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RESOLUTION OF A FIXED POINT PROBLEM BY AN INCREMENTAL METHOD AND APPLICATION IN NONLINEAR ELASTICITY (*)

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Abstract — In this paper, we solve a fixed point problem by proving that the solution is the value at the point 1 of the solution of an appropriate ordinary differential equation. This approach is applied in nonlinear elasticity to the pure traction boundary-value problem with live load. An incremental method is then used in approximating the solution. The number of successive linearizations is considerably reduced as compared to that used in [19].

Résumé — Dans cet article, nous résolvons un problème de point fixe en montrant que la solution est la valeur au point 1 de la solution d'une équation différentielle appropriée. Cette approche est alors appliquée en élasticité non linéaire au problème de traction pure avec charge vive. La solution est approchée par une méthode incrémentale. Le nombre de linéarisations successives est considérablement réduit comparativement à la méthode présentée dans [19].

NOTATIONS

We shall use the following notations

Ω	: a smooth bounded domain in \mathbb{R}^3 ,
\bar{B}	: the closure of a set B ,
Γ	: the boundary of Ω ,
$\partial_i = \frac{\partial}{\partial x_i}$: usual partial derivatives,
$u_{,i} = \partial_i u$,	
$\underline{\nu} = (\nu_i)$: unit outer normal vector to the boundary of a domain,
$A = (A_{ij})$: matrix with element A_{ij} (i = row index, j = column index),

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$\nabla \phi = (\partial_i \phi_j) \in \mathbb{M}^3$: gradient of a mapping $\phi : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$,
 \mathbb{M}^3 : set of all matrices of order 3,
 skew : set of skew-symmetric matrices of order 3,
 sym : set of symmetric matrices of order 3,
 O_+^3 : set of positive orthogonal matrices,
 \hat{T} : first Piola-Kirchhoff stress,
 $\text{div } T = (\partial_i T_{ij}) \in \mathbb{R}^3$: divergence of a tensor field $T : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{M}^3$,
 $\underline{L}^p = (L^p(\Omega))^3$,
 $\underline{W}^{m,p} = (W^{m,p}(\Omega))^3$ for some integer $m \geq 0$ and $p \geq 1$,

$$\underline{W}^{m-\frac{1}{p},p} = \left(W^{m-\frac{1}{p},p}(\Gamma) \right)^3,$$

$$\mathcal{C}_{\text{sym}} = \{ \phi \in \underline{W}^{m+2,p}, \phi(0) = 0, \nabla \phi(0) = \nabla \phi(0)^T \},$$

$$\underline{L} = \left\{ (\underline{b}, \underline{\tau}) \in \underline{W}^{m,p} \times \underline{W}^{m+1-\frac{1}{p},p}, \int_{\Omega} \underline{b} + \int_{\Gamma} \underline{\tau} = 0 \right\},$$

$\underline{L} = (\underline{b}, \underline{\tau})$ a loading operator,

$$k(\underline{L}) = \int_{\Omega} \underline{b} \otimes x + \int_{\Gamma} \underline{\tau} \otimes x \in \mathbb{M}^3,$$

$$\underline{L}_e = \{ \underline{L} \in \underline{L}, k(\underline{L}) \in \text{sym} \},$$

$$\text{Skew} = \{ \underline{L} \in \underline{L}, k(\underline{L}) \in \text{skew} \},$$

$\underline{\theta}', \underline{\theta}''$: first and second Frechet derivatives of an operator $\theta : X \rightarrow Y$,
 $\theta'(x) \in L(X, Y)$, $\theta''(x) \in L_2(X, Y)$, X and Y being two normed vector spaces,

$\| \cdot \|$: norms of vectors in the different spaces or norms of operators.

We shall use the repeated index convention and denote by C , any constant which is independent of the various functions found in a given inequality.

INTRODUCTION

Solutions to an important class of nonlinear equations are obtained via the fixed point theorem. The solution x satisfies an equation of the form

$$x = \psi(x). \quad (0.1)$$

The sequence $x_{n+1} = \psi(x_n)$, $n = 0, 1, \dots, N$ tends to x as $N \rightarrow \infty$. Each term x_n of the above sequence is the solution of a nonlinear equation which can be solved by an M -step incremental method [19], [5] if ψ possesses the adequate

properties. In this case MN linearizations are necessary. This number may be very important and the approximation very expensive.

In this paper we study the case where

$$\psi = G^{-1} \circ F \quad (0.2)$$

for some nonlinear operators G and F . The solution then satisfies the equation

$$\mathcal{G}(x) = F(x). \quad (0.3)$$

Under some hypotheses which will be specified in Section 1, we consider the problem of finding a curve $(x(\lambda), y(\lambda))$ such that

$$G(x(\lambda)) - \lambda y(\lambda) = 0 \quad (0.4)$$

$$F(x(\lambda)) - y(\lambda) = 0 \quad (0.5)$$

for $\lambda \in [0, \lambda_M]$. The curve $(x(\lambda), y(\lambda))$ is the solution of the differential equation

$$\begin{bmatrix} G'(x(\lambda)) - \lambda I \\ F'(x(\lambda)) - I \end{bmatrix} \begin{pmatrix} \frac{dx}{d\lambda} \\ \frac{dy}{d\lambda} \end{pmatrix} = \begin{pmatrix} y(\lambda) \\ 0 \end{pmatrix} \quad (0.6)$$

$$x(0) = x_0, \quad y(0) = F(x_0). \quad (0.7)$$

It is shown that $\lambda_M = 1$ and one has

$$G(x(1)) = F(x(1)). \quad (0.8)$$

It suffices therefore to apply to the differential equations (0.6)-(0.7), Euler's approximation which is nothing but the incremental method. The number of linearizations is therefore reduced to M instead of MN .

Solutions to the successive linear problems obtained in the Euler's scheme can be computed easily if the matrix on the left of (0.6) is well conditioned (for example has a dominant diagonal). This shows the importance of the operator G . These conditions also hold even if $G' = I$ ($G(x) = x - x_0$). In this case the norm of the operator $F' = \psi'$ must be bounded by 0.5 as shown further in relation (1.35). This leads to a greater restriction on the class of operators ψ unlike the case where $F' \neq I$ may give a greater bound.

In Section 1 we specify conditions on the operators G and F and prove the existence of an integral curve $(x(\lambda), y(\lambda))$ for $\lambda \in [0, 1]$.

In Section 2 we consider the traction boundary-value problem in nonlinear elasticity with live load described by the equations

$$-\operatorname{div} \hat{T} = \underline{b}(\phi) \quad \text{in } \Omega, \quad (0.9)$$

$$\hat{T} \nu = \tau(\phi) \quad \text{on } \Gamma, \quad (0.10)$$

$$\det \nabla \phi > 0 \quad \text{in } \bar{\Omega}, \quad (0.11)$$

$$\hat{T}(x) = \underline{q}(x, \nabla \phi(x)), \quad (0.12)$$

$$\underline{q}(x, A) \in \bar{\Omega} \times \mathbb{M}^3 \rightarrow \mathbb{M}^3, \quad (0.13)$$

$$\int_{\Omega} \underline{b}(\phi) + \int_{\Gamma} \tau(\phi) = 0 \quad (0.14)$$

The constitutive law \underline{q} of the first Piola-Kirchhoff stress tensor \hat{T} satisfies the principle of material frame-indifference,

$$\underline{q}(x, QA) = Q\underline{q}(x, A), \quad Q \in O_+^3 \quad (0.15)$$

Assuming the constitutive law is a C^∞ matrix-valued function, the problem is equivalent to finding in a neighborhood of $\phi = \iota \underline{q}$, the solution of the equation

$$\underline{\theta}(\phi) = (-\operatorname{div} \underline{q}(\cdot, \nabla \phi(\cdot))), \quad \underline{q}(\cdot, \nabla \phi(\cdot)) \nu = \underline{l}(\phi) \quad (0.16)$$

where the nonlinear operators $\underline{\theta}$ and \underline{l} map the function space $\mathcal{W}^{m+2,p}$ into \underline{L} for some integer $m \geq 0$ and $p > 3$

In [19], R. Nzengwa proved the existence of a solution, via the fixed point theorem provided the loading operator \underline{l} is Lipschitz-continuous, $\underline{l}(\iota \underline{q}) = \underline{l}_0$ is a load in \underline{L}_e without axis of equilibrium and $\det k(\underline{l}_0) > 0$. Approximation consisted in solving using incremental methods the sequence of dead load traction boundary-value problem

$$\underline{\theta}(\phi_n) = \underline{l}(\phi_{n-1}) = \underline{h}_{n-1} \quad (0.17)$$

A solution of the dead load problem consists in finding the rotation $Q(\underline{l}_e)$ such that

$$\underline{l} = Q(\underline{l}_e) R(\underline{h})^T \underline{h} \in \mathbf{N}, \quad \underline{l}_e = R(\underline{h})^T \underline{h} \quad (0.18)$$

where \mathbf{N} is the image of $\hat{\underline{\theta}}$, $\hat{\underline{\theta}}$ is the restriction of $\underline{\theta}$ in \mathcal{C}_{sym} ,

$$R(\underline{h}) = k(\underline{h})[k(\underline{h})^T k(\underline{h})]^{-1/2}, \quad (0.19)$$

and by solving

$$\hat{\ell}(\varphi) = \underline{\ell}. \quad (0.20)$$

The solution ϕ is

$$\phi = RQ^T \varphi. \quad (0.21)$$

The existence of this solution is essentially based on Chillingworth-Marsden-Wan's approach which considers the load $\underline{\ell}_e$ as an argument [16].

The rotation Q and the solution φ are computed respectively by an M -step incremental method [19]. Therefore the necessary number of linearizations is $2M$ for each ϕ_n of the sequence and $2NM$ when N terms are considered.

In [19] it was also proved that one has

$$\|\bar{\phi}^N - \phi\| \leq C \frac{(N+1)}{M} \quad (0.22)$$

where $\bar{\phi}^N$ is the approximation computed by an incremental method. One concludes that convergence of the method is guaranteed only if $M > N + 1$. Therefore at least $2N^3$ linearizations are needed to obtain a $O\left(\frac{1}{N}\right)$ error estimate.

The novelty in this paper is to consider the load $\underline{\ell}_e$ and the rotation $Q(\underline{\ell}_e)$ as arguments and to approximate simultaneously $\underline{\ell}_e$, $Q(\underline{\ell}_e)$ and φ in order to calculate the fixed point

$$\phi = RQ^T \varphi. \quad (0.23)$$

We obtain this result by considering a differential equation on $(Q, \varphi, \underline{\ell}_e)$. We prove that the solution is defined in $[0, 1]$. Euler's method is then applied to approximate the solution.

In order to make this paper self-contained we shall recall without further proofs some results in [19].

1. APPROXIMATION OF A FIXED POINT

We consider two Banach spaces X and Y and three operators $\psi : X \rightarrow X$, $G : X \rightarrow Y$ and $F : X \rightarrow Y$. We suppose that in a bounded neighborhood of a point x_0 , a ball $B(x_0, \rho)$ for example, there exists a fixed point for the operator ψ . We next suppose that G is a diffeomorphism in $B(x_0, \rho)$ such that

$$\psi = G^{-1} \circ F. \quad (1.1)$$

Then if \bar{x} is the fixed point of ψ ,

$$\bar{x} = \psi(\bar{x}) \quad (1.2)$$

is equivalent to

$$G(\bar{x}) = F(\bar{x}). \quad (1.3)$$

The usual method for solving equation (1.2) consists in considering the sequence (x_n) , $n = 1, \dots, N$

$$x_n = \psi(x_{n-1}) \quad (1.4)$$

and by approximating each x_n by an M -step incremental method as defined in [5]. One has to compute NM successive linearizations if N terms are considered. In order to reduce the number of linearizations we prove, under hypotheses which will be specified below, that there exists a curve $x(\lambda)$, $\lambda \in [0, 1]$ which is the solution of an appropriate differential equation, and

$$x(1) = \bar{x}. \quad (1.5)$$

We begin by proving some theorems which will be useful in the sequel.

THEOREM 1 : *Let the operator G be twice differentiable between the spaces X and Y ($G \in C^2(X, Y)$) and have locally bounded derivatives. We assume $G'(x_0)$ is an isomorphism, ($G'(x_0) \in I_{\text{som}}(X, Y)$). Then there exist positive real numbers ρ , γ_ρ and L_ρ such that*

$$G'(x) \text{ is invertible in the ball } \bar{B}(x_0, \rho), \quad (1.6)$$

$$\| \{G'(x)\}^{-1} \| \leq \gamma_\rho, \quad (1.7)$$

and

$$\| \{G'(x)\}^{-1} - \{G'(y)\}^{-1} \| \leq L_\rho \|x - y\|. \quad (1.8)$$

Proof : We deduce from the mean-value theorem that in a ball $B(x_0, \rho)$, $\rho > 0$ there exists \bar{x} such that

$$G'(x) - G'(x_0) = G''(\bar{x})(x - x_0). \quad (1.9)$$

Then there exists a constant $M_\rho > 0$ such that

$$\|G'(x) - G'(x_0)\| \leq \rho M_\rho. \quad (1.10)$$

We also have

$$G'(x) = G'(x_0)(I - G'(x_0)^{-1}(G'(x_0) - G'(x))). \quad (1.11)$$

Let

$$\gamma_0 = \|G'(x_0)^{-1}\|. \quad (1.12)$$

Then there exists a number $\rho_0 > 0$ such that

$$\gamma_0 \rho_0 M_{\rho_0} < 1 \quad (1.13)$$

since the function ρM_ρ is non decreasing in $[0, +\infty)$. It follows from (1.9)-(1.11) that $G'(x)$ is invertible for each $x \in \bar{B}(x_0, \rho)$, $\rho < \rho_0$ and one has

$$G'(x)^{-1} = [I - G'(x_0)^{-1} (G'(x_0)^{-1} - G'(x)^{-1})] G'(x_0)^{-1}. \quad (1.14)$$

We deduce that

$$\|G'(x)^{-1}\| \leq \frac{\gamma_0}{1 - \rho M_\rho \gamma_0} = \gamma_\rho < +\infty. \quad (1.15)$$

We can also write

$$G'(x)^{-1} - G'(y)^{-1} = G'(x)^{-1} (G'(y) - G'(x)) G'(y)^{-1} \quad (1.16)$$

and deduce from (1.15) and the mean-value theorem the existence of $L_\rho > 0$ such that

$$\|G'^{-1}(x) - G'^{-1}(y)\| \leq L_\rho \|x - y\|. \quad \blacksquare \quad (1.17)$$

THEOREM 2 : Let $A_1(x)$ and $A_2(y)$ be such that $A = (A_1, A_2)$ be an isomorphism between $X \times Y$ and $Y \times Y$ for each $(x, y) \in B(x_0, \rho_1) \times B(y_0, \rho_2)$. Let the operator $B = (B_1, B_2)$ be such that

$$\|B(x, y)\| < \frac{1}{\gamma_{(\rho_1, \rho_2)}} \quad (1.18)$$

in $B(x_0, \rho_1) \times B(y_0, \rho_2)$, where

$$\gamma_{(\rho_1, \rho_2)} = \sup_{B(x_0, \rho_1) \times B(y_0, \rho_2)} \|A^{-1}(x, y)\|, \quad (1.19)$$

then the operator $A + B$ is also an isomorphism for each (x, y) in $B(x_0, \rho_1) \times B(y_0, \rho_2)$.

Proof : It suffices to write

$$A + B = A(I + A^{-1}B) \quad (1.20)$$

and deduce from (1.18) that $(I + A^{-1}B)$ is invertible. \blacksquare

THEOREM 3 : We consider the family of operators

$$\bar{A}(\lambda) = A(\lambda) + B(\lambda), \quad \lambda \in [0, 1] \quad (1.21)$$

where each operator $A(\lambda)$ satisfies the conditions of theorem 2 and

$$\|B(\lambda)\| < \frac{1}{\gamma_\rho}, \quad \gamma_\rho = \sup_{(\lambda, x, y) \in [0, 1] \times B(x_0, y_0, \rho)} \|(\lambda)^{-1}\|.$$

Then $\bar{A}(\lambda)$ is an isomorphism between $X \times Y$ and $Y \times Y$ for each $\lambda \in [0, 1]$. ■

THEOREM 4 : Let the operator $\bar{A}(\lambda, x, y) = A(\lambda, x, y) + B(\lambda, x, y)$ satisfy the conditions of theorem 3, in the ball $B(x_0, \rho_1) \times B(y_0, \rho_2)$ for each $\lambda \in [0, 1]$. Then the solution to the differential equation

$$\frac{dX}{d\lambda} = \bar{A}(\lambda, X)^{-1} (f(\lambda, X)) \quad (1.23)$$

$$X(0) = X_0 \quad (1.24)$$

$$X = (x, y) \quad (1.25)$$

is defined for all $\lambda \in [0, 1]$ if $f(\lambda, X) \neq 0$, locally Lipschitz-continuous and

$$\|f(\lambda, X)\| < \rho \gamma^{-1}(\rho_1, \rho_2), \quad \rho = \min(\rho_1, \rho_2). \quad (1.26)$$

Proof : We consider the autonomous differential equation associated to (1.23)-(1.25)

$$\frac{dX}{ds} = \bar{A}(\lambda, X)^{-1} (f(\lambda, X)) \quad (1.27)$$

$$\frac{d\lambda}{ds} = 1 \quad (1.28)$$

$$X(0) = X_0, \quad \lambda(0) = 0. \quad (1.29)$$

The vector field

$$V(\lambda, X) = (\bar{A}(\lambda, X)^{-1} (f(\lambda, X)), 1) \quad (1.30)$$

is Lipschitz-continuous in the ball $B(x_0, \rho_1) \times B(y_0, \rho_2) \times [0, 1]$. The existence and uniqueness of a maximal solution of the differential equation for $0 \leq \lambda \leq \lambda_M$ is classical (see [9]). But one deduces from (1.26) that

$$\|X(\lambda_M) - X_0\| < \rho \quad (1.31)$$

and consequently a maximal solution is global, therefore defined for all $0 \leq \lambda \leq 1$. ■

We now consider the equation (1.3) and suppose the operator G satisfies in a neighborhood of x_0 the conditions of theorem 1. Then there exists a ball $B(x_0, \rho_0)$ in which $G'(x)$ is an isomorphism for each x . We suppose F is twice differentiable and has all its derivatives bounded in this ball.

THEOREM 5: *Let G satisfy the hypotheses of theorem 1 in the ball $\bar{B}(x_0, \rho_1)$, i.e.,*

$$\sup_{x \in \bar{B}(x_0, \rho_1)} \|G'(x)^{-1}\| \leq \gamma_{\rho_1} \quad (1.32)$$

and

$$G(x_0) = 0. \quad (1.33)$$

Let $B_1(x) = F'(x)$ be such that

$$\beta = \sup_{x \in \bar{B}(x_0, \rho_1)} \|B_1(x)\| \quad (1.34)$$

and

$$\beta < \frac{1}{\gamma_{\rho_1} + 1}. \quad (1.35)$$

Let

$$A = \begin{bmatrix} G'(x) & -\lambda I \\ 0 & -I \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ B_1 & 0 \end{bmatrix}, \quad (1.36)$$

$$\alpha = \sup_{x \in \bar{B}(x_0, \rho_1), \lambda \in [0, 1]} \|A^{-1}B\| \quad (1.37)$$

and

$$\rho_2 = \sup_{x \in \bar{B}(x_0, \rho_1)} \|F(x)\| < \min \left(\rho_1, \rho_1 \frac{1 - \alpha}{1 + \gamma_{\rho_1}} \right). \quad (1.38)$$

Then there exists a curve $x(\lambda)$, $0 \leq \lambda \leq 1$ such that

$$G(x(1)) = F(x(1)) \quad (1.39)$$

and $x(1)$ is the fixed point.

Proof: It comes from theorem 1 that A is invertible in the ball $B(x_0, \rho_1)$ for $\lambda \in [0, 1]$ and one has

$$\sup_{x \in \bar{B}(x_0, \rho_1), \lambda \in [0, 1]} \|A^{-1}\| \leq \gamma_{\rho_1} + 1. \quad (1.40)$$

Then using (1.35) we deduce that $\bar{A} = A + B$ is invertible and

$$\sup_{x \in \bar{B}(x_0, \rho_1), \lambda \in [0, 1]} \|\bar{A}^{-1}\| \leq \frac{\gamma_{\rho_1} + 1}{1 - \alpha}. \quad (1.41)$$

We can therefore consider the solution of the differential equation

$$\frac{dX}{d\lambda} = \bar{A}^{-1}(\lambda, X) \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad 0 \leq \lambda \leq 1 \quad (1.42)$$

$$X = (x, y), \quad X(0) = (x_0, F(x_0)), \quad (1.43)$$

which is well defined in $[0, 1] \times B(x_0, \rho_1) \times B(y_0, \rho_2)$ and has a global solution because of (1.38) according to theorems 3, and 4. Then one has

$$\bar{A} \begin{pmatrix} \frac{dx}{d\lambda} \\ \frac{dy}{d\lambda} \end{pmatrix} - \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} G'(x) \frac{dx}{d\lambda} - \lambda \frac{dy}{d\lambda} - y \\ F'(x) \frac{dx}{d\lambda} - \frac{dy}{d\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \quad (1.44)$$

consequently

$$\frac{d}{d\lambda} (G(x(\lambda)) - \lambda y(\lambda)) = 0, \quad (1.45)$$

$$\frac{d}{d\lambda} (F(x(\lambda)) - y(\lambda)) = 0. \quad (1.46)$$

Therefore for each λ in $[0, 1]$ one has

$$G(x(\lambda)) - \lambda y(\lambda) = G(x(0)) = 0, \quad (1.47)$$

$$F(x(\lambda)) - y(\lambda) = 0, \quad (1.48)$$

and finally

$$G(x(1)) = y(1) = F(x(1)). \quad \blacksquare \quad (1.49)$$

We can therefore approximate the point $\bar{x} = x(1)$ by applying Euler's method on the differential equation (1.42)-(1.43). The fixed point x is approximated by an N -step incremental method as follows : let there be given a regular partition

$$0 = \lambda^0 < \lambda^1 < \dots < \lambda^N = 1$$

of the interval $[0, 1]$, i.e.,

$$\lambda^{n+1} - \lambda^n = \frac{1}{N}. \quad (1.50)$$

We let

$$x^0 = x_0 \quad \text{and} \quad y^0 = y_0, \quad (1.51)$$

then assuming x^n and y^n known, we solve the linear problem

$$\bar{A}(\lambda^n, x^n, y^n) \begin{pmatrix} \delta x^n \\ \delta y^n \end{pmatrix} = \frac{1}{N} \begin{pmatrix} y^n \\ 0 \end{pmatrix} \quad (1.52)$$

and compute the $(n+1)$ -st approximate term x^{n+1} , y^{n+1} by

$$x^{n+1} = x^n + \delta x^n, \quad y^{n+1} = y^n + \delta y^n. \quad (1.53)$$

The approximate fixed point is

$$\bar{x}^N = x^N. \quad (1.54)$$

One has

$$\|\bar{x} - \bar{x}^N\| \leq \frac{C}{N} \quad (1.55)$$

for a certain constant C . Therefore we only have to compute N successive linearizations to have a $O\left(\frac{1}{N}\right)$ error estimate.

We shall apply this approach to the live load traction boundary-value problem in nonlinear elasticity.

2. THE PURE TRACTION BOUNDARY VALUE PROBLEM WITH LIVE LOAD

The problem is described by the equations :

$$-\operatorname{div} \hat{T} = b(\phi) \quad \text{in} \quad \Omega, \quad (2.1)$$

$$\hat{T}\nu = \tau(\phi) \quad \text{on} \quad \Gamma, \quad (2.2)$$

$$\det \nabla \phi > 0 \quad \text{in} \quad \Omega, \quad (2.3)$$

$$\hat{T}(x) = \underline{q}(x, \nabla \phi(x)), \quad (2.4)$$

$$\underline{q} : (x, F) \in \bar{\Omega} \times \mathbb{M}^3 \rightarrow \mathbb{M}^3 \quad (2.5)$$

and

$$\int_{\Omega} b(\phi) + \int_{\Gamma} \tau(\phi) = 0. \quad (2.6)$$

In the above equations we assume the following hypotheses on the data as in [19],

- The constitutive law of the first Piola-Kirchhoff tensor $\hat{\underline{T}}$, defined by

$$\underline{q} : \bar{\Omega} \times \mathbb{M}^3 \rightarrow \mathbb{M}^3 \text{ is of class } C^\infty, \quad (2.7)$$

- the principle of material frame indifference is satisfied i.e.

$$\underline{q}(QF) = Q\underline{q}(F), \text{ for } Q \text{ in } O_+^3, \quad (2.8)$$

- the reference configuration is a natural state, i.e.

$$\underline{q}(I) = 0, \quad (2.9)$$

- letting $\underline{C} = \partial_A \underline{q}(I)$, there exists a real number $\beta > 0$ such that

$$C_{ijkl} e_{ij} e_{kl} \geq \beta e_{ij} e_{kl} \quad (2.10)$$

where $\underline{e} = (e_{ij})$ is a symmetric matrix. ■

In hypotheses (2.8)-(2.10) we have voluntarily omitted the dependence on x . From these hypotheses we deduce that the equations (2.1)-(2.6) are equivalent to solving in a neighborhood of $i\underline{d}$ the equation

$$\underline{\theta}(\underline{\phi}) = \underline{l}(\underline{\phi}) \quad (2.11)$$

where the nonlinear operators $\underline{\theta}$ and \underline{l} are defined by

$$\underline{\theta} : \underline{\phi} \in W^{m+2,p} \rightarrow (-\operatorname{div} \underline{q}(\cdot, \nabla \underline{\phi}(\cdot)), \underline{q}(\cdot, \nabla \underline{\phi}(\cdot)) \underline{\nu}) \in \underline{L} \quad (2.12)$$

$$\underline{l} : \underline{\phi} \in W^{m+2,p} \rightarrow (\underline{b}(\underline{\phi}), \underline{\tau}(\underline{\phi})) \in \underline{L} \quad (2.13)$$

for some integer $m \geq 0$ and real number $p > 3$. The above spaces have been defined in the notations. The nonlinear operator $\underline{\theta}$ is a well defined C^∞ mapping because of hypothesis (2.7) and all its derivatives are bounded [13], [26]. We deduce from (2.8) that

$$\underline{\theta}(Q\underline{\phi}) = Q\underline{\theta}(\underline{\phi}) \quad (2.14)$$

and from (2.9) that

$$\underline{\theta}(i\underline{d}) = \underline{0}. \quad (2.15)$$

In order to eliminate indeterminations due to rigid body motions we restrict the operator to the set $\underline{C}_{\text{sym}}$. We therefore consider the second operator

$$\hat{\underline{\theta}} : \underline{\varphi} \in \underline{C}_{\text{sym}} \rightarrow \underline{\theta}(\underline{\varphi}) \in \underline{L} \quad (2.16)$$

whose derivative at $i\mathcal{d}$ is

$$\hat{\theta}'(i\mathcal{d}): \mathcal{v} \in \mathcal{C}_{\text{sym}} \rightarrow (-\operatorname{div} \mathcal{C} \in (\mathcal{v}), \mathcal{C} \in (\mathcal{v}) \mathcal{v}) \in \mathcal{L}_e, \quad (2.17)$$

$$2 \varepsilon_{ij}(\mathcal{v}) = (v_{i,j} + v_{j,i}). \quad (2.18)$$

It is a classical result in linear elasticity, because of the strong ellipticity (2.10) [24], [6] that $\hat{\theta}'(i\mathcal{d})$ is an isomorphism between \mathcal{C}_{sym} and \mathcal{L}_e and consequently in a neighborhood of $i\mathcal{d}$ the image of $\hat{\theta}$ is a C^∞ submanifold \mathbf{N} in \mathcal{L} [12], [6]. This submanifold is in fact the graph of the function

$$G: \mathcal{L}_e \in \cup \subset \mathcal{L}_e \rightarrow \Pi_{\hat{\theta}} \{ \Pi_e \hat{\theta} \}^{-1} (\mathcal{L}_e) \in \text{Skew}, \quad (2.19)$$

where \cup is a neighborhood of Q in \mathcal{L}_e , say a ball $B(O, \rho_0)$ for simplicity, Π and Π_e are the canonical projections of \mathcal{L} onto Skew and \mathcal{L}_e respectively [19]. One also has

$$G(Q) = 0 \quad \text{and} \quad G'(Q) = 0. \quad (2.20)$$

It is clear that there exists a neighborhood $B(i\mathcal{d}, \rho_0)$ of $i\mathcal{d}$ in \mathcal{C}_{sym} in which the operator

$$\hat{\theta}_e = \Pi_e \hat{\theta} \quad (2.21)$$

is a diffeomorphism.

Let \mathcal{l} lies in \mathbf{N} , then

$$\varphi = \hat{\theta}_e^{-1}(\mathcal{L}_e), \quad \mathcal{L}_e = \Pi_e \mathcal{L} \quad (2.22)$$

satisfies

$$\hat{\theta}(\varphi) = \mathcal{L} \quad (2.23)$$

and is therefore the solution to the pure traction boundary-value problem with the dead load \mathcal{L} .

Let us recall the following definition. A load \mathcal{L} in \mathcal{L}_e is without axis of equilibrium if the following equivalent conditions are satisfied

$$\det(k(\mathcal{L}) - \operatorname{tr} k(\mathcal{L}) I) \neq 0 \quad (2.24)$$

or the mapping

$$w \in \text{Skew} \rightarrow k(\mathcal{L}) w + w k(\mathcal{L}) \in \text{Skew} \quad (2.25)$$

is an isomorphism.

This equivalence has been established in [6]. Let $\underline{l}_0 = \underline{l}(id) \in \cup \subset \underline{L}_e$ be without axis of equilibrium, the following results have been proved in [6], [19] :

there exists a neighborhood of \underline{l}_0 , $O_0 \subset \cup = B(0, \beta)$ in which for each load $\underline{l} \in O_0$ there exists a unique rotation Q in a neighborhood V_I of I in O_+^3 such that $Q\underline{l} \in N$. The rotation Q is the value at the point 1 of the implicit equation

$$H(\lambda, Q(\lambda), \underline{l}) = 0, \quad 0 \leq \lambda \leq 1, \quad Q(0) = I \quad (2.26)$$

$$H : (\lambda, Q, \underline{l}) \in [0, 1] \times O_+^3 \times \underline{L}_e \rightarrow \Pi Q \underline{l} - \frac{1}{\lambda} G(\lambda \Pi_e Q \underline{l}) \in \text{Skew}. \quad (2.27)$$

Letting

$$Q = \exp w, \quad Q \in V_I \quad \text{and} \quad w \in V_0 \quad (2.28)$$

where $V_0 = B(0, \alpha)$ is a neighborhood of the matrix 0 in skew, the implicit equation is defined by the function

$$\bar{H}(\lambda, w, \underline{l}) = H(\lambda, \exp w, \underline{l}), \quad (2.29)$$

which is such that

$$\frac{\partial \bar{H}}{\partial w}(\lambda, w, \underline{l}) \text{ is an isomorphism for}$$

$$(\lambda, w, \underline{l}) \in [0, 1] \times V_0 \times O_0. \quad (2.30)$$

Therefore for each load $\underline{l} \in O_0$, there exists a rotation $Q(\underline{l}) = \exp w(1)$ such that $Q(\underline{l}) \underline{l} \in N$, where $w(\lambda)$ is the solution of the implicit equation

$$\bar{H}(\lambda, w(\lambda), \underline{l}) = 0, \quad 0 \leq \lambda \leq 1, \quad w(0) = 0. \quad (2.31)$$

A solution to the traction problem then satisfies the equation

$$\hat{\theta}_e(\varphi) = \Pi_e \exp w(1) \underline{l} \quad (2.32)$$

or equivalently

$$\hat{\theta}(\varphi) = \exp w(1) \underline{l}. \quad (2.33)$$

In [19], the rotation $Q = \exp w(1)$ and the solution φ were computed separately by applying incremental methods on the implicit equation (2.31) and next to a differential equation equivalent to (2.32) in which the rotation $Q = \exp w(1)$ was replaced by its approximated value.

In the present approach we consider simultaneously the equation

$$\bar{H}(\lambda, w, \underline{l}) = 0, \quad (2.34)$$

$$\hat{\theta}_e(\varphi(\lambda)) - \lambda \Pi_e \exp w(\lambda) \underline{l} = 0, \quad (2.35)$$

$$w(0) = 0, \quad \varphi(0) = i\underline{d}, \quad 0 \leq \lambda \leq 1. \quad (2.36)$$

For a live load \underline{l} we consider the equations

$$\bar{H}(\lambda, w(\lambda), \underline{l}(\lambda)) = 0 \quad (2.37)$$

$$\hat{\theta}_e(\varphi(\lambda)) - \lambda \Pi_e \exp w(\lambda) \underline{l}(\lambda) = 0 \quad (2.38)$$

$$\underline{l}(\varphi(\lambda)) - \exp w(\lambda) \underline{l}(\lambda) = 0 \quad (2.39)$$

$$w(0) = 0, \quad \underline{l}(0) = \underline{l}_0 = \underline{l}(i\underline{d}), \quad \varphi(0) = i\underline{d}, \quad 0 \leq \lambda \leq 1. \quad (2.40)$$

It is clear from (2.14), (2.15), (2.32) and (2.33) that if a curve $(w(\lambda), \varphi(\lambda), \underline{l}(\lambda))$, $0 \leq \lambda \leq 1$ is the solution of the equation (2.37)-(2.40), then necessary $\varphi(1)$ is a solution of the problem

$$\hat{\theta}(\varphi) = \underline{l}(\varphi). \quad (2.41)$$

In order to prove the existence of an integral curve to (2.37)-(2.40) we assume the following hypotheses on the loading operator \underline{l} .

We assume that the loading operator \underline{l} is at least C^1 with \underline{l}' Lipschitz-continuous in $B(i\underline{d}, \rho_0)$ and $\underline{l}_0 = \underline{l}(i\underline{d})$ is a load in O_0 without axis of equilibrium.

If an integral curve $(w(\lambda), \varphi(\lambda), \underline{l}(\lambda))$ exists for $0 \leq \lambda \leq \lambda_m$, then it is at least C^1 and satisfies the equations

$$\frac{\partial \bar{H}}{\partial \lambda} + \frac{\partial \bar{H}}{\partial w} \frac{dw}{d\lambda} + \frac{\partial \bar{H}}{\partial \underline{l}} \frac{d\underline{l}}{d\lambda} = 0 \quad (2.42)$$

$$\begin{aligned} \hat{\theta}'_e(\varphi) \frac{d\varphi}{d\lambda} - \Pi_e \exp w \underline{l}(\lambda) - \lambda \exp' w \frac{dw}{d\lambda} \underline{l}(\lambda) - \\ - \lambda \Pi_e \exp w \frac{d\underline{l}}{d\lambda} = 0 \end{aligned} \quad (2.43)$$

$$\underline{l}'(\varphi) \frac{d\varphi}{d\lambda} - \exp' w \frac{dw}{d\lambda} \underline{l}(\lambda) - \exp w \frac{d\underline{l}}{d\lambda} = 0 \quad (2.44)$$

$$w(0) = 0, \quad \varphi(0) = i\underline{d}, \quad \underline{l}(0) = \underline{l}_0. \quad (2.45)$$

Using (2.20), we consider a second order Taylor expansion of \bar{H} ,

$$\bar{H}(\lambda, w, \underline{l}) = \Pi \exp w \underline{l} - \frac{\lambda}{2} G''(0)(\Pi_e \exp w \underline{l})^2 + \lambda^2 T(w, \underline{l}) \quad (2.46)$$

and define

$$A(\lambda, X) = \begin{bmatrix} \frac{\partial \bar{H}}{\partial w}(\lambda, w, \varphi) & 0 & \Pi \exp w \\ 0 & \hat{\varrho}'_e(\varphi) & -\lambda \Pi_e \exp w \\ 0 & 0 & -\exp w \end{bmatrix} \quad (2.47)$$

$B(\lambda, X) =$

$$= \begin{bmatrix} 0 & 0 & -\lambda G''(0)(\Pi_e \exp w \underline{l}, \Pi_e \exp w(\cdot)) + \frac{\lambda^2 \partial T}{\partial \underline{l}}(w, \underline{l}) \\ -\lambda \exp' w(\cdot) \underline{l} & 0 & 0 \\ -\exp' w(\cdot) \underline{l} & \underline{l}' & 0 \end{bmatrix} \quad (2.48)$$

$$F(\lambda, X) = \begin{pmatrix} -\frac{\partial \bar{H}}{\partial \lambda} \\ -\Pi_e \exp w \underline{l} \\ 0 \end{pmatrix} \quad (2.49)$$

where

$$X = (w, \varphi, \underline{l}). \quad (2.50)$$

The operator A , B and F are defined in $V_0 \times B(\underline{ld}, \rho_0) \times O_0$. Let us recall that in this domain $\frac{\partial \bar{H}}{\partial w}$, $\hat{\varrho}_e$ and $\exp w$ are all isomorphisms for each $\lambda \in [0, 1]$. It is easy to conclude that $A(\lambda, X)$ is invertible for all $(\lambda, X) \in [0, 1] \times V_0 \times B(\underline{ld}, \rho_0) \times O_0$, $V_0 = B(0, \alpha)$, $O_0 = B(\underline{l}_0, \beta')$. Let

$$\gamma(\alpha, \rho_0, \beta') = \sup_{(\lambda, X) \in [0, 1] \times V_0 \times B(\underline{ld}, \rho_0) \times O_0} \|A^{-1}(\lambda, X)\|. \quad (2.51)$$

There exists a constant C such that

$$\|B(\lambda, X)\| \leq C \|\underline{l}\| + \|\underline{l}'\| \quad (2.52)$$

and

$$\|F(\lambda, X)\| \leq C \|\underline{l}\|. \quad (2.53)$$

We can choose the neighborhood $\cup = B(0, \beta)$, ($O_0 = B(\underline{l}_0, \beta') \subset \cup$) such that

$$\begin{aligned} C \|\underline{l}\| &\leq C\beta < \rho \gamma^{-1}(\alpha, \rho_0, \beta'), \\ \rho &= \min(\alpha, \rho_0, \beta'). \end{aligned} \quad (2.54)$$

We assume the loading operator is such that

$$C\beta + \|\underline{l}'\| < \gamma^{-1}(\alpha, \rho_0, \beta'). \quad (2.55)$$

It is clear that $F(\lambda, X) \neq 0$ in $B(0, \alpha) \times B(\underline{i}\underline{d}, \rho_0) \times B(\underline{l}_0, \beta')$.

We can now prove the existence of a solution to the pure traction boundary-value problem with a live load.

THEOREM 6 : *Under the additional hypotheses (2.54) and (2.55), the pure traction boundary-value problem (2.11) has at least one solution.*

Proof : It suffices to prove the existence of an integral curve defined in $[0, 1]$ for equations (2.37)-(2.40) or equivalently a solution to the differential equation

$$[A(\lambda, X) + B(\lambda, X)] \frac{dX}{d\lambda} = F(\lambda, X), \quad (2.56)$$

$$X(0) = (0, \underline{i}\underline{d}, \underline{l}_0).$$

The operator $A(\lambda, X)$ and $B(\lambda, X)$ satisfy the conditions of theorems 3 and 4. We also deduce from (2.54)-(2.55) that the operator $F(\lambda, X)$ satisfies the condition in theorem 4.

Consequently the existence of an integral curve in $[0, 1]$ is guaranteed and equation (2.41) is satisfied. We have thus proved the existence of a solution of (2.11) in \mathcal{C}_{sym} . ■

We can therefore apply Euler's method to the differential equation

$$\frac{dX}{ds} = \bar{A}(\lambda, X)^{-1} F(\lambda, X), \quad 0 \leq s \leq 1 \quad (2.58)$$

$$\frac{d\lambda}{ds} = 1 \quad (2.59)$$

$$X(0) = (w(0), \varphi(0), \underline{l}(0)) = (0, \underline{i}\underline{d}, \underline{l}_0), \quad \lambda(0) = 0 \quad (2.60)$$

$$\bar{A}(\lambda, X) = A(\lambda, X) + B(\lambda, X) \quad (2.61)$$

for A , B and F defined respectively by (2.47), (2.48) and (2.49).

Let the integer N be the number of steps and $x^N = (w^N, \varphi^N, \underline{l}^N)$ the N -th approximate term, then there exists a constant C such that

$$\|w^N - w(1)\| \leq C \frac{1}{N} \quad (2.62)$$

$$\|\varphi^N - \varphi(1)\| \leq C \frac{1}{N} \quad (2.63)$$

$$\|\underline{l}^N - \underline{l}(1)\| \leq C \frac{1}{N}. \quad (2.64)$$

This result is classical.

Remark : Only N linearizations are necessary to obtain a $O\left(\frac{1}{N}\right)$ error estimate. In [19], $2N^3$ linearizations were needed when N terms of the sequence converging to the fixed point were considered. This considerable gain is due to the fact that the loading operator \underline{l} is assumed to be C^1 with a Lipschitz-continuous, \underline{l}' unlike in [19] where it was only assumed to be Lipschitz-continuous.

Practically a second order truncated Taylor expansion

$$\bar{H}_2 = \Pi \exp w \underline{l} - \frac{\lambda}{2} G''(0)(\Pi_e \exp w \underline{l}, \Pi_e \exp w \underline{l}) \quad (2.65)$$

of \bar{H} is preferable. Indeed unlike \bar{H} , the expression of \bar{H}_2 is deduced from linear elasticity and can be computed [18], [6]. In this case we still have the same $A(\lambda, X)$. The expressions of $B(\lambda, X)$ and $F(\lambda, X)$ become simple and still satisfy the conditions of theorems 3 and 4 which guarantee the existence of an integral curve $X_2(\lambda) = (w_2(\lambda), \varphi_2(\lambda), \underline{l}_2(\lambda))$, $0 \leq \lambda \leq 1$.

An N -step incremental method can still be applied to approximate $X_2(1)$. The equivalent equations to (2.42)-(2.45) on the curve $X_2(\lambda)$ are obtained by substituting \bar{H}_2 , w_2 , φ_2 , and \underline{l}_2 to \bar{H} , w , φ , \underline{l} respectively. By subtracting both sets of equations and by using the mean-value theorem we deduce easily that there exists a constant C such that

$$\|X_2(1) - X(1)\| \leq C [T] \quad (2.66)$$

where

$$[T] = \sup_{(w, \underline{l}) \in V_0 \times O_0} \|T(w, \underline{l})\|. \quad (2.67)$$

It therefore follows that if φ_2^N is the N -th approximate of $\varphi_2(1)$ and $\varphi(1)$ the real solution, then there exists a constant C such that

$$\|\varphi_2^N - \varphi(1)\| \leq C \frac{1}{N} + [T] \quad (2.68)$$

hence the smaller is $[T]$, the better is the approximation.

We can therefore approximate equation (2.11) by an incremental method applied on equivalent equations to (2.58)-(2.53) obtained from the second order truncated Taylor expansion \bar{H}_2 .

In theorem 6 we proved the existence of a solution in \mathcal{C}_{sym} to the pure traction boundary-value problem with live load (2.11). In fact the solution is also that of a pure traction problem with the dead load $\underline{l}(\varphi(1))$. There exists at least four solutions [6], each being deduced from the other by applying an appropriate rotation. If we assume as in [20], [23] that the loading operator \underline{l} satisfies the condition

$$\underline{l}(Q\phi) = Q\underline{l}(\phi), \quad Q \in O_+^3, \quad (2.69)$$

then we can deduce all the solutions of (2.11) from that of (2.41).

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