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A GALERKIN SPECTRAL APPROXIMATION IN LINEARIZED BEAM THEORY (*)

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Abstract — *Spectral methods are well adapted for numerically approximating the displacement field of a thin beam In this paper, following a technique introduced by M Vogelius and I Babuška [1981] and already extended by the first author to plates (see B Miara [1989]), we show how to select the basis functions of the spectral approximation in order to minimize the approximation error with respect to a parameter that characterizes the geometry For special loadings and for particular geometries, we prove that these basis functions are polynomials Finally, we show on an example how to compute the spectral approximation*

Résumé — *Les méthodes spectrales sont bien adaptées à l'approximation numérique du champ de déplacement d'une poutre mince En adaptant une technique introduite par M Vogelius et J Babuška [1981] et déjà utilisée par le premier auteur pour des plaques (B Miara [1989]) nous montrons comment construire une base de fonctions de l'approximation spectrale de façon à minimiser l'erreur d'approximation en fonction d'un paramètre lié à la géométrie de la poutre Pour des chargements et des géométries particulières, nous montrons que ces fonctions de base sont des polynômes Enfin, sur un exemple nous indiquons comment s'effectue le calcul de l'approximation spectrale*

INTRODUCTION

In this work, the summation convention on repeated indices is used. Latin indices take values in $\{1, 2, 3\}$ while Greek indices take values in $\{1, 2\}$.

Let ε be a « small » positive parameter and let $\mathbf{u}^\varepsilon : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ denote the displacement field of a thin clamped beam that occupies the cylindrical volume $\bar{\Omega}^\varepsilon = \bar{\omega}^\varepsilon \times [0, L]$, (the beam is « thin » because the diameter of the

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cross section ω^{ε} is of the order of ε, which is « small » when compared with the length of the beam). In section 1 we give the three-dimensional variational formulation of the equilibrium equations, whose solution is the displacement field $\mathbf{u}^{\varepsilon}$, in the framework of linearized elasticity. In recent works, L. Trabucho and J. Viaño [1987, 1988*a*, *b*, 1989] have shown (following the early works of P. G. Ciarlet and P. Destuynder [1979*a*, *b*] for plates ; A. Bermudez and J. Viaño [1984] and I. Aganovič and Z. Tutek [1987] for beams) that, in order to study the behaviour of $\mathbf{u}^{\varepsilon}$ when ε becomes very small, it is convenient to give **an equivalent formulation** of the three-dimensional elasticity problem posed **in a fixed domain** $\bar{\Omega} = \bar{\omega} \times [0, L]$ whose cross section ω is independent of ε. We have found that this approach, presented in section 1, is also appropriate for our purpose of spectral approximation. Then the corresponding displacement field $\mathbf{u}(\varepsilon) : \bar{\Omega} \to \mathbb{R}^3$ is obtained as the unique solution of the variational problem :

Find $\mathbf{u}(\varepsilon) \in V$ such that

$$B(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v}) = F(\varepsilon)(\mathbf{v}) \quad \text{for all } \mathbf{v} \in V ,$$

where $B(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v})$ denotes the internal virtual work associated with the transformed displacement field $\mathbf{u}(\varepsilon)$, and $F(\varepsilon)(\mathbf{v})$ stands for the total potential of the applied forces. The space V of admissible displacements will be specified later. In section 2, the solution of this three-dimensional problem is replaced by a **finite number of problems** posed on **lower dimensional** spaces by applying a Galerkin spectral technique. For each $N \geq 0$, the spectral approximation $\tilde{\mathbf{u}}_N(\varepsilon)$ of $\mathbf{u}(\varepsilon)$ is, by definition, the projection of $\mathbf{u}(\varepsilon)$ onto an approximation space V_N of V, of the form :

$$V_N = \left\{ \mathbf{v} \in V : \mathbf{v} = \left(\sum_{k=0}^{N} T_i^k(x_1, x_2)\, v_i^k(x_3) \right)_{1 \leq i \leq 3}, \right.$$

$$\left. (x_1, x_2) \in \omega,\ x_3 \in [0, L] \right\} .$$

Thus, if the basis functions T_i^k are known, the coefficients $u_i^k(\varepsilon)$ of $\tilde{\mathbf{u}}_N(\varepsilon) = \left(\sum_{k=0}^{N} T_i^k(x_1, x_2)\, u_i^k(\varepsilon)(x_3) \right)_{1 \leq i \leq 3}$, are solutions of one-dimensional problems. This significantly simplifies the computation of $\mathbf{u}(\varepsilon)$. Moreover, using an argument from M. Vogelius and I. Babuška [1981], we show that, for special loadings, the basis functions T_i^k can be chosen so as to minimize the approximation error $\left\| \mathbf{u}(\varepsilon) - \tilde{\mathbf{u}}_N(\varepsilon) \right\|_{H^1(\Omega)}$ with respect to the parameter ε. In section 3 we give an example of how to compute the coefficients $u_i^k(\varepsilon)$ of $\tilde{\mathbf{u}}_N(\varepsilon)$. For thin clamped plates the same kind of results have already been obtained by B. Miara [1989].

1. STATEMENT OF THE PROBLEM

1.1. The three-dimensional linearized elasticity problem

Let ε and L be two positive scalars and let ω^{ε} denote an open, bounded, simply connected subset of $\mathbb{R}^2$, with a Lipschitz boundary γ^{ε}. The beam is then identified with the three-dimensional body occupying the volume $\bar{\Omega}^{\varepsilon}$, where $\Omega^{\varepsilon} = \omega^{\varepsilon} \times (0, L)$. The boundary Γ^{ε} of Ω^{ε} is the union of the end faces $\Gamma_0^{\varepsilon} = \omega^{\varepsilon} \times \{0, L\}$ and of the lateral surface $\Gamma_1^{\varepsilon} = \gamma^{\varepsilon} \times (0, L)$. Let $\mathbf{x}^{\varepsilon} = (x_i^{\varepsilon})$ denote a generic point in $\bar{\Omega}^{\varepsilon}$, and let $\partial_i^{\varepsilon} \mathbf{u}^{\varepsilon} = \partial \mathbf{u}^{\varepsilon} / \partial x_i^{\varepsilon}$. Let $\mathbf{u}^{\varepsilon} = (u_i^{\varepsilon}) : \bar{\Omega}^{\varepsilon} \to \mathbb{R}^3$ denote the displacement field and $\sigma^{\varepsilon} = (\sigma_{ij}^{\varepsilon}) : \bar{\Omega}^{\varepsilon} \to S^3$ the stress field. Assume that the beam is subjected to body forces $\mathbf{f}^{\varepsilon} = (f_i^{\varepsilon}) : \Omega^{\varepsilon} \to \mathbb{R}^3$ and to surface forces $\mathbf{g}^{\varepsilon} = (g_i^{\varepsilon}) : \Gamma_1^{\varepsilon} \to \mathbb{R}^3$. In linearized elasticity the equilibrium equations are :

$$\begin{cases} -\,\mathbf{div}^{\varepsilon}\, \sigma^{\varepsilon} = \mathbf{f}^{\varepsilon} & \text{in } \Omega^{\varepsilon}, \\ \sigma^{\varepsilon} . \mathbf{n}^{\varepsilon} = \mathbf{g}^{\varepsilon} & \text{on } \Gamma_1^{\varepsilon}, \end{cases}$$

where $(\mathbf{div}^{\varepsilon}\, \sigma^{\varepsilon})_i = \partial_j^{\varepsilon}\, \sigma_{ij}^{\varepsilon}$ and $\mathbf{n}^{\varepsilon}$ is the outward unit normal to Γ_1^{ε}. The beam being clamped at both ends, the displacement field satisfies the **boundary condition of place** $\mathbf{u}^{\varepsilon} = \mathbf{0}$ on Γ_0^{ε}. For an isotropic, homogeneous, linear elastic material, whose reference configuration $\bar{\Omega}^{\varepsilon}$ is a natural state, the **linearized constitutive equation** in Ω^{ε} is given by $\sigma_{ij}^{\varepsilon} = \lambda\, e_{kk}^{\varepsilon}\, \delta_{ij} + 2\, \mu\, e_{ij}^{\varepsilon}$, where the Lamé constants λ and μ of the material constituting the beam are assumed to be independent of ε, and where the components e_{ij}^{ε} of the linearized elasticity strain tensor are related to the displacement field by $e_{ij}^{\varepsilon} = (\partial_i^{\varepsilon}\, u_j^{\varepsilon} + \partial_j^{\varepsilon}\, u_i^{\varepsilon})/2$.

Let us now introduce the **variational formulation** associated to this problem. Consider the Hilbert space $V^{\varepsilon} = \{\mathbf{v}^{\varepsilon} = (v_i^{\varepsilon}) \in [H^1(\Omega^{\varepsilon})]^3 ;\ \mathbf{v}^{\varepsilon} = \mathbf{0}$ on $\Gamma_0^{\varepsilon}\}$, equipped with the H^1-norm. Let $B^{\varepsilon} : V^{\varepsilon} \times V^{\varepsilon} \to \mathbb{R}$ denote the symmetric, continuous, bilinear form :

$$B^{\varepsilon}(\mathbf{u}^{\varepsilon}, \mathbf{v}^{\varepsilon}) = \int_{\Omega^{\varepsilon}} [\lambda\, e_{pp}^{\varepsilon}(\mathbf{u}^{\varepsilon})\, e_{qq}^{\varepsilon}(\mathbf{v}^{\varepsilon}) + 2\, \mu\, e_{ij}^{\varepsilon}(\mathbf{u}^{\varepsilon})\, e_{ij}^{\varepsilon}(\mathbf{v}^{\varepsilon})]\, d\mathbf{x}^{\varepsilon},$$

and let $F^{\varepsilon} : V^{\varepsilon} \to \mathbb{R}$ denote the continuous, linear form :

$$F^{\varepsilon}(\mathbf{v}^{\varepsilon}) = \int_{\Omega^{\varepsilon}} \mathbf{f}^{\varepsilon} . \mathbf{v}^{\varepsilon}\, d\mathbf{x}^{\varepsilon} + \int_{\Gamma_1^{\varepsilon}} \mathbf{g}^{\varepsilon} . \mathbf{v}^{\varepsilon}\, d\mathbf{a}^{\varepsilon} .$$

The following result, which relies on Korn's inequality and the Lax-Milgram lemma, is well known (for a proof see e.g. P. G. Ciarlet [1988], pp. 288-292).

THEOREM 1.1 : *Let* $\mathbf{f}^\varepsilon \in [L^2(\Omega^\varepsilon)]^3$ *and* $\mathbf{g}^\varepsilon \in [L^2(\Gamma_1^\varepsilon)]^3$; *then there exists a unique displacement field* $\mathbf{u}^\varepsilon \in V^\varepsilon$ *that solves the variational equations :*

$$B^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) = F^\varepsilon(\mathbf{v}^\varepsilon), \quad \forall \mathbf{v}^\varepsilon \in V^\varepsilon . \quad \blacksquare$$

1.2. Equivalent formulation of the three-dimensional elasticity problem over a fixed set Ω

Let ω be an open bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary γ. Define $\Omega = \omega \times (0, L)$ to be the **reference domain** of $\mathbb{R}^3$ whose boundary Γ is the union of the end faces $\Gamma_0 = \omega \times \{0, L\}$ and of the lateral surface $\Gamma_1 = \gamma \times (0, L)$. For $\varepsilon > 0$ the physical domain Ω^ε is then the image of Ω by the transformation which associates to each point $\mathbf{x} = (x_1, x_2, x_3) \in \Omega$ the point $\mathbf{x}^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (\varepsilon x_1, \varepsilon x_2, x_3) \in \Omega^\varepsilon$. According to this transformation the image of the section ω is the section ω^ε.

Following Trabucho and Viaño [1987, 1988*a*, *b*, 1989] we associate with the displacement field $\mathbf{u}^\varepsilon \in \mathbb{R}^3$, the function $\mathbf{u}(\varepsilon) = (u_i(\varepsilon)) : \bar{\Omega} \to \mathbb{R}^3$ defined by the **scalings :**

$$\begin{cases} u_\alpha^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon u_\alpha(\varepsilon)(\mathbf{x}), & \forall \mathbf{x}^\varepsilon \in \bar{\Omega}^\varepsilon, \\ u_3^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^2 u_3(\varepsilon)(\mathbf{x}), & \forall \mathbf{x}^\varepsilon \in \bar{\Omega}^\varepsilon, \end{cases}$$

We also assume that there exist functions $\mathbf{f} \in [L^2(\Omega)]^3$ and $\mathbf{g} \in [L^2(\Gamma^1)]^3$ independent of ε such that :

$$\begin{cases} f_\alpha^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^3 f_\alpha(\mathbf{x}), & f_3^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^2 f_3(\mathbf{x}), & \forall \mathbf{x}^\varepsilon \in \bar{\Omega}^\varepsilon, \\ g_\alpha^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^4 g_\alpha(\mathbf{x}), & g_3^\varepsilon(\mathbf{x}^\varepsilon) = \varepsilon^3 g_3(\mathbf{x}), & \forall \mathbf{x}^\varepsilon \in \bar{\Omega}^\varepsilon . \end{cases}$$

We can thus reformulate the variational problem of section 1.1 in an equivalent form. Consider the Hilbert space $V = \{\mathbf{v} = (v_i) \in [H^1(\Omega)]^3 ; \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\}$, equipped with the H^1-norm. Let $B(\varepsilon)(\cdot,\cdot) : V \times V \to \mathbb{R}$ denote the symmetric, bilinear form defined by :

$$B(\varepsilon)(\mathbf{u}(\varepsilon), (\mathbf{v})) =$$

$$\varepsilon^2 \int_\Omega [\lambda\, e_{\alpha\alpha}(\mathbf{u}(\varepsilon))\, e_{\beta\beta}(\mathbf{v}) + 2\,\mu\, e_{\alpha\beta}(\mathbf{u}(\varepsilon))\, e_{\alpha\beta}(\mathbf{v})]\, d\mathbf{x}$$

$$+ \varepsilon^4 \int_\Omega \{\lambda\, [e_{33}(\mathbf{u}(\varepsilon))\, e_{\beta\beta}(\mathbf{v}) + e_{\alpha\alpha}(\mathbf{u}(\varepsilon))\, e_{33}(\mathbf{v})] + 4\,\mu\, e_{3\alpha}(\mathbf{u}(\varepsilon))\, e_{3\alpha}(\mathbf{v})\}\, d\mathbf{x}$$

$$+ \varepsilon^6 \int_\Omega [(\lambda + 2\,\mu)\, e_{33}(\mathbf{u}(\varepsilon))\, e_{33}(\mathbf{v})]\, d\mathbf{x} ,$$

and let $F(\varepsilon)(\,.\,): V \to \mathbb{R}$ denote the continuous, linear form, defined by :

$$F(\varepsilon)(\mathbf{v}) = \varepsilon^6\left[\int_\Omega \mathbf{f}\,.\,\mathbf{v}\,d\mathbf{x} + \int_{\Gamma_1} \mathbf{g}\,.\,\mathbf{v}\,d\mathbf{a}\right].$$

The scaled displacement field $\mathbf{u}(\varepsilon)$ is then the solution of the variational problem

$$B(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v}) = F(\varepsilon)(\mathbf{v}), \quad \forall \mathbf{v} \in V.$$

LEMMA 1.1 : *For each $\varepsilon > 0$ the bilinear form $B(\varepsilon)(\,.,\,.\,)$ is continuous on $V \times V$ and V-elliptic. This means that there exists two constants m and M (independent of ε) such that :*

$$B(\varepsilon)(\mathbf{u}, \mathbf{v}) \leqslant \varepsilon^2 M \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)}, \quad \forall \mathbf{u}, \mathbf{v} \in V,$$
$$B(\varepsilon)(\mathbf{v}, \mathbf{v}) \geqslant \varepsilon^6 m \|\mathbf{v}\|^2_{H^1(\Omega)}, \quad \forall \mathbf{v} \in V. \blacksquare$$

2. SPECTRAL APPROXIMATION

2.1. Principle of Galerkin Approximation

A **spectral method** consists in seeking the solution of a variational problem in terms of a truncated series of known, smooth, functions (for example, polynomials, trigonometric functions) taken from an approximation space V_N. More precisely, let $V_N \subset V$ be the approximation space and $\tilde{\mathbf{u}}_N(\varepsilon) \in V_N$ be the Galerkin spectral approximation to the unique solution $\mathbf{u}(\varepsilon) \in V$ to the variational equation :

$$B(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v}) = F(\varepsilon)(\mathbf{v}), \quad \forall \mathbf{v} \in V.$$

By definition, the approximation $\tilde{\mathbf{u}}_N(\varepsilon)$ is the unique solution of the problem :

$$B(\varepsilon)(\tilde{\mathbf{u}}_N(\varepsilon), \mathbf{v}) = F(\varepsilon)(\mathbf{v}), \quad \forall \mathbf{v} \in V_N,$$

which can also be characterized as the unique solution of the minimization problem :

$$B(\varepsilon)(\mathbf{u}(\varepsilon) - \tilde{\mathbf{u}}_N(\varepsilon), \mathbf{u}(\varepsilon) - \tilde{\mathbf{u}}_N(\varepsilon)) = \\ = \inf\left\{B(\varepsilon)(\mathbf{u}(\varepsilon) - \mathbf{z}_N, \mathbf{u}(\varepsilon) - \mathbf{z}_N) : \mathbf{z}_N \in V_N\right\},$$

thus, $\tilde{\mathbf{u}}_N(\varepsilon)$ is the projection of $\mathbf{u}(\varepsilon)$ onto the approximation space V_N with respect to the inner product associated with the quadratic form $B(\varepsilon)(\,.,\,.\,)$.

In this section we give the structure of a possible approximation space V_N and we state a convergence theorem when N goes to infinity.

2.2. Another variational formulation of the three-dimensional problem posed over the fixed set Ω

As it will be seen later, it is convenient to split the bilinear form $B(\varepsilon)$ into an « horizontal part » $B_H(\varepsilon)$ (this means that $B_H(\varepsilon)(\mathbf{u}(\varepsilon), .\,)$ acts only on the « horizontal » component $\mathbf{v}_H = (v_\alpha)$ of any test function $\mathbf{v} = (v_i) = (\mathbf{v}_H, v_3)$ and a « vertical » part $B_3(\varepsilon)$ (this means that $B_3(\varepsilon)(\mathbf{u}(\varepsilon), .\,)$ acts only on the « vertical » component v_3). More specifically, we write,

$$B(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v}) = B_H(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v}_H) + B_3(\varepsilon)(\mathbf{u}(\varepsilon), v_3), \quad \forall \mathbf{v} = (\mathbf{v}_H, v_3) \in V$$

with the following explicit expressions for $B_H(\varepsilon)(\,.,\,.\,)$ and $B_3(\,.,\,.\,)$:

$$B_H(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v}_H) = \varepsilon^2 \int_\Omega [\lambda\, e_{\alpha\alpha}(\mathbf{u}(\varepsilon))\, e_{\beta\beta}(\mathbf{v}) + 2\,\mu\, e_{\alpha\beta}(\mathbf{u}(\varepsilon))\, e_{\alpha\beta}(\mathbf{v})]\, d\mathbf{x} +$$

$$+ \varepsilon^4 \int_\Omega \left\{\lambda\, e_{33}(\mathbf{u}(\varepsilon)\, e_{\beta\beta}(\mathbf{v}) + \mu\, [\partial_\alpha u_3(\varepsilon) + \partial_3 u_\alpha(\varepsilon)]\, \partial_3 v_\alpha\right\} d\mathbf{x},$$

$$B_3(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v}_3) =$$

$$= \varepsilon^4 \int_\Omega \left\{\lambda\, e_{\alpha\alpha}(\mathbf{u}(\varepsilon))\, e_{33}(\mathbf{v}) + \mu\, [\partial_\alpha u_3(\varepsilon) + \partial_3 u_\alpha(\varepsilon)]\, \partial_\alpha v_3\right\} d\mathbf{x}$$

$$+ \varepsilon^6 \int_\Omega [(\lambda + 2\,\mu)\, e_{33}(\mathbf{u}(\varepsilon))\, e_{33}(\mathbf{v})]\, d\mathbf{x}.$$

Similarly, the linear form $F(\varepsilon)(\,.\,)$ can be written as :

$$F(\varepsilon)(\mathbf{v}) = F_H(\varepsilon)(\mathbf{v}_H) + F_3(\varepsilon)(v_3),$$

$$F_H(\varepsilon)(\mathbf{v}_H) = \varepsilon^6 \left[\int_\Omega f_\alpha\, v_\alpha\, d\mathbf{x} + \int_{\Gamma_1} g_\alpha\, v_\alpha\, d\mathbf{a}\right],$$

$$F_3(\varepsilon)(\mathbf{v}_3) = \varepsilon^6 \left[\int_\Omega f_3\, v_3\, d\mathbf{x} + \int_{\Gamma_1} g_3\, v_3\, d\mathbf{a}\right], \quad \forall \mathbf{v} = (\mathbf{v}_H, v_3) \in V.$$

Consequently, the unknown $\mathbf{u}(\varepsilon)$ solves the system :

$$\begin{cases} B_H(\varepsilon)(\mathbf{u}(\varepsilon), \mathbf{v}_H) = F_H(\varepsilon)(\mathbf{v}_H), \\ B_3(\varepsilon)(\mathbf{u}(\varepsilon), v_3) = F_3(\varepsilon)(v_3), \quad \forall \mathbf{v} = (\mathbf{v}_H, v_3) \in V. \end{cases} \tag{2.1}$$

2.3. Choice of the Approximation Space

As in section 2.2, let us distinguish the « horizontal » part (P_α^k) and the « vertical » part Q^k of the basis functions $\mathbf{T}^k$ mentioned in the introduction, that is, $\mathbf{T}^k = (P_1^k, P_2^k, Q^k)$ for $k \geqslant 0$. The approximation space V_N of the space V, is therefore defined by :

$$V_N = \Big\{ \mathbf{v} = (v_i) \in V : v_\alpha = \sum_{k=0}^{N} P_\alpha^k(x_1, x_2)\, v_\alpha^k(x_3),$$

$$v_\alpha^k \in H_0^1(0, L),\ P_\alpha^k \in H^1(\omega),$$

$$v_3 = \sum_{k=0}^{N} Q^k(x_1, x_2)\, v_3^k(x_3),\ v_3^k \in H_0^1(0, L),\ Q^k \in H^1(\omega),$$

$$\forall 0 \leqslant k \leqslant N, \quad \text{(no summation on } \alpha \text{)} \Big\} .$$

Following the results obtained for plates by Miara [1989], one can choose as follows a particular element $\boldsymbol{\Psi}_N \in V_N$ that simplifies both the expressions of $B_H(\boldsymbol{\Psi}_N, \mathbf{v}_H)$ and $B_3(\boldsymbol{\Psi}_N, v_3)$ for all $\mathbf{v} = (\mathbf{v}_H, v_3) \in V$.

LEMMA 2.1 : *For* $N \geqslant 0$ *let* $\Psi_1^0 = \Psi_2^0 \in H_0^{2N+2}(0, L)$ *and let* $\boldsymbol{\Psi}^k = (\Psi_1^k, \Psi_2^k, \Psi_3^k)$, $k \geqslant 0$, *be defined by :*

$$\begin{cases} \Psi_3^0(x_3) = \partial_3 \Psi_1^0(x_3) = \partial_3 \Psi_2^0(x_3), & \Psi_3^k(x_3) = \partial_{33} \Psi_3^{k-1}(x_3), \quad k \geqslant 1, \\ \Psi_1^k(x_3) = \Psi_2^k(x_3) = \partial_3 \Psi_3^{k-1}(x_3) = \partial_{33} \Psi_1^{k-1}(x_3) = \partial_{33} \Psi_2^{k-1}(x_3), & k \geqslant 1. \end{cases}$$

Then, the element $\boldsymbol{\Psi}_N = \sum_{k=0}^{N} \varepsilon^{2k} (\Psi_1^k P_1^k, \Psi_2^k P_2^k, \Psi_3^k Q^k)$ *belongs to* V_N *and solves the following equations, valid for all* $\mathbf{v} = (\mathbf{v}_H, v_3) \in V$:

$$B_H(\boldsymbol{\Psi}_N, \mathbf{v}_H) = \varepsilon^{2N+4} \int_\omega \{ \lambda Q^N e_{\alpha\alpha}(\mathbf{s}^{N+1}) - \mu (P_\alpha^N + \partial_\alpha Q^N) s_\alpha^{N+1} \} \, d\omega +$$

$$+ \sum_{k=0}^{N} \varepsilon^{2k+2} \int_\omega \{ \lambda e_{\alpha\alpha}(\mathbf{P}^k) e_{\beta\beta}(\mathbf{s}^k) + 2 \mu e_{\alpha\beta}(\mathbf{P}^k) \partial_\alpha s_\beta^k$$

$$+ \lambda Q^{k-1} e_{\alpha\alpha}(\mathbf{s}^k) - \mu (\partial_\alpha Q^{k-1} + P_\alpha^{k-1}) s_\alpha^k \} \, d\omega ,$$

where (with no summation on α*),*

$$s_\alpha^k(x_1, x_2) = \int_0^L \Psi_\alpha^k(x_3)\, v_\alpha(x_1, x_2, x_3)\, dx_3 \in H^1(\omega), \quad 0 \leqslant k \leqslant N,$$

$$s_\alpha^{N+1}(x_1, x_2) = \int_0^L \partial_3 \Psi_3^N(x_3)\, v_\alpha(x_1, x_2, x_3)\, dx_3 \in H^1(\omega),$$

and

$$B_3(\boldsymbol{\Psi}_N, v_3) = \varepsilon^{2N+6} \int_\omega (\lambda + 2\mu) Q^N r^{N+1} \, d\omega +$$

$$+ \sum_{k=0}^{N} \varepsilon^{2k+4} \int_\omega \{\mu (\partial_\alpha Q^k + P_\alpha^k) \partial_\alpha r^k - [(\lambda + 2\mu) Q^{k-1} + \lambda \partial_\alpha P_\alpha^k] r^k\} \, d\omega$$

where

$$r^k(x_1, x_2) = \int_0^L \Psi_3^k(x_3) v_3(x_1, x_2, x_3) \, dx_3 \in H^1(\omega), \quad 0 \leq k \leq N,$$

$$r^{N+1}(x_1, x_2) = \int_0^L \partial_3 \Psi_3^N(x_3) \partial_3 v_3(x_1, x_2, x_3) \, dx_3 \in H^1(\omega),$$

and where, by convention, $\mathbf{P}^{-1} = \mathbf{0}$, $Q^{-1} = 0$.

Proof: The proof is straightforward if we replace $\mathbf{u}(\varepsilon)$ by $\boldsymbol{\Psi}_N$ in system (2.1). ■

2.4. Selection of the basis functions $(\mathbf{P}^k, Q^k)$

The idea now is to select the basis functions $(\mathbf{P}^k, Q^k)$ so that $B(\varepsilon)(\mathbf{u}(\varepsilon) - \boldsymbol{\Psi}_N, \mathbf{v})$ be as small as possible with respect to ε, for all $\mathbf{v} \in V$. For simplicity, we suppose hereafter, and with no loss of generality (see remark 2.2), that all the forces vanish except g_3 (the axial component of the surface force), and that it depends only on x_3. Accordingly, the construction of the basis functions is achieved by the following scheme:

i) in $B(\varepsilon)(\boldsymbol{\Psi}_N, \mathbf{v})$, cancel all the coefficients of ε^{2k+2}, $k \leq N+1$, except that of ε^6,

ii) in $B(\varepsilon)(\boldsymbol{\Psi}_N, \mathbf{v})$, set the coefficient of ε^6 equal to $\int_{\Gamma_1} g_3 v_3 \, d\mathbf{a}$.

This yields:

$$B(\mathbf{u}(\varepsilon) - \boldsymbol{\Psi}_N, v) =$$
$$- \varepsilon^{2N+4} \int_\omega \int_0^L [\lambda Q^N \partial_3 \Psi_3^N \partial_\beta v_\beta + \mu (P_\beta^N \partial_3 \Psi_\beta^N + \partial_\beta Q^N \Psi_3^N) \partial_3 v_\beta] \, d\mathbf{x}$$
$$- \varepsilon^{2N+6} \int_\omega \int_0^L [(\lambda + 2\mu) Q^N \partial_3 \Psi_3^N \partial_3 v_3] \, d\mathbf{x},$$

and as a consequence the basis functions $(\mathbf{P}^k, Q^k)$ are given by the following recursion formulas (1^k), (2^k) for $k \geq 0$, with $\mathbf{P}^{-1} = \mathbf{0}$, $Q^{-1} = 0$:

$$\left\{\begin{array}{l}\displaystyle\int_{\omega} \{\lambda e_{\alpha\alpha}(\mathbf{P}^k)\, e_{\beta\beta}(\mathbf{s}) + 2\,\mu e_{\alpha\beta}(\mathbf{P}^k)\, \partial_\alpha s_\beta + \\ + \lambda Q^{k-1} e_{\alpha\alpha}(\mathbf{s}) - \mu\,(\partial_\alpha Q^{k-1} + P_\alpha^{k-1})\, s_\alpha\}\, d\omega = 0\,, \forall \mathbf{s} \in [H^1(\omega)]^2\,,\end{array}\right. \qquad (1^k)$$

$$\left\{\begin{array}{l}\displaystyle\int_{\omega} \{\mu\,(\partial_\alpha Q^k + P_\alpha^k)\, \partial_\alpha r - [(\lambda + 2\,\mu)\, Q^{k-1} + \lambda\, \partial_\alpha P_\alpha^k]\, r\}\, d\omega \\ \displaystyle\qquad\qquad = \delta_1^k \int_{\gamma} r\, d\gamma\,, \quad \forall r \in H^1(\omega)\,.\end{array}\right. \qquad (2^k)$$

Remark 2.1 : **Interpretation of equations (1^k) and (2^k)**

The vector valued function $\mathbf{P}^k = (P_1^k, P_2^k)$ can be interpreted as a weak solution of a **plane deformation problem** of linearized elasticity in ω. The two-dimensional displacement field $\mathbf{P}^k$ expresses the deformation of a body with Lamé constants λ and μ subjected to volume forces $((\lambda + \mu)\, \partial_\alpha Q^{k-1} + \mu P_\alpha^{k-1})$ in ω and to surface forces $-\lambda Q^{k-1} n_\alpha$ on γ where $\mathbf{n} = (n_\alpha)$ is the outward unit normal along the boundary γ. This effect is not present in the classical engineering beam theories. The **compatibility conditions** for equation (1^k) express the fact that these applied forces are in equilibrium, namely the resultant and total moment vanish :

$$\int_{\omega} (\partial_\alpha Q^{k-1} + P_\alpha^{k-1})\, d\omega = 0\,, \qquad (3^k)$$

$$\int_{\omega} [(\partial_1 Q^{k-1} + P_1^{k-1})\, x_2 - (\partial_2 Q^{k-1} + P_2^{k-1})\, x_1]\, d\omega = 0\,. \qquad (4^k)$$

Therefore, if these conditions are satisfied, there exists a function $\mathbf{P}^k$, unique up to an infinitesimal rigid displacement, that solves the variational equations (1^k).

The function Q^k can be interpreted as a weak solution of a plane **membrane-torsion problem** of linearized in elasticity in ω. This takes into account the warping of the cross section associated with Saint Venant's torsion theory. The two-dimensional body with shear modulus μ and cross section ω is subjected to volume forces $(\lambda + 2\,\mu)\, Q^{k-1} - (\lambda + \mu)\, \partial_\alpha P_\alpha^k$ in ω and to surface forces $-\mu P_\alpha^k n_\alpha + \delta_1^k$ on γ. The **compatibility conditions** for equation (2^k) express the fact that the resultant of the applied forces vanishes :

$$\int_{\omega} [(\lambda + 2\,\mu)\, Q^{k-1} + \lambda\, \partial_\alpha P_\alpha^k]\, d\omega = \delta_1^k\, |\gamma|\,. \qquad (5^k)$$

Therefore, if this condition is satisfied, there exists a solution Q^k of equations (2^k), defined up to an additive constant. ■

Remark 2.2 : **Extension to other loadings.**

We considered that the system of applied forces reduces to $g_3(x_1, x_2, x_3) = g_3(x_3)$. This is not a restriction since other types of loadings are possible. For example :

i) If $f_3(x_1, x_2, x_3) = f_3(x_3)$ then we must replace the right hand-side of (2^k) by $\delta_1^k \int_\omega r\, d\omega$ and the right hand-side of (1^k) by zero.

ii) If $f_3(x_1, x_2, x_3) = f_3^1(x_1, x_2)\, f_3^2(x_3)$, (resp. $g_3(x_1, x_2, x_3) = g_3^1(x_1, x_2)\, g_3^2(x_3)$), we must replace the right hand-side of (2^k) by $\delta_1^k \int_\omega f_3^1(x_1, x_2)\, r\, d\omega$, (resp. $\delta_1^k \int_\gamma g_3^1(x_1, x_2)\, r\, d\gamma$), and the right hand-side of (1^k) by zero.

iii) If $f_\alpha(x_1, x_2, x_3) = f_\alpha(x_3)$, (resp. $g_\alpha(x_1, x_2, x_3) = g_\alpha(x_3)$), then we must replace the right hand-side of (1^k) by $\delta_2^k \int_\omega (s_1 + s_2)\, d\omega$, (resp. $\delta_2^k \int_\gamma (s_1 + s_2)\, d\gamma$), and the right hand-side of (2^k) by zero. Moreover, the following compatibility condition must hold :

$$f_1(x_1, x_2, x_3) = f_2(x_1, x_2, x_3)\,,$$
$$(\text{resp. } g_1(x_1, x_2, x_3) = g_2(x_1, x_2, x_3))\,.$$

iv) If (with no summation on α) $f_\alpha(x_1, x_2, x_3) = f_\alpha^1(x_1, x_2)\, f_\alpha^2(x_3)$, (resp. $g_\alpha(x_1, x_2, x_3) = g_\alpha^1(x_1, x_2)\, g_\alpha^2(x_3)$), we must replace the right hand-side of (1^k) by $\delta_2^k \int_\omega f_\alpha^1(x_1, x_2)\, s_\alpha\, d\omega$, (resp. $\delta_2^k \int_\omega g_\alpha^1(x_1, x_2)\, s_\alpha\, d\gamma$), and the right hand-side of (2^k) by zero. Moreover, the following compatibility condition must hold :

$$f_1(x_1, x_2, x_3) = f_2(x_1, x_2, x_3)\,, (\text{resp. } g_1(x_1, x_2, x_3) = g_2(x_1, x_2, x_3)). \quad \blacksquare$$

2.4.1. *Some notations*

We now introduce some notations for the statement of the next Lemma. Let the vector field $\tilde{\mathbf{P}}^k$ be a translation in $\mathbb{R}^2$ and let $\tilde{Q}^k = -\tilde{P}_\alpha^k x_\alpha$; then $\tilde{Q}^k$ is solution of the problem :

$$\int_\omega (\partial_\alpha \tilde{Q}^k + \tilde{P}_\alpha^k)\, \partial_\alpha r\, d\omega = 0\,, \quad \forall r \in H^1(\omega)\,.$$

Let $\mathbf{R}^k = (R^k_\alpha)$ be an infinitesimal rigid body displacement. It has the general form, $R^k_1 = -r^k_0 x_2 + \tilde{P}^k_1$, $R^k_2 = r^k_0 x_1 + \tilde{P}^k_2$, with $r^k_0 \in \mathbb{R}$. It is thus the solution of the two-dimensional elasticity problem :

$$\int_\omega [\lambda\, e_{\alpha\alpha}(\mathbf{R}^k)\, e_{\beta\beta}(\mathbf{s}) + 2\,\mu\, e_{\alpha\beta}(\mathbf{R}^k)\, \partial_\alpha s_\beta]\, d\omega = 0\,, \quad \forall \mathbf{s} \in [H^1(\omega)]^2\,.$$

Let the vector field $\mathbf{P}^{*k} = (P^{*k}_\alpha)$ be of the form $P^{*k}_\alpha = [\lambda/2(\lambda+\mu)]\, \Phi_{\alpha\beta}\, \tilde{P}^k_\beta$, where the function valued matrix $\Phi = (\Phi_{\alpha\beta})$ is given by $\Phi_{11} = -\Phi_{22} = (x_1^2 - x_2^2)/2$, $\Phi_{12} = \Phi_{21} = x_1 x_2$. Consequently, $\mathbf{P}^{*k}$ is solution of the two-dimensional elasticity problem :

$$\int_\omega [\lambda\, e_{\alpha\alpha}(\mathbf{P}^{*k})\, e_{\beta\beta}(\mathbf{s}) + \mu\, e_{\alpha\beta}(\mathbf{P}^{*k})\, \partial_\alpha s_\beta - \lambda \tilde{P}^k_\beta\, x_\beta\, e_{\alpha\alpha}(\mathbf{s})]\, d\omega = 0\,,$$

$$\forall \mathbf{s} \in [H^1(\omega)]^2\,.$$

Let η_α, θ_α and H be the unique solutions of the following second order problems :

$$\begin{cases} -\Delta \eta_\alpha = -2\,x_\alpha \text{ in } \omega\,, \\ \partial_n \eta_\alpha = 0 \text{ on } \gamma\,, \\ \displaystyle\int_\omega \eta_\alpha\, d\omega = 0\,, \end{cases}$$

$$\begin{cases} -\Delta \theta_\alpha = 2\,x_\alpha \text{ in } \omega\,, \\ \partial_n \theta_\alpha = -\Phi_{\alpha\beta}\, n_\beta \text{ on } \gamma\,, \\ \displaystyle\int_\omega \theta_\alpha\, d\omega = 0\,, \end{cases}$$

$$\begin{cases} -\Delta H = -\,|\gamma|/|\omega| \text{ in } \omega\,, \\ \partial_n H = 1 \text{ on } \gamma\,, \\ \displaystyle\int_\omega H\, d\omega = 0\,. \end{cases}$$

Let $Q^{*k} = [\lambda \tilde{P}^k_\alpha\, \theta_\alpha + (3\,\lambda + 2\,\mu)\, \tilde{P}^k_\alpha\, \eta_\alpha]/2(\lambda+\mu)$; then Q^{*k} is a solution of the following membrane-torsion type problem :

$$\int_\omega \left\{\mu\,(\partial_\alpha Q^{*k} + \tilde{P}^{*k}_\alpha)\, \partial_\alpha r - [(\lambda + 2\,\mu)\, \tilde{Q}^k + \lambda\, \partial_\alpha P^{*k}_\alpha]\, r\right\} d\omega = 0\,,$$

$$\forall r \in H^1(\omega)\,.$$

Let w (the warping function) and φ (the torsion function) be the unique solutions of the following second order problems :

$$\begin{cases} -\Delta w = 0 \text{ in } \omega , \\ \partial_n w = x_2 n_1 - x_1 n_2 \text{ on } \gamma , \\ \displaystyle\int_\omega w \, d\omega = 0 , \end{cases}$$

$$\begin{cases} -\Delta \varphi = 2 \text{ in } \omega , \\ \varphi = 0 \text{ on } \gamma , \end{cases}$$

and finally define the following constants,

$$I_\alpha^w = 2 \int_\omega x_\alpha \, w \, d\omega , \qquad I_1^\varphi = 2 \int_\omega x_2 \, \varphi \, d\omega ,$$

$$I_2^\varphi = -2 \int_\omega x_1 \, \varphi \, d\omega , \quad J = 2 \int_\omega \varphi \, d\omega ,$$

$$I_\alpha = \int_\omega x_\alpha^2 \, d\omega , \qquad H_\alpha = \int_\omega x_\alpha (x_1^2 + x_2^2)/2 \, d\omega .$$

2.4.2. *Existence of the basis function* $(\mathbf{P}^k, Q^k)$

Some results concerning the existence and uniqueness of the basis functions are given in the following lemma.

LEMMA 2.2 :

i) *Equations* (1^k) *and* (2^k) $0 \leqslant k \leqslant j+2$ *uniquely define the basis functions* $(\mathbf{P}^k, Q^k)_{0 \leqslant k \leqslant j}$. *Moreover,*

ii) *the displacement field* $\mathbf{P}^{j+1}$ *is uniquely defined up to any arbitrary translation vector* $\tilde{\mathbf{P}}^{j+1}$,

iii) *function* Q^{j+1} *is uniquely defined up to the additive function* $\tilde{Q}^{j+1}$,

iv) *the displacement field* $\mathbf{P}^{j+2}$ *is uniquely defined up to an infinitesimal rigid displacement* $\mathbf{R}^{j+2}$ *and up to the additive function* $\mathbf{P}^{*j+1}$,

v) *function* Q^{j+2} *is uniquely defined up to an additive constant* q^{j+2} *and up to the additive functions* $\tilde{Q}^{j+2} + Q^{*j+1} + r_0^{j+2} w$.

Proof : The proof is done by induction. For $j = 0$ the result is true, since equations $(1^k, 2^k)_{0 \leqslant k \leqslant 2}$ give the following expressions for the basis functions $(\mathbf{P}^k, Q^k)_{0 \leqslant k \leqslant 2}$.

$$\begin{cases} P_\alpha^0 = C_\alpha^0/\mu , \\ Q^0 = (D^0 - C_\alpha^0 x_\alpha)/\mu , \end{cases}$$

where the constants C_α^0 and D^0 are defined by :

$$C_\alpha^0 = [(\lambda + \mu)/(3\lambda + 2\mu)] \int_\gamma x_\alpha \, d\gamma / I_\alpha , \quad \text{(no summation on } \alpha) ,$$

$$D^0 = - [(\lambda + \mu)/(3\lambda + 2\mu)] \, |\gamma| / |\omega| ,$$

$$\begin{cases} P_1^1 = \lambda(-D^0 x_1 + C_\alpha^0 \Phi_{1\alpha})/2\mu(\lambda + \mu) - \lambda K^1 x_2/\mu^2 + \tilde{P}_1^1 , \\ P_2^1 = \lambda(-D^0 x_2 + C_\alpha^0 \Phi_{2\alpha})/2\mu(\lambda + \mu) + \lambda K^1 x_1/\mu^2 + \tilde{P}_2^1 , \\ Q^1 = \lambda K^1 w/\mu^2 + \lambda D^1/\mu^2 + (3\lambda + 2\mu) C_\alpha^0 \eta_\alpha / 2\mu(\lambda + \mu) \\ \qquad + \lambda C_\alpha^0 \theta_\alpha / 2\mu(\lambda + \mu) + \lambda D^0 [(x_1^2 + x_2^2)/2 - \\ \qquad - (I_1 + I_2)/2\,|\omega|\,]/[2\mu(\lambda + \mu)] + \tilde{Q}^1 , \end{cases}$$

where the constants K^1 and D^1 are given by :

$$K^1 = -\mu\,[(3\lambda + 2\mu) C_\alpha^0 I_\alpha^w + \lambda C_\alpha^0 I_\alpha^\Psi]/2\lambda J(\lambda + \mu) + \mu \int_\gamma w \, d\gamma/\lambda J ,$$

$$D^1 = \mu \Big[2 H_\alpha \int_\gamma x_\alpha \, d\gamma / I_\alpha - \int_\gamma (x_1^2 + x_2^2)\, d\gamma + \\ + |\gamma| \, (I_1 + I_2)/|\omega| \, \Big] / [4(3\lambda + 2\mu)\,|\omega|\,]$$

and where $\tilde{\mathbf{P}}^1$ is an arbitrary vector in $\mathbb{R}^2$ and $\tilde{Q}^1$ is an arbitrary constant in $\mathbb{R}$. Moreover,

$$P_\beta^2 = -\lambda^2 D^1 x_\beta / 2\mu^2(\lambda + \mu) + \bar{P}_\beta^2 + P_\beta^{*1} + R_\beta^2 ,$$

where the vector $\bar{\mathbf{P}}^2$ is defined as follows :

$$\bar{P}_\alpha^2 = \lambda^2 K^1 P_\alpha^{2,1}/\mu^2 + \lambda P_\alpha^{2,2}/2\mu + \lambda^2 P_\alpha^{2,3}/2\mu(3\lambda + 2\mu) \\ - \lambda^2 |\gamma| \, P_\alpha^{2,4}/2\mu(3\lambda + 2\mu)\,|\omega| + \lambda P_\alpha^{2,5}/\mu + \mu P_\alpha^{2,6}$$

and where $P_\alpha^{2,k}$, $1 \leq k \leq 6$, are the unique solutions of plane deformation elasticity problems of the form :

$$\begin{cases} -(\lambda + \mu)\,\partial_{\beta\alpha} P_\alpha^{2,k} - \mu\,\Delta P_\beta^{2,k} = F_\beta^k \text{ in } \omega , \\ \lambda\,\partial_\alpha P_\alpha^{2,k} n_\beta + \mu(\partial_n P_\beta^{2,k} + \partial_\beta P_\alpha^{2,k} n_\alpha) = G_\beta^k \text{ on } \gamma , \\ \int_\omega P_\alpha^{2,k}\, d\omega = \int_\omega (P_1^{2,k} x_2 - P_2^{2,k} x_1)\, d\omega = 0 \end{cases}$$

and the data F_β^k, G_β^k associated with $P_\beta^{2,k}$, $1 \leq k \leq 6$, are given by (with summation on α) :

$$\begin{cases} F_\beta^1 = \partial_\beta w \,, \\ G_\beta^1 = - w n_\beta \,, \end{cases}$$

$$\begin{cases} F_\beta^2 = \partial_\beta \eta_\alpha \displaystyle\int_\gamma x_\alpha \, d\gamma / I_\alpha \,, \\ G_\beta^2 = - \left(\eta_\alpha \displaystyle\int_\gamma x_\alpha \, d\gamma / I_\alpha \right) n_\beta \,, \end{cases}$$

$$\begin{cases} F_\beta^3 = \partial_\beta \theta_\alpha \displaystyle\int_\gamma x_\alpha \, d\gamma / I_\alpha \,, \\ G_\beta^3 = - \left(\theta_\alpha \displaystyle\int_\gamma x_\alpha \, d\gamma / I_\alpha \right) n_\beta \,, \end{cases}$$

$$\begin{cases} F_\beta^4 = x_\beta \,, \\ G_\beta^4 = - \left[(x_1^2 + x_2^2)/2 - (I_1 + I_2)/|\omega| \right] n_\beta \,, \end{cases}$$

$$\begin{cases} F_\beta^5 = \partial_\beta H \,, \\ G_\beta^5 = - H n_\beta \,, \end{cases}$$

$$\begin{cases} F_1^6 = \lambda C_\alpha^0 (\partial_1 \theta_\alpha + \Phi_{1\alpha})/2\,\mu(\lambda + \mu) \\ \qquad + (3\lambda + 2\mu)\, C_\alpha^0\, \partial_1 \eta_\alpha / 2\,\mu(\lambda + \mu) + \partial_1 H/\mu + (\lambda K^1/\mu^2)\, \partial_2 \varphi \,, \\ F_2^6 = \lambda C_\alpha^0 (\partial_2 \theta_\alpha + \Phi_{2\alpha})/2\,\mu(\lambda + \mu) \\ \qquad + (3\lambda + 2\mu)\, C_\alpha^0\, \partial_2 \eta_\alpha / 2\,\mu(\lambda + \mu) + \partial_2 H/\mu - (\lambda K^1/\mu^2)\, \partial_1 \varphi \,, \\ G_\beta^6 = 0 \,. \end{cases}$$

The expression for $\mathbf{Q}^2$, which is defined by (2^2), is omitted because it is too lengthy in the general case. We shall nevertheless write it down for the simpler case of a circular cross section (see § 2.4.3). Let us now suppose that the lemma has been proved for $0 \leq k \leq j + 2$, we shall prove that it also holds for $k = j + 3$ according to the following five steps :

First step (part i) : Let us prove that the compatibility equations 3^{j+3} determine the translation vector $\tilde{\mathbf{P}}^{j+1}$ and consequently that $\mathbf{P}^{j+1}$ and Q^{j+1} are uniquely defined. In fact, when we replace the test function r by x_β in equation 2^{j+2} we get, using the compatibility equations 3^{j+3} :

$$\int_\omega \left[(\lambda + 2\mu)\, Q^{j+1} + \lambda\, \partial_\alpha P_\alpha^{j+2} \right] x_\beta \, d\omega = \delta_1^{j+2} \int_\gamma x_\beta \, d\gamma \,.$$

This represents a nonsingular system of two equations in the two unknowns $\tilde{P}_\alpha^{J+1}$ which are thus uniquely defined. This implies that the vector field $\mathbf{P}^{*J+1}$ and the function Q^{*J+1} are known so that, first the displacement field $\mathbf{P}^{J+2}$ is defined up to any infinitesimal rigid displacement $\mathbf{R}^{J+2}$, and secondly, the function Q^{J+2} is defined up to the constant q^{J+2} and up to the additive function $\tilde{Q}^{J+2} - r_0^{J+2} w$.

Second step (part ii) : Let us prove that the compatibility equation 4^{J+3} determines r_0^{J+2} and consequently that the displacement field $\mathbf{P}^{J+2}$ is defined up to a translation $\tilde{\mathbf{P}}^{J+2}$. Since

$$\int_\omega [(\partial_1 \tilde{Q}^{J+2} + R_1^{J+2}) x_2 - (\partial_2 \tilde{Q}^{J+2} + R_2^{J+2}) x_1] \, d\omega = \\ = - r_0^{J+2} \int_\omega (x_2^2 + x_1^2) \, d\omega ,$$

the coefficient of r_0^{J+2} in equation 4^{J+3} does not vanish and therefore r_0^{J+2} is determined by this compatibility equation. Thus the function Q^{J+2} is defined up to $q^{J+2} + \tilde{Q}^{J+2}$.

Third step : The vector field $\mathbf{P}^{J+3}$ can be computed from equation 1^{J+3}, as follows :

$$\int_\omega [\lambda e_{\alpha\alpha}(\mathbf{P}^{J+3}) e_{\beta\beta}(\mathbf{s}) + 2 \mu e_{\alpha\beta}(\mathbf{P}^{J+3}) e_{\alpha\beta}(\mathbf{s})] \, d\omega = \\ = \int_\omega [\mu (\partial_\alpha Q^{J+2} + P_\alpha^{J+2}) s_\alpha - \lambda Q^{J+2} e_{\alpha\alpha}(\mathbf{s})] \, d\omega ,$$

valid for all $\mathbf{s} \in (H^1(\omega))^2$. The only unknown on the right-hand side of this equation is

$$\int_\omega [\mu (\partial_\alpha \tilde{Q}^{J+2} + \tilde{P}_\alpha^{J+2}) s_\alpha - \lambda (q^{J+2} + \tilde{Q}^{J+2}) e_{\alpha\alpha}(\mathbf{s})] \, d\omega = \\ = \int_\omega [\lambda (q^{J+2} + \tilde{P}_\beta^{J+2} x_\beta) e_{\alpha\alpha}(\mathbf{s})] \, d\omega .$$

Therefore $\mathbf{P}^{J+3}$ is uniquely defined up to the additive function $\mathbf{P}^{*J+2} - [\lambda/2(\lambda + \mu)] q^{J+2} \mathbf{x} + \mathbf{R}^{J+3}$.

Fourth step (parts iii and iv) : Let us prove that the compatibility condition 5^{J+3} determines q^{J+2}. Equation 5^{J+3} can also be written as :

$$\int_\omega \{(\lambda + 2\mu) q^{J+2} + \lambda \partial_\alpha [- \lambda q^{J+2} x_\alpha / 2(\lambda + \mu)]\} \, d\omega = \delta_1^{J+3} |\gamma| .$$

Then q^{j+2} is uniquely determined and therefore the displacement flied $\mathbf{P}^{j+3}$ is uniquely determined up to the function $\mathbf{R}^{j+3} + \mathbf{P}^{*j+2}$ and Q^{j+2} is defined up to the additive function $\tilde{Q}^{j+2}$.

Fifth step (part v) : The function Q^{j+2} can be computed from equation $\mathcal{2}^{j+3}$:

$$\int_\omega \mu\,(\partial_\alpha Q^{j+3} + P_\alpha^{j+3})\,\partial_\alpha r\,d\omega = \int_\omega [(\lambda + 2\mu)\,Q^{j+2} + \lambda\,\partial_\alpha P_\alpha^{j+2}]\,r\,d\omega + \\ + \delta_1^{j+3}\int_\gamma r\,d\gamma\,,\quad \forall r \in H^1(\omega)$$

in which the only unknowns are on the right-hand side $\int_\omega [(\lambda + 2\mu)\,\tilde{Q}^{j+2} + \lambda\,\partial_\alpha P_\alpha^{*j+2}]\,r\,d\omega$, and on the left-hand side $\int_\omega \mu\,(\partial_\alpha Q^{j+3} + P_\alpha^{*j+2} + R_\alpha^{j+3})\,\partial_\alpha r\,d\omega$. Therefore the function Q^{j+3} is uniquely defined up to a constant q^{j+3} and up to the additive function $\tilde{Q}^{j+3} + Q^{*j+2} - r_0^{j+3} w$. ■

2.4.3. *Basis functions for the circular cross section*

For the particular case of a **circular cross section of radius** R the basis functions are **polynomials** since the elementary functions w, φ, η_α θ_α, and H introduced previously are polynomials. In fact for this case we have :

$$w = 0\,,\quad \varphi = [R^2 - (x_1^2 + x_2^2)]/2\,,$$
$$\eta_\alpha = [x_1^2 + x_2^2 - 3R^2]\,x_\alpha/4\,,\qquad \theta_\alpha = -\,[x_1^2 + x_2^2 - R^2]\,x_\alpha/4\,,$$
$$H = [x_1^2 + x_2^2 - R^2/2]/2R\,,$$
$$I_\alpha = \pi R^4/4\,,\quad J = \pi R^4/2\,,\qquad I_\alpha^w = 0\,,\quad I_\alpha^\varphi = 0\,,$$
$$H_\alpha = 0\,,\qquad H_3 = \pi R^6/12\,,\qquad \int_\gamma \chi_a\,d\gamma = 0\,.$$

For example, taking into account the first four equations ($j = 1$ in Lemma 2.2), we get the following expressions for $\mathbf{P}^0$, $\mathbf{P}^1$, Q^0, Q^1:

$$\begin{cases} P_\beta^0 = 0\,, \\ Q^0 = -2(\lambda + \mu)/\mu(3\lambda + 2\mu)R\,, \end{cases}$$

$$\begin{cases} P_\beta^1 = \lambda x_\beta/\mu(3\lambda + 2\mu)R \\ Q^1 = -R/4\mu + (\lambda + \mu)(x_1^2 + x_2^2)/\mu(3\lambda + 2\mu)R \end{cases}$$

and for $\mathbf{P}^2$, Q^2 (which are defined up to additive polynomial functions which can be determined using higher order terms) :

$$P_\beta^2 = \lambda^2 R x_\beta / 8\,\mu(3\lambda + 2\mu)(\lambda + \mu) -$$
$$- (2\lambda + \mu)(x_1^2 + x_2^2)\, x_\beta / 8\,\mu(3\lambda + 2\mu) R +$$
$$+ (2\lambda^2 + 6\lambda\mu + 3\mu^2) R x_\beta / 8\,\mu(\lambda + \mu)(3\lambda + 2\mu) + \tilde{P}_\beta^1 ,$$
$$Q^2 = (2\lambda + \mu)(x_1^2 + x_2^2) R / 16\,\mu(3\lambda + 2\mu) +$$
$$+ \lambda^2 R^3 / 32\,\mu(\lambda + \mu)(3\lambda + 2\mu) -$$
$$- 3(\lambda + \mu)(x_1^2 + x_2^2)^2 / 32\,\mu(3\lambda + 2\mu) R + q_2 + \tilde{q}_2 . \quad \blacksquare$$

2.5. Convergence of the spectral approximation

For a sufficiently smooth data g_3 (all other forces vanish by assumption), the spectral approximation $\tilde{\mathbf{u}}_N(\varepsilon)$ gives a « good approximation » of the three-dimensional solution $\mathbf{u}(\varepsilon)$ when ε goes to zero. This is the result we shall state next.

Let $\quad G_3 = \{ g_3 : x \in (0, L) \to g_3(x) = \partial_3 h , \quad h \in H_0^4(0, L) \}$,

the following result then holds.

THEOREM 2.1 : *If the system of applied forces is such that* $\mathbf{f} = \underset{\sim}{0}$, $g_\alpha = 0$ *and the component* $g_3 = g_3(x_3) \in G_3 \cap H_0^{2N-1}(0, L)$ *for* $N \geq 1$, *there exists a constant* C_N, *independent of* ε, *such that :*

$$\| \mathbf{u}(\varepsilon) - \tilde{\mathbf{u}}_N(\varepsilon) \|_{H^1(\Omega)} \leq C_N\, \varepsilon^{2N-2} , \quad \| \mathbf{u}^\varepsilon - \tilde{\mathbf{u}}_N^\varepsilon \|_{H^1(\Omega^\varepsilon)} \leq C_N\, \varepsilon^{2N-1} .$$

Proof : By assumption the element

$$\tilde{\Psi}_N = \sum_{k=0}^{N} \varepsilon^{2k} (P_1^k(x_1, x_2)\, \partial_3^{2k-3} g_3(x_3),\ P_2^k(x_1, x_2)\, \partial_3^{2k-3} g_3(x_3) ,$$
$$Q^k(x_1, x_2)\, \partial_3^{2k-2} g_3(x_3))$$

belongs to V_N and satisfies Lemma 2.1 (this element is obtained choosing $\Psi_1^3(x_3) = g_3(x_3)$). Then, using the definition of the basis functions, we get for all $\mathbf{v} \in V$,

$$B(\varepsilon)\,(\mathbf{u}(\varepsilon) - \tilde{\Psi}_N, \mathbf{v}) =$$
$$= - \varepsilon^{2N+4} \int_0^L \int_\omega \{ \lambda Q^N\, \partial_3^{2N-1} g_3\, \partial_\beta v_\beta + \mu\, (P_\beta^N\, \partial_3^{2N-2} g_3 +$$
$$+ \partial_\beta Q^N\, \partial_3^{2N-2} g_3)\, \partial_3 v_\beta \}\, d\mathbf{x}$$
$$- \varepsilon^{2N+6} \int_0^L \int_\omega \{ (\lambda + 2\mu)\, Q^N\, \partial_3^{2N-1} g_3\, \partial_3 v_3 \}\, d\mathbf{x} ,$$

therefore, $\left| B(\varepsilon)(\mathbf{u}(\varepsilon) - \tilde{\Psi}_N, \mathbf{v}) \right| \leq C_N\, \varepsilon^{2N+4} \| \mathbf{v} \|_{H^1(\Omega)}$. Since $\tilde{\mathbf{u}}_N(\varepsilon)$ is the

Galerkin approximation of $\mathbf{u}(\varepsilon)$ the result is a consequence of the coerciveness of $B(\varepsilon)(.,.)$ and of the scalings defined in section 1.2. ■

Remark 2.3 : **Extension to other loadings**

With reference to remark 2.2 we have, in addition :

i) If $f_3(x_1, x_2, x_3) = f_3(x_3)$, we set $\Psi_3^1(x_3) = f_3(x_3)$.

ii) If $f_3(x_1, x_2, x_3) = f_3^1(x_1, x_2)\, f_3^2(x_3)$, (resp. $g_3(x_1, x_2, x_3) = g_3^1(x_1, x_2)\, g_3^2(x_3)$), we set $\Psi_3^1(x_3) = f_3(x_3)$, (resp. $\Psi_3^1(x_3) = g_3(x_3)$).

iii) If $f_\alpha(x_1, x_2, x_3) = f_\alpha(x_3)$, (resp. $g_\alpha(x_1, x_2, x_3) = g_\alpha(x_3)$), then we choose $\Psi_\alpha^2(x_3) = f_\alpha(x_3)$, (resp. $\Psi_\alpha^2(x_3) = g_\alpha(x_3)$).

iv) If (with no summation on α) $f_\alpha(x_1, x_2, x_3) = f_\alpha^1(x_1, x_2)\, f_\alpha^2(x_3)$, (resp. $g_\alpha(x_1, x_2, x_3) = g_\alpha^1(x_1, x_2)\, g_\alpha^2(x_3)$), we choose $\Psi_\alpha^2(x_3) = f_\alpha(x_3)$, (resp. $\Psi_\alpha^2(x_3) = g_\alpha(x_3)$).

We finally remark that in cases ii) and iv), the basis functions for the circular case are not necessarily polynomials. ■

Remark 2.4

For ε sufficiently small, M. L. Mascarenhas and L. Trabucho [1990] have shown that $C_N\, \varepsilon^{2N-2}$ goes to zero as N goes to infinity. ■

3. AN EXAMPLE. SPECTRAL APPROXIMATION OF ORDER ONE

The spectral approximation $\tilde{\mathbf{u}}_1(\varepsilon)$, as we defined it previously, is expressed in terms of Q^0, Q^1, $\mathbf{P}^0$, $\mathbf{P}^1$ by :

$$\tilde{\mathbf{u}}_1(\varepsilon) = \begin{bmatrix} u_1^0 P_1^0 + \varepsilon^2 u_1^1 P_1^1 \\ u_2^0 P_2^0 + \varepsilon^2 u_2^1 P_2^1 \\ u_3^0 Q^0 + \varepsilon^2 u_3^1 Q^1 \end{bmatrix}$$

and it is a solution of the following variational equations :

$$B(\varepsilon)(\tilde{\mathbf{u}}_1(\varepsilon), \mathbf{v}) = F(\varepsilon)(\mathbf{v}), \quad \forall \mathbf{v} \in V_1 .$$

3.1. General case

Specifically, if $P_\alpha^0 \neq 0$, $P_\alpha^1 \neq 0$, $Q^0 \neq 0$, $Q^1 \neq 0$, we have for all $v \in H_0^1(0, L)$:

$$\left\{ \begin{aligned} & \int_0^L \mu\, [\partial_3 u_1^0 (P_1^0, P_1^0) + u_3^0 (P_1^0, \partial_1 Q^0)]\, \partial_3 v\, dx_3 + \\ & + \varepsilon^2 \int_0^L \mu\, [\partial_3 u_1^1 (P_1^0, P_1^1) + u_3^1 (P_1^0, \partial_1 Q^1)]\, \partial_3 v\, dx_3 = \varepsilon^2 \int_0^L F_1^0 v\, dx_3 , \end{aligned} \right. \quad (3.1)$$

$$\left\{\begin{aligned}
&\int_0^L \{(\lambda + 2\mu)(\partial_1 P_1^1, \partial_1 P_1^1)\, u_1^1 v + \lambda (\partial_1 P_1^1, \partial_2 P_2^1)\, u_2^1 v + \\
&+ \mu [(\partial_2 P_1^1, \partial_2 P_1^1)\, u_1^1 + (\partial_1 P_2^1, \partial_2 P_1^1)\, u_2^1]\, v + \\
&+ \lambda (\partial_1 P_1^1, Q^0)\, \partial_3 u_3^0 v + \mu [(P_1^1, P_1^0)\, \partial_3 u_1^0 + (P_1^1, \partial_1 Q^0)\, u_3^0]\, \partial_3 v\}\, dx_3 + \\
&+ \varepsilon^2 \int_0^L \{\lambda (\partial_1 P_1^1, Q^1)\, \partial_3 u_3^1 v + \mu [(P_1^1, P_1^1)\, \partial_3 u_1^1 + \\
&+ (P_1^1, \partial_1 Q^1)\, u_3^1]\, \partial_3 v\, dx_3\} = \\
&= \varepsilon^2 \int_0^L F_1^1 v\, dx_3 ,
\end{aligned}\right. \tag{3.2}$$

$$\left\{\begin{aligned}
&\int_0^L \mu [\partial_3 u_2^0 (P_2^0, P_2^0) + u_3^0 (P_2^0, \partial_2 Q^0)]\, \partial_3 v\, dx_3 + \\
&+ \varepsilon^2 \int_0^L \mu [\partial_3 u_2^1 (P_2^0, P_2^1) + u_3^1 (P_2^0, \partial_2 Q^1)]\, \partial_3 v\, dx_3 = \varepsilon^2 \int_0^L F_2^0 v\, dx_3 ,
\end{aligned}\right. \tag{3.3}$$

$$\left\{\begin{aligned}
&\int_0^L \{(\lambda + 2\mu)(\partial_2 P_2^1, \partial_2 P_2^1)\, u_2^1 v + \lambda (\partial_1 P_1^1, \partial_2 P_2^1)\, u_1^1 v + \\
&+ \mu [(\partial_2 P_1^1, \partial_1 P_2^1)\, u_1^1 + (\partial_1 P_2^1, \partial_1 P_2^1)\, u_2^1]\, v + \\
&+ \lambda (\partial_2 P_2^1, Q^0)\, \partial_3 u_3^0 v + \mu [(P_2^1, P_2^0)\, \partial_3 u_1^0 + (P_2^1, \partial_2 Q^0)\, u_3^0]\, \partial_3 v\}\, dx_3 + \\
&+ \varepsilon^2 \int_0^L \{\lambda (\partial_2 P_2^1, Q^1)\, \partial_3 u_3^1 v + \\
&+ \mu [(P_2^1, P_2^1)\, \partial_3 u_2^1 + (P_2^1, \partial_2 Q^1)\, u_3^1]\, \partial_3 v\}\, dx_3 = \\
&= \varepsilon^2 \int_0^L F_2^1 v\, dx_3 ,
\end{aligned}\right. \tag{3.4}$$

$$\left\{\begin{aligned}
&\int_0^L \mu [(\partial_1 Q^0, P_1^0)\, \partial_3 u_1^0 + (\partial_2 Q^0, P_2^0)\, \partial_3 u_2^0 + (\partial_\beta Q^0, \partial_\beta Q^0)\, u_3^0]\, v\, dx_3 + \\
&+ \varepsilon^2 \int_0^L \{(\lambda + 2\mu)(Q^0, Q^0)\, \partial_3 u_3^0\, \partial_3 v + \lambda [(Q^0, \partial_1 P_1^1)\, u_1^1 + \\
&+ (Q^0, \partial_2 P_2^1)\, u_2^1]\, \partial_3 v + \\
&+ \mu [(\partial_1 Q^0, P_1^1)\, \partial_3 u_1^1 + (\partial_2 Q^0, P_2^1)\, \partial_3 u_2^1 + (\partial_\beta Q^0, \partial_\beta Q^1)\, u_3^1]\, v\}\, dx_3 + \\
&+ \varepsilon^4 \int_0^L (\lambda + 2\mu)(Q^0, Q^1)\, \partial_3 u_3^1\, \partial_3 v\, dx_3 = \varepsilon^2 \int_0^L F_3^0 v\, dx_3 ,
\end{aligned}\right. \tag{3.5}$$

$$
\left\{
\begin{aligned}
&\int_0^L \mu\,[(\partial_1 Q^1, P_1^0)\,\partial_3 u_1^0 + (\partial_2 Q^1, P_2^0)\,\partial_3 u_2^0 + (\partial_\beta Q^1, \partial_\beta Q^0)\, u_3^0]\, v\, dx_3 + \\
&\quad + \varepsilon^2 \int_0^L \{(\lambda + 2\mu)(Q^1, Q^0)\,\partial_3 u_3^0\,\partial_3 v + \lambda\,[(Q^1, \partial_1 P_1^1)\, u_1^1 + \\
&\quad + (Q^1, \partial_2 P_2^1)\, u_2^1]\,\partial_3 v + \\
&\quad + \mu\,[(\partial_1 Q^1, P_1^1)\,\partial_3 u_1^1 + (\partial_2 Q^1, P_2^1)\,\partial_3 u_2^1 + (\partial_\beta Q^1, \partial_\beta Q^1)\, u_3^1]\, v\}\, dx_3 + \\
&\quad + \varepsilon^4 \int_0^L (\lambda + 2\mu)(Q^1, Q^1)\,\partial_3 u_3^1\,\partial_3 v\, dx_3 = \varepsilon^2 \int_0^L F_3^1 v\, dx_3 ,
\end{aligned}
\right.
\tag{3.6}
$$

where (with no summation on α) ; $m = 0,1$, $F_\alpha^m = \int_\omega f_\alpha P_\alpha^m \, d\omega + \int_\gamma g_\alpha P_\alpha^m \, d\gamma$; $F_3^m = \int_\omega f_3 Q^m \, d\omega + \int_\gamma g_3 Q^m \, d\gamma$, and where $(.,.)$ denotes the $L^2(\omega)$ inner product.

We shall now consider the problem of existence and uniqueness of solution of system (3.1)-(3.6). To this end, we substitute this problem by an equivalent one obtained by considering the following linear combinations of (3.1)-(3.6) :

$$
\left\{
\begin{aligned}
&(3.1)\ (P_1^1, P_1^1)/(P_1^0, P_1^1) - (3.2) && (3.7)\\
&-(3.1)\ (P_1^1, P_1^0)/(P_1^0, P_1^0) + (3.2) && (3.8)\\
&(3.3)\ (P_2^1, P_2^1)/(P_2^0, P_2^1) - (3.4) && (3.9)\\
&-(3.3)\ (P_2^1, P_2^0)/(P_2^0, P_2^0) + (3.4) && (3.10)\\
&(3.5)\ (Q^1, Q^1)/(Q^0, Q^1) - (3.6) && (3.11)\\
&-(3.5)\ (Q^0, Q^1)/(Q^0, Q^0) + (3.6) . && (3.12)
\end{aligned}
\right.
$$

Let W denote the Sobolev space

$$
W = \{ w = (w_\alpha^0, w_3^0, w_\alpha^1, w_3^1) \in [H_0^1(0, L)]^6 \} .
$$

Let $C(.,.) : W \times W \to \mathbb{R}$ denote the bilinear form associated with the variational formulation of problem (3.7)-(3.12), and let $M(.) : W \to \mathbb{R}$ denote the linear form associated with the variational formulation of the same system. We then have :

LEMMA 3.1 : *The bilinear form $C(.,.)$ is continuous on W and satisfies the following inequality of Garding type on W : there exist six positive constants (independent of ε) A_k, $1 \leqslant k \leqslant 6$ such that for any $w \in W$:*

$$\left\{\begin{aligned} C(w,w) \geq & A_1(|\partial_3 w_1^0|^2 + |\partial_3 w_2^0|^2 + |w_1^1|^2 + |w_2^1|^2 + |w_3^0|^2) + \\ & + \varepsilon^2 A_2(|\partial_3 w_1^0|^2 + |\partial_3 w_2^0|^2 + |\partial_3 w_1^1|^2 + |\partial_3 w_2^1|^2 + |\partial_3 w_3^0|^2 + |w_3^1|^2) + \\ & + \varepsilon^4 A_3 |\partial_3 w_3^1|^2 - \\ & - A_4(|w_1^0|^2 + |w_2^0|^2 + |w_3^0|^2 + |w_1^1|^2 + |w_2^1|^2 + |w_3^1|^2) - \\ & - \varepsilon^2 A_5(|w_3^0|^2 + |w_1^1|^2 + |w_2^1|^2 + |w_3^1|^2) - \\ & - \varepsilon^{-2} A_6(|w_1^0|^2 + |w_2^0|^2 + |w_1^1|^2 + |w_2^1|^2)\,. \end{aligned}\right.$$

The proof is done using Young's inequality that we recall here. Denoting by $|\,.\,|$ the $L^2(0, L)$ norm, then for all a, $b \in L^2(0, L)$ and all $\delta \in \mathbb{R}^+$, $(a, b)_{L^2(0,L)} \geq -\delta |a|^2/2 - |b|^2/2\,\delta$. ■

Since the linear form $M(\,.\,)$ is continuous on W, we then have.

THEOREM 3.1 : *For any* $\varepsilon \neq 0$, *if* 0 *is not an eigenvalue associated with the bilinear form* $C(\,.,\,.\,)$, *system* (3.7)-(3.12) *and consequently system* (3.1)-(3.6) *have a unique solution on* W.

For a proof see Nečas [1967, pp. 53]. ■

3.2. Case of the circular cross section

When the beam's cross section is circular (of radius R), the first basis function $\mathbf{P}^0$ vanishes and the previous equations (3.1) and (3.3) are identically satisfied. It is easy to show that when $\varepsilon = 0$ the system of three equations (3.2), (3.4), (3.5) gives immediately $u_\alpha^1 = \partial_3 u_3^0$ and u_3^0 as solution of the variational problem :

$$\mu\,[(3\lambda + 2\mu)/(\lambda + \mu)]\,\pi R^2 \int_0^L Q^0\, \partial_3 u_3^0\, \partial_3 v\, dx_3 = \\ = \int_0^L \left[\int_\omega f_3\, d\omega + \int_\gamma g_3\, d\gamma\right] v\, dx_3\,,$$

valid for all $v \in H_0^1(0, L)$, which is exactly the leading term obtained via the asymptotic expansion method (see Trabucho and Viaño [1978]). Equation (3.6) gives u_3^1 as a solution of the variational problem (also for all $v \in H_0^1(0, L)$:

$$(\lambda + \mu)^2\, \pi R^2 \int_0^L Q^0\, \partial_3 u_3^0\, \partial_3 v\, dx_3 = \\ = [\lambda \mu R^2 (3\lambda + 2\mu)/(\lambda + \mu)] \int_0^L F_3^0 v\, dx_3 + \int_0^L F_3^1 v\, dx_3\,,$$

4. EXTENSIONS

The technique introduced here can be used in order to compute the basis functions for **multilayered beams.** For the circular cross section case studied before it is easy to show that these basis functions are (as for the plate case (Miara [1989]) **piecewise polynomials.**

These results also apply to the anisotropic case.

For the multicellular cross section case we refer to Mascarenhas and Trabucho [1990], where different approximations are studied resulting from the noncommutativity between the Galerkin approximation and the homogenization technique. ■

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