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## PARAMETRIC REPRESENTATION OF QUADRIC SURFACES

by W. BOEHM <sup>(1)</sup> and D. HANSFORD <sup>(2)</sup>

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*Abstract.* — This paper briefly considers some cases where triangular and quadrangular quadratic Bézier patches represent a quadric, which are used in geometric modeling. A detailed discussion may be found in [3] and [8].

*Résumé.* — Représentation paramétrique de surfaces quadriques. Cet article considère brièvement quelques cas utilisés en modélisation géométrique où des carreaux de Bézier triangulaires ou quadrilatères représentent une quadrique. Les détails sont donnés en [3] et [8].

### INTRODUCTION

Quadric surfaces, such as circular cylinders, cones, and spheres, are frequently used in geometric modeling. On the other hand, in free-form design with predominantly parametric Bernstein-Bézier representations, implicitly defined quadrics are not used frequently. Therefore, as a step toward integrating quadrics surfaces into free-form design, this paper examines the parametric representation of quadrics with quadratic and biquadratic patches from a geometric point of view.

### QUADRATIC BÉZIER PATCHES AND QUADRICS

An integral triangular quadratic  $B$ -patch,  $\mathbf{b}(\mathbf{u})$  where  $\mathbf{u} = (u, v, w)$  denotes barycentric coordinates in the domain, is controlled by six  $B$ -points,  $\mathbf{b}_{0,0}$ ,  $\mathbf{b}_{0,1}$ ,  $\mathbf{b}_{0,2}$ ,  $\mathbf{b}_{1,1}$ ,  $\mathbf{b}_{2,0}$ ,  $\mathbf{b}_{1,0}$ . An integral rectangular biquadratic  $B$ -patch,  $\mathbf{b}(s)$ , where  $\mathbf{s} = (s, t)$  denotes affine coordinates in the domain, is controlled by nine  $B$ -points,  $\mathbf{b}_{0,0}$ , ...,  $\mathbf{b}_{2,2}$  [2, 4], as illustrated in figure 1.

In the case of a rational patch, each  $B$ -point  $\mathbf{b}_{i,k}$  has associated with it a weight  $\beta_{i,k}$ . The boundaries of the quadratic and biquadratic patches are conic sections, thus each is defined by three  $B$ -points. Note that the degree

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of a triangular quadratic  $B$ -patch is four or less, and the degree of a biquadratic patch is eight or less [9], therefore the  $B$ -points of quadratic and biquadratic patches on a quadric are not independent.

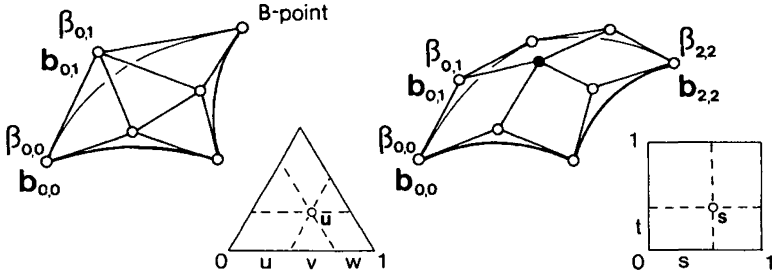


Figure 1. — Quadratic and biquadratic patches.

A quadric  $\mathcal{Q}$  can be defined as a surface which carries two families of straight lines, the so-called generatrices. If both families are real and different, the quadric is said to be doubly ruled or annular; it is a hyperboloid of one sheet or a hyperbolic paraboloid. If both families are non-real but different, the quadric is said to be non-ruled or oval; it is an ellipsoid, a hyperboloid of two sheets, or an elliptic paraboloid. In the special case, where both families coincide to one family, the quadric is singly ruled and degenerates to a quadratic cone or a cylinder.

Any plane section of a quadric is a conic section. Note that this conic section is not necessarily real, or if the plane is a tangent plane it decomposes into a pair of coinciding or different lines. Any conic section which contacts the infinite plane is a parabola. Any quadric which contacts the infinite plane is a paraboloid. Let  $\mathbf{a}$  denote such a point of contact at infinity. Viewed as a vector, it is called an axis direction of the parabola or the paraboloid under consideration [1].

A conic section is determined by five points or five tangents in the plane which may coincide in pairs, thus being a point with a tangent. Therefore six points or tangents are dependent, they are related by the configuration of Pascal and Brianchon [11].

A quadric is defined by nine points in space, where two or three may coincide to a point with tangent or to a point with tangent plane, respectively. However, three tangent planes at three points of  $\mathcal{Q}$  are related by Brianchon's configuration in the plane spanned by the three points.

Finally, any irreducible quadric as a whole can be viewed as the real projection of one of the three paraboloids shown in figure 2, below. More interesting properties of quadrics may be found in [1, 3].

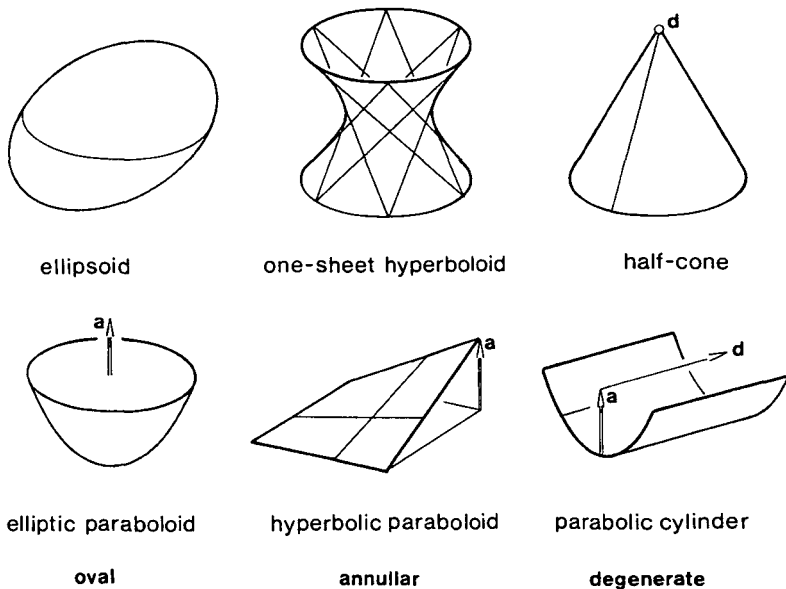


Figure 2. — Some quadrics in  $\mathbb{R}^3$ .

INTEGRAL TRIANGULAR PATCHES ON A QUADRIC

All integral quadratic curves represent parabolas. Let  $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$  denote the  $B$ -points of such a parabola, and its axis direction is  $\mathbf{a} = \mathbf{b}_2 - 2\mathbf{b}_1 + \mathbf{b}_0$  [1, 3, 7]. Now consider an integral triangular quadratic  $B$ -patch,  $\mathbf{b}(\mathbf{u})$ . Its three boundary curves are parabolas, and thus the quadric  $\mathcal{Q}$  under consideration must be a paraboloid. Assuming that  $\mathcal{Q}$  is an elliptic or a hyperbolic paraboloid, any parabola lies in a plane parallel to the axis. Consequently, the axes of the three boundary parabolas must be parallel to each other. In effect, this gives conditions on the  $B$ -points, as illustrated in figure 3.

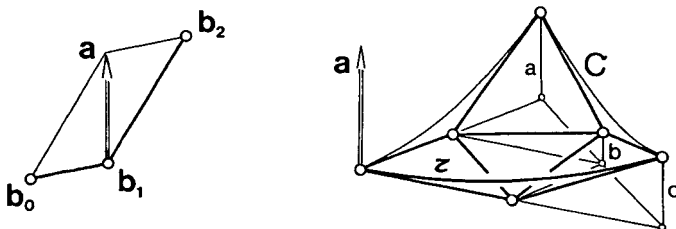


Figure 3. — The general integral quadratic patch.

There is a simple method for classifying the paraboloid. As in figure 3, let  $\tau$  denote the tangent plane at  $\mathbf{b}_{0,0}$ , and let  $C$  denote the opposite boundary parabola. If  $C$  intersects  $\tau$  in two different real points, in two non-real points, or  $\tau$  is tangent to  $C$ , the quadric is a hyperbolic paraboloid, an elliptic paraboloid, or a parabolic cylinder, respectively.

Now assuming the quadric above is degenerate, it is a parabolic cylinder. Let  $\mathbf{d}$  denote the direction from  $\mathbf{b}_{0,0}$  to the point of contact of  $C$  and  $\tau$ . Any translation of the  $B$ -points in the direction of  $\mathbf{d}$  will not effect the parabolic cylinder. Hence, the axes of all parabolas are parallel to a plane spanned by  $\mathbf{a}$  and  $\mathbf{d}$ .

**RATIONAL QUADRATIC TRIANGULAR PATCHES**

Any rational  $B$ -patch in  $\mathbb{R}^3$  can be defined as the projection of an integral  $B$ -patch in  $\mathbb{R}^4$  [1, 4]. This projection can easily be realized by the simple procedure of inhomogeneizing.

Moreover, any rational triangular quadratic patch on a non-degenerate quadric  $\mathcal{Q}$  can be viewed as the projection of an integral triangular patch on a non-degenerate paraboloid.

As a corollary, a rational triangular quadratic patch lies on a quadric if the three boundaries meet in one point,  $\mathbf{q}$ , where their three tangents,  $U, V, W$ , are coplanar, and  $\mathbf{q}$  corresponds three times to the parameter value  $\infty$ , as illustrated in figure 4 [10]. An example where these conditions are violated and a quadratic patch is not defined is discussed in [5], a rational quartic patch is needed to represent an octant of the sphere.

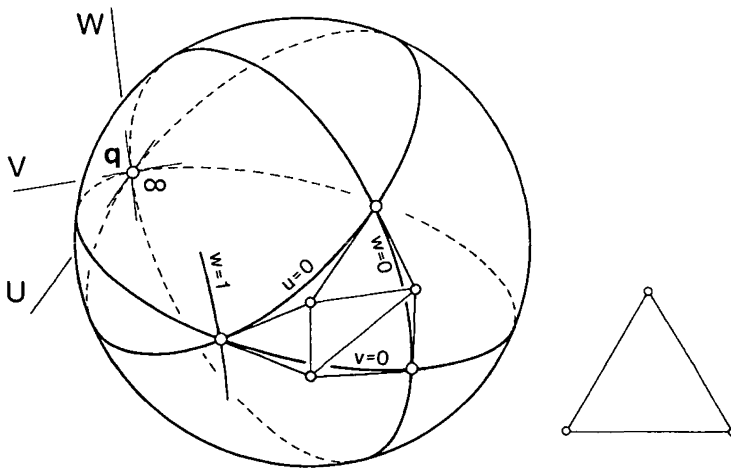


Figure 4. — The rational triangular patch on a quadric

Analogously one has that any rational triangular quadratic patch on a cone or cylinder, can be viewed as the projection of an integral patch on a parabolic cylinder, as illustrated in figure 5. Note that only in a special case do all three boundaries meet in a point  $\mathbf{q}$ . Note also that each of the three boundaries have to contact the opposite tangent plane. In effect this gives conditions on the  $B$ -points of the net and their weights.

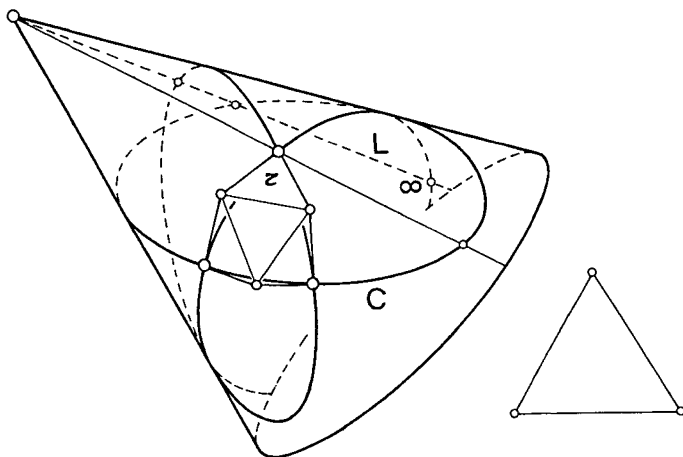


Figure 5. — The rational triangular patch on a cone.

A ONE TO ONE CORRESPONDENCE

There is a simple one to one correspondence between finite points  $\mathbf{u}$  in the domain and the finite points  $\mathbf{b}(\mathbf{u})$  of the nondegenerate paraboloids, as illustrated in figure 3. This correspondence can be generalized by perspectivity, and is also known as a stereographic projection.

Consider a triangular patch on a quadric  $\mathcal{Q}$  and let  $\mathbf{q}$  denote the point where the three boundary curves meet. Let  $\rho$  denote the tangent plane of  $\mathcal{Q}$  at  $\mathbf{q}$ , and let  $\sigma$  be a plane parallel to  $\rho$ , as in figure 6. Consider the projection which projects the points of  $\mathcal{Q}$  not lying in  $\rho$  from  $\mathbf{q}$  onto  $\sigma$ . The  $\infty^2$  lines of  $\sigma$  correspond to the  $\infty^2$  conic sections through  $\mathbf{q}$  and vice versa. Note that such a one to one correspondence is a necessary and sufficient condition for a quadratic triangular patch to lie on a non-degenerate quadric [9].

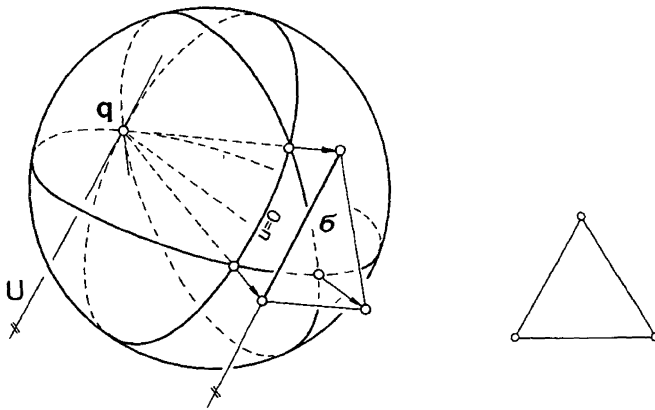


Figure 6. — The one to one correspondence.

QUADRANGULAR FROM TRIANGULAR PATCHES

Any integral quadratic triangular patch can easily be extended to a biquadratic patch by the construction of the *B*-points of the isolines  $u = 1$  and  $v = 1$  via the de Casteljau algorithm [2, 4]. Note that all isolines are parabolas too. Moreover, all isolines of the same family are parallel and congruent in space. The interior *B*-point must be adjusted so that the twist of the rectangular patch is identical to the twist of the triangular patch.

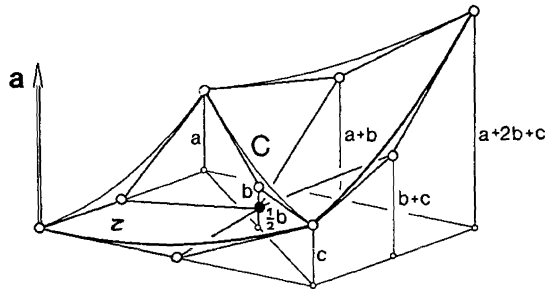


Figure 7. — Extension of an integral triangular patch.

The extension to a rational patch is shown in figure 8. The boundaries  $u = 0$  and  $u = 1$  have a common tangent *U* at *q*, and similarly the boundaries  $v = 0$  and  $v = 1$  have a common tangent *V* at *q*. Note that a final linear rational change of the parametrization will not change the surface.

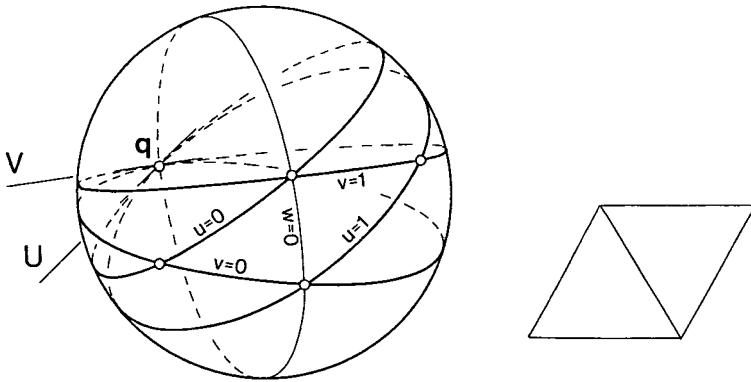


Figure 8. — Extension of rational triangular patch.

This can be generalized. Let the barycentric coordinates  $\mathbf{u}$  be expressed as the bilinear interpolant,  $\mathbf{s}$ , of four points,  $\mathbf{u}_1, \dots, \mathbf{u}_4$ , in the  $\mathbf{u}$ -plane, as in figure 8. This is done by substituting into  $\mathbf{s}$ , the barycentric definitions of  $\mathbf{u}_1, \dots, \mathbf{u}_4$  and rearranging. Now  $\mathbf{u}$  has been expressed bilinearly in terms of  $\mathbf{s}$ . Substituting this expression of  $\mathbf{u}$  into the representation of the quadratic patch  $\mathbf{b}(\mathbf{u})$  one gets a biquadratic patch in  $\mathbf{s}$ .

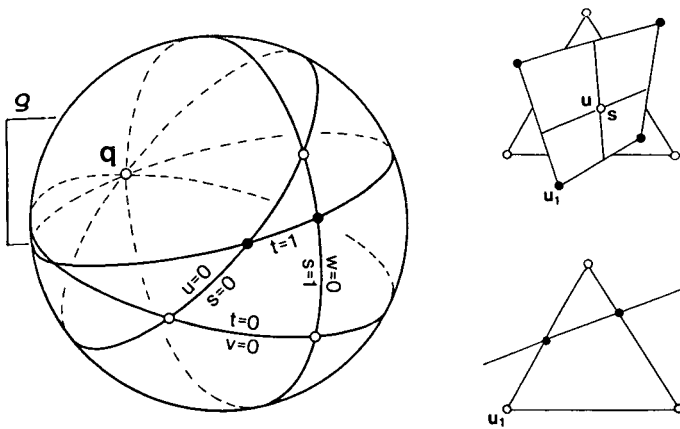


Figure 9. — Corner cutting of a triangular patch.

In a simple case, the fourth side of the rectangle can be constructed by a corner cutting construction of the triangle, as illustrated in figure 9. The fourth boundary also meets  $\mathbf{q}$ .



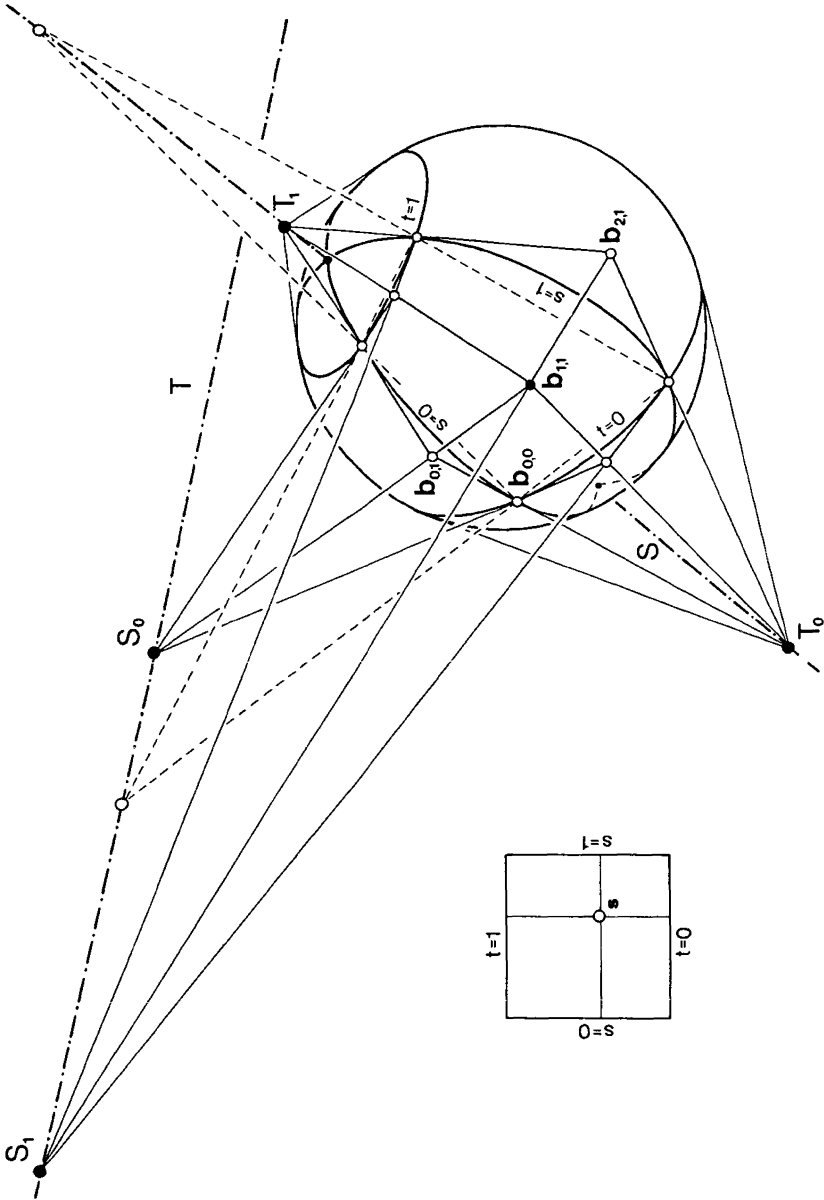


Figure 10. — The rational biquadratic patch.

### THE RATIONAL BIQUADRATIC PATCH

A simple example of a rational biquadratic patch can be constructed on the sphere, where the isoparametric lines are formed by parallels and meridians. As before, its projection gives a rational biquadratic patch, illustrated in figure 10.

Let the planes through  $s = 0$  and  $s = 1$  intersect in  $S$ , and let  $S_0$  and  $S_1$  denote the poles of these planes with respect to  $\mathcal{Q}$ , both determining an axis  $T$ . The planes through  $t = 0$  and  $t = 1$  intersect in  $T$ . The poles  $T_0$  and  $T_1$  of both planes lie on  $S$ . Moreover, the isolines of the patch lie in the two pencils of planes supported by  $S$  and  $T$ , while their poles vary on  $T$  and  $S$ , respectively. The chords of opposite boundaries meet on  $S$  and  $T$ , respectively. Consequently the four patch corners lie in a plane. Moreover, the tangents to  $s = 0$  and  $s = 1$  at  $t = 0$  meet in  $T_0$ , and similarly for the other sets of tangents. Finally, the connection of  $\mathbf{b}_{0,1}$ ,  $\mathbf{b}_{1,1}$  meets  $T$  in  $S_0$ . These relations may be used to construct the weighted  $B$ -points of the patch.

It should be mentioned that two arbitrary plane intersections of a quadric intersect each other in two (not necessarily real) points. Consequently, by the correspondence of  $\mathbf{s}$  to  $\mathbf{b}(\mathbf{s})$  the quadric  $\mathcal{Q}$  will be covered twice. It should be mentioned that special rational tensor product patches are considered in [6].

### BILINEAR PATCHES

There is an interesting special case, where the boundary parabolas of a quadrangular patch degenerate to straight lines. In the case of an integral patch the resulting surface is well-known as a bilinear interpolant [2], representing a hyperbolic paraboloid. Its projection gives a ruled quadric. Let this ruled quadric be defined by two skew pairs of its generatrices and a point  $\mathbf{p}$ . There are two further generatrices through  $\mathbf{p}$ . Where they intersect the boundary lines, may serve as points corresponding to parameter values  $1/2$  for a rational parametrization of the four boundaries, *cf.* [4], followed by bilinear rational interpolation [3].

### ACKNOWLEDGEMENTS

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