M2AN - Modélisation mathématique et analyse numérique

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M2AN - Modélisation mathématique et analyse numérique, tome 27, nº 5 (1993), p. 565-589
<http://www.numdam.org/item?id=M2AN_1993__27_5_565_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 27, n° 5, 1993, p. 565 à 589)

EXTERNAL FINITE ELEMENT APPROXIMATIONS OF EIGENVALUE PROBLEMS (*)

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Communicated by J. DESCLOUX

Abstract. — *The paper is devoted to the finite element analysis of second order elliptic eigenvalue problems in the case when the approximate domains* Ω_h *are not subdomains of the original domain* $\Omega \subset \mathbb{R}^2$. *The considerations are restricted to piecewise linear approximations and in the case of eigenfunctions to simple eigenvalues. The optimum rates of convergence for both the approximate eigenvalues and the approximate eigenfunctions are obtained.*

Résumé. — *Cet article est consacré à l'analyse des problèmes elliptiques spectraux du second ordre par la méthode des éléments finis dans le cas où l'ouvert approché* Ω_h *n'est pas contenu dans l'ouvert original* $\Omega \subset \mathbb{R}^2$. *Les développements sont faits pour des approximations linéaires par morceaux et dans le cas des fonctions propres pour des valeurs propres simples. On obtient des ordres optimaux de convergence pour l'approximation des valeurs propres et des fonctions propres à la fois.*

1. INTRODUCTION

In [11, Chapter 6] the second order elliptic eigenvalue problems are approximated by the finite element method in the case of domains Ω which are such that $\Omega_h \subseteq \Omega$, Ω_h being the approximate polygonal domain.

The aim of our paper is to generalize the results of [11, Chapter 6] to the case where approximate domains Ω_h are not subdomains of the given bounded domain $\Omega \subset \mathbb{R}^2$. We analyse piecewise linear finite element approximations and (similarly as in [11]) we do not take into account the numerical integration and restrict our considerations concerning the eigen-

(*) Manuscrit reçu le 21 juillet 1992.

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functions to the case of simple exact eigenvalues. To our knowledge this paper is the first one devoted to eigenvalue problems where the situation $\Omega_h \not\subset \Omega$ is considered.

When the exact eigenfunctions w_m belong to $V \equiv H_0^1(\Omega)$ we prove the convergence of both the approximate eigenvalues and the approximate eigenfunctions. In the case when the exact eigenfunctions belong to $V \cap H^2(\Omega)$ we prove the optimum rates of convergence, i.e.

$$|\lambda_m - \lambda_{m,h}| \leq C(m)\, h^2 ,$$

$$\|\tilde{w}_m - w_{m,h}\|_{i,\Omega_h} \leq C(m) \cdot h^{2-i} \quad (i = 0, 1),$$

$\tilde{w}_m$ being an extension of w_m.

The convergence of approximate eigenfunctions corresponding with multiple exact eigenvalues and the effect of numerical integration will be considered in a subsequent paper.

2. FORMULATION OF THE PROBLEM

Let Ω be a bounded two-dimensional domain with a Lipschitz-continuous boundary which is piecewise of class C^3. (This means that Ω may have corners.) Let $\Omega_h \subset \mathbb{R}^2$ be a polygonal domain approximating Ω. Let us assume that all vertices of the polygonal boundary $\partial\Omega_h$ are lying on $\partial\Omega$ and that all points where the condition of C^3-smoothness of $\partial\Omega$ is not fulfilled are vertices of $\partial\Omega_h$. We have $\Omega_h \not\subset \Omega$.

Further, let $\tilde{\Omega} \subset \mathbb{R}^2$ be a bounded domain satisfying $\tilde{\Omega} \supset \bar{\Omega} \cup \bar{\Omega}_h$, regardless of h. Next, let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be a bilinear form given by

$$a(u, v) = \int_\Omega \sum_{i,j=1}^{2} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx , \quad \forall u, v \in H^1(\Omega), \tag{2.1}$$

where the coefficients k_{ij}, $i, j = 1, 2$, are defined on $\tilde{\Omega}$ and satisfy

$$\text{(i)} \quad k_{ij} \in W^{1,\infty}(\tilde{\Omega}),\ i, j = 1, 2 , \tag{2.2}$$

$$\text{(ii)} \quad k_{12} = k_{21} \quad \text{a.e. in} \quad \tilde{\Omega} , \tag{2.3}$$

$$\text{(iii)} \quad \exists \mu_0 > 0 : \sum_{i,j=1}^{2} k_{ij}(x)\, \xi_i\, \xi_j \geq \mu_0(\xi_1^2 + \xi_2^2) \quad \text{a.e. in} \quad \tilde{\Omega} , \quad \forall \xi_1, \xi_2 \in \mathbb{R}. \tag{2.4}$$

We consider then the following eigenvalue problem, associated with $a(\cdot\,,\cdot)$:

2.1. Problem

Find $\lambda \in \mathbb{R}$, $w \in V \equiv H_0^1(\Omega)$ $(w \neq 0)$:

$$a(w, v) = \lambda (w, v) \quad \forall v \in V \equiv H_0^1(\Omega), \tag{2.5}$$

with $(. , .)$ the inner-product in $L_2(\Omega)$.

The pairs $\{\lambda, w\}$ are called eigenpairs, λ is an eigenvalue and w is an eigenfunction.

2.2. *Remark :* $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$, $H^m(\Omega) = W^{m,2}(\Omega)$ and $H_0^m(\Omega)$ are the usual notations for the Sobolev spaces (see, e.g., [4], [11], [13]). The symbols $\| . \|_{k,\Omega}$ and $| . |_{k,\Omega}$ denote the norm and the semi-norm, respectively, in $H^k(\Omega)$ [we have $L_2(\Omega) = H^0(\Omega)$].

Assumptions (2.2)-(2.4), properties of $L_2(\Omega)$ and $H_0^1(\Omega)$, and [11, Theorem 6.2-1] imply the following theorem :

2.3 THEOREM : *The eigenvalues of Problem* 2.1 *form an increasing sequence,*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots \to +\infty .$$

The corresponding eigenfunctions $\{w_m\}_{m=1}^{\infty}$ *form a Hilbertian orthonormal basis of* $L_2(\Omega)$.

In this paper we shall approximate Problem 2.1 by the finite element method using triangular finite C^0-elements with polynomials of first degree.

2.4. Triangulations

We consider a triangulation ([11, § 1.5]) $\mathcal{T}_h$ of the polygonal domain $\bar{\Omega}_h$, consisting of a finite number of closed triangles $\bar{T}$. Denoting by σ_h the set of all vertices (nodal points) of $\mathcal{T}_h$, we assume :

(i) $\sigma_h \subset \bar{\Omega}$, $\sigma_h \cap \partial\Omega_h \subset \partial\Omega$,

(ii) the points of $\partial\Omega$, where the condition of C^3-smoothness of $\partial\Omega$ is not satisfied, are elements of σ_h.

The symbols h_T and θ_T denote the length of the maximum side and the magnitude of the minimum angle of $\bar{T} \subset \mathcal{T}_h$, respectively. We set

$$h = \max_{\bar{T} \in \mathcal{T}_h} h_T .$$

We assume that the family of triangulations $\{\mathcal{T}_h\}_{h \in (0, h_0)}$, $h_0 > 0$, satisfies *the minimum angle condition* (see [11, (5.1-21)]) and *the inverse assumption* (see [4, (3.2.28)]).

We shall consider only such triangulations $\mathcal{T}_h$ that at most two vertices of each $\bar{T} \in \mathcal{T}_h$ lie on $\partial\Omega$. A *straight* triangle with two vertices on $\partial\Omega$ is called a *boundary* triangle.

2.5. Ideal triangulations.

Let $\bar{T} \in \mathcal{T}_h$ be a *boundary* triangle. We associate with it an *ideal* triangle $\bar{T}^{id}$. This closed curved triangle is obtained from $\bar{T}$ replacing the side of $\bar{T}$ which approximates a part of $\partial\Omega$, by this part of $\partial\Omega$. When we replace all boundary triangles in $\mathcal{T}_h$ by the associated ideal triangles $\bar{T}^{id}$ we obtain the *ideal* triangulation $\mathcal{T}_h^{id}$ of the domain $\bar{\Omega}$ associated with $\mathcal{T}_h$.

2.6. *Remark :* For simplicity, we shall assume in Sections 4, 5 and 6 that the triangulations $\mathcal{T}_h \in \{\mathcal{T}_h\}$ are constructed in such a way that for all boundary triangles lying along the boundary $\partial\Omega$ we have either $\bar{T} \subset \bar{T}^{id}$ or $\bar{T} \supset \bar{T}^{id}$.

2.7. *Remark :* Following the terminology of [6] we call Ω_h an *internal* approximation of Ω if $\Omega_h \subset \Omega$. In the opposite case $\Omega_h \not\subset \Omega$ we call Ω_h an *external* approximation of Ω. To our knowledge the finite element approximations of Problem 2.1 have been studied till now only in the case of domains Ω having internal approximations Ω_h (see [1, 2, 3], [11] and the references in [1, 2, 3]). □

With the triangulation of $\bar{\Omega}_h$ we associate the finite element spaces

$$X_h = \left\{v_h \in C^0(\bar{\Omega}_h) : v_h|_{\bar{T}} = \text{a linear polynomial } \ \forall \bar{T} \in \mathcal{T}_h\right\} \subset H^1(\Omega_h), \tag{2.6}$$

$$V_h = \{v_h \in X_h : v_h = 0 \quad \text{on} \quad \partial\Omega_h\} \subset H_0^1(\Omega_h). \tag{2.7}$$

We have in general $X_h \not\subset H^1(\Omega)$ and $V_h \not\subset V \equiv H_0^1(\Omega)$.

Let us set

$$a_h(u, v) = \sum_{i,j=1}^{2} \int_{\Omega_h} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \quad \forall u, v \in H^1(\Omega_h), \tag{2.8}$$

$$(u, v)_h = \int_{\Omega_h} uv \, dx \quad \forall u, v \in L_2(\Omega_h). \tag{2.9}$$

From (2.2)-(2.4) and [13, p. 221, case $\Gamma_1 = \partial\Omega$] it is seen that $a_h(\,.\,,\,.\,)$ has the following properties :

(i) $a_h(u, v) = a_h(v, u) \quad \forall u, v \in H^1(\Omega_h)$; (2.10)

(ii) $\exists C_1 > 0 : |a_h(u, v)| \leq C_1 |u|_{1, \Omega_h} |v|_{1, \Omega_h}$

$$\forall u, v \in H^1(\Omega_h), \forall h ; \quad (2.11)$$

(iii) $\exists C_2 > 0 : a_h(v, v) \geq \mu_0 |v|^2_{1, \Omega_h} \geq C_2 \|v\|^2_{1, \Omega_h}$

$$\forall v \in H^1_0(\Omega_h), \forall h . \quad (2.12)$$

The first inequality in (2.12) holds also for all $v \in H^1(\Omega_h)$.

With $a_h(.,.)$ we associate the continuous eigenvalue problem on Ω_h :

2.8. Problem

Find $\lambda^{(h)} \in \mathbb{R}$, $w^{(h)} \in H^1_0(\Omega_h)$ $(w^{(h)} \neq 0)$:

$$a_h(w^{(h)}, v) = \lambda^{(h)}(w^{(h)}, v)_h \quad \forall v \in H^1_0(\Omega_h) .$$

The consistent mass (internal) approximation of it reads :

2.9. Problem

Find $\lambda_h \in \mathbb{R}$, $w_h \in V_h$ $(w_h \neq 0)$:

$$a_h(w_h, v_h) = \lambda_h(w_h, v_h)_h \quad \forall v_h \in V_h . \quad (2.13)$$

The properties (2.10)-(2.12) of a_h, the properties of $L_2(\Omega_h)$ and of $H^1_0(\Omega_h)$, and [11, Theorem 6.4-1] imply the following theorem :

2.10. THEOREM : *The eigenvalues of Problem 2.9 form an increasing finite sequence*

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{I,h} \quad (I(h) \equiv \dim V_h) .$$

There exists a basis of V_h, consisting of eigenfunctions $w_{m,h}(1 \leq m \leq I(h))$, which satisfy

$$(w_{i,h}, w_{j,h})_h = \delta_{ij} . \quad (2.14)$$

2.11. *Remarks :* *a*) In accordance with Remark 2.7, in the case $\Omega_h \not\subset \Omega$ we call the eigenpair $\{\lambda_{m,h}, w_{m,h}\}$ an *external* finite element approximation of $\{\lambda_m, w_m\}$.

b) As this paper is a generalization of [11, Chapter 6] we use the same notations as in [11, Chapter 6].

c) The main tool in generalizing [11, Chapter 6] will be the simultaneous consideration of $v_h \in V_h$ and $\hat{v}_h \in V$, where $\hat{v}_h$ is the function associated with v_h [a generalization of the techniques developed in [12], [5] and [13]].

d) As usual, the symbol C will denote a generic constant independent of h with generally different values at any two different places.

3. EXTENSION THEOREM

In the case of external finite element approximations we estimate the expressions of the form $\left\| \tilde{w}_m - w_{m,h} \right\|_{i, \Omega_h}$ $(i = 0, 1)$, where $\tilde{w}_m$ is an extension of w_m. As we restrict our considerations to two-dimensional problems we can use the following theorem proved in [10, Section I.3] to define this extension :

3.1. THEOREM : *Let Ω be a bounded two-dimensional domain with a Lipschitz-continuous boundary $\partial\Omega$ which is piecewise of class C^k, $k \geq 1$. Then there exists a linear and bounded extension operator $\mathscr{E} : H^k(\Omega) \to H^k(\mathbb{R}^2)$. The operator $\mathscr{E}$ is also a linear and bounded extension operator from $H^{k-i}(\Omega)$ into $H^{k-i}(\mathbb{R}^2)$, $1 \leq i \leq k$.*

3.2. *Remark :* For a given k we set $\tilde{u} = \mathscr{E}(u)$ for all $u \in H^k(\Omega)$ (usually $k = 2$).

4. SOME AUXILIARY RESULTS

Let $\bar{T}_0$ be the reference triangle which lies in the (ξ_1, ξ_2)-plane and has the vertices $P_1^*(0, 0)$, $P_2^*(1, 0)$ and $P_3^*(0, 1)$. We consider the usual affine invertible mapping F_T which maps the reference triangle $\bar{T}_0$ onto the triangle $\bar{T} \in \mathcal{T}_h$ (see [13, Theorem 9.1]). In addition we need a one-to-one mapping $F_{T^{id}} : \bar{T}_0 \to \bar{T}^{id}$. For this we lean on a suitable analytical representation of the curved side of $\bar{T}^{id} \in \mathcal{T}_h^{id}$. (For details see [13, Section 22].)

4.1. LEMMA : *To every $v_h \in V_h$ there exists a function $\hat{v}_h \in V$ (called to be associated with v_h) with the following properties :*

a) $\hat{v}_h \in C^0(\bar{\Omega})$;

b) $\hat{v}_h(P_i) = v_h(P_i) \quad \forall P_i \in \sigma_h$;

c) *$\hat{v}_h$ is linear on each triangle $\bar{T} \in (\mathcal{T}_h \cap \mathcal{T}_h^{id})$* ;

d) *if $\bar{T} \subset \bar{T}^{id}$ then $\hat{v}_h = 0$ on $\bar{T}^{id} - \bar{T}$ and $\hat{v}_h = v_h$ on $\bar{T}$* ;

e) *if $\bar{T}^{id} \subset \bar{T}$ then $\hat{v}_h = v_h \circ F_T \circ F_{T^{id}}^{-1}$ on $\bar{T}^{id}$.*

Proof : This follows from the definition and the properties of $F_{T^{id}}$ (see [13, Sections 22, 23]). □

4.2. LEMMA : *Let $\hat{v}_h \in V$ be associated with $v_h \in V_h$ and let $\bar{T} \supset \bar{T}^{id}$. Assume that the Lipschitz-continuous boundary $\partial\Omega$ of Ω is piecewise of class C^3. Then*

$$\left\| v_h - \hat{v}_h \right\|_{i, T^{id}} \leq C h^{2-i} \left\| v_h \right\|_{2, T^{id}} = C h^{2-i} \left\| v_h \right\|_{1, T^{id}} \quad (i = 0, 1).$$

Proof : This follows immediately from [14, Theorem 2] (see also [13, Section 25]), Lemma 4.1 and the linearity of v_h on $\bar{T}$. □

4.3. *Notation :* We denote

$$\tau_h = \Omega_h - \bar{\Omega}\,, \quad \omega_h = \Omega - \bar{\Omega}_h\,, \tag{4.1}$$

$$M_h = \left\{\bar{T}^{id} \in \mathcal{C}_h^{id} : \bar{T}^{id} \subset \bar{T}\right\}\,, \tag{4.2}$$

$$S_h = \bigcup_{\bar{T} \in N_h} \bar{T}\,, \quad N_h = \left\{\bar{T} \in \mathcal{C}_h : \bar{T} \supset \bar{T}^{id},\ \bar{T}^{id} \in \mathcal{C}_h^{id}\right\}\,. \tag{4.3}$$

4.4. LEMMA : *We have*

$$\|v\|_{0,\varepsilon_h} \leq Ch\|v\|_{1,\tilde{\Omega}}\,, \quad \forall v \in H^1(\tilde{\Omega})\ (\varepsilon = \tau, \omega)\,, \tag{4.4}$$

$$\|v\|_{0,\varepsilon_h} \leq Ch^2|v|_{1,\varepsilon_h}\,, \quad \forall v \in H^1(\tilde{\Omega}) \text{ with } \operatorname{tr} v = 0 \text{ on } \partial\Omega\,. \tag{4.5}$$

Proof : For the proof of relations (4.4) see [13, Lemma 28.3] ; estimates (4.5) are a simple consequence of the cited proof. □

4.5. LEMMA : *We have*

$$|v|_{1,\tau_h} \leq Ch^{1/2}|v|_{1,S_h} \leq Ch^{1/2}|v|_{1,\Omega_h} \quad \forall v \in X_h\,, \tag{4.6}$$

$$\|v\|_{0,\tau_h} \leq Ch\|v\|_{1,\Omega_h} \quad \forall v \in X_h\,, \tag{4.7}$$

$$\|v\|_{0,\tau_h} \leq Ch^2|v|_{1,\tau_h} \quad \forall v \in V_h\,. \tag{4.8}$$

Proof : Relations (4.6)-(4.7) are proved in [5, Lemma 3.3.12] ; estimate (4.8) is a consequence of the proof of (4.7). □

4.6. LEMMA : *We have*

$$\|v\|_{0,S_h} \leq Ch^{1/2}\|v\|_{1,\tilde{\Omega}} \quad \forall v \in H^1(\tilde{\Omega})\,, \tag{4.9}$$

$$\|v\|_{0,S_h-\tau_h} \leq Ch|v|_{1,S_h-\tau_h} \quad \forall v \in H_0^1(\Omega)\,, \tag{4.10}$$

where the set S_h is defined by (4.3).

Proof : The proof is a simple modification of the proof of [5, Lemma 3.3.11]. We use the fact that the distance of the vertex of $\bar{T} \subset S_h$, which lies in the interior of Ω, to the opposite side is $O(h)$. □

4.7. DEFINITION : *Let $u \in V \cap H^k(\Omega)$. We define the elliptic projection (also called the Ritz approximation) $\Pi_h u \in V_h$ of u by*

$$a_h(\tilde{u} - \Pi_h u, v_h) = 0 \quad \forall v_h \in V_h\,, \tag{4.11}$$

where $\tilde{u} = \mathscr{E}(u)$ *and* $\mathscr{E} : H^k(\Omega) \to H^k(\mathbb{R}^2)$ *is an extension operator from Theorem 3.1.*

4.8. THEOREM : *We have*

$$\left\| \tilde{u} - \Pi_h u \right\|_{1, \Omega_h} \leq C h \| u \|_{2, \Omega} \quad \forall u \in V \cap H^2(\Omega) . \tag{4.12}$$

In addition, if the boundary $\partial \Omega$ *is of class* $\mathscr{C}^{1,1}$ *(for the notation* $\mathscr{C}^{1,1}$ *see [4, p. 12]) and the coefficients* $k_{ij} \in C^{0,1}(\bar{\Omega})$ *then*

$$\left\| \tilde{u} - \Pi_h u \right\|_{0, \Omega_h} \leq C h^2 \| u \|_{2, \Omega} \quad \forall u \in V \cap H^2(\Omega) . \tag{4.13}$$

Proof : A) Let $u_h^{int} \in V_h$ be the interpolant of u, i.e. $u_h^{int}(P_i) = u(P_i)$ for all $P_i \in \sigma_h$. Using (2.12), (4.11) and (2.11), we obtain

$$\left\| u_h^{int} - \Pi_h u \right\|_{1, \Omega_h} \leq \frac{C_1}{C_2} \left\| u_h^{int} - \tilde{u} \right\|_{1, \Omega_h} . \tag{4.14}$$

Combining (4.14) with the standard finite element interpolation theorem and Theorem 3.1 we obtain (4.12).

B) The form of V and the assumptions concerning $\partial \Omega$ and k_{ij} enable us to use [8, Theorem 4.2.1] and we see that relations [9, (45)] are satisfied. Thus we can repeat all considerations introduced in [9, pp. 416-419]. To obtain estimate (4.13) we note only that [9, (62)] can be improved to the form

$$\| \tilde{u} \|_{0, \partial \Omega_h} \leq C h^2 \| \tilde{u} \|_{2, \tilde{\Omega}} \quad \forall u \in V \cap H^2(\Omega) . \qquad \square$$

4.9. THEOREM : *Let* $\{h\} = \{h_j\}_{j=1}^{\infty}$ *with* $\lim_{j \to \infty} h_j = 0$. *Then*

$$\lim_{h_j \to 0} \left\| \tilde{u} - \Pi_{h_j} u \right\|_{1, \Omega_{h_j}} = 0 \quad \forall u \in V . \tag{4.15}$$

Proof : According to [13, Theorem 31.4] for every $u \in V$ there exists a sequence $\{u_{h_j}\}$, where $u_{h_j} \in V_{h_j}$, such that

$$\lim_{h_j \to 0} \left\| \tilde{u} - u_{h_j} \right\|_{1, \Omega_{h_j}} = 0 . \tag{4.16}$$

The functions u_{h_j} satisfy an estimate similar to (4.14) (replace u_h^{int} by u_{h_j}) which combined with (4.16) leads to (4.15). $\square$

5. CONVERGENCE OF APPROXIMATE EIGENVALUES

To extend the results from [11, Section 6.4] we first present some lemmas.

5.1. LEMMA : *Let* w_i, $i \geqslant 1$, *be eigenfunctions of* (2.5), *orthonormal in* $L_2(\Omega)$. *Then*

$$(\tilde{w}_i, \tilde{w}_j)_h = \delta_{ij} + D_{i,j,h} \quad (i, j \geqslant 1) \tag{5.1}$$

with

$$D_{i,j,h} = (\tilde{w}_i, \tilde{w}_j)_{\tau_h} - (w_i, w_j)_{\omega_h}, \tag{5.2}$$

where

$$(u, v)_{\varepsilon_h} = \int_{\varepsilon_h} uv \, dx \quad (\varepsilon = \tau, \omega).$$

Further, we have

$$|D_{i,j,h}| \leqslant C h^4 \sqrt{\lambda_i \lambda_j} \quad (i, j \geqslant 1). \tag{5.3}$$

Proof : Relations (5.1)-(5.2) follow immediately from Theorem 2.3, (4.1) and Theorem 3.1. Further, we see that

$$|D_{i,j,h}| \leqslant \sum_{\varepsilon = \tau, \omega} \left| (\tilde{w}_i, \tilde{w}_j)_{\varepsilon_h} \right| . \tag{5.4}$$

We estimate the terms on the right-hand side. Using on the one hand (4.5) and Theorem 3.1, and on the other hand Friedrichs' inequality and (2.5), relation (5.3) follows. □

5.2. COROLLARY : *For* $v \in V$ *satisfying* $\|v\|_{0, \Omega} = 1$ *we can write*

$$\|\tilde{v}\|^2_{0, \Omega_h} = 1 + B_{m,h} \tag{5.5}$$

with

$$B_{m,h} = \|\tilde{v}\|^2_{0, \tau_h} - \|v\|^2_{0, \omega_h}. \tag{5.6}$$

If $v \in V_m$, $\|v\|_{0, \Omega} = 1$, *with*

$$V_m = \text{span}\,(w_1, \ldots, w_m),\ w_i \text{ eigenfunction of } (2.5)\ (1 \leqslant i \leqslant m), \tag{5.7}$$

then we have

$$|B_{m,h}| \leqslant C h^4 \lambda_m . \tag{5.8}$$

Proof : It is easily seen that the relations (5.5)-(5.8) follow from Lemma 5.1 and Theorem 2.3. We only note that for $v \in V_m$, $\|v\|_{0, \Omega} = 1$ we have

$$V_m \ni v = \sum_{i=1}^{m} \alpha_i w_i , \quad \sum_{i=1}^{m} \alpha_i^2 = 1 . \tag{5.9}$$

□

5.3. LEMMA : *Let $\tilde{V}_m^h$ be the space spanned on $\tilde{w}_i^h := \tilde{w}_i \big|_{\Omega_h}$ $(i = 1, \dots, m)$. Then we have*

$$\dim \tilde{V}_m^h = \dim \Pi_h V_m = m , \quad h \leqslant h_0(m) ,$$

where Π_h is the elliptic projector defined by (4.11) and V_m is given by (5.7).

Proof : A) From Lemma 5.1 it follows that

$$(\tilde{w}_i, \tilde{w}_j)_h = \delta_{ij} + O(h^4) \quad (i, j = 1, \dots, m) .$$

This implies that the set $(\tilde{w}_i^h)_{1 \leqslant i \leqslant m}$ is linearly independent. Hence, $\dim \tilde{V}_m^h = m$.

B) To prove the second equality, we show that $(\Pi_h w_i)_{1 \leqslant i \leqslant m}$ is a linearly independent set. From

$$(\Pi_h w_i, \Pi_h w_j)_h = (\Pi_h w_i - \tilde{w}_i + \tilde{w}_i, \Pi_h w_j - \tilde{w}_j + \tilde{w}_j)_h , \tag{5.10}$$

it follows that

$$\begin{aligned} \left|(\Pi_h w_i, \Pi_h w_j)_h\right| \leqslant \left\|\Pi_h w_i - \tilde{w}_i\right\|_{0, \Omega_h} [C \left\|w_j\right\|_{0, \Omega} + \left\|\Pi_h w_j - \tilde{w}_j\right\|_{0, \Omega_h}] + \\ + \left|(\tilde{w}_i, \tilde{w}_j)_h\right| + C \left\|w_i\right\|_{0, \Omega} \left\|\Pi_h w_j - \tilde{w}_j\right\|_{0, \Omega_h} . \end{aligned}$$

Using (5.1)-(5.3) and (4.15) we see that for every $\varepsilon > 0$ we can find $h_0(m, \varepsilon)$ such that

$$\left|(\Pi_h w_i, \Pi_h w_j)_h\right| < \varepsilon \quad (i \neq j) , \quad h \leqslant h_0(m, \varepsilon) .$$

On the other hand (5.10), combined with (5.1)-(5.3) and (4.15) (or (4.12)), leads to

$$\begin{aligned} \left\|\Pi_h w_i\right\|_{0, \Omega_h}^2 \geqslant \frac{1}{2} \left\|\tilde{w}_i\right\|_{0, \Omega_h}^2 - \left\|\tilde{w}_i - \Pi_h w_i\right\|_{0, \Omega_h}^2 \geqslant \\ \geqslant \frac{1}{2} (1 - \left|D_{i, i, h}\right|) - \left\|\tilde{w}_i - \Pi_h w_i\right\|_{0, \Omega_h}^2 \geqslant \frac{1}{4} . \end{aligned}$$

Thus, the matrix $((\Pi_h w_i, \Pi_h w_j)_h)_{1 \leqslant i, j \leqslant m}$ is non-singular. □

5.4. LEMMA : *For $v \in V$ satisfying $\|v\|_{0, \Omega} = 1$ we have*

$$\left\|\Pi_h v\right\|_{0, \Omega_h}^2 \geqslant 1 - \delta_{m, h}(v) \tag{5.11}$$

with

$$\delta_{m, h}(v) = \left|B_{m, h}\right| + 2 \left\|\tilde{v} - \Pi_h v\right\|_{0, \Omega_h} (1 + \left|B_{m, h}\right|)^{1/2} . \tag{5.12}$$

Let $\{h\} = \{h_j\}_{j=1}^{\infty}$ *with* $\lim_{j \to \infty} h_j = 0$. *Then, if* $v \in V_m$, $\|v\|_{0,\Omega} = 1$ *we have*

$$\lim_{h_j \to 0} \delta_{m, h_j}(v) = 0 . \tag{5.13}$$

[The convergence is uniform with respect to $v \in V_m$, $\|v\|_{0,\Omega} = 1$.*]*
Moreover, if $V_m \subset H^2(\Omega)$ *then*

$$|\delta_{m,h}(v)| \leq C(m)\, h \quad \forall h \leq h_0(m) , \tag{5.14}$$

where $C(m)$ *is a constant depending on* m.

Proof: Let $v \in V$. We have

$$\|\tilde{v}\|_{0,\Omega_h}^2 - \|\Pi_h v\|_{0,\Omega_h}^2 = (\tilde{v} - \Pi_h v, \tilde{v} + \Pi_h v)_h =$$

$$= 2(\tilde{v} - \Pi_h v, \tilde{v})_h - \|\tilde{v} - \Pi_h v\|_{0,\Omega_h}^2 ;$$

hence

$$\|\Pi_h v\|_{0,\Omega_h}^2 \geq \|\tilde{v}\|_{0,\Omega_h}^2 - 2\|\tilde{v} - \Pi_h v\|_{0,\Omega_h} \|\tilde{v}\|_{0,\Omega_h} .$$

Then, if $\|v\|_{0,\Omega} = 1$, relation (5.5) leads to (5.11)-(5.12).

For $v \in V_m$, $\|v\|_{0,\Omega} = 1$ we find using (5.9) and the Cauchy inequality that

$$\|\tilde{v} - \Pi_h v\|_{0,\Omega_h}^2 \leq \sum_{i=1}^{m} \|\tilde{w}_i - \Pi_h w_i\|_{0,\Omega_h}^2 ; \tag{5.15}$$

hence, due to (4.15),

$$\lim_{h_j \to 0} \|\tilde{v} - \Pi_{h_j} v\|_{0,\Omega_{h_j}} = 0 .$$

Taking into account (5.12) and (5.8), relation (5.13) follows.

If $V_m \subset H^2(\Omega)$, (4.12) is applicable on the terms of (5.15) ; hence

$$\|\tilde{v} - \Pi_h v\|_{0,\Omega_h} \leq C(m)\, h .$$

Substituting this estimate and (5.8) into (5.12) we obtain (5.14). □

Let λ_m and $\lambda_{m,h}$ be the m-th eigenvalue of (2.5) and (2.13) respectively. Analogously to [11, Lemmas 6.4-1, 6.4-2 and Theorem 6.4-2], we first estimate $\lambda_{m,h}$ from above.

5.5. THEOREM : *We have*

$$\lambda_{m,h} \leqslant \left(\lambda_m + C \sum_{\varepsilon=\tau,\,\omega} G_m(\varepsilon_h)\right)\left[1 + \sup_{\substack{v \in V_m \\ \|v\|_{0,\Omega}=1}} \frac{\delta_{m,h}(v)}{1-\delta_{m,h}(v)}\right], \tag{5.16}$$

with $\delta_{m,h}(v)$ *given by (5.12) and with*

$$G_m(\varepsilon_h) = \sum_{i=1}^{m} \|\tilde{w}_i\|^2_{1,\varepsilon_h}. \tag{5.17}$$

Here w_i *is eigenfunction of (2.5) and* $\tilde{w}_i = \mathscr{E}(w_i)$ *(see Remark 3.2)* $(1 \leqslant i \leqslant m)$.

Proof : Let $\mathscr{V}_{m,h}$ denote the set of all m-dimensional subspaces of V_h. According to the min-max characterization, Lemma 5.3 and Lemma 5.4, we find

$$\lambda_{m,h} = \min_{E_m \in \mathscr{V}_{m,h}} \max_{\substack{v_h \in E_m \\ v_h \neq 0}} \frac{a_h(v_h, v_h)}{\|v_h\|^2_{0,\Omega_h}} \leqslant \max_{\substack{v_h \in \Pi_h V_m \\ v_h \neq 0}} \frac{a_h(v_h, v_h)}{\|v_h\|^2_{0,\Omega_h}} =$$

$$= \max_{\substack{v \in V_m \\ \|v\|_{0,\Omega}=1}} \frac{a_h(\Pi_h v, \Pi_h v)}{\|\Pi_h v\|^2_{0,\Omega_h}}. \tag{5.18}$$

Using the definition of $\Pi_h v$ we can write

$$a_h(\Pi_h v, \Pi_h v) \leqslant a_h(\Pi_h v, \Pi_h v) + a_h(\tilde{v} - \Pi_h v, \tilde{v} - \Pi_h v) = a_h(\tilde{v}, \tilde{v}) =$$

$$= a(v, v) + a_{\tau_h}(\tilde{v}, \tilde{v}) - a_{\omega_h}(v, v) \leqslant a(v, v) + \sum_{\varepsilon=\tau,\,\omega} a_{\varepsilon_h}(\tilde{v}, \tilde{v}), \tag{5.19}$$

where we denote

$$a_{\varepsilon_h}(\tilde{v}, \tilde{v}) = \sum_{i,j=1}^{2} \int_{\varepsilon_h} k_{ij} \frac{\partial \tilde{v}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_j}\, dx \quad (\varepsilon = \tau, \omega),$$

τ_h, ω_h being defined in (4.1).

Using (2.2), (5.9) and (5.17), we obtain

$$\left|a_{\varepsilon_h}(\tilde{v}, \tilde{v})\right| \leqslant C\,\|\tilde{v}\|^2_{1,\varepsilon_h} \leqslant C G_m(\varepsilon_h) \quad \forall v \in V_m,\ \|v\|_{0,\Omega} = 1. \tag{5.20}$$

We have

$$\max_{\substack{v \in V_m \\ \|v\|_{0,\Omega}=1}} a(v, v) = \max_{\substack{w \in V_m \\ w \neq 0}} \frac{a(w, w)}{\|w\|^2_{0,\Omega}} = \max_{\substack{w \in V_m \\ w \neq 0}} \mathscr{R}(w) = \lambda_m \tag{5.21}$$

where the last relation holds according to [11, (6.2-22)].

Relations (5.19)-(5.21) give

$$a_h(\Pi_h v, \Pi_h v) \leq \lambda_m + C \sum_{\varepsilon = \tau, \omega} G_m(\varepsilon_h). \tag{5.22}$$

Substituting (5.22) and (5.11) into (5.18) we easily obtain (5.16). □

5.6. LEMMA : *For* $G_m(\varepsilon_h)$, (5.17), *we have*

$$\lim_{h \to 0} G_m(\varepsilon_h) = 0 \quad (\varepsilon = \tau, \omega). \tag{5.23}$$

Let $V_m \subset H^2(\Omega)$. *Then we have*

$$G_m(\varepsilon_h) \leq C(m) h^2. \tag{5.24}$$

Proof: Relation (5.23) follows from the absolute continuity of the Lebesgue integral. If $V_m \subset H^2(\Omega)$ then (5.24) is proved by means of Lemma 4.4. □

Now, we estimate $\lambda_{m,h}$ from below.

A. Case $\Omega_h \subseteq \Omega$ (internal approximation)

Let $\hat{v}_h \in V$ be the function associated with $v_h \in V_h$, according to Lemma 4.1. Then (5.18) can be rewritten as

$$\lambda_{m,h} = \min_{\hat{E}_m \in \mathscr{V}_{m,h}} \max_{\substack{\hat{v}_h \in \hat{E}_m \\ \hat{v}_h \neq 0}} \frac{a(\hat{v}_h, \hat{v}_h)}{\|\hat{v}_h\|_{0,\Omega}}, \tag{5.25}$$

where $\hat{E}_m$ is the space of functions from V which are associated with functions belonging to E_m. As $\hat{v}_h = v_h$ in the case of a polygonal domain Ω and $\hat{v}_h$ is the extension of v_h by zero in the case of internal approximations of Ω, it is obvious that $\dim \hat{E}_m = m$.

As the set $\mathscr{V}_{m,h}$ of all spaces $\hat{E}_m$ is a proper subset of $\mathscr{V}_m$ (the set of all m-dimensional subspaces of V) we see that

$$\lambda_{m,h} \geq \lambda_m. \tag{5.26}$$

B. Case $\Omega_h \not\subset \Omega$ (external approximation)

In this case estimating $\lambda_{m,h}$ from below is much more difficult.
We define

$$A_0(v_h) := \|v_h\|^2_{0,\Omega_h} - \|\hat{v}_h\|^2_{0,\Omega} = A_{01}(v_h) + A_{02}(v_h) \quad \forall v_h \in V_h \tag{5.27}$$

with (see (4.1), (4.2) and Lemma 4.1)

$$A_{01}(v_h) = \sum_{\bar{T}^{id} \in M_h} \int_{T^{id}} (v_h^2 - \hat{v}_h^2)\, dx\,, \quad A_{02}(v_h) = \|v_h\|^2_{0, \tau_h}. \tag{5.28}$$

Further we define

$$A_1(v_h) := a_h(v_h, v_h) - a(\hat{v}_h, \hat{v}_h) \quad \forall v_h \in V_h\,. \tag{5.29}$$

For simplicity, we first assume that $k_{12} = 0$ in (2.1) and (2.8). Then (5.29) is reduced to

$$A_1(v_h) = A_{11}(v_h) + A_{12}(v_h) \tag{5.30}$$

with (using the short notation ∂_i for $\partial/\partial x_i$)

$$A_{11}(v_h) = \sum_{\bar{T}^{id} \in M_h} \sum_{i=1}^{2} \int_{T^{id}} k_{ii}[(\partial_i v_h)^2 - (\partial_i \hat{v}_h)^2]\, dx\,, \tag{5.31}$$

$$A_{12}(v_h) = \sum_{\bar{T}^{id} \in M_h} \sum_{i=1}^{2} \int_{T - T^{id}} k_{ii}(\partial_i v_h)^2\, dx\,. \tag{5.32}$$

5.7. LEMMA : *For* A_0, (5.27)-(5.28), *and* A_1, (5.29)-(5.32), *we have*

$$|A_0(v_h)| \leq Ch^2 |v_h|^2_{1, \Omega_h} \quad \forall v_h \in V_h\,, \tag{5.33}$$

$$|A_1(v_h)| \leq Ch |v_h|^2_{1, \Omega_h} \quad \forall v_h \in V_h\,. \tag{5.34}$$

Proof : Using (4.1)-(4.2), Lemma 4.2 and Lemma 4.5, we obtain from (5.28) that

$$|A_{01}(v_h)| \leq \sum_{\bar{T}^{id} \in M_h} \|v_h - \hat{v}_h\|_{0, T^{id}} (2\|v_h\|_{0, T^{id}} + \|v_h - \hat{v}_h\|_{0, T^{id}}) \leq$$

$$\leq Ch^2 \|v_h\|^2_{1, \Omega_h}\,,$$

$$|A_{02}(v_h)| \leq Ch^4 |v_h|^2_{1, \tau_h} \leq Ch^5 |v_h|^2_{1, \Omega_h}\,.$$

Substituting both inequalities in (5.27) and using (2.12), we obtain (5.33). Using (2.2), relation (5.34) can be proved by similar devices as above. □

5.8. *Remark :* If $k_{12} \neq 0$ then we can derive the same estimate (5.34). For this, observe that for the additional term in A_1, (5.29), we have

$$2(\partial_i v_h \partial_j v_h - \partial_i \hat{v}_h \partial_j \hat{v}_h) = (\partial_i v_h - \partial_i \hat{v}_h)\, \partial_j v_h + \partial_i v_h (\partial_j v_h - \partial_j \hat{v}_h) -$$

$$- (\partial_i \hat{v}_h - \partial_i v_h)\, \partial_j \hat{v}_h - \partial_i \hat{v}_h (\partial_j \hat{v}_h - \partial_j v_h). \quad \square$$

According to (5.27) and (5.29), the quotient in (5.18) can be rewritten as

$$\frac{a_h(v_h, v_h)}{\|v_h\|_{0,\Omega_h}^2} = \frac{a(\hat{v}_h, \hat{v}_h)}{\|\hat{v}_h\|_{0,\Omega}^2} H(v_h) \text{ with } H(v_h) = \frac{1 + A_1(v_h)/a(\hat{v}_h, \hat{v}_h)}{1 + A_0(v_h)/\|\hat{v}_h\|_{0,\Omega}^2}. \tag{5.35}$$

Making use of Lemma 5.7 and (2.12) we obtain

$$\frac{|A_1(v_h)|}{a(\hat{v}_h, \hat{v}_h)} \leqslant Ch \frac{|v_h|_{1,\Omega_h}^2}{\|\hat{v}_h\|_{0,\Omega}^2}, \quad \frac{|A_0(v_h)|}{\|\hat{v}_h\|_{0,\Omega}^2} \leqslant Ch^2 \frac{|v_h|_{1,\Omega_h}^2}{\|\hat{v}_h\|_{0,\Omega}^2}. \tag{5.36}$$

Relations (5.35)-(5.36) form a starting point for estimating the approximate eigenvalue $\lambda_{m,h}$ from below. According to (5.13) and (5.23), the right-hand side of (5.16) tends to λ_m with $h \to 0$. This means that $\lambda_{m,h} \leqslant 2\lambda_m$ for $h \leqslant h_0(m)$. Hence, (5.18) can be rewritten as

$$\lambda_{m,h} = \min_{E_m^* \in \mathcal{V}_{m,h}} \max_{\substack{v_h \in E_m^* \\ v_h \neq 0}} \frac{a_h(v_h, v_h)}{\|v_h\|_{0,\Omega_h}^2}, \tag{5.37}$$

where only the m-dimensional subspaces E_m^* of V_h are (needed to be) considered, satisfying

$$\max_{\substack{v_h \in E_m^* \\ v_h \neq 0}} \frac{a_h(v_h, v_h)}{\|v_h\|_{0,\Omega_h}^2} \leqslant 2\lambda_m. \tag{5.38}$$

5.9. LEMMA : *We have*

$$\frac{|v_h|_{1,\Omega_h}^2}{\|\hat{v}_h\|_{0,\Omega}^2} \leqslant C\lambda_m \quad \forall v_h \in E_m^*,\ E_m^* \in \mathcal{V}_{m,h}. \tag{5.39}$$

Proof : Using (5.38) and (2.12) we find that

$$\frac{|v_h|_{1,\Omega_h}^2}{\|v_h\|_{0,\Omega_h}^2} \leqslant 2\mu_0^{-1}\lambda_m \quad \forall v_h \in E_m^*,\ E_m^* \in \mathcal{V}_{m,h}.$$

By Lemma 4.1 and (4.3) we have

$$\|\hat{v}_h\|_{0,\Omega - S_h} = \|v_h\|_{0,\Omega_h - S_h} \quad \forall v_h \in V_h.$$

In view of the definition of V_h (see (2.7)), the minimum angle condition and the quasi-uniformity of $\mathcal{T}_h$ (guaranteed by the inverse assumption) it can be proved that

$$\|v_h\|_{0,\Omega_h}^2 \leqslant C \|v_h\|_{0,\Omega_h - S_h}^2 \quad \forall v_h \in V_h, \forall h \leqslant h_0 .$$

Using these three relations for $v_h \in E_m^*$ we obtain (5.39). □

5.10. LEMMA : *For $H(v_h)$ defined in* (5.35) *we have*

$$H(v_h) \geqslant 1 - C(m)\, h \quad \forall v_h \in E_m^*, E_m^* \in \mathcal{V}_{m,h}, \forall h \leqslant h_0(m) . \quad (5.40)$$

Proof: Combination of $(5.35)_2$ with (5.36) and Lemma 5.9 yields the desired result. □

We now give a lower bound for $\lambda_{m,h}$.

5.11. THEOREM : *We have*

$$\lambda_{m,h} \geqslant \lambda_m (1 - C(m)\, h) \quad \forall h \leqslant h_0(m) . \quad (5.41)$$

Proof: From (5.35) and (5.37) it follows that

$$\lambda_{m,h} \geqslant \left(\min_{E_m^* \in \mathcal{V}_{m,h}} \max_{\substack{v_h \in E_m^* \\ v_h \neq 0}} \frac{a(\hat{v}_h, \hat{v}_h)}{\|\hat{v}_h\|_{0,\Omega}^2} \right) \left(\min_{E_m^* \in \mathcal{V}_{m,h}} \min_{\substack{v_h \in E_m^* \\ v_h \neq 0}} H(v_h) \right) . \quad (5.42)$$

Denote by $\hat{E}_m^*$ the subspace of functions from V which are associated with functions belonging to E_m^*. The correspondence $v_h \mapsto \hat{v}_h$ is a linear bijection. This implies that $\dim \hat{E}_m^* = m$ and $\hat{\mathcal{V}}_{m,h} \subset \mathcal{V}_m$; the meaning of $\hat{\mathcal{V}}_{m,h}$ and of $\mathcal{V}_m$ were explained in relation with (5.25). Hence

$$\min_{E_m^* \in \mathcal{V}_{m,h}} \max_{\substack{v_h \in E_m^* \\ v_h \neq 0}} \frac{a(\hat{v}_h, \hat{v}_h)}{\|\hat{v}_h\|_{0,\Omega}^2} =$$

$$= \min_{\hat{E}_m^* \in \hat{\mathcal{V}}_{m,h}} \max_{\substack{\hat{v}_h \in \hat{E}_m^* \\ \hat{v}_h \neq 0}} \frac{a(\hat{v}_h, \hat{v}_h)}{\|\hat{v}_h\|_{0,\Omega}^2} \geqslant \min_{\hat{E}_m \in \hat{\mathcal{V}}_{m,h}} \max_{\substack{\hat{v}_h \in \hat{E}_m \\ \hat{v}_h \neq 0}} \frac{a(\hat{v}_h, \hat{v}_h)}{\|\hat{v}_h\|_{0,\Omega}^2} \geqslant \lambda_m ,$$

by the definition of λ_m (see [11, (6.2-21)]), and (5.41) follows from (5.42) and (5.40). □

We formulate the main result as

5.12. THEOREM : *Let λ_m be an eigenvalue of Problem* 2.1 *and $\lambda_{m,h}$ the corresponding eigenvalue of Problem* 2.9. *Then it holds that*

$$\lim_{h_j \to 0} \lambda_{m,h_j} = \lambda_m . \quad (5.43)$$

Moreover, if $V_m \subset H^2(\Omega)$ *then we have*

$$|\lambda_m - \lambda_{m,h}| \leqslant C(m)\, h \quad \forall h \leqslant h_0(m). \tag{5.44}$$

Proof: Combine Theorems 5.5 and 5.11 and apply Lemmas 5.4 and 5.6. □

6. CONVERGENCE OF APPROXIMATE EIGENFUNCTIONS

As in [11, Chapter 6] we restrict ourselves in this section to the case of eigenfunctions corresponding to simple exact eigenvalues. We start our considerations with some lemmas.

Let us set

$$\rho_{m,h} = \max_{\substack{1 \leqslant i \leqslant I(h) \\ i \neq m}} \frac{\lambda_m}{|\lambda_{i,h} - \lambda_m|}, \quad \kappa_{m,h} = \max_{\substack{1 \leqslant i \leqslant I(h) \\ i \neq m}} \frac{1}{\left|1 - \dfrac{\lambda_m}{\lambda_{i,h}}\right|}.$$

We note that $\rho_{m,h}$ is defined in [11, (6.4-21)]. As $\lambda_{i,h} \to \lambda_i$, it is easily seen that

6.1. LEMMA:

$$\rho_{m,h} \leqslant \rho_m = \max\left\{\frac{\lambda_m}{\lambda_m - \lambda_{m-1} - \varepsilon}, \frac{\lambda_m}{\lambda_{m+1} - \lambda_m - \varepsilon}\right\} \quad \forall h \leqslant h_0(m),$$

$$\kappa_{m,h} \leqslant \kappa_m = \max\left\{\frac{\lambda_{m-1} + \varepsilon}{\lambda_m - \lambda_{m-1} - \varepsilon}, \frac{\lambda_{m+1} - \varepsilon}{\lambda_{m+1} - \lambda_m - \varepsilon}\right\} \quad \forall h \leqslant h_0(m),$$

with

$$0 < \varepsilon < \min\{\lambda_{m+1} - \lambda_m, \lambda_m - \lambda_{m-1}\}.$$

6.2. LEMMA: *We have*

$$\sum_{\substack{i=1 \\ i \neq m}}^{I(h)} |(\Pi_h w_m, w_{i,h})_h|^2 \leqslant$$

$$\leqslant C(m) \sum_{\substack{i=1 \\ i \neq m}}^{I(h)} \left[|(\tilde{w}_m - \Pi_h w_m, w_{i,h})_h|^2 + \lambda_{i,h}^{-2} |K_j + \lambda_m K_0|^2\right] \tag{6.1}$$

where $j = 1$ *or, if* $w_m \in H^2(\Omega)$, $j = 2$ *and where we denote*

$$K_0 \equiv K_0(m, i, h) = (w_m, \hat{w}_{i,h}) - (\tilde{w}_m, w_{i,h})_h, \tag{6.2}$$

$$K_1 \equiv K_1(m, i, h) = a_h(\tilde{w}_m, w_{i,h}) - a(w_m, \hat{w}_{i,h}), \tag{6.3}$$

$$K_2 \equiv K_2(m, i, h) = (L\tilde{w}_m, w_{i,h})_h - (Lw_m, \hat{w}_{i,h}), \tag{6.4}$$

$$L = -\sum_{j,\ell=1}^{2} \frac{\partial}{\partial x_j}\left(k_{\ell j} \frac{\partial}{\partial x_\ell}\right). \tag{6.5}$$

Proof : As $\Pi_h w_m \in V_h$ we have by (2.13), Definition 4.7, (2.5) and (6.2)-(6.3)

$$\begin{aligned}(\Pi_h w_m, w_{i,h})_h &= \lambda_{i,h}^{-1} a_h(\Pi_h w_m, w_{i,h}) = \lambda_{i,h}^{-1} a_h(\tilde{w}_m, w_{i,h}) \\ &= \lambda_{i,h}^{-1}[a(w_m, \hat{w}_{i,h}) + K_1] \\ &= \lambda_m \lambda_{i,h}^{-1}[(\tilde{w}_m, w_{i,h})_h + K_0] + \lambda_{i,h}^{-1} K_1 . \qquad (6.6)\end{aligned}$$

Subtracting from both sides of (6.6) the term $\lambda_m \lambda_{i,h}^{-1}(\Pi_h w_m, w_{i,h})_h$ and using Lemma 6.1, we obtain

$$|(\Pi_h w_m, w_{i,h})_h| \leq \rho_m |(\tilde{w}_m - \Pi_h w_m, w_{i,h})_h| + \frac{\kappa_m}{\lambda_{i,h}} |K_1 + \lambda_m K_0| \quad (i \neq m).$$

This implies (6.1) with $j = 1$.

If $w_m \in H^2(\Omega)$ then we have, according to Green's theorem [8, Theorem 3.1.1] (which can be used because $k_{ij}\, \partial\tilde{w}_m/\partial x_j \in H^1(\tilde{\Omega})$),

$$a_h(\tilde{w}_m, w_{i,h}) = (L\tilde{w}_m, w_{i,h})_h , \quad a(w_m, \hat{w}_{i,h}) = (Lw_m, \hat{w}_{i,h}) .$$

Hence $K_1 = K_2$. □

6.3. LEMMA : *We have*

$$|K_0(m, i, h)| \leq C h^3 \|w_{i,h}\|_{1, S_h} \|w_m\|_{1,\Omega} \leq C h^3 \sqrt{\lambda_{i,h} \lambda_m} , \qquad (6.7)$$

$$|K_1(m, i, h)| \leq C \|w_{i,h}\|_{1, S_h} (h |w_m|_{1, S_h - \tau_h} + h^{1/2} |\tilde{w}_m|_{1, \tau_h}) \leq$$
$$\leq C h^{1/2} \sqrt{\lambda_{i,h} \lambda_m} . \qquad (6.8)$$

If $w_m \in H^2(\Omega)$ *then*

$$|K_0(m, i, h)| \leq C h^{7/2} \|w_{i,h}\|_{1, S_h} (\|\tilde{w}_m\|_{2, \tilde{\Omega}} + h \|\tilde{w}_m\|_{1, \tilde{\Omega}}) \leq$$
$$\leq C(m)\, h^{7/2} \sqrt{\lambda_{i,h}} , \qquad (6.9)$$

$$|K_2(m, i, h)| \leq C h^2 \|w_{i,h}\|_{1, S_h} (\|Lw_m\|_{0, S_h - \tau_h} + h^{1/2} \|L\tilde{w}_m\|_{0, \tau_h}) \leq$$
$$\leq C(m)\, h^{5/2} \sqrt{\lambda_{i,h}} . \qquad (6.10)$$

Proof : A) Making use of (4.1)-(4.2) and Lemma 4.1, we can rewrite (6.2) as

$$K_0(m, i, h) = \sum_{T^{id} \in M_h} \int_{T^{id}} w_m(\hat{w}_{i,h} - w_{i,h})\, dx - \int_{\tau_h} \tilde{w}_m w_{i,h}\, dx \equiv K_{01} - K_{02} . \qquad (6.11)$$

Notations (4.1)-(4.3) and Lemmas 4.2, 4.4, 4.5 and 4.6 imply

$$|K_{01}| \leqslant \|w_m\|_{0, S_h - \tau_h} \left(\sum_{\bar{T}^{id} \in M_h} \|\hat{w}_{i,h} - w_{i,h}\|^2_{0, T^{id}} \right)^{1/2} \leqslant$$

$$\leqslant C h^3 |w_m|_{1, S_h - \tau_h} \|w_{i,h}\|_{1, S_h}, \quad (6.12)$$

$$|K_{02}| \leqslant \|\tilde{w}_m\|_{0, \tau_h} \|w_{i,h}\|_{0, \tau_h} \leqslant C h^{9/2} |\tilde{w}_m|_{1, \tau_h} |w_{i,h}|_{1, S_h}. \quad (6.13)$$

Combination of these estimates with Theorem 3.1 and with the relations

$$\|w_m\|_{1, \Omega} \leqslant C \sqrt{a(w_m, w_m)} = C \sqrt{\lambda_m},$$
$$\|w_{i,h}\|_{1, S_h} \leqslant \|w_{i,h}\|_{1, \Omega_h} \leqslant C \sqrt{a_h(w_{i,h}, w_{i,h})} = C \sqrt{\lambda_{i,h}}$$

and substitution in (6.11) leads to (6.7). If $w_m \in H^2(\Omega)$ we first apply (4.9) on (6.12) and (4.4) on (6.13).

B) For K_1 we derive in an analogous way

$$K_1(m, i, h) =$$
$$= \sum_{\bar{T}^{id} \in M_h} \sum_{j, \ell = 1}^{2} \left\{ \int_{T - T^{id}} k_{\ell j} \frac{\partial \tilde{w}_m}{\partial x_\ell} \frac{\partial w_{i,h}}{\partial x_j} dx + \int_{T^{id}} k_{\ell j} \frac{\partial w_m}{\partial x_\ell} \frac{\partial (w_{i,h} - \hat{w}_{i,h})}{\partial x_j} dx \right\}.$$

Using similar devices as in part A) we obtain (6.8).

C) If $w_m \in H^2(\Omega)$ then relation $K_1 = K_2$ holds with

$$K_2(m, i, h) = \sum_{\bar{T}^{id} \in M_h} \int_{T^{id}} (w_{i,h} - \hat{w}_{i,h}) L w_m \, dx + \int_{\tau_h} w_{i,h} L \tilde{w}_m \, dx.$$

According to Green's theorem [8, Theorem 3.1.1] and the density of $H_0^1(\Omega)$ in $L_2(\Omega)$, the eigenvalue equation (2.5) for the eigenpair $\{\lambda_m, w_m\}$ can be rewritten :

$$L w_m = \lambda_m w_m \text{ a.e. in } \Omega.$$

It is easily seen that these equalities directly lead to (6.10). □

6.4. LEMMA : *For the eigenvalues* $\lambda_n^{(h)}$ *of Problem* 2.8 *we have*

$$c_1 n \leqslant \lambda_n^{(h)} \leqslant c_2 n \quad \forall h \leqslant h_0 \quad (n = 1, 2, \dots),$$

where $c_1, c_2 > 0$ *are constants independent of* n *and* h.

Proof : The proof follows from [7, pp. 375-378]. □

6.5. THEOREM : *Let λ_m be a simple eigenvalue of Problem 2.1. Then, choosing conveniently the sign of every w_{m, h_j}, we have*

$$\lim_{h_j \to 0} \left\| \tilde{w}_m - w_{m, h_j} \right\|_{0, \Omega_{h_j}} = 0 . \tag{6.14}$$

If $w_m \in H^2(\Omega)$ then, choosing conveniently the sign of $w_{m, h}$, it holds that

$$\left\| \tilde{w}_m - w_{m, h} \right\|_{0, \Omega_h} \leq C(m)\, h \quad \forall h \leq h_0(m) . \tag{6.15}$$

Moreover, if $\partial\Omega$ is of class $\mathscr{C}^{1,1}$ and $k_{ij} \in C^{0,1}(\bar{\Omega})$ then we have

$$\left\| \tilde{w}_m - w_{m, h} \right\|_{0, \Omega_h} \leq C(m)\, h^2 \quad \forall h \leq h_0(m) . \tag{6.16}$$

Proof : We estimate the second term in

$$\left\| \tilde{w}_m - w_{m, h} \right\|_{0, \Omega_h} \leq \left\| \tilde{w}_m - \Pi_h w_m \right\|_{0, \Omega_h} + \left\| \Pi_h w_m - w_{m, h} \right\|_{0, \Omega_h} . \tag{6.17}$$

Using (2.14) we obtain

$$V_h \ni \Pi_h w_m = \sum_{i=1}^{I(h)} (\Pi_h w_m, w_{i, h})_h\, w_{i, h} \tag{6.18}$$

and

$$\left\| \Pi_h w_m - w_{m, h} \right\|_{0, \Omega_h}^2 = \sum_{\substack{i=1 \\ i \neq m}}^{I(h)} \left| (\Pi_h w_m, w_{i, h})_h \right|^2 + \left| (\Pi_h w_m, w_{m, h})_h - 1 \right|^2 . \tag{6.19}$$

The first term of the right-hand side is bounded by the right-hand side of (6.1), where we separately estimate each term.

(i) Let $z_{m, h}$ denote the orthogonal projection of the function $\tilde{w}_m^h - \Pi_h w_m$ on the subspace of $L_2(\Omega_h)$ with the orthonormal basis $\{w_{i, h}\}_{i=1}^{I(h)}$ (for the notation $\tilde{w}_m^h$ see Lemma 5.3). Then we have

$$\sum_{i=1}^{I(h)} \left| (\tilde{w}_m - \Pi_h w_m, w_{i, h})_h \right|^2 = \left\| z_{m, h} \right\|_{0, \Omega_h}^2 = \\ = \left\| \tilde{w}_m - \Pi_h w_m \right\|_{0, \Omega_h}^2 - \left\| \tilde{w}_m - \Pi_h w_m - z_{m, h} \right\|_{0, \Omega_h}^2 . \tag{6.20}$$

(ii) For the second term in (6.1) we use Lemma 6.3. Hence we obtain

$$\sum_{\substack{i=1 \\ i \neq m}}^{I(h)} \lambda_{i, h}^{-2} \left| K_j + \lambda_m K_0 \right|^2 \leq C(m)\, h^{4j-5} \left(h^2 \sum_{i=1}^{I(h)} \lambda_{i, h}^{-1} \right) \quad (j = 1, 2) , \tag{6.21}$$

where the case $j = 2$ requires $w_m \in H^2(\Omega)$. (This assumption will be made in every case where $j = 1, 2$.) Now we estimate $h^2 \sum \lambda_{i,h}^{-1}$. By (5.26) we have $\lambda_n^{(h)} \leqslant \lambda_{n,h}$. This result combined with Lemma 6.4 gives

$$\lambda_{n,h}^{-1} \leqslant (\lambda_n^{(h)})^{-1} \leqslant c_1^{-1} n^{-1} < c_1^{-1} n^{-\alpha(p)}, \quad \alpha(p) = \frac{p}{p+1} \; (p \geqslant 1).$$

Hence, observing that $I(h) = O(h^{-2})$,

$$h^2 \sum_{i=1}^{I(h)} \lambda_{i,h}^{-1} \leqslant c_1^{-1} h^2 \int_0^{I(h)} t^{-\alpha(p)} \, dt \leqslant (p+1) C h^{2\alpha(p)}.$$

Substitution of this estimate in (6.21) gives

$$\sum_{\substack{i=1 \\ i \neq m}}^{I(h)} \lambda_{i,h}^{-2} |K_j + \lambda_m K_0|^2 \leqslant C(m) h^{4j-5} (p+1) h^{2\alpha(p)}.$$

Combining this estimate and (6.20) in (6.1), we arrive at

$$\sum_{\substack{i=1 \\ i \neq m}}^{I(h)} |(\Pi_h w_m, w_{i,h})_h|^2 \leqslant C(m) \left[\|\tilde{w}_m - \Pi_h w_m\|_{0,\Omega_h}^2 + (p+1) h^{2\alpha(p)+4j-5} \right] \quad (j = 1, 2) \qquad (6.22)$$

Let us denote

$$v_{m,h} = (\Pi_h w_m, w_{m,h})_h w_{m,h}.$$

As to the second term of (6.19) we find

$$\left| \|\tilde{w}_m\|_{0,\Omega_h} - |(\Pi_h w_m, w_{m,h})_h| \right| \leqslant \|\tilde{w}_m - v_{m,h}\|_{0,\Omega_h} \leqslant$$
$$\leqslant \|\tilde{w}_m - \Pi_h w_m\|_{0,\Omega_h} + \|\Pi_h w_m - v_{m,h}\|_{0,\Omega_h} \leqslant$$
$$\leqslant C(m) \left[\|\tilde{w}_m - \Pi_h w_m\|_{0,\Omega_h} + \sqrt{p+1}\, h^{\alpha(p)+2j-\frac{5}{2}} \right] \quad (j = 1, 2),$$

using (6.18) and (6.22) in the last inequality. In view of Lemma 5.1 we obtain

$$\left| 1 - \|\tilde{w}_m\|_{0,\Omega_h} \right| \leqslant \left| 1 - \|\tilde{w}_m\|_{0,\Omega_h}^2 \right| = |D_{m,m,h}| \leqslant C h^4 \lambda_m.$$

Combination of this estimate with the former one results in

$$\left| 1 - |(\Pi_h w_m, w_{m,h})_h| \right| \leqslant$$
$$\leqslant C(m) \left[\|\tilde{w}_m - \Pi_h w_m\|_{0,\Omega_h} + \sqrt{p+1}\, h^{\alpha(p)+2j-\frac{5}{2}} \right] \quad (j = 1, 2). \qquad (6.23)$$

We can always choose the sign of the function $w_{m,h}$ in such a way that

$$(\Pi_h w_m, w_{m,h})_h \geq 0 .$$

Combining (6.23) and (6.22) with (6.19) and then with (6.17) we find

$$\|\tilde{w}_m - w_{m,h}\|_{0,\Omega_h} \leq C(m)\left[\|\tilde{w}_m - \Pi_h w_m\|_{0,\Omega_h} + \sqrt{p+1}\, h^{\alpha(p)+2j-\frac{5}{2}}\right] \quad (j = 1, 2). \qquad (6.24)$$

Applying Theorem 4.9 on (6.24) ($j = 1$, $p \geq 2$) leads to (6.14). Estimates (6.15) and (6.16) follow from the combination of (6.24) ($j = 2$, $p \geq 1$) with Theorem 4.8. □

By means of Theorem 6.5 it is possible to obtain the error estimate (5.44) under weaker conditions for the exact eigenfunctions $w_1, \ldots, w_{m-1}$. Moreover this estimate can be improved to $O(h^2)$ if $\partial\Omega$ is of class $\mathscr{C}^{1,1}$ and the coefficients $k_{\ell j} \in C^{0,1}(\bar{\Omega})$.

6.6. THEOREM : *Let λ_m be a simple eigenvalue of Problem 2.1. If $w_m \in H^2(\Omega)$ then*

$$|\lambda_{m,h} - \lambda_m| \leq C(m)\, h^{1+i} \quad \forall h \leq h_0(m) \quad (i = 0, 1),$$

where $i = 1$ if $\partial\Omega$ is of class $\mathscr{C}^{1,1}$ and $k_{\ell j} \in C^{0,1}(\bar{\Omega})$.

Proof : As in the proof of Lemma 6.2 we find

$$(\lambda_{m,h} - \lambda_m)(\Pi_h w_m, w_{m,h})_h =$$
$$= \lambda_m(\tilde{w}_m - \Pi_h w_m, w_{m,h})_h + K_2(m, m, h) + \lambda_m K_0(m, m, h)$$

Relation (6.23) ($j = 2$ and $p \geq 1$) combined with (4.12) implies

$$|(\Pi_h w_m, w_{m,h})_h| \geq \frac{1}{2} \quad \forall h \leq h_0(m).$$

Hence

$$|\lambda_{m,h} - \lambda_m| \leq 2\lambda_m \|\tilde{w}_m - \Pi_h w_m\|_{0,\Omega_h} + 2(|K_2| + \lambda_m |K_0|).$$

Combination of this estimate with (6.9)-(6.10) and with (4.12) or, if $\partial\Omega$ is of class $\mathscr{C}^{1,1}$ and $k_{\ell j} \in C^{0,1}(\bar{\Omega})$, with (4.13) leads to the desired result. □

To prove the convergence and an error estimate for the approximate eigenfunctions in the $H^1(\Omega_h)$-norm, we first estimate $\Pi_h w_m - w_{m,h}$.

6.7. LEMMA : *Let λ_m be a simple eigenvalue of Problem 2.1. Then we have*

$$\lim_{h_j \to 0} \left\| \Pi_{h_j} w_m - w_{m, h_j} \right\|_{1, \Omega_{h_j}} = 0 \tag{6.25}$$

provided that the signs of w_{m, h_j} are conveniently chosen. If $w_m \in H^2(\Omega)$ then

$$\| \Pi_h w_m - w_{m, h} \|_{1, \Omega_h} \leqslant C(m) h^{1+i} \quad \forall h \leqslant h_0(m) \quad (i = 0, 1), \tag{6.26}$$

where $i = 1$ if $\partial\Omega$ is of class $\mathscr{C}^{1,1}$ and $k_{\ell j} \in C^{0,1}(\bar{\Omega})$.

Proof : Let us set $u_h = \Pi_h w_m - w_{m, h}$. Using (2.12), Definition 4.7, (2.5) and (2.13), we get

$$\begin{aligned} C_2 \| u_h \|_{1, \Omega_h}^2 &\leqslant a_h(u_h, u_h) = a_h(\tilde{w}_m, u_h) - a_h(w_{m, h}, u_h) \\ &= \lambda_m [(\tilde{w}_m, u_h)_h + (w_m, \hat{u}_h) - (\tilde{w}_m, u_h)_h] \\ &\quad - \lambda_{m, h}(w_{m, h}, u_h)_h + [a_h(\tilde{w}_m, u_h) - a(w_m, \hat{u}_h)] . \end{aligned} \tag{6.27}$$

In the first place, we see that

$$\begin{aligned} &|\lambda_m(\tilde{w}_m, u_h)_h - \lambda_{m, h}(w_{m, h}, u_h)_h| = \\ &\qquad = |\lambda_m(\tilde{w}_m - w_{m, h}, u_h)_h + (\lambda_m - \lambda_{m, h})(w_{m, h}, u_h)_h| \\ &\qquad \leqslant [\lambda_m \| \tilde{w}_m - w_{m, h} \|_{0, \Omega_h} + |\lambda_m - \lambda_{m, h}|] \| u_h \|_{1, \Omega_h} . \end{aligned}$$

Using similar tricks as in the proof of Lemma 6.3 we obtain

$$|(w_m, \hat{u}_h) - (\tilde{w}_m, u_h)_h| \leqslant C(m) \sqrt{h^{j+5}} \| u_h \|_{1, \Omega_h} \qquad (j = 1, 2),$$

$$|a_h(\tilde{w}_m, u_h) - a(w_m, \hat{u}_h)| \leqslant C(m) \sqrt{h^{4j-3}} \| u_h \|_{1, \Omega_h} \qquad (j = 1, 2).$$

Substitution of the last three inequalities in (6.27) results in

$$C_2 \| u_h \|_{1, \Omega_h} \leqslant \lambda_m \| \tilde{w}_m - w_{m, h} \|_{0, \Omega_h} + |\lambda_m - \lambda_{m, h}| + C(m) \sqrt{h^{4j-3}} \qquad (j = 1, 2), \tag{6.28}$$

with the estimate for $j = 2$ being valid only if $w_m \in H^2(\Omega)$. Combination with Theorem 6.5 and Theorem 5.12 proves the convergence (6.25). Estimates (6.26) follow from (6.28) combined with Theorems 6.5 and 6.6. The results are of course evident in the case $u_h \equiv 0$. □

6.8. THEOREM : Let λ_m *be a simple eigenvalue of Problem* 2.1. *Then we have*

$$\lim_{h_j \to 0} \left\| \tilde{w}_m - w_{m, h_j} \right\|_{1, \Omega_{h_j}} = 0 . \tag{6.29}$$

provided that the signs of w_{m, h_j} *are conveniently chosen. If* $w_m \in H^2(\Omega)$ *then*

$$\left\| \tilde{w}_m - w_{m, h} \right\|_{1, \Omega_h} \leq C(m) h \quad \forall h \leq h_0(m) . \tag{6.30}$$

Proof : Relations (6.29) and (6.30) follow from (4.15), (6.25) and (4.12), (6.26), respectively. □

6.9. *Concluding remark :* In the case of eigenfunctions we have obtained results of the same quality as in [11, Chapter 6].

In the case of eigenvalues, in order to obtain the rate of convergence $O(h^2)$, we cannot avoid the assumption that $\partial\Omega$ is of class $\mathscr{C}^{1,1}$.

The error estimates for both the eigenvalues and the eigenfunctions only require $w_m \in H^2(\Omega)$, while in [11, Theorem 6.5-1] this $H^2(\Omega)$-regularity is also required for the eigenfunctions $w_1, \dots, w_{m-1}$.

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