde{k}+1)}} [\Lambda_{\tilde{k}+1}(K(\tilde{t}, . ; a, s) f)(\tau) - K(t, \tau ; a, s)(\Lambda_{\tilde{k}+1} f)(\tau)] \, d\tau\right|$$
$$+ \ell |f|_\infty \int_0^t \left|K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s)\right| d\tau$$
$$+ \ell |f|_\infty \int_t^{\tilde{t}} \left|K(\tilde{t}, \tau ; a, s)\right| d\tau$$
$$:= I + II + III + IV + V + VI .$$

The last term is estimated using (2.15) in [8]

$$VI \leq 4 \, \mu \ell \tilde{\kappa} \, |f|_\infty (\tilde{t} - t)^{\frac{1}{2}} .$$

Using Proposition 2.1 one concludes

$$V \leq \ell M_1 |f|_\infty (\tilde{t} - t)^{\frac{1}{q}} , \quad q > 2 .$$

Next consider a typical term in I,

$$\left|\Lambda_j([K(\tilde{t}, . ; a, s) - K(t, . ; a, s)] f)\right|_\infty \leqslant$$
$$\leqslant \ell |f|_\infty \max_{\tau \in I_j} \left|K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s)\right| , \quad j = 1, \ldots, k+1 .$$

In view of (5.4) we infer the existence of a constant $M_2 > 0$, independent of $0 < h \leqslant \frac{\delta}{6}$, $(a, s) \in \mathscr{A} \times \mathscr{S}_{T, K}$ such that

$$\left|\frac{\partial}{\partial t} K(t, \tau ; a, s)\right| \leqslant M_2 \, \delta^{-\frac{3}{2}}, \quad 0 \leqslant \tau < t - \frac{5\,\delta}{6}, \quad t > \delta .$$

As a consequence we obtain

$$I \leqslant \sum_{j=1}^{k+1} 2\, h\ell |f|_\infty \max_{\tau \in I_j} \left|K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s)\right|$$
$$\leqslant \ell M_2 \, \delta^{-3/2} T |f|_\infty |\tilde{t} - t| .$$

Now we must distinguish between the cases $\tilde{k} > k+1$ and $\tilde{k} = k+1$. The case $\tilde{k} > k+1$ occurs when $2h < \tilde{t} - t$, from which we deduce that $2(\tilde{k} - k) h \leqslant 2(\tilde{t} - t)$, and thus $2(\tilde{k} - k + 1) h \leqslant 3(\tilde{t} - t)$. Then we bound the term $II + III + IV$ as

$$II + III + IV \leqslant$$
$$\leqslant \sum_{j=k+1}^{\tilde{k}+1} \int_{t_{2j-2}}^{t_{2j}} \left|\Lambda_j(K(\tilde{t}, . ; a, s) f)(\tau) - K(t, \tau ; a, s)(\Lambda_j f)(\tau)\right| d\tau$$
$$\leqslant \ell |f|_\infty \, 2\, \mu \tilde{\kappa} \left[\left(\frac{5}{6}\delta\right)^{-1/2} 2(\tilde{k} - k + 1) h + \int_{t_{2k}}^{t_{2(\tilde{k}+1)}} (t - \tau)^{-1/2} d\tau \right] .$$

Because of (5.6) we have $t - t_{2(\tilde{k}+1)} \geqslant 0$; using (5.1) gives $t - t_{2k} > \delta$; the assumption $2h < \tilde{t} - t$ gives $t_{2(\tilde{k}+1)} - t_{2k} = 2h(\tilde{k} - k + 1) \leqslant 3(\tilde{t} - t)$; thus we obtain the estimate

$$\sqrt{t - t_{2k}} - \sqrt{t - t_{2(\tilde{k}+1)}} \leqslant 3\, \delta^{-1/2}(\tilde{t} - t) .$$

Hence there is a constant $M_3 > 0$ independent of $0 < h \leqslant \frac{\delta}{6}$, $(a, s) \in \mathscr{A} \times \mathscr{S}_{T, K}$ such that

$$II + III + IV \leqslant M_3 \, \delta^{-1/2}(\tilde{t} - t)$$

holds. Next suppose that $\tilde{k} = k + 1$; this happens in case $\tilde{t} - t \leq 2h$. We must distinguish three different possibilities :

$$t \in [t_{2(j^* + k + 2) - 1}, t_{2(j^* + k + 2)}), \quad \tilde{t} \in [t_{2(j^* + k + 2)}, t_{2(j^* + k + 2) + 1}]$$

$$t \in [t_{2(j^* + k + 1)}, t_{2(j^* + k + 1) + 1}], \quad \tilde{t} \in [t_{2(j^* + k + 2)}, t_{2(j^* + k + 2) + 1}]$$

$$t \in [t_{2(j^* + k + 2) - 1}, t_{2(j^* + k + 2)}), \quad \tilde{t} \in [t_{2(j^* + k + 2) + 1}, t_{2(j^* + k + 3)}).$$

Since their discussion is similar we indicate the argument just for the first alternative. In this case term II in (5.8) is zero because of the definition of $\chi_{j^* + k + 1}$ and term III vanishes because of our convention on void sums. We note the estimates

$$\psi_{j^* + 1 + \tilde{k}}(\tilde{t}) = \frac{1}{h}(\tilde{t} - t_{2(j^* + k + 2)}) \leq \frac{1}{h}(\tilde{t} - t)$$

and

$$t - t_{2\tilde{k}} \geq 2j^* h - 2(\tilde{k} - k)h + 3h \geq 3h .$$

Hence we bound IV in (5.8) by

$$\Big| \psi_{j^* + \tilde{k} + 1}(\tilde{t}) \times$$

$$\times \int_{t_{2\tilde{k}}}^{t_{2(\tilde{k}+1)}} [\Lambda_{\tilde{k}+1}(K(\tilde{t}, \cdot\,; a, s) f)(\tau) - K(t, \tau\,; a, s)(\Lambda_{\tilde{k}+1} f)(\tau)] \, d\tau \Big|$$

$$\leq \frac{1}{h}(\tilde{t} - t) \ell \, 2 \mu \tilde{\kappa} |f|_\infty \left[2h \left(\frac{5\delta}{6} \right)^{-1/2} + \int_{t_{2\tilde{k}}}^{t_{2(\tilde{k}+1)}} (t - \tau)^{-1/2} \, d\tau \right]$$

$$\leq \ell \, 2 \mu \tilde{\kappa} |f|_\infty \left[2 \left(\frac{5\delta}{6} \right)^{-1/2} (\tilde{t} - t) + \frac{4}{\sqrt{3}} (\tilde{t} - t) h^{-1/2} \right]$$

$$\leq M_4 \, \delta^{-1/2} |f|_\infty (\tilde{t} - t)^{1/2} .$$

The last inequality is a consequence of $\tilde{t} - t \leq 2h$.

Finally, if $k = \tilde{k}$ the expression corresponding to (5.7) is

$$[\mathcal{K}_h(a, s) f](\tilde{t}) - [\mathcal{K}_h(a, s) f](t) =$$

$$= \sum_{j=1}^{k} \int_{t_{2j-2}}^{t_{2j}} \Lambda_j([K(\tilde{t}, \cdot\,; a, s) - K(t, \cdot\,; a, s)] f)(\tau) \, d\tau$$

$$+ \psi_{j^* + 1 + k}(\tilde{t}) \int_{t_{2k}}^{t_{2(k+1)}} \Lambda_{k+1}(K(\tilde{t}, \cdot\,; a, s) f)(\tau) \, d\tau$$

$$- \psi_{j^*+1+k}(t) \int_{t_{2k}}^{t_{2(k+1)}} \Lambda_{k+1}(K(t, . ; a, s) f)(\tau) d\tau$$

$$+ \chi_{j^*+1+k}(\tilde{t}) \int_{t_{2k}}^{t_{2(k+1)}} K(\tilde{t}, \tau ; a, s)(\Lambda_{k+1} f)(\tau) d\tau$$

$$- \chi_{j^*+1+k}(t) \int_{t_{2k}}^{t_{2(k+1)}} K(t, \tau ; a, s)(\Lambda_{k+1} f)(\tau) d\tau$$

$$+ \sum_{j=k+2}^{k+j^*+1} \int_{t_{2(j-1)}}^{t_{2j}} [K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s)] (\Lambda_j f)(\tau) d\tau$$

$$+ \int_{t_{2(k+j^*+1)}}^{t} [K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s)] (\Lambda_{k+j^*+2} f)(\tau) d\tau$$

$$+ \int_{t}^{\tilde{t}} K(\tilde{t}, \tau ; a, s)(\Lambda_{k+j^*+2} f)(\tau) d\tau ,$$

which we estimate as

$$\left| [\mathscr{K}_h(a, s) f](\tilde{t}) - [\mathscr{K}_h(a, s) f](t) \right| \leqslant$$

$$\leqslant \sum_{j=1}^{k} \int_{t_{2j-2}}^{t_{2j}} \left| \Lambda_j([K(\tilde{t}, . ; a, s) - K(t, . ; a ; s)] f)(\tau) \right| d\tau$$

$$+ \psi_{j^*+1+k}(\tilde{t}) \int_{t_{2k}}^{t_{2(k+1)}} \left| \Lambda_{k+1}[K(\tilde{t}, . ; a, s) - K(t, . ; a, s)] f \right| d\tau$$

$$+ \left| \psi_{j^*+1+k}(\tilde{t}) - \psi_{j^*+1+k}(t) \right| \int_{t_{2k}}^{t_{2(k+1)}} \left| \Lambda_{k+1}(K(t, . ; a, s) f)(\tau) \right| d\tau$$

$$+ \chi_{j^*+1+k}(\tilde{t}) \int_{t_{2k}}^{t_{2(k+1)}} \left| K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s) \right| \left| (\Lambda_{k+1} f)(\tau) \right| d\tau$$

$$+ \left| \chi_{j^*+1+k}(\tilde{t}) - \chi_{j^*+1+k}(t) \right| \int_{t_{2k}}^{t_{2(k+1)}} \left| K(t, \tau ; a, s)(\Lambda_{k+1} f)(\tau) \right| d\tau$$

$$+ \sum_{j=k+2}^{k+j^*+1} \int_{t_{2(j-1)}}^{t_{2j}} \left| K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s) \right| \left| (\Lambda_j f)(\tau) \right| d\tau$$

$$+ \int_{t_{2(k+j^*+1)}}^{t} \left| K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s) \right| \left| (\Lambda_{k+j^*+2} f)(\tau) \right| d\tau$$

$$+ \int_{t}^{\tilde{t}} \left| K(\tilde{t}, \tau ; a, s)(\Lambda_{k+j^*+2} f)(\tau) \right| d\tau$$

$$\leq \sum_{j=1}^{k+1} \int_{t_{2j-2}}^{t_{2j}} \left| \Lambda_j([K(\tilde{t}, . ; a, s) - K(t, . ; a ; s)] f)(\tau) \right| d\tau$$

$$+ \left| \psi_{j^*+1+k}(\tilde{t}) - \psi_{j^*+1+k}(t) \right| \int_{t_{2k}}^{t_{2(k+1)}} \left| \Lambda_{k+1}(K(t, . ; a, s) f)(\tau) \right| d\tau$$

$$+ \left| \chi_{j^*+1+k}(\tilde{t}) - \chi_{j^*+1+k}(t) \right| \int_{t_{2k}}^{t_{2(k+1)}} \left| K(t, \tau ; a, s)(\Lambda_{k+1} f)(\tau) \right| d\tau$$

$$+ \sum_{j=k+1}^{k+j^*+1} \int_{t_{2(j-1)}}^{t_{2j}} \left| K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s) \right| \left| (\Lambda_j f)(\tau) \right| d\tau$$

$$+ \int_{t_{2(k+j^*+1)}}^{t} \left| K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s) \right| \left| (\Lambda_{k+j^*+2} f)(\tau) \right| d\tau$$

$$+ \int_{t}^{\tilde{t}} \left| K(\tilde{t}, \tau ; a, s)(\Lambda_{k+j^*+2} f)(\tau) \right| d\tau$$

$$\leq \sum_{j=1}^{k+1} \int_{t_{2j-2}}^{t_{2j}} \left| \Lambda_j([K(\tilde{t}, . ; a, s) - K(t, . ; a ; s)] f)(\tau) \right| d\tau$$

$$+ \left| \psi_{j^*+1+k}(\tilde{t}) - \psi_{j^*+1+k}(t) \right| \int_{t_{2k}}^{t_{2(k+1)}} \left| \Lambda_{k+1}(K(t, . ; a, s) f)(\tau) \right| d\tau$$

$$+ \left| \chi_{j^*+1+k}(\tilde{t}) - \chi_{j^*+1+k}(t) \right| \int_{t_{2k}}^{t_{2(k+1)}} \left| K(t, \tau ; a, s)(\Lambda_{k+1} f)(\tau) \right| d\tau$$

$$+ \ell \int_0^t \left| K(\tilde{t}, \tau ; a, s) - K(t, \tau ; a, s) \right| d\tau \, |f|_\infty$$

$$+ \ell \int_t^{\tilde{t}} \left| K(\tilde{t}, \tau ; a, s) \right| d\tau \, |f|_\infty$$

$$=: I + II + III + IV + V .$$

Terms I, IV, and V are estimated in exactly the same way as were terms I, V, and VI of (5.8). To estimate terms II and III, consider that

$$\left| \psi_{j^*+k+1}(\tilde{t}) - \psi_{j^*+k+1}(t) \right| \leq \frac{1}{h} \left| \tilde{t} - t \right|$$

and

$$\left| \chi_{j^*+k+1}(\tilde{t}) - \chi_{j^*+k+1}(t) \right| \leq \frac{1}{h} \left| \tilde{t} - t \right| .$$

Thus, we see that

$$II + III \leqslant \frac{2}{h} |\tilde{t} - t| \, \ell |f|_\infty 2 \mu \tilde{\kappa} \delta^{-\frac{1}{2}} 2h \leqslant 8 \mu \tilde{\kappa} \ell |f|_\infty \delta^{-\frac{1}{2}} |\tilde{t} - t| ,$$

and this completes the proof.

Remark : The proof shows that

$$\left\{ \mathcal{K}_h(a, s) f \mid (a, s) \in \mathcal{A} \times S_{T, K}, \; 0 < h \leqslant \frac{\delta}{6}, |f|_\infty \leqslant 1 \right\}$$

is equi-Hölder-continuous with exponent α, $0 < \alpha < \frac{1}{2}$.

LEMMA 5.2 : *There is a constant* $M_5 > 0$, *independent of* $(a, s) \in \mathcal{A} \times \mathcal{S}_{T, K}$ *such that*

$$|K(t, \tau ; a, s) - K(t, \tau ; \tilde{a}, \tilde{s})| \leqslant$$
$$\leqslant M_5((t - \tau)^{-1/2} |a - \tilde{a}|_\infty + (t - \tau)^{-3/2} |s - \tilde{s}|_\infty) ,$$

holds for $0 < \tau < t$, (a, s), $(\tilde{a}, \tilde{s}) \in \mathcal{A} \times \mathcal{S}_{T, K}$.

Proof : Fix $0 < \tau < t$, (a, s), $(\tilde{a}, \tilde{s}) \in \mathcal{A} \times \mathcal{S}_{T, K}$ and consider

$$|K(t, \tau ; a, s) - K(t, \tau ; \tilde{a}, \tilde{s})| =$$
$$= |a(\tau) N_x(s(t), t ; s(\tau), \tau) - \tilde{a}(\tau) \tilde{N}_x(\tilde{s}(t), t ; \tilde{s}(\tau), \tau)|$$
$$\leqslant \tilde{\kappa} (t - \tau)^{-1/2} |a - \tilde{a}|_\infty + \mu |N_x(s(t), t ; s(\tau), \tau) - \tilde{N}_x(\tilde{s}(t), t ; \tilde{s}(\tau), \tau)|$$

(above $\tilde{N}$ indicates the dependence of N on $(\tilde{a}, \tilde{s})$). A short manipulation yields the estimate

$$|N_x(s(t), t ; s(\tau), \tau) - \tilde{N}_x(\tilde{s}(t), t ; \tilde{s}(\tau), \tau)| \leqslant$$
$$\leqslant \frac{1}{16\sqrt{\pi}} \nu^{-3/2} K (t - \tau)^{-1/2} \left| \frac{(\tilde{s}(t) - \tilde{s}(\tau))^2}{\tilde{\alpha}(t) - \tilde{\alpha}(\tau)} - \frac{(s(t) - s(\tau))^2}{\alpha(t) - \alpha(\tau)} \right|$$
$$+ \frac{1}{4\sqrt{\pi}} \left| \frac{\tilde{s}(t) - \tilde{s}(\tau)}{(\tilde{\alpha}(t) - \tilde{\alpha}(\tau))^{3/2}} - \frac{s(t) - s(\tau)}{(\alpha(t) - \alpha(\tau))^{3/2}} \right| + \frac{1}{4\sqrt{\pi}}$$
$$\times \left| \frac{s(t) + s(\tau)}{(\alpha(t) - \alpha(\tau))^{3/2}} e^{-\frac{(s(t) + s(\tau))^2}{4(\alpha(t) - \alpha(\tau))}} + \frac{\tilde{s}(t) + \tilde{s}(\tau)}{(\tilde{\alpha}(t) - \tilde{\alpha}(\tau))^{3/2}} e^{-\frac{(\tilde{s}(t) + \tilde{s}(\tau))^2}{4(\tilde{\alpha}(t) - \tilde{\alpha}(\tau))}} \right|$$
$$= I + II + III .$$

Observe that for (a, s), $(\tilde{a}, \tilde{s}) \in \mathcal{A} \times \mathcal{S}_{T,K}$

$$\left| \frac{(\tilde{s}(t) - \tilde{s}(\tau))^2}{\tilde{\alpha}(t) - \tilde{\alpha}(\tau)} - \frac{(s(t) - s(\tau))^2}{\alpha(t) - \alpha(\tau)} \right| \leqslant$$

$$\leqslant \nu^{-1}(t-\tau)^{-1} \left| (\tilde{s}(t) - \tilde{s}(\tau))^2 - (s(t) - s(\tau))^2 \right| + \\ + K^2 (t-\tau)^2 \left| (\tilde{\alpha}(t) - \tilde{\alpha}(\tau))^{-1} - (\alpha(t) - \alpha(\tau))^{-1} \right| \\ \leqslant 4 K \nu^{-1} |s - \tilde{s}|_\infty + K^2 \nu^{-2} (t - \tau) |a - \tilde{a}|_\infty ,$$

where in the last inequality we used (3.16) in [8]. Likewise we obtain

$$\left| \frac{\tilde{s}(t) - \tilde{s}(\tau)}{(\tilde{\alpha}(t) - \tilde{\alpha}(\tau))^{3/2}} - \frac{s(t) - s(\tau)}{(\alpha(t) - \alpha(\tau))^{3/2}} \right| \leqslant$$

$$\leqslant 2 \nu^{-3/2} (t - \tau)^{-3/2} |s - \tilde{s}|_\infty + \\ + K(t-\tau) \left| (\tilde{\alpha}(t) - \tilde{\alpha}(\tau))^{-3/2} - (\alpha(t) - \alpha(\tau))^{-3/2} \right| \\ \leqslant 2 \nu^{-3/2} (t-\tau)^{-3/2} |s - \tilde{s}|_\infty + K \frac{3}{2} \frac{\mu^2}{2 \nu^{9/2}} (t - \tau)^{-1/2} |a - \tilde{a}|_\infty ,$$

where in the last inequality we used (3.18) in [8]. Combining the above estimates we deduce the existence of a constant $\tilde{M}_5$ such that

$$I + II \leqslant \tilde{M}_5 [(t - \tau)^{-3/2} |\tilde{s} - s|_\infty + (t - \tau)^{-1/2} |a - \tilde{a}|_\infty] .$$

The proof is completed since the thir term admits the same bound. □

The above proof shows that the following is true :

COROLLARY 5.3 : *There is a constant $M_6 > 0$, independent of $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$, such that*

$$|K(t, \tau ; a, s) - K(t, \tau ; \tilde{a}, \tilde{s})| \leqslant M_6 (t - \tau)^{-1/2} (|a - \tilde{a}|_\infty + |s - \tilde{s}|_\beta)$$

holds for $0 \leqslant \tau < t$, (a, s), $(\tilde{a}, \tilde{s}) \in \mathcal{A} \times \mathcal{S}_{T,K}$.

PROPOSITION 5.4 : *There is a constant $M_7 > 0$, independent of $0 < h \leqslant \frac{\delta}{6}$, $(a, s) \in \mathcal{A} \times \mathcal{S}_{T,K}$, such that*

$$|\mathcal{K}_h(a, s) - \mathcal{K}_h(\tilde{a}, \tilde{s})| \leqslant M_7 (|a - \tilde{a}|_\infty + |s - \tilde{s}|_\beta)$$

holds for (a, s), $(\tilde{a}, \tilde{s}) \in \mathcal{A} \times \mathcal{S}_{T,K}$.

Proof : Choose $f \in C([0, T])$ and assume $t \in I_k$ for some $k \in \{-(j^* + 1), \dots, n - j^* - 2\}$. In view of (5.3) and (5.2),

$$[\mathscr{K}_h(a, s) f](t) - [\mathscr{K}_h(\tilde{a}, \tilde{s}) f](t) =$$

$$= \sum_{j=1}^{k} \int_{I_j} \Lambda_j([K(t, . ; a, s) - K(t, . ; \tilde{a}, \tilde{s})] f)(\tau) \, d\tau$$

$$+ \psi_{j^*+1+k}(t) \int_{I_{k+1}} \Lambda_{k+1}([K(t, . ; a, s) - K(t, . ; \tilde{a}, \tilde{s})] f)(\tau) \, d\tau$$

$$+ \chi_{j^*+1+k}(t) \int_{I_{k+1}} [K(t, \tau ; a, s) - K(t, \tau ; \tilde{a}, \tilde{s})](\Lambda_{k+1} f)(\tau) \, d\tau$$

$$+ \sum_{j=1}^{j^*} \int_{I_{k+j+1}} [K(t, \tau ; a, s) - K(t, \tau ; \tilde{a}, \tilde{s})](\Lambda_{k+j+1} f)(\tau) \, d\tau$$

$$+ \int_{t_{2(k+j^*+1)}}^{t} [K(t, \tau ; a, s) - K(t, \tau ; \tilde{a}, \tilde{s})](\Lambda_{k+j^*+2} f)(\tau) \, d\tau$$

which together with Corollary 5.3 results in

$$\left| [\mathscr{K}_h(a, s) f](t) - [\mathscr{K}_h(\tilde{a}, \tilde{s}) f](t) \right| \leq M_6 \, \ell \, |f|_\infty \times$$

$$\times \left[\sum_{j=1}^{k} (t - t_{2j})^{-1/2} \, 2h + 2h(t - t_{2(k+1)})^{-1/2} + \int_{t_{2k}}^{t} (t - \tau)^{-1/2} \, d\tau \right]$$

$$\times (|a - \tilde{a}|_\infty + |s - \tilde{s}|_\beta) \leq \tilde{M}_7 \, \delta^{-1/2} (|a - \tilde{a}|_\infty + |s - \tilde{s}|_\beta) . \qquad \square$$

PROPOSITION 5.5 : *Let* $\tilde{\mathscr{A}} \subset C[0, T]$ *be compact. Then* $\mathscr{K}_h(a, s)$ *converges strongly in* $C[0, T]$ *to* $\mathscr{K}(a, s)$*, the convergence being uniform in* $(a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T, K}$.

Proof : Because of Proposition 5.1 it suffices to show convergence on a dense subset of $C[0, T]$. Fix $\varepsilon > 0$ and $f \in C^{(3)}([0, T])$. For $\tau \in I_j$, $j = 1, \ldots, n$ we have

$$|\Lambda_j f(\tau) - f(\tau)| \leq \frac{1}{3} |f^{(3)}|_\infty h^3 .$$

Because of Propositions 5.1 and 2.1 there are finitely many $\hat{t}_i$, $i = 1, \ldots, m(\varepsilon)$ such that for any $t \in [0, T]$ there is some $\hat{t}_i$ such that

$$\left| (\mathscr{K}_h(a, s) f)(t) - (\mathscr{K}_h(a, s) f)(\hat{t}_i) \right| +$$

$$+ \left| (\mathscr{K}(a, s) f)(t) - (\mathscr{K}(a, s) f)(\hat{t}_i) \right| \leq \frac{\varepsilon}{4} ,$$

where the time instances $\hat{t}_i$ are independent of $(a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T, K}$. Because of compactness of $\tilde{\mathscr{A}} \times \mathscr{S}_{T, K}$ in $C([0, T]) \times C([0, T])$ it is also possible to choose a finite net $(\hat{a}_r, \hat{s}_r) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T,K}$, $r = 1, \dots, M(\varepsilon)$, such that for any $(a, s) \in \tilde{\mathscr{A}} \times \mathscr{S}_{T,K}$ there is some $(\hat{a}_r, \hat{s}_r)$ with

$$\left|a - \hat{a}_r\right|_\infty + \left|s - \hat{s}_r\right|_\infty \leq M_9\, \delta^{-3/2} \frac{\varepsilon}{4}$$

where M_9 is a constant depending on M_5 and $|\Lambda_j|$ and will be specified below. Fix $t \in [0, T]$ and choose $\hat{t}_i$ as above. We determine a unique index $k_i \in \{-(j^* + 1), \dots, n - j^* - 2\}$ by $\hat{t}_i \in [t_{2(j^* + k_i + 1)}, t_{2(j^* + k_i + 2)})$. As a consequence we obtain the estimate

$$\begin{aligned}
&\left|[\mathscr{K}_h(a, s) f](t) - [\mathscr{K}(a, s) f](t)\right| \leq \frac{3}{4} + \\
&\quad + \left|[\mathscr{K}_h(a, s) f](\hat{t}_i) - [\mathscr{K}(a, s) f](\hat{t}_i)\right| \\
&\leq \frac{\varepsilon}{4} + \left|\sum_{j=1}^{k_i}\left[\mathscr{S}_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \cdot\,; a, s) f - \int_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \tau\,; a, s) f(\tau)\, d\tau\right]\right| \\
&\quad + \int_{t_{2k_i}}^{t_{2(k_i+1)}} \left|\Lambda_{k_i+1}(K(\hat{t}_i, \cdot\,; a, s) f)(\tau) - K(\hat{t}_i, \tau\,; a, s) f(\tau)\right| d\tau \\
&\quad + \int_{t_{2k_i}}^{t_{2(k_i+1)}} \left|K(\hat{t}_i, \tau\,; a, s)\right| \left|(\Lambda_{k_i+1} f)(\tau) - f(\tau)\right| d\tau \\
&\quad + \sum_{j=1}^{j^*} \int_{t_{2(k_i+j)}}^{t_{2(k_i+j+1)}} \left|K(\hat{t}_i, \tau\,; a, s)\right| \left|(\Lambda_{k_i+j+1} f)(\tau) - f(\tau)\right| d\tau \\
&\quad + \int_{t_{2(k_i+j^*+1)}}^{\hat{t}_i} \left|K(\hat{t}_i, \tau\,; a, s)\right| \left|(\Lambda_{k_i+j^*+2} f)(\tau) - f(\tau)\right| d\tau \\
&\leq \frac{\varepsilon}{4} + \left|\sum_{j=1}^{k_i}\left[\mathscr{S}_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \cdot\,; a, s) f - \int_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \tau\,; a, s) f(\tau)\, d\tau\right]\right| \\
&\quad + 2\,\mu\,\tilde{\kappa}\,|f|_\infty\, 2\,h(\hat{t}_i - t_{2(k_i+1)})^{-1/2}\, 2(\ell + 1) \\
&\quad + 2\,\mu\,\frac{\tilde{\kappa}}{3}\,|f^{(3)}|_\infty\, h^3 \int_0^{\hat{t}_i} (\hat{t}_i - \tau)^{-1/2}\, d\tau \\
&\leq \frac{\varepsilon}{4} + M_8 |f^{(3)}|_\infty\, \delta^{-1/2}\, h \\
&\quad + \left|\sum_{j=1}^{k_i}\left[\mathscr{S}_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \cdot\,; a, s) f - \int_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \tau\,; a, s) f(\tau)\, d\tau\right]\right|.
\end{aligned}$$

Finally, the last term is estimated exploiting the compactness of $\tilde{\mathscr{A}} \times \mathscr{S}_{T,K}$

$$\left| \sum_{j=1}^{k_i} \left[\mathscr{S}_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, . ; a, s) f - \int_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \tau ; a, s) f(\tau) \, d\tau \right] \right| \leqslant$$

$$\leqslant \sum_{j=1}^{k_i} \int_{t_{2j-2}}^{t_{2j}} \left| A_j([K(\hat{t}_i, . ; a, s) - K(\hat{t}_i, . ; \hat{a}_r, \hat{s}_r)] f)(\tau) \right| d\tau$$

$$+ \sum_{j=1}^{k_i} \int_{t_{2j-2}}^{t_{2j}} \left| [K(\hat{t}_i, \tau ; a, s) - K(\hat{t}_i, \tau ; \hat{a}_r, \hat{s}_r)] f(\tau) \right| d\tau$$

$$+ \left| \sum_{j=1}^{k_i} \left[\mathscr{S}_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, . ; \hat{a}_r, \hat{s}_r) f - \int_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \tau ; \hat{a}_r, \hat{s}_r) f(\tau) \, d\tau \right. \right|$$

$$\leqslant M_9 \, \delta^{-3/2} \, |f|_\infty (|a - \hat{a}_r|_\infty + |s - \hat{s}_r|_\infty)$$

$$+ \left| \sum_{j=1}^{k_i} \left[\mathscr{S}_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, . ; \hat{a}_r, \hat{s}_r) f - \int_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \tau ; \hat{a}_r, \hat{s}_r) f(\tau) \, d\tau \right. \right| .$$

The last inequality follows from Lemma 5.2 and (5.4) and defines the constant M_9 which was used in the construction of the finite covering of $\tilde{\mathscr{A}} \times \mathscr{S}_{T,K}$. The final term may be viewed as a Simpson approximation to $\int_0^{\hat{t}_i - \delta} K(\hat{t}_i, \tau ; \hat{a}_r, \hat{s}_r) f(\tau) \, d\tau$, $i = 1, \ldots, m$, $r = 1, \ldots, M$. Because of continuity of the integrands and convergence of Simpson's rule on continuous functions it is possible to choose $0 < h_0 \leqslant \frac{\delta}{6}$ such that $0 < h \leqslant h_0$ ensures

$$\left| \sum_{j=1}^{k_i} \left[\left[\int_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, . ; \hat{a}_r, \hat{s}_r) f - \int_{t_{2j-2}}^{t_{2j}} K(\hat{t}_i, \tau ; \hat{a}_r, \hat{s}_r) f(\tau) \, d\tau \right] \right| < \frac{\varepsilon}{4}$$

$i = 1, \ldots, m(\varepsilon)$, $r = 1, \ldots, M(\varepsilon)$, which completes the proof of Proposition 5.5. □

THEOREM 5.6 : *Hypothesis* (H) *of section 3 holds for* $\mathscr{K}_h$ *as defined by* (5.3), *with* $h_0 = \frac{\delta}{6}$.

Remark : As a final comment we mention that the numerical solution of (3.1) based on the approximation (5.3) results in a linear system of order 2 n, the coefficient matrix of which is of lower Hessenberg form.

6. EXAMPLES

Following [7], we find that a systematic way to generate a class of test examples is to choose $a \in \mathscr{A}$ and define

$$s(t) = 1 + \int_0^t a(\tau)\, d\tau \quad t \in [0, T]$$

$$u(x, t) = e^{s(t) - x} - 1 \quad (t, x) \in [0, T] \times [0, 1].$$

It can be verified that (u, s) is a classical solution of the Stefan problem (1.1), (1.2) with g and φ determined by u. In the example which is discussed below, we took the coefficient

$$a^*(t) = 0.5 \arctan(50(t - 0.6)) + 2\,, \quad t \in [0, 1].$$

We note that a^* has a steep slope at $t = 0.6$. The data were obtained by evaluating the true u^* at $(x, t) = (0, t_i)$, $t_i = \frac{i}{2n}$, $i = 1, \dots, 2n$ (i.e., $m = 2n$). In the approximation technique described in Section 5 we took $\delta = 0.01$ and $m = 2n = 20$. The coefficient a in the cost functional $J1_h^N(a)$ was replaced by its interpolating spline based on a partitioning of $[0, 1]$ into $na = 16$ equally spaced subintervals. This gridsize for a guarantees that there is at least one mesh point in the region where a^* changes rapidly. The approximate integral equation (3.1) is equivalent to the linear system

$$(I - \tilde{K})\, x = g$$

where $x \in \mathbb{R}^{2n}$ is an approximation to the solution v of (2.3), evaluated at t_i, $i = 1, \dots, 2n$, $g \in \mathbb{R}^{2n}$ is the vector obtained by evaluating $\mathscr{G}$ at the same time instances. The entries of the matrix $\tilde{K} \in \mathbb{R}^{2n \times 2n}$ involve either the evaluation of K at certain quadrature points (from Simpson's rule) or « singular » integrals arising from the quadratic interpolation terms. The Volterra structure of the kernel K leads to a lower Hessenberg form of $\tilde{K}$. To solve $\mathrm{ID}1_h^N$ we consider

$$\operatorname*{Min}_{\alpha \in \mathbb{R}^{na+1}} J1_h^N(\alpha)\,, \tag{6.1}$$

where $= (\alpha_0, \dots, \alpha_{na}) \in \mathbb{R}^{na+1}$ with $\alpha_i = a\left(\frac{i}{na}\right)$, $i = 0, \dots, na$ is the coordinate vector of a piecewise linear spline function. The optimization problem (6.1) was solved using LMDIF, an implementation of the Levenberg-Marquardt algorithm in MINPACK. Numerically we did not realize the constraints involved in the definition of $\tilde{\mathscr{A}}$. Positivity of the coordinates of $\mathfrak{a}$ was guaranteed by restricting the initial stepsize of the search algorithm. The fixed point iteration as well as the iteration involved in the iterative solution of (6.1) were terminated once two successive iterates of s and a respectively coincided in a specified number of digits (5 in our particular example). Table 1 summarizes our numerical findings for different initial guesses for a. All calculations were done on a STARDENT 3000.

TABLE 1.

	CPU-sec	#MBVP	$J1_h^N(\mathfrak{a})$	$\lvert a - a^* \rvert_\infty$	$\lvert s - s^* \rvert_\infty$	$\lvert u - u^* \rvert_\infty$
$a_0 \equiv 2$	248	834	$0.124.10^{-2}$	0.104	0.104	0.022
$a_0 \equiv 3$	488	1 603	$0.124.10^{-2}$	0.104	0.014	0.022
$a_0 \equiv 5$	482	1 515	$0.124.10^{-2}$	0.104	0.014	0.022
$a_0 \equiv 10$	587	1 756	$0.124.10^{-2}$	0.104	0.014	0.022

#MBVP denotes the number of Volterra integral equations (3.1) which were solved. The above results show that the quality of the identification is maintained for a wide range of initial guesses for a. Figure 1 compares the identified parameter with the discretized true parameter a^*. It is apparent that almost all of the error is concentrated in the small interval of rapid change of a^*.

Each evaluation of the cost functional $J1_h^N$ requires the approximate solution of a Stefan problem. This suggests an alternative algorithm which interchanges the optimization loops in the iterative solution of (6.1) and thereby eliminates the iteration (3.3) from the evaluation of the cost functional. At each fixed level of this alternative process the variables (u, s) are not solutions of a Stefan problem anymore.

More precisely, define for each $s \in \mathscr{S}_{T,K}$ an approximate cost functional

$$J3_h(a\,; s) = \sum_{i=1}^{m} \left| z_i - u_h(0, t_i\,; a, s) \right|^2$$

where u_h is obtained from (2.2) with the given a and s, and $v_h(a, s)$ the corresponding solution of (3.1).

The associated optimization problem is

$$(\mathrm{ID3}_h) \qquad \operatorname*{Min}_{\mathfrak{a} \in \mathbb{R}^{na+1}} J3_h(\mathfrak{a}\,; s)\,.$$

The alternative algorithm consists of the following iteration :

Guess a^0, s^0.
For $k = 1, 2, \ldots, N$
1) Solve ($\mathrm{ID3}_h$) (with s^{k-1} held fixed) to obtain a^k.
2) Solve the approximate integral equation (3.1) :

$$(\mathfrak{I} - \mathscr{K}_h(a^k, s_h^{k-1}))\, v_h(a^k, s_h^{k-1}) = \mathscr{G}(a^k, s_h^{k-1})$$

3) Update the boundary : (6.2)

$$s_h^k(t) = b - \int_0^t p[a^k(\tau)\, v_h(\tau\,;\, a^k, s_h^{k-1})]\, d\tau$$

4) Repeat 2) and 3) until a specified tolerance stol is reached and go to 1).

At this point we have not carried out a theoretical analysis of algorithm (6.2). However, we can report about successful numerical experiments. We reran the same example as above with the same parameters for stopping criteria, dimensions of approximating subspaces and $stol = 10^{-3}$. The boundary was initialized by choosing s^0 the Stefan boundary corresponding to a^0.

TABLE 2.

	CPU-sec	#MBVP	$J3_h(a)$	$\lvert a - a^* \rvert_\infty$	$\lvert s - s^* \rvert_\infty$	$\lvert u - u^* \rvert_\infty$
$a_0 \equiv 2$	180	324	$0.124.10^{-2}$	0.104	0.015	0.022
$a_0 \equiv 3$	309	531	$0.124.10^{-2}$	0.104	0.015	0.022
$a_0 \equiv 5$	245	434	$0.124.10^{-2}$	0.104	0.014	0.022
$a_0 \equiv 10$	364	587	$0.124.10^{-2}$	0.104	0.015	0.022

Table 2 shows that using the alternative algorithm one obtains virtually the same estimates for a as with the standard algorithm. However, the computational effort involved is considerably smaller. Additional numerical experiments show that the alternative algorithm is also more robust with respect to the initial guess for a than the original algorithm. Figure 2 depicts the first 5 iterates of the boundary s^k as defined in (6.2) (with $a^0 \equiv 3$). Figure 3 displays the corresponding minimizers a^k of $\mathrm{ID3}_h$ with s replaced by s^{k-1}. We note that iterates 2-5 cannot be distinguished on the plot and are very close to a^*. If in algorithm (6.2) we only carry out 1 update of s then the iterates a^k and s^k approach a^*, s^* in an alternating monotonic fashion with the even iterates converging from above and the odd iterates converging from below. We also carried out calculations with relative, uniformly distributed noise σ added to the data at the grid points while the remaining parameters *na*, *n* and the stopping criteria remained unchanged. The results for the original and for the alternative algorithm with $a^0 \equiv 3$ are shown in Table 3 and Table 4 respectively.

TABLE 3.

	CPU-sec	#MBVP	$J1_h^N(a)$	$\mid a - a^* \mid_\infty$	$\mid s - s^* \mid_\infty$	$\mid u - u^* \mid_\infty$
$\sigma \equiv 0.01$	540	1 781	$0.126.10^{-2}$	0.127	0.031	0.162
$\sigma \equiv 0.05$	488	1 599	$0.135.10^{-2}$	0.250	0.102	0.812
$\sigma \equiv 0.10$	449	1 447	$0.124.10^{-2}$	0.467	0.185	1.624

TABLE 4.

	CPU-sec	#MBVP	$J3_h(a)$	$\mid a - a^* \mid_\infty$	$\mid s - s^* \mid_\infty$	$\mid u - u^* \mid_\infty$
$\sigma \equiv 0.01$	263	468	$0.126.10^{-2}$	0.129	0.030	0.162
$\sigma \equiv 0.05$	226	394	$0.135.10^{-2}$	0.250	0.103	0.812
$\sigma \equiv 0.10$	301	514	$0.146.10^{-2}$	0.467	0.186	1.624

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