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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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MIXED FINITE ELEMENT METHODS FOR QUASILINEAR SECOND ORDER ELLIPTIC PROBLEMS : THE *p*-VERSION (*)

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Abstract — *The p-version of the finite element method is analyzed for quasilinear second order elliptic problems in mixed weak form Approximation properties of the Raviart-Thomas projection are demonstrated and L^2-error bounds for the three relevant variables in the mixed method are derived*

Résumé — *Nous analysons la version-p de la méthode d'éléments finis mixtes pour des problèmes quasilinéaires elliptiques du second ordre en forme faible mixte Nous démontrons des propriétés d'approximation de la projection de Raviart-Thomas et on dérive des bornes de l'erreur dans $L^2(\Omega)$ pour les trois variables d'intérêt dans la méthode mixte*

I. INTRODUCTION

We consider here the numerical solution of the following boundary-value problem :

$$\begin{cases} \mathscr{D}(u) = -\underset{\sim}{\nabla} \cdot (a(u)\underset{\sim}{\nabla} u + \underset{\sim}{b}(u)) + c(u) = 0 & \text{in} \quad \Omega\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = -g & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a convex polygon with boundary $\partial\Omega$, $\underset{\sim}{\nabla} w$ denotes the gradient of the scalar function w and $\underset{\sim}{\nabla} \cdot \underset{\sim}{v}$ and div $\underset{\sim}{v}$ denote the divergence of the vector

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function $\underset{\sim}{v}$. We shall assume that for $r \geq 2$ and for each $g \in H^{r-1/2}(\partial\Omega)$ there exists a unique isolated solution $u \in H^r(\Omega)$ of (1.1) (that is, a solution not situated at a bifurcation point). Note that Sobolev's embedding theorem implies then that $u \in W^{r-1-\varepsilon, \infty}(\Omega)$, $\varepsilon > 0$, $\varepsilon \ll 1$, which will be needed throughout the paper.

We shall also assume that the coefficients $a : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, $\underset{\sim}{b} : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^2$ and $c : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ are twice continuously differentiable with bounded derivatives through second order, and that $a(\underset{\sim}{x}, q) \geq a_1 > 0$. The variable $\underset{\sim}{x}$ will be omitted as explicit argument of all functions, except when necessary to avoid ambiguity.

For $1 \leq s \leq \infty$ and k any nonnegative integer, we let

$$W^{k,s}(\Omega) = \{f \in L^s(\Omega) : D^\alpha f \in L^s(\Omega),\ |\alpha| \leq k\}$$

denote the Sobolev space endowed with its standard norm

$$\|f\|_{k,s,\Omega} = \Big(\sum_{|\alpha| \leq k} \|D^\alpha f\|^s_{L^s(\Omega)} \Big)^{1/s}, \quad s \leq \infty,$$

$$\|f\|_{k,\infty,\Omega} = \max_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)}.$$

The subscript Ω in the norms will be omitted. Let $H^k(\Omega) = W^{k,2}(\Omega)$ with norm $\|\cdot\|_k = \|\ \|_{k,2}$. In particular, the notation $\|\cdot\|_0$ will mean $\|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|_{L^2(\Omega)^2}$. For $0 \leq r < \infty$ let $W^{r,s}(\Omega)$, $W^{r,s}(\partial\Omega)$, $H^r(\Omega)$, and $H^r(\partial\Omega)$ denote the fractional order Sobolev spaces with norms $\|\cdot\|_{r,s,\Omega}$, $\|\cdot\|_{r,s,\partial\Omega}$, $\|\cdot\|_{r,\Omega}$ and $\|\cdot\|_{r,\partial\Omega}$, respectively, defined by interpolation [7].

We shall denote by $(\cdot,\cdot)$ the Hilbert inner product in either $L^2(\Omega)$ or $L^2(\Omega)^2$ and by $\langle \cdot,\cdot \rangle$ the L^2-inner product on the boundary of Ω. The same notation will be used to indicate the dualities between $W^{r,s}(\Omega)$ and $W^{r,s}(\Omega)'$ and $H^s(\partial\Omega)$ and $H^{-s}(\partial\Omega)$, respectively. Throughout the paper, C, Q, and K will denote generic positive constants which need not have the same value in all their occurrences.

Let

$$\underset{\sim}{V} = \underset{\sim}{H}(\mathrm{div}\,;\Omega) = \{\underset{\sim}{v} \in L^2(\Omega)^2 : \mathrm{div}\,\underset{\sim}{v} \in L^2(\Omega)\},$$

normed by

$$\|\underset{\sim}{v}\|^2_{H(\mathrm{div},\Omega)} = \|\underset{\sim}{v}\|_0^2 + \|\mathrm{div}\,\underset{\sim}{v}\|_0^2,$$

and

$$W = L^2(\Omega).$$

The mixed finite element method we shall consider seeks simultaneous approximations of the solution of (1.1), u, and of the flux

$$\underset{\sim}{z} = - a(u) \underset{\sim}{\nabla} u - \underset{\sim}{b}(u) . \tag{1.2}$$

The mixed weak formulation of (1.1) consists of finding $(\underset{\sim}{z}, u) \in \underset{\sim}{V} \times W$ such that

$$\begin{cases} (\alpha(u) \underset{\sim}{z}, \underset{\sim}{v}) - (u, \operatorname{div} \underset{\sim}{v}) + (\underset{\sim}{\beta}(u), \underset{\sim}{v}) = \langle g, \underset{\sim}{v} \cdot \underset{\sim}{\nu} \rangle , & \underset{\sim}{v} \in \underset{\sim}{V} , \\ (\operatorname{div} \underset{\sim}{z}, w) + (c(u), w) = 0 , & w \in W , \end{cases} \tag{1.3}$$

where we have set

$$\alpha(u) = 1/a(u) , \quad \underset{\sim}{\beta}(u) = \alpha(u) \underset{\sim}{b}(u) , \tag{1.4}$$

and $\underset{\sim}{\nu}$ is the outward unit normal vector on $\partial \Omega$. Our mixed finite element method is a discrete form of (1.3).

Let $\mathcal{T}$ be a decomposition of Ω by parallelograms which will be the « elements » E and let $\mathcal{P}_{p,q}(E) = \{$polynomials $f(x, y)$ on E, of degree $\leq p$ in x and degree $\leq q$ in $y\}$, $\mathcal{Q}_p(E) = \{$polynomials of degree $\leq p$ on $E\}$; next define, for each element E,

$$\underset{\sim}{V}^p(E) = \mathcal{P}_{p+1,p}(E) \times \mathcal{P}_{p,p+1}(E) ,$$

and let

$$\underset{\sim}{V}^p \times W^p \subset \underset{\sim}{V} \times W$$

be the Raviart-Thomas-Nedelec space of index $p \geq 0$ associated with this decomposition [3, 5], given by

$$\begin{cases} \underset{\sim}{V}^p = \left(\prod_{E \in \mathcal{T}} \underset{\sim}{V}^p(E) \right) \cap \{ \underset{\sim}{f} : \Omega^2 \to \mathbb{R} \,|\, \underset{\sim}{f} \cdot \nu_E \\ \quad = \underset{\sim}{f} \cdot \nu_{E'} \text{ on } E \cap E', E, E' \in \mathcal{T} \} \\ W^p = \prod_{E \in \mathcal{T}} \mathcal{Q}_p(E) , \end{cases}$$

where $\underset{\sim}{\nu}_E$ denotes the outward unit normal vector along ∂E, $E \in \mathcal{T}$. It is known [3, 5] that $\operatorname{div} \underset{\sim}{V}^p \subset W^p$, a property we shall exploit later.

We seek $(\underset{\sim}{z}^p, u^p) \in \underset{\sim}{V}^p \times W^p$ so that

$$\begin{cases} (\alpha(u^p) \underset{\sim}{z}^p, \underset{\sim}{v}) - (u^p, \operatorname{div} \underset{\sim}{v}) + (\underset{\sim}{\beta}(u^p), \underset{\sim}{v}) = \langle g, \underset{\sim}{v} \cdot \underset{\sim}{\nu} \rangle , & \underset{\sim}{v} \in \underset{\sim}{V}^p , \\ (\operatorname{div} \underset{\sim}{z}^p, w) + (c(u^p), w) = 0 , & w \in W^p . \end{cases} \tag{1.5}$$

Equations (1.5) define the p-version of the mixed finite element approximation for (1.3). This version is based on using a fixed mesh and increasing the degree of the finite elements (as opposed to the h-version that keeps the degree fixed and refines the mesh). The p-version has been analyzed for the linearized version of (1.1) in terms of the standard variational form in [1] and in terms of the mixed variational form in [6]. In this paper, we extend the results obtained in [6] for the linear problem to the quasilinear case. We also obtain an improved version of lemma 3.1 of [6] by reducing the regularity assumed there. We restrict our attention to the mixed method, the corresponding generalization for the standard finite element method is more straightforward.

Milner [3] described the h-version of this method for the same problem, demonstrated the unique solvability (for small h) of the nonlinear algebraic system (1.5) and derived error estimates in $L^s(\Omega)$, $2 \leq s \leq +\infty$, for the error in u, and in $\underset{\sim}{H}(\text{div}\,;\,\Omega)$ for the error in $\underset{\sim}{z}$. The assumption there was that the solution of (1.1) was in the space $H^{2+\varepsilon}(\Omega)$. In contrast, for the present paper we shall need an extra half derivative, that is, $u \in H^{5/2+\varepsilon}(\Omega)$.

We shall follow very closely the analysis of [3]. In order to do so we shall use the L^2-projection onto W^p, $P^p : L^2 \to W^p$, given by

$$(P^p w - w, \chi) = 0\,, \quad \chi \in W^p\,, \quad w \in W\,, \tag{1.6}$$

for which the following approximation properties follow by repeating the arguments of [4] in two dimensions and using interpolation from the cases $s = 2$ and $s = \infty$:

$$\|P^p w - w\|_{0,s} \leq Q p^{-m+3/2-3/s} \|w\|_m\,, \quad s \geq 2\,, \quad 3/2 - 3/s \leq m\,, \tag{1.7}$$

if $w \in H^m(\Omega)$. We shall also use the Raviart-Thomas projection of $\underset{\sim}{V}$ onto $\underset{\sim}{V}^p$, $\pi^p : \underset{\sim}{V} \to \underset{\sim}{V}^p$, [5] for which we shall demonstrate in Section 2 the following approximation property :

$$\|\pi^p \underset{\sim}{v} - \underset{\sim}{v}\|_0 \leq Q p^{1/2-r} \|\underset{\sim}{v}\|_r\,, \quad r > 1/2\,, \quad \underset{\sim}{v} \in H^r(\Omega)^2 \cap \underset{\sim}{V}\,. \tag{1.8}$$

Our proof of (1.8) improves upon the one presented in [6], which imposed extra regularity on $\underset{\sim}{v}$ by requiring that $r > 1$. In contrast, the condition $r > 1/2$ is optimal (see remark 2.1). We also obtain estimates for the approximation properties of π^p in the $W^{0,s}(\Omega)$-norm.

We shall find very useful the following inverse-type inequalities, the two dimensional form of the ones found in [4] :

$$\|\chi\|_{0,s} \leq Q p^{4/r-4/s} \|\chi\|_{0,r}\,, \quad 1 \leq r \leq s \leq \infty\,,$$
$$\chi \in L^s(\Omega) \cap W^p \ (\text{or } \chi \in L^s(\Omega)^2 \cap \underset{\sim}{V}^p)\,. \tag{1.9}$$

The plan of the paper is as follows : in Section 2 we demonstrate (1.8), in Section 3 we prove that, for p sufficiently large, (1.5) is uniquely solvable and its solution $(\underset{\sim}{z}^p, u^p)$ converges to $(\underset{\sim}{z}, u)$ in $\underset{\sim}{V} \cap L^{2+\varepsilon}(\Omega)^2 \times L^{(2+4\varepsilon)/\varepsilon}(\Omega)$ for any fixed ε, $0 \ll \varepsilon \ll 1$, and in Section 4 we establish the rate of convergence of the approximation to the exact solution.

II. THE APPROXIMATION PROPERTIES OF π^p

We recall that $\pi^p \underset{\sim}{v}$ is given locally (on every element E) by the following relations (2.1) and (2.2) (see [5]) :

$$\left\langle [\pi^p \underset{\sim}{v} - \underset{\sim}{v}] \cdot \underset{\sim}{\nu}_E, \varphi \right\rangle_{S_i} = 0 , \quad \varphi \in \mathscr{P}_p , \tag{2.1}$$

where $\langle \cdot , \cdot \rangle_{S_i}$, $1 \leq i \leq 4$, denotes the line integral along each side S_i of the element E and $\mathscr{P}_p$ is the set of all polynomials in one variable of degree less than or equal to p,

$$(\pi^p \underset{\sim}{v} - \underset{\sim}{v}, \underset{\sim}{\psi})_E = 0 , \quad \underset{\sim}{\psi} \in \underset{\sim}{V}^p(E) , \tag{2.2}$$

where $(\cdot , \cdot)_E$ denotes the standard $L^2(E)$-inner product.

Now, let $R = [-1, 1] \times [-1, 1]$ and let $\{P_i\}_{i \geq 0}$ denote the $L^2([-1, 1])$-complete orthogonal Legendre polynomials. For any $\underset{\sim}{v} \in \underset{\sim}{H}(\mathrm{div} ; R)$, let

$$\underset{\sim}{v}(x, y) = \left[\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} P_i(x) P_j(y), \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j} P_i(x) P_j(y) \right] , \tag{2.3}$$

and let

$$\pi^p \underset{\sim}{v}(x, y) = \left[\sum_{i=0}^{p+1} \sum_{j=0}^{p} \tilde{a}_{i,j} P_i(x) P_j(y), \sum_{i=0}^{p} \sum_{j=0}^{p+1} \tilde{b}_{i,j} P_i(x) P_j(y) \right] . \tag{2.4}$$

It follows from (2.2)-(2.4) that

$$\begin{cases} a_{i,j} = \tilde{a}_{i,j} , & 0 \leq i \leq p-1 , \quad 0 \leq j \leq p , \\ b_{kl} = \tilde{b}_{kl} , & 0 \leq k \leq p , \quad 0 \leq l \leq p-1 . \end{cases} \tag{2.5}$$

Next, we see that (2.1), (2.3)-(2.5) imply that

$$\begin{cases} \sum_{i=p}^{p+1} \tilde{a}_{i,j} P_i(\pm 1) = \sum_{i=p}^{\infty} a_{i,j} P_i(\pm 1) , & 0 \leq j \leq p , \\ \sum_{j=p}^{p+1} \tilde{b}_{i,j} P_j(\pm 1) = \sum_{j=p}^{\infty} b_{i,j} P_i(\pm 1) , & 0 \leq i \leq p . \end{cases} \tag{2.6}$$

Since $P_i(-1) = (-1)^i$ and $P_i(1) = 1$, (2.6) implies that

$$\begin{cases} \tilde{a}_{pj} = \sum_{i=0}^{\infty} a_{2i+p,j}, & \tilde{a}_{p+1,j} = \sum_{i=0}^{\infty} a_{2i+p+1,j}, \quad 0 \leq j \leq p, \\ \tilde{b}_{ip} = \sum_{j=0}^{\infty} b_{i,p+2j}, & \tilde{b}_{i,p+1} = \sum_{j=0}^{\infty} b_{i,p+1+2j}, \quad 0 \leq i \leq p. \end{cases} \tag{2.7}$$

PROPOSITION 2.1 : *Let $\underset{\sim}{v} \in \underset{\sim}{V}$ and let $\pi^p \underset{\sim}{v}$ be its Raviart-Thomas projection in $\underset{\sim}{V}^p$ given by* (2.1)-(2.2). *Then, if $\underset{\sim}{v} \in H^r(\Omega)^2$, we have*

$$\|\underset{\sim}{v} - \pi^p \underset{\sim}{v}\|_0 \leq Q p^{1/2 - r} \|\underset{\sim}{v}\|_r, \quad r > 1/2,$$

where $Q > 0$ is a constant independent of p and $\underset{\sim}{v}$ but depending on r.

Proof : We first assume that $\Omega = R$ and that the decomposition consists of just one element. Then, $\underset{\sim}{v} \in \underset{\sim}{V}$ and $\pi^p \underset{\sim}{v} \in \underset{\sim}{V}^p$ can be given, respectively, by (2.3) and (2.4).

The following relation is a trivial consequence of well known properties of the Legendre polynomials,

$$\begin{aligned} \|\underset{\sim}{v} - \pi^p \underset{\sim}{v}\|_0^2 = & \sum_{i=p}^{p+1} \sum_{j=0}^{p} \frac{4(a_{i,j} - \tilde{a}_{i,j})^2}{(2i+1)(2j+1)} + \sum_{i=0}^{p} \sum_{j=p}^{p+1} \frac{4(b_{i,j} - \tilde{b}_{i,j})^2}{(2i+1)(2j+1)} + \\ & + \sum_{i=0}^{p+1} \sum_{j=p+1}^{\infty} \frac{4a_{i,j}^2}{(2i+1)(2j+1)} + \sum_{i=p+1}^{\infty} \sum_{j=0}^{p+1} \frac{4b_{i,j}^2}{(2i+1)(2j+1)} \\ & + \sum_{i=p+2}^{\infty} \sum_{j=0}^{p} \frac{4a_{i,j}^2}{(2i+1)(2j+1)} + \sum_{i=0}^{p} \sum_{j=p+2}^{\infty} \frac{4b_{i,j}^2}{(2i+1)(2j+1)} \\ & + \sum_{i=p+2}^{\infty} \sum_{j=p+1}^{\infty} \frac{4a_{i,j}^2}{(2i+1)(2j+1)} + \sum_{i=p+1}^{\infty} \sum_{j=p+2}^{\infty} \frac{4b_{i,j}^2}{(2i+1)(2j+1)} \end{aligned}$$

$= \mathrm{I} + \mathrm{II} + \cdots + \mathrm{VIII}$.

Note that III-VIII can be bounded as follows :

$$\mathrm{III} + \mathrm{V} + \mathrm{VII} \leq Q p^{-2r} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_{i,j}^2 (1 + i^2 + j^2)^r}{(2i+1)(2j+1)}, \quad r \geq 0,$$

while

$$\mathrm{IV} + \mathrm{VI} + \mathrm{VIII} \leq Q p^{-2r} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{b_{i,j}^2 (1 + i^2 + j^2)^r}{(2i+1)(2j+1)}, \quad r \geq 0,$$

which implies that (see [6])

$$\mathrm{III} + \cdots + \mathrm{VIII} \leq Q p^{-2r} \|\underset{\sim}{v}\|_r^2, \quad r \geq 0. \tag{2.9}$$

On the other hand, it follows from (2.7) that

$$\mathrm{I} = \frac{4}{2p+1} \sum_{j=0}^{p} \frac{\left(a_{p,j} - \sum_{j=0}^{\infty} a_{2i+p,j}\right)^2}{2j+1} + \frac{4}{2p+3} \sum_{j=0}^{p} \frac{\left(a_{p+1,j} - \sum_{i=0}^{\infty} a_{2i+p+1,j}\right)^2}{2j+1}$$

$$= \frac{4}{2p+1} \sum_{j=0}^{p} \frac{\left(\sum_{i=1}^{\infty} a_{2i+p,j}\right)^2}{2j+1} + \frac{4}{2p+3} \sum_{j=0}^{p} \frac{\left(\sum_{i=1}^{\infty} a_{2i+p+1,j}\right)^2}{2j+1} \tag{2.10}$$

and

$$\mathrm{II} = \frac{4}{2p+1} \sum_{i=0}^{p} \frac{\left(b_{ip} - \sum_{j=0}^{\infty} b_{i,2j+p}\right)^2}{2i+1}$$

$$+ \frac{4}{2p+3} \sum_{i=0}^{p} \frac{\left(b_{i,p+1} - \sum_{j=0}^{\infty} b_{i,2j+p+1}\right)^2}{2i+1}$$

$$= \frac{4}{2p+1} \sum_{i=0}^{p} \frac{\left(\sum_{j=1}^{\infty} b_{i,2j+p}\right)^2}{2i+1} + \frac{4}{2p+3} \sum_{i=0}^{\infty} \frac{\left(\sum_{j=1}^{\infty} b_{i,2j+p+1}\right)^2}{2i+1}. \tag{2.11}$$

Next observe that bounding the series $\sum_{k=p+1}^{\infty} (c+k^2)^{-s}(1+2k)$ using the integral method for $\int_p^{\infty} (c+t^2)^{-s}(1+2t)\,dt \leqslant \frac{K}{s-1} p^{2-2s}$ $(s>1)$, and using the Cauchy-Schwarz inequality, we see that, for s bounded away from 1,

$$\left(\sum_{i=1}^{\infty} a_{2i+p,j}\right)^2 \leqslant$$

$$\leqslant \sum_{i=1}^{\infty} \frac{a^2_{2i+p,j}}{4i+2p+1} [1+(2i+p)^2+j^2]^s \sum_{i=1}^{\infty} [1+(2i+p)^2+j^2]^{-s} \times$$

$$\times (1+4i+2p)$$

$$\leqslant \sum_{i=0}^{\infty} \frac{a^2_{i,j}(1+i^2+j^2)^s}{2i+1} \sum_{i=p+2}^{\infty} (1+i^2+j^2)^{-s}(1+2i)$$

$$\leqslant Qp^{2-2s} \sum_{i=0}^{\infty} \frac{a^2_{i,j}(1+i^2+j^2)^s}{2i+1}, \tag{2.12}$$

with exactly the same final bound holding for $\left(\sum_{i=1}^{\infty} a_{2i+p+1,j}\right)^2$. It follows from (2.10) and (2.12) that, for s bounded away from 1,

$$\mathrm{I} \leq Q p^{1-2s} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{a_{i,j}^2 (1+i^2+j^2)^s}{(2i+1)(2j+1)} . \tag{2.13}$$

In an entirely analogous way (replacing $a_{i,j}$ by $b_{i,j}$ and reversing the roles of i and j) we deduce from (2.11) that, for s bounded away from 1,

$$\mathrm{II} \leq Q p^{1-2s} \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{b_{i,j}^2 (1+i^2+j^2)^s}{(2i+1)(2j+1)} . \tag{2.14}$$

Combining (2.13) and (2.14) results, for s bounded away from 1, in

$$\mathrm{I} + \mathrm{II} \leq Q p^{1-2s} \| \underset{\sim}{v} \|_s^2 . \tag{2.15}$$

Next note that

$$\begin{aligned} \underset{\sim}{v} \cdot \underset{\sim}{\nu} \big|_{\partial R} = & \\ v_1(\pm 1, y) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} P_i(\pm 1) P_j(y) , \quad -1 \leq y \leq 1 , \\ v_2(x, \pm 1) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{i,j} P_i(y) P_j(\pm 1) , \quad -1 \leq x \leq 1 . \end{aligned} \tag{2.16}$$

The trace theorem (Sobolev's embedding theorem) implies that $v_1, v_2 \in L^2(\partial R)$ for $s > 1/2$. Consequently, since $P_i(1) = 1$ and $P_i(-1) = (-1)^i$, we see from (2.16) that

$$\begin{aligned} \| v_1(\pm 1, .) \|_{0,\partial\Omega}^2 &= 2 \sum_{j=0}^{\infty} \frac{\left[\sum_{i=0}^{\infty} (\pm 1)^i a_{i,j}\right]^2}{2j+1} < \infty , \\ \| v_2(., \pm 1) \|_{0,\partial\Omega}^2 &= 2 \sum_{j=0}^{\infty} \frac{\left[\sum_{j=0}^{\infty} (\pm 1)^j b_{i,j}\right]^2}{2i+1} < \infty . \end{aligned} \tag{2.17}$$

Let now $\underset{\sim}{v} \in H^{1/2+\varepsilon}(\Omega)^2$. We shall prove that

$$\| \pi^p \underset{\sim}{v} - \underset{\sim}{v} \|_0 \leq Q p^{-\varepsilon} \| \underset{\sim}{v} \|_{1/2+\varepsilon} . \tag{2.18}$$

In view of (2.8) and (2.9) it is sufficient to prove that I, II $< Qp^{-2\varepsilon}\|\underset{\sim}{v}\|^2_{1/2+\varepsilon}$. It follows from (2.10) that

$$\mathrm{I} \leqslant 4\,p^{-1}\sum_{j=0}^{p}(2j+1)^{-1}\left[\left(\sum_{i=1}^{\infty}a_{2i+p,j}\right)^2+\left(\sum_{i=1}^{\infty}a_{2i+p+1,j}\right)^2\right]$$

$$= 2\,p^{-1}\sum_{j=0}^{p}(2j+1)^{-1}\left[\left(\sum_{i=p+2}^{\infty}a_{i,j}\right)^2+\left(\sum_{i=p+2}^{\infty}(-1)^i a_{i,j}\right)^2\right]$$

$$= 2\,p^{-1}\sum_{j=0}^{p}(2j+1)^{-1}\left[\left(\sum_{i=0}^{\infty}a_{i,j}-\sum_{i=0}^{p+1}a_{i,j}\right)^2+\right.$$

$$\left.+\left(\sum_{i=0}^{\infty}(-1)^i a_{i,j}-\sum_{i=0}^{p+1}(-1)^i a_{i,j}\right)^2\right]$$

$$\leqslant 4\,p^{-1}\sum_{j=0}^{p}(2j+1)^{-1}\left[\left(\sum_{i=0}^{\infty}a_{i,j}\right)^2+\left(\sum_{i=0}^{\infty}(-1)^i a_{i,j}\right)^2+\left(\sum_{i=0}^{p+1}a_{i,j}\right)^2+\right.$$

$$\left.+\left(\sum_{i=0}^{p+1}(-1)^i a_{i,j}\right)^2\right]$$

$$\leqslant 4\,p^{-1}(\|v_1(1,\,.\,)\|^2_{0,\partial\Omega}+\|v_1(-1,\,.\,)\|^2_{0,\partial\Omega})+$$

$$+4\,p^{-1}\sum_{j=0}^{p}(2j+1)^{-1}\left[\left(\sum_{i=0}^{p+1}a_{i,j}\right)^2+\left(\sum_{i=0}^{p+1}(-1)^i a_{i,j}\right)^2\right]. \tag{2.19}$$

Note that the next to last term on the right hand side of (2.19) can be bounded, using the integral method for

$$\int_0^{p+1}(2\,i+1)(1+i^2+j^2)^{-1/2-\varepsilon}=O(p^{1-2\varepsilon})$$

as $p\to\infty$, as follows :

$$\sum_{j=0}^{p}(2j+1)^{-1}\left(\sum_{i=0}^{p+1}a_{i,j}\right)^2 \leqslant \sum_{j=0}^{p}(2j+1)^{-1}\sum_{i=0}^{p+1}\frac{a^2_{i,j}(1+i^2+j^2)^{1/2+\varepsilon}}{2\,i+1}$$

$$\times\sum_{i=0}^{p+1}(2\,i+1)(1+i^2+j^2)^{-1/2-\varepsilon}$$

$$\leqslant Q\,\|\underset{\sim}{v}\|^2_{1/2+\varepsilon}\,p^{1-2\varepsilon}\,, \tag{2.20}$$

with an identical bound holding for the last term of (2.19). Combining (2.19) and (2.20) and using Sobolev's embedding theorem yields

$$\mathrm{I}\leqslant Qp^{-2\varepsilon}\|\underset{\sim}{v}\|^2_{1/2+\varepsilon}\,. \tag{2.21}$$

In an entirely analogous fashion we can see that

$$\mathrm{II} \leqslant Q p^{-2\varepsilon} \|\underset{\sim}{v}\|^2_{1/2+\varepsilon},$$

which together with (2.21) yields (2.18). Using interpolation [7], it follows from (2.15) and (2.18) that, for s bounded away from 1/2,

$$\mathrm{I} + \mathrm{II} \leqslant Q p^{1-2s} \|\underset{\sim}{v}\|^2_s,$$

which together with (2.8) and (2.9) concludes the proof for the case $\Omega = R$. For the case when Ω is a disjoint union of parallelograms the result follows on each element by using affine mappings onto R. The proposition then follows by summing over all the elements (see [6] for details).

Remark 2.1 : This result differs from the one known for the h-version of the finite element method [3, (1.5)],

$$\|\underset{\sim}{v} - \pi^h \underset{\sim}{v}\|_0 \leqslant Q h^r \|\underset{\sim}{v}\|_r, \quad r > 1/2 . \tag{2.22}$$

The constraint $r > 1/2$ (or $r \geqslant 1/2 + \varepsilon$) stems from the fact that, according to the trace theorem, this is the minimal requirement to ensure that $\underset{\sim}{v}$ has a trace on the boundary which is an L^2-function (not just a distribution). In [6] the corresponding result required an additional half derivative on $\underset{\sim}{v}\,(r > 1)$. In contrast, proposition 2.1 assumes the minimum regularity necessary. It is possible, however, that the bound still holds with the exponent of p replaced by $-r$, as suggested by (2.22).

COROLLARY 2.1 : *For* $s \geqslant 2$, $r > \max\ \{1/2\,;\,3/2 - 3/s\}$,

$$\|\underset{\sim}{v} - \pi^p \underset{\sim}{v}\|_{0,s} \leqslant Q p^{5/2 - r - 4/s} \|\underset{\sim}{v}\|_r .$$

Proof : Let $\underset{\sim}{P}^p \underset{\sim}{v}$ be the L^2-projection $P^p \times P^p : \underset{\sim}{V} \to \underset{\sim}{V}^p$. Then the following analogue of (1.7) holds :

$$\|\underset{\sim}{P}^p \underset{\sim}{v} - \underset{\sim}{v}\|_{0,s} \leqslant Q p^{-r+3/2-3/s} \|\underset{\sim}{v}\|_r, \quad s \geqslant 2\,, \quad 3/2 - 3/s \leqslant r\,. \tag{2.23}$$

Also,

$$\|\pi^p \underset{\sim}{v} - \underset{\sim}{v}\|_{0,s} \leqslant \|\underset{\sim}{P}^p \underset{\sim}{v} - \underset{\sim}{v}\|_{0,s} + \|\pi^p \underset{\sim}{v} - \underset{\sim}{P}^p \underset{\sim}{v}\|_{0,s} . \tag{2.24}$$

The second term in this expression may be bounded using the inverse inequality (1.9) as follows :

$$\begin{aligned} \|\pi^p \underset{\sim}{v} - \underset{\sim}{P}^p \underset{\sim}{v}\|_{0,s} &\leqslant Q p^{2-4/s} \|\pi^p \underset{\sim}{v} - \underset{\sim}{P}^p \underset{\sim}{v}\|_0 \\ &\leqslant Q p^{2-4/s} (\|\underset{\sim}{P}^p \underset{\sim}{v} - \underset{\sim}{v}\|_0 + \|\pi^p \underset{\sim}{v} - \underset{\sim}{v}\|_0) . \end{aligned} \tag{2.25}$$

Combining (2.23)-(2.25) and using proposition 2.1, we obtain the corollary.

III. SOLVABILITY OF THE DISCRETE PROBLEM

Following [3] we introduce, for $\rho \in W^p$, the notation

$$\alpha(\rho) - \alpha(u) = -\tilde{\alpha}_u(\rho)(u-\rho) = -\alpha_u(u)(u-\rho) + \tilde{\alpha}_{uu}(\rho)(u-\rho)^2, \tag{3.1}$$

where

$$\tilde{\alpha}_u(\rho) = \int_0^1 \alpha_u(\rho + t[u-\rho])\, dt,$$

and

$$\tilde{\alpha}_{uu}(\rho) = \int_0^1 (1-t)\, \alpha_{uu}(u + t[\rho - u])\, dt,$$

are bounded functions in $\bar{\Omega}$. Similarly, we write

$$\underset{\sim}{\beta}(\rho) - \underset{\sim}{\beta}(u) = -\tilde{\underset{\sim}{\beta}}_u(\rho)(u-\rho) = -\underset{\sim}{\beta}_u(u)(u-\rho) + \tilde{\underset{\sim}{\beta}}_{uu}(\rho)(u-\rho)^2, \tag{3.2}$$

and

$$c(\rho) - c(u) = -\tilde{c}_u(\rho)(u-\rho) = -c_u(u)(u-\rho) + \tilde{c}_{uu}(\rho)(u-\rho)^2, \tag{3.3}$$

where $\tilde{\underset{\sim}{\beta}}_u(\rho)$, $\tilde{\underset{\sim}{\beta}}_{uu}(\rho)$, $\tilde{c}_u(\rho)$, and $\tilde{c}_{uu}(\rho)$ are bounded functions in $\bar{\Omega}$. Also, let

$$\underset{\sim}{\Gamma} = \alpha_u(u)\, \underset{\sim}{z} + \underset{\sim}{\beta}_u(u), \qquad \gamma = c_u(u). \tag{3.4}$$

With the notation of (3.1)-(3.4), the following error equations follow from (1.3) and (1.5), [3] :

$$\begin{aligned}(\alpha(u)[\pi^p \underset{\sim}{z} - \underset{\sim}{z}^p], \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, P^p u - u^p) + ([P^p u - u^p]\, \underset{\sim}{\Gamma}, \underset{\sim}{v}) &= \\ = (\underset{\sim}{q}(u^p, \underset{\sim}{z}^p), \underset{\sim}{v}), \quad \underset{\sim}{v} \in \underset{\sim}{V}^p&, \\ (\operatorname{div}[\pi^p \underset{\sim}{z} - \underset{\sim}{z}^p], w) + (\gamma[P^p u - u^p], w) = (\eta(u^p), w), \quad w \in W^p&,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\underset{\sim}{q}(u^p, \underset{\sim}{z}^p) &= \alpha(u)[\pi^p \underset{\sim}{z} - \underset{\sim}{z}] + [P^p u - u]\, \underset{\sim}{\Gamma} + \\ &+ (u-u^p)^2 [\tilde{\alpha}_{uu}(u^p)\, \underset{\sim}{z} + \tilde{\underset{\sim}{\beta}}_{uu}(u^p)] + \tilde{\alpha}_u(u^p)(u-u^p)(\underset{\sim}{z} - \underset{\sim}{z}^p),\end{aligned} \tag{3.6}$$

and

$$\eta(u^p) = \gamma [P^p u - u] + \tilde{c}_{uu}(u^p)(u - u^p)^2 . \qquad (3.7)$$

Just as in [3], we let

$$\Phi : \underset{\sim}{V}^p \times W^p \to \underset{\sim}{V}^p \times W^p$$

be given by $\Phi((\underset{\sim}{\mu}, \rho)) = (\underset{\sim}{\lambda}, \kappa)$, $(\underset{\sim}{\lambda}, \kappa)$ being the (unique) solution of the system

$$(\alpha(u)[\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}], \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, P^p u - \kappa) + ([P^p u - \kappa] \underset{\sim}{\Gamma}, \underset{\sim}{v}) = = (\underset{\sim}{q}(\rho, \underset{\sim}{\mu}), \underset{\sim}{v}) , \quad \underset{\sim}{v} \in \underset{\sim}{V}^p ,$$

$$(\operatorname{div} [\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}], w) + (\gamma [P^p u - \kappa], w) = (\eta(\rho), w) , \quad w \in W^p , \qquad (3.8)$$

where $\underset{\sim}{q}(\rho, \mu)$ and $\eta(\rho)$ are given by (3.6) and (3.7), respectively, replacing u^p by ρ and $\underset{\sim}{z}^p$ by $\underset{\sim}{\mu}$. The unique solvability of this (linear) system follows, for p sufficiently large, from [2], since the left hand side of (3.8) corresponds to the mixed method for the operator $M : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ given by

$$Mw = -\underset{\sim}{\nabla} \cdot (a(u) \underset{\sim}{\nabla} w + a(u) w \underset{\sim}{\Gamma}) + \gamma \omega ,$$

which has a bounded inverse. In fact, note that (1.2), (1.4) and (3.4) give

$$\begin{aligned} Mw &= -\underset{\sim}{\nabla} \cdot [a(u) \underset{\sim}{\nabla} w + a(u) w(\alpha_u(u) \underset{\sim}{z} + \underset{\sim}{\beta}_u(u))] + c_u(u) w \\ &= -\underset{\sim}{\nabla} \cdot \left[a(u) \underset{\sim}{\nabla} w + a(u) w \left[-\frac{a_u(u)}{a^2(u)} (-a(u) \nabla u) + \alpha(u) \underset{\sim}{b}_u(u) \right] \right] + \\ &\quad + c_u(u) w \\ &= -\underset{\sim}{\nabla} \cdot [a(u) \underset{\sim}{\nabla} w + (a_u(u) \nabla u + \underset{\sim}{b}_u(u)) w] + c_u(u) w , \end{aligned}$$

which shows that M is the linearization of the operator $\mathscr{D}$ in (1.1) about the function u, and, thus, it has a bounded inverse since we have assumed that (1.1) admits unique isolated solutions.

The solvability of (1.5) is now equivalent to showing that Φ has a fixed point. This will follow from the Brouwer fixed point theorem if we show that Φ maps a ball of $\underset{\sim}{V}^p \times W^p$ into itself. We shall need the following technical result, a p-version of lemma 2.1 of [3]. Let $\varepsilon > 0$ be fixed for the rest of the paper, $\varepsilon \ll 1$.

LEMMA 3.1 : *Let* $2 \leqslant \theta \leqslant 4 - \varepsilon$. *Let* $\underset{\sim}{\omega} \in \underset{\sim}{V}$, $\underset{\sim}{q} \in L^2(\Omega)^2$, *and* $\eta \in L^2(\Omega)$. *If* $\tau \in W^p$ *satisfies*

$$\begin{cases} (\alpha(u)\,\underset{\sim}{\omega}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \tau) + (\tau \underset{\sim}{\Gamma}, \underset{\sim}{v}) = (\underset{\sim}{q}, \underset{\sim}{v}), & \underset{\sim}{v} \in \underset{\sim}{V}^p, \\ (\operatorname{div} \underset{\sim}{\omega}, w) + (\gamma\tau, \omega) = (\eta, w), & w \in W^p, \end{cases}$$

then, there exists a constant $C = C(\theta, u, \alpha, \underset{\sim}{\Gamma}, \gamma, \Omega, \varepsilon)$ *such that, for p sufficiently large, depending upon* ε,

$$\|\tau\|_{0,\theta} \leqslant C\,[p^{1/2 - 2/\theta}\, \|\underset{\sim}{\omega}\|_0 + p^{-1-2/\theta}\, \|\operatorname{div} \underset{\sim}{\omega}\|_0 + \|\underset{\sim}{q}\|_0 + \|\eta\|_0]\,.$$

Proof : We follow the proof of lemma 2.1 of [3]. Let $\theta' = \theta/(\theta - 1)$ be the conjugate exponent of θ. For $\psi \in L^{\theta'}(\Omega)$ let $\phi \in W^{2,\theta'}(\Omega)$ be the (unique) solution of $M^* \phi = \psi$ in Ω, $\psi = 0$ on $\partial\Omega$, where M^* is the formal adjoint of M. It follows that $\|\phi\|_{2,\theta'} \leqslant Q\,\|\psi\|_{0,\theta'}$. We then have [3],

$$\begin{aligned} (\tau, \psi) = {} & (\underset{\sim}{q}, a(u)\,\underset{\sim}{\nabla}\phi) + (\underset{\sim}{q}, \pi^p a(u)\,\underset{\sim}{\nabla}\phi - a(u)\,\underset{\sim}{\nabla}\phi) + \\ & + (\operatorname{div} \underset{\sim}{\omega} + \gamma\tau, \phi - P^p \phi) \\ & + (\alpha(u)\,\underset{\sim}{\omega} + \tau\underset{\sim}{\Gamma}, a(u)\,\underset{\sim}{\nabla}\phi - \pi^p a(u)\,\underset{\sim}{\nabla}\phi) + (\eta, \phi) + (\eta, P^p \phi - \phi)\,. \end{aligned} \tag{3.9}$$

Note that Sobolev's embedding theorem implies that

$$(\underset{\sim}{q}, a(u)\,\underset{\sim}{\nabla}\phi) \leqslant C\,\|\underset{\sim}{q}\|_0\, \|\phi\|_1 \leqslant C\,\|\underset{\sim}{q}\|_0\, \|\phi\|_{2,\theta'}\,. \tag{3.10}$$

Next, (1.8) and Sobolev's embedding theorem imply that

$$\begin{aligned} (\underset{\sim}{q} - \alpha(u)\,\underset{\sim}{\omega}, \pi^p a(u)\,\underset{\sim}{\nabla}\phi - a(u)\,\underset{\sim}{\nabla}\phi) & \leqslant \\ & \leqslant C\,(\|\underset{\sim}{q}\|_0 + \|\underset{\sim}{\omega}\|_0)\,p^{1/2 - 2/\theta}\, \|\underset{\sim}{\nabla}\phi\|_{2/\theta} \\ & \leqslant C\,(\|\underset{\sim}{q}\|_0 + \|\underset{\sim}{\omega}\|_0)\,p^{1/2 - 2/\theta}\, \|\phi\|_{2,\theta'}\,, \end{aligned} \tag{3.11}$$

and that

$$\begin{aligned} (\tau\underset{\sim}{\Gamma}, a(u)\,\underset{\sim}{\nabla}\phi - \pi^p a(u)\,\underset{\sim}{\nabla}\phi) & \leqslant C\,\|\tau\|_{0,\theta}\, \|a(u)\,\underset{\sim}{\nabla}\phi - \pi^p a(u)\,\underset{\sim}{\nabla}\phi\|_0 \\ & \leqslant C\,\|\tau\|_{0,\theta}\, p^{-\varepsilon/8}\, \|\phi\|_{2,\theta'}\,. \end{aligned} \tag{3.12}$$

On the other hand, (1.7) and Sobolev's embedding theorem lead to

$$(\operatorname{div} \underset{\sim}{\omega}, \phi - P^p \phi) \leqslant K\,\|\operatorname{div} \underset{\sim}{\omega}\|_0\, p^{-1-2/\theta}\, \|\phi\|_{2,\theta'}\,, \tag{3.13}$$

$$(\gamma\tau, \phi - P^p \phi) \leqslant K\,\|\tau\|_{0,\theta}\, p^{-1-2/\theta}\, \|\phi\|_{2,\theta'}\,, \tag{3.14}$$

and

$$(\eta, \phi) + (\eta, P^p \phi - \phi) \leq K \|\eta\|_0 \|\phi\|_0 \leq K \|\eta\|_0 \|\phi\|_{2, \theta'} . \quad (3.15)$$

Collecting (3.9)-(3.15) we see that

$$(\tau, \psi) \leq K \|\psi\|_{0, \theta'} [p^{1/2 - 2/\theta} \|\underset{\sim}{\omega}\|_0 + p^{-1-2/\theta} \|\operatorname{div} \underset{\sim}{\omega}\|_0 + $$

$$+ p^{-\varepsilon/8} \|\tau\|_{0, \theta} + \|\underset{\sim}{q}\|_0 + \|\eta\|_0] ,$$

which, for p sufficiently large, yields the desired estimate.

Now let $\underset{\sim}{\mathscr{V}}^p = \underset{\sim}{V}^p$ with the stronger norm $\|\underset{\sim}{v}\|_{\underset{\sim}{\mathscr{V}}^p} = \|\underset{\sim}{v}\|_{0, 2+\varepsilon} + \|\operatorname{div} \underset{\sim}{v}\|_0$ and let $\mathscr{W}^p = W^p$ with the stronger norm $\|w\|_{\mathscr{W}^p} = \|w\|_{0, t}$, where $t = \dfrac{4 + 2\varepsilon}{\varepsilon}$. We can prove now the existence of a solution of (1.5).

THEOREM 3.1 : *For $\delta > 0$ sufficiently small (dependent on p) and for p sufficiently large, Φ maps a ball of radius δ centered at $(\pi^p \underset{\sim}{z}, P^p u)$ of $\underset{\sim}{\mathscr{V}}^p \times \mathscr{W}^p$ into itself.*

Proof : Note that $1/t + 1/(2 + \varepsilon) = 1/2$. Let

$$\|\pi^p \underset{\sim}{z} - \underset{\sim}{\mu}\|_{\underset{\sim}{\mathscr{V}}^p} \leq \delta \quad \text{and} \quad \|P^p u - \rho\|_{\mathscr{W}^p} \leq \delta < 1 .$$

Let us use lemma 3.1 on (3.8) with $\tau = P^p u - \kappa$, $\underset{\sim}{w} = \pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}$, $\underset{\sim}{q} = \underset{\sim}{q}(\rho, \underset{\sim}{\mu})$, $\eta = \eta(\rho)$ and $\theta = 4 - \varepsilon$. Observe that (1.7)-(1.9) and corollary 2.1 imply that, for $r > 1/2$, $m = r + 1$,

$$\begin{aligned}
\|\underset{\sim}{q}(\rho, \underset{\sim}{\mu})\|_0 + \|\eta(\rho)\|_0 &\leq \mathscr{Q} [p^{1/2 - r} \|\underset{\sim}{z}\|_r + p^{-m} \|u\|_m + \|u - \rho\|_{0,4}^2 + \\
&\quad + \|u - \rho\|_{0,t} \|\underset{\sim}{z} - \underset{\sim}{\mu}\|_{0, 2+\varepsilon}] \\
&\leq \mathscr{Q} [p^{1/2 - r} \|u\|_{r+1} + (\|u - P^p u\|_{0,4} + \|P^p u - \rho\|_{0,4})^2 + \\
&\quad + (\|u - P^p u\|_{0,t} + \|P^p u - \rho\|_{0,t}) \times \\
&\quad \times (\|\underset{\sim}{z} - \pi^p \underset{\sim}{z}\|_{0, 2+\varepsilon} + \|\pi^p \underset{\sim}{z} - \underset{\sim}{\mu}\|_{0, 2+\varepsilon})] \\
&\leq \mathscr{Q} [p^{1/2 - r} \|u\|_{r+1} + (p^{-m + 3/4} \|u\|_m + \delta)^2 + \\
&\quad + (p^{5/2 - r - 4/(2+\varepsilon)} \|u\|_{r+1} + \delta)(p^{-m + 3/2 - 3/t} \|u\|_m + \delta)] \\
&\leq \mathscr{Q} (\delta^2 + p^{1/2 - r} \|u\|_{r+1}) ,
\end{aligned} \quad (3.16)$$

where $\mathcal{Q}$ depends on $\|u\|_m$. Therefore,

$$\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q}[p^{-\varepsilon/8}\|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_0 + p^{-1-2/(4-\varepsilon)} \times \\ \times \|\operatorname{div}(\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda})\|_0 + \delta^2 + p^{1/2-r}] . \quad (3.17)$$

On the other hand, taking $\underset{\sim}{v} = \pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}$ and $w = P^p u - \kappa$ in (3.8), we see that

$$\|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_0 \leq \mathcal{Q}[\|P^p u - \kappa\|_0 + \|\underset{\sim}{q}\|_0 + \|\eta\|_0] , \quad (3.18)$$

and, taking $w = \operatorname{div}(\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda})$ in the second equation of (3.8) results in

$$\|\operatorname{div}(\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda})\|_0 \leq \mathcal{Q}[\|P^p u - \kappa\|_0 + \|\underset{\sim}{q}\|_0 + \|\eta_0\|] . \quad (3.19)$$

Combining (3.17)-(3.19) yields the relation

$$\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q}[p^{-\varepsilon/8}\|P^p u - \kappa\|_0 + \delta^2 + p^{1/2-r}] ,$$

which, for p sufficiently large and $r = 5/2$, implies that

$$\|P^p u - \kappa\|_{0,4-\varepsilon} \leq \mathcal{Q}[\delta^2 + p^{-2}] , \quad (3.20)$$

where the constant $\mathcal{Q}$ depends on $\|u\|_{7/2}$. Combining (3.20) with (1.9) we see that

$$\begin{aligned} \|P^p u - \kappa\|_{0,t} &\leq \mathcal{Q} p^{\frac{4}{4-\varepsilon} - \frac{2t}{\varepsilon+2}} \|P^p u - \kappa\|_{0,4-\varepsilon} \\ &\leq \mathcal{Q}(p^{1-\varepsilon/4}\delta^2 + p^{-1-\varepsilon/4}) , \end{aligned} \quad (3.21)$$

while (1.9), (3.18), (3.16), and (3.20) imply that

$$\begin{aligned} \|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_{0,2+\varepsilon} &\leq \mathcal{Q} p^{2\varepsilon/(2+\varepsilon)}\|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_0 \\ &\leq \mathcal{Q}(p^\varepsilon \delta^2 + p^{-2+\varepsilon}) . \end{aligned} \quad (3.22)$$

Combining (3.19) and (3.22) yields

$$\|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_{\underset{\sim}{\mathcal{V}}^p} \leq \mathcal{Q}(p^\varepsilon \delta^2 + p^{-2+\varepsilon}) . \quad (3.23)$$

We can now combine (3.21) and (3.23) in the bound

$$\|P^p u - \kappa\|_{\mathcal{W}^p} + \|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_{\underset{\sim}{\mathcal{V}}^p} \leq \mathcal{Q}_1(p^{1-\varepsilon/4}\delta^2 + p^{-1-\varepsilon/4}) . \quad (3.24)$$

We want to choose p and δ so that $\mathcal{Q}_1 p^{1-\varepsilon/4}\delta^2 \leq \frac{\delta}{2}$ and $\mathcal{Q}_1 p^{-1-\varepsilon/4} \leq \frac{\delta}{2}$.

Let $p \geqslant (2\,\mathcal{Q}_1)^{4/\varepsilon}$, so that $I = \left[2\,\mathcal{Q}_1\,p^{-1-\varepsilon/4}, \frac{p^{\varepsilon/4-1}}{2\,\mathcal{Q}_1}\right]$ is not empty. Then, for $\delta \in I$, (3.24) implies that

$$\|P^p u - \kappa\|_{\mathcal{W}^p} \leqslant \delta \quad \text{and} \quad \|\pi^p \underset{\sim}{z} - \underset{\sim}{\lambda}\|_{\underset{\sim}{\mathcal{V}}^p} \leqslant \delta\,,$$

as we needed.

Remark 3.1 : Note that the choice $\delta = 2\,\mathcal{Q}_1\,p^{-1-\varepsilon/4}$ in theorem 3.1 shows (using (1.7) and (1.8)) not only that (1.5) is solvable but also that, for $p \to \infty$, the solution of (1.5), $(\underset{\sim}{z}^p, u^p)$, differs from $(\underset{\sim}{z}, u)$ in the $\underset{\sim}{\mathcal{V}}^p \times \mathcal{W}^p$ norm by $0(p^{-1-\varepsilon/4})$ at most. We shall need this observation in order to arrive at the correct error estimates.

4. THE L^2-ERROR BOUNDS

Just as in [3], using (3.1)-(3.3) we now rewrite (3.5) in the form

$$\begin{cases} (\alpha(u)\,\underset{\sim}{\zeta}, \underset{\sim}{v}) - (\operatorname{div} \underset{\sim}{v}, \tau) + (\tau\tilde{\underset{\sim}{\Gamma}}, \underset{\sim}{v}) = (\underset{\sim}{q}, \underset{\sim}{v})\,, & \underset{\sim}{v} \in \underset{\sim}{V}^p\,, \\ \qquad (\operatorname{div} \underset{\sim}{\zeta}, w) + (\tilde{\gamma}\tau, w) = (\eta, w)\,, & w \in W^p\,, \end{cases} \tag{4.1}$$

where $\underset{\sim}{\zeta} = \underset{\sim}{z} - \underset{\sim}{z}^p$, $\tau = P^p u - u^p$, $\tilde{\underset{\sim}{\Gamma}} = \tilde{\alpha}_u(u^p)\,\underset{\sim}{z}^p + \tilde{\underset{\sim}{\beta}}_u(u^p)$, $\tilde{\gamma} = \tilde{c}_u(u^p)$, $\underset{\sim}{q} = (P^p u - u)\,\tilde{\underset{\sim}{\Gamma}}$, and $\eta = (P^p u - u)\,\tilde{\gamma}$. Note that the left hand side of (4.1) corresponds to the mixed method for the operator $N : H^2(\Omega) \to L^2(\Omega)$ given by

$$Nw = -\underset{\sim}{\nabla} \cdot (a(u)\,\underset{\sim}{\nabla} w + a(u)\,w\tilde{\underset{\sim}{\Gamma}}) + \tilde{\gamma} w\,.$$

Therefore, if we show that its formal adjoint, N^*, has a bounded inverse $L^2 \to H^2(\Omega) \cap H_0^1(\Omega)$, then lemma 3.1 would apply to (4.1) without any change in the proof. Since we know that M^* has a bounded inverse, all we need to do is to check that the operator norm of $M^* - N^*$ can be made arbitrarily small by taking p large enough.

LEMMA 4.1 : *There exists a positive integer p_0 such that, for all $p \geqslant p_0$, N^* has a bounded inverse $L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega)$. ($N^*$ depends on p through $\tilde{\gamma}$ and $\tilde{\underset{\sim}{\Gamma}}$).*

Proof : Just as in [3], we have

$$(M^* - N^*)\chi = a(u)\Big\{[\bar{\alpha}_{uu}\,\underset{\sim}{z} + \bar{\underset{\sim}{\beta}}_{uu}](u - u^p) + \tilde{\alpha}_u(u^p)(\underset{\sim}{z} - \underset{\sim}{z}^p)\Big\} \times \\ \times \underset{\sim}{\nabla}\chi + \bar{c}_{uu}(u - u^p)\,\chi\,, \quad \chi \in L^2(\Omega)\,,$$

where $\bar{\alpha}_{uu} = \dfrac{\alpha_u(u) - \tilde{\alpha}_u(u^p)}{u - u^p}$ and $\bar{\beta}_{uu}$, and $\bar{c}_{uu}$, defined by analogous relations, are bounded functions in $\bar{\Omega}$. It follows from remark 3 1 and Sobolev's embedding theorem that

$$\begin{aligned}\|(M^* - N^*)\chi\|_0 &\leq K[\|z\|_{0\ \infty}\ \|u - u^p\|_{0\ t}\ \|\nabla\chi\|_{0\ 2+\varepsilon} + \\ &\quad + \|z - z^p\|_{0\ 2+\varepsilon}\ \|\nabla\chi\|_{0\ t} + \|u - u^p\|_0\ \|\chi\|_{0\ \infty}] \\ &\leq K(\|\nabla\chi\|_1 + \|\chi\|_{1+\varepsilon})\,p^{-1-\varepsilon/4} \\ &\leq Kp^{-1}\|\chi\|_2 ,\end{aligned}$$

as needed

To conclude, we establish the rate of convergence of (z^p, u^p) to (z, u).

THEOREM 4.1 : *Assume that the solution u of* (1.1) *is in* $H^{7/2}(\Omega)$ *There is a positive constant* Q, *independent of* p *but dependent on* $\|u\|_{7/2+2\varepsilon}$, *such that, for* p *sufficiently large and* $m \geq 7/2$,

i) $$\|u - u^p\|_0 \leq Qp^{1-m}\|u\|_m ,$$

ii) $$\|z - z^p\|_0 \leq Qp^{3/2-m}\|u\|_m ,$$

iii) $$\|\operatorname{div}(z - z^p)\|_0 \leq Qp^{2-m}\|u\|_m$$

Proof In view of remark 3 1 and lemma 4.1, we can use lemma 3.1 on (4 1) with $\theta = 2$. Thus,

$$\|\tau\|_0 \leq C\,[p^{-1/2}\|\zeta\|_0 + p^{-2}\|\operatorname{div}\zeta\|_0 + \|q\|_0 + \|\eta\|_0] \qquad (4.2)$$

Note that remark 3.1 together with (1.7) lead to the following estimate for $r \geq 0$, $m > 3/2$,

$$\begin{aligned}\|q\|_0 + \|\eta\|_0 &= \|(P^p u - u)\tilde{\Gamma}\| + \|(P^p u - u)\tilde{\gamma}\| \leq \\ &\leq K(\|P^p u - u\|_0 + \|(P^p u - u)z^p\|_0) \\ &\leq K[p^{-r}\|u\|_r(1 + \|z\|_{0\ \infty}) + \|z^p - z\|_{0,2+\varepsilon}\ \|P^p u - u\|_{0,t}] \\ &\leq K(p^{-r}\|u\|_r + p^{-1-\varepsilon/4}\,p^{3/2-3\varepsilon/[2(2+\varepsilon)]-m}\ \|u\|_m) \\ &\leq Kp^{1-m-\varepsilon}\|u\|_m .\end{aligned} \qquad (4\ 3)$$

Combining (4.2), (4.3), (3.18), (3.19), (1.7) and (1.8) yields,

$$\|\tau\|_0 \leq C\,[p^{-1/2}(\|z - \pi^p z\|_0 + \|\pi^p z - z^p\|_0) + p^{-2}(\|\text{div}\, z - P^p \,\text{div}\, z\|_0 + \|\text{div}\,(\pi^p z - z^p)\|_0) + p^{1-m-\varepsilon}\|u\|_m]$$

$$\leq C\,[p^{-1/2}\|\tau\|_0 + p^{-1/2}p^{1/2-r}\|u\|_{r+1} + p^{-2}p^{-s}\|u\|_{s+2} + p^{-1/2}p^{-1-\varepsilon/4} + p^{1-m-\varepsilon}\|u\|_m]\,,\quad r > 1/2\,,\quad s \geq 0\,,\quad m > 3/2\,,$$

which, for p sufficiently large, leads to

$$\|\tau\|_0 \leq Cp^{1-m}\|u\|_m\,,\quad m \geq 2\,, \tag{4.4}$$

where the constant C depends on $\|u\|_{7/2}$. The first part of the theorem is an immediate consequence of (1.7) and (4.4). On the other hand, it follows from (1.8), (3.18), (4.3) and (4.4), that

$$\|z - z^p\|_0 \leq \|z - \pi^p z\|_0 + \|\pi^p z - z^p\|_0 \leq C\,[p^{3/2-m}\|u\|_m + p^{1-m}\|u\|_m]\,,$$

which proves the second part of the theorem.

Finally, we deduce from (3.19), (1.7), (4.3) and (4.4) that

$$\|\text{div}\,(z - z^p)\|_0 \leq \|\text{div}\, z - P^p\,\text{div}\, z\|_0 + \|\text{div}\,(\pi^p z - z^p)\|_0 \leq C\,[p^{2-m}\|u\|_m + p^{1-m}\|u\|_m]\,,$$

which gives iii).

Remark 4.1 : The estimate for the error in z is the best we could hope for in view of (1.8). The estimate for the error in div z is optimal in rate and regularity, while the one for u is probably not sharp in view of (1.7).

REFERENCES

[1] I. BABUSKA, I. N. KATZ and B. SZABO, *The p-version of the finite element method,* SIAM Num. Anal., 18 (1981), pp. 515-545.

[2] J. DOUGLAS Jr. and J. E. ROBERTS, *Global estimates for mixed methods for second order elliptic equations,* Math. Comp., 44 (1985), pp. 39-52.

[3] F. A. MILNER, *Mixed finite element methods for quasilinear second order elliptic problems,* Math. Comp., 44 (1985), pp. 303-320.

[4] A. QUARTERONI, *Some results of Bernstein and Jackson type for polynomial approximation in L^p spaces,* Jap. J. Appl. Math., 1 (1984), pp. 173-181.

[5] P. A. RAVIART and J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems, Proceed. Conf. on Mathematical aspects of finite elements methods,* Lecture notes in mathematics, vol. 606, Springer-Verlag, Berlin, 1977, pp. 292-315.

[6] M. SURI, *On the stability and convergence of higher order mixed finite element methods for second order elliptic problems,* Math. Comp., 54 (1990), pp. 1-19.

[7] H. TRIEBEL, *Interpolation Theory, Function Spaces, Differential Operators,* North Holland, Amsterdam, 1978.