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CURVE MESH FAIRING AND GC^2 SURFACE INTERPOLATION

by H. NOWACKI [1], P. D. KAKLIS [1] and J. WEBER [1]

Abstract. — *A two-stage method for generating a fair surface from discrete, noisy data is presented. First, an approximating regular curve mesh is fitted through the given $m \times n$ point data set minimizing a fairness functional subject to maximum deviation inequality constraints. This results in a faired orthogonal mesh of curves. Second, a curvature continuous surface is interpolated through the curve mesh by means of a Boolean sum construction. The Bézier surface equivalent of this method was implemented and verified by test examples, which demonstrated the effectiveness of this surface fairing approach.*

Résumé. — Réseau de courbe d'ajustement et interpolation GC^2 de surface. *On présente une méthode à deux étapes pour générer une surface ajustée à partir de données discrètes en présence de bruit. En premier on ajuste un réseau régulier de courbes par rapport aux $m \times n$ points donnés par minimisation d'une fonctionnelle de « régularité » sous des contraintes de type distances maxima. Cela donne un réseau régulier et orthogonal de courbes. Ensuite une surface à courbure continue est interpolée par ce réseau de courbes au moyen d'une construction de sommes Booléennes. La surface de Bézier équivalente obtenue par cette méthode a été implémentée et vérifiée à l'aide d'exemples tests, qui démontrent l'efficacité de cette approche d'ajustement de surface.*

1. INTRODUCTION

This article presents a two-stage method for achieving a fair surface from noisy data : first, a mesh of curves, interconnected at their intersection points and treated as an elastic continuum, is faired on the basis of a strain energy criterion. Second, a Boolean sum surface is interpolated through the resulting mesh curves, ensuring curvature continuity at cross patch boundaries. Thus a fair, GC^2 continuous, surface is generated.

This work was motivated by the experience of the first of the authors regarding the difficulty in achieving fair surfaces by single-stage procedures. In earlier work by Nowacki and Reese (1983) it had been attempted to develop a fair surface from given data points by applying a strain energy

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minimization to the surface patches and using a Coons Cartesian product interpolation of the data. The experience was that, whenever the data points were not prefaired and exhibited tendencies that caused unfair boundary curves, or mesh lines, then the resulting surface would also suffer from fairness flaws. The application of the surface fairness criterion alone did not prevent local deficiencies. This suggested that the quality of the input curve set should be improved by a fairing process before interpolating a surface.

This approach is further in good conformance with manual design practice where surface definitions are usually produced from drawings in which suitable sets of curves are faired and adjusted until the curves are consistent with each other and promise to yield a fair surface. In the current approach, all curves in the mesh are faired simultaneously so that the problem of consistency does not arise.

The Boolean sum approach of interpolation with curvature continuity (GC^2) was newly developed by Weber (1990). It necessitates fairly high degrees in the polynomial surface representation, strictly degrees of 15 by 15 in the regular mesh case, but it ensures that fairness qualities present in the curve mesh are retained by the surface. Thus the two stages of mesh fairing and surface interpolation combined promise to yield surfaces of very high fairness quality. This hypothesis was tested and verified by the methods implemented and the examples examined in this work.

2. CURVE MESH FAIRING

2.1. History, State of the Art

In the spline literature the problem of fairing (= smoothing) 3D noisy data by using bivariate splines has been addressed by numerous authors. Not attempting a complete bibliography, one should refer to the works of Anselone and Laurent (1968), Nielson (1973, 1974), Hayes and Halliday (1974), Wahba (1979, 1983, 1984), Utreras (1979, 1987), Wahba and Wendelberger (1980), Dierckx (1981, 1982), Dyn and Wahba (1982), Cox (1984), Hu and Schumaker (1985, 1986) and the references cited therein [1]. In the majority of the afore-mentioned works, the fairing problem is formulated as a minimization problem of the form : find the unique minimizer $s \in X$ of the objective functional : $I_{\text{fair}}(s) + \lambda I_{\text{near}}(s)$, where $I_{\text{fair}}(s)$ is a functional measuring the fairness of the fairing bivariate spline s, e.g., $I_{\text{fair}} = \iint (s_{xx}^2 + s_{xy}^2 + s_{yy}^2)\, dx\, dy$ in the case of the so-called « thin-plate » or « Laplacian » smoothing splines, $I_{\text{near}}(s)$ is a functional measuring

[1] See also the well known survey papers of Schumaker (1976) and Barnhill (1977).

the nearness between the noisy and the faired data (I_{near} is usually the average sum of the square Euclidean distances between the noisy and the faired data), and X is a function space sufficiently rich to work with, e.g., X is the Hilbert space H^2. Finally, λ is a real parameter controlling the tradeoff between the fairness of the solution and the nearness to the noisy data. This parameter can be either user specified, in which case the fairing problem is a linear one, or determined in the context of a statistically oriented criterion, such as the GCV (= Generalised Cross-Validation) criterion (see, e.g., Wahba (1979)), which incorporates both the noisy and the faired data set, thus rendering the fairing problem non-linear.

The curve mesh fairing approach, apparently introduced by Hosaka (1969), is a two stage fairing process. At the first stage, the noisy data set is faired by constructing the so-called fairing curve mesh h, which, in analogy with the fairing bivariate spline, minimizes a functional of the form $I_{\text{fair}}(h) + \lambda I_{\text{near}}(h)$. At the second stage, the faired bivariate spline is obtained by interpolating the so obtained faired data. In Hosaka's work (2), which adopts a mechanical interpretation of the curve mesh fairing problem, $I_{\text{fair}}(h)$ represents the elastic strain energy contained in the mesh curves, considered as elastic beams of constant stiffness. Furthermore, $I_{\text{near}}(h)$ is the weighted sum of the square Euclidean distances between the noisy and the faired data, the weights being interpreted as the stiffness factors of elastic springs attached to the data points, and, finally, the smoothing parameter λ is but the inverse of the constant stiffness of the mesh curves. It is not known to the authors whether Hosaka published any numerical results with his method. In this connection, Kakishita (1970), Goult (1985) and Nowacki *et al.* (1989) have developed and computationally implemented linearized versions of Hosaka's work.

2.2. A curve mesh fairing criterion: formulation and well-posedness

This section deals with the presentation and theoretical investigation of a new curve mesh fairing criterion for fairing three-dimensional noisy data defined on a rectangular and noise free grid in the physical (x, y)-plane. The apparent novelty of this criterion resides in the combination of the curve mesh concept and the concept of fairing in a statistical framework introduced by Reinsch (1967, 1971). Let us begin with the formulation of the fairing criterion as a constrained minimization problem, henceforth referred to as

The curve mesh fairing problem. Given a set of data $\{(x_j, y_i, z_{ij}),\ i = 1, \dots, N, j = 1, \dots, M, x_1 < x_2 < \cdots < x_M, y_1 < y_2 < \cdots < y_N\}$, with the

(2) Hosaka's method is also described in detail in the textbooks by Ding and Davies (1987) and Su and Liu (1989).

z-components of the interior data ($i = 2, \ldots, N-1$, $j = 2, \ldots, M-1$) (3) being considered as noisy, the noise having normal density distribution, zero mean and equal variance σ^2. Find a curve mesh $h = \{(x, y_i, h_{xi}(x)),$ $h_{xi}(x) \in C^2[x_1, x_M]$, $i = 1, \ldots, N$, $(x_j, y, h_{yj}(y))$, $h_{yj}(y) \in C^2[y_1, y_N]$, $j = 1, \ldots, M\}$, which minimizes *the fairness functional*

$$I_{\text{fair}} = \sum_{i=1}^{N} \int_{x_1}^{x_M} h''_{xi}(x)\, h''_{xi}(x)\, dx + \sum_{j=1}^{M} \int_{y_1}^{y_N} h''_{yj}(y)\, h''_{yj}(y)\, dy\,, \tag{1}$$

and satisfies the following constraints :

(i) *The accuracy constraint.*

$$I_{\text{acc}}(h) = \sum_i \sum_j (h_{xi}(x_j) - z_{ij})^2 - \varepsilon \leq 0\,, \tag{2a}$$

$$\sigma^2(\kappa - \sqrt{2\,\kappa}) \leq \varepsilon \leq \sigma^2(\kappa + \sqrt{2\,\kappa})\,, \quad \kappa = (N-2)(M-2) \geq 2\,. \tag{2b}$$

(ii) *The interior compatibility constraints :*

$$h_{xi}(x_j) = h_{yj}(y_i)\,. \tag{3}$$

(iii) *The type-I (or -II′) boundary constraints :*

$$h_{xi}(x_j) = z_{ij}\,, \quad h'_{xi}(x_j) = d_{xij} (\text{or } h''_{xi}(x_j) = 0)\,, \quad i = 1, \ldots, N\,, \quad j = 1, M\,, \tag{4a}$$

$$h_{yj}(y_i) = z_{ij}\,, \quad h'_{yj}(y_i) = d_{yij} (\text{or } h''_{yj}(y_i) = 0)\,, \quad i = 1, N\,, \quad j = 1, \ldots, M\,, \tag{4b}$$

d_{xij}, $i = 1, \ldots, N$, $j = 1, M$, and d_{yij}, $i = 1, N$, $j = 1, \ldots, M$, being given finite real numbers.

In order to examine the well-posedness of the above problem we appeal to the following result due to Wong (1984).

LEMMA 2.2.1 : *Let V a reflexive Banach space, K a closed convex set in V with non-empty interior, S a strictly normed Banach space, $T : V \to S$ a bounded linear operator such that $T(V)$ is closed in S, and $A : V \to \mathbf{R}^p$ a continuous linear map from V onto $\mathbf{R}^p$, such that $Null(A) \cap Null(T) = \{0\}$. Then the constrained minimization problem :*

$$\textit{minimize } \|Th\|_S^2 \textit{ subject to} : \; h \in X = K \cap \{h \in V : Au = r\}\,, \tag{5}$$

admits of a unique solution if $X \neq 0$.

(3) Within sections 2 2 and 2 3 the following notational convention is adopted if the range of i and/or j is not explicitly given then $i = 2, \ldots, N-1$ and/or $j = 2, \ldots, M-1$

In order to apply the above lemma, we first define the working space $V = V_x \times V_y$, where $V_\bullet = H^2(I_\bullet) \times \cdots \times$ (N or M times) $\times \cdots \times H^2(I_\bullet)$, $I_\bullet = (\bullet_1, \bullet_L)$ with $\bullet = x$ or y, and $L = M$ or N, respectively. Space V is endowed with a norm as follows: let $h = (h_x, h_y)$ with $h_\bullet = (h_{\bullet_1}, ..., h_{\bullet_L}) \in V$, then $\|h\|_V = \|h_x\|_{V_x} + \|h_y\|_{V_y}$ with $\|h_\bullet\|_{V_\bullet} = \sum_{\ell=1}^{L} \|h_{\bullet_\ell}\|_{H^2(I_\bullet)}$ where $\|h\|_{H^2}^2 = \sum_{k=0}^{2} \|h^{(k)}\|_{L^2}$ The space V is a reflexive Banach space for it is defined as the cartesian product of a finite number of reflexive Banach spaces (see, e.g., Kufner *et al.* (1977, § 0.16 and § 5.4)).

Second, we introduce the set

$$K = \left\{ h \in V : h_{xi}(x_j) = h_{yj}(y_i)\,, \quad \sum_i \sum_j (h_{xi}(x_j) - z_{ij})^2 - \varepsilon \leq 0 \right\}. \quad (6)$$

The set K is non-empty for any curve-mesh consisting of cubic splines which interpolate the set of the interior noisy data belongs to K. Furthermore, by virtue of the Cauchy-Schwarz inequality and the fact that V is the cartesian product of a finite number of proper subspaces of $C^0(I_\bullet)(H^2(I_\bullet) \subset C^1(I_\bullet))$, one can easily prove that K is also convex and closed.

Third, we define the space $S = S_x \times S_y$, with $S_\bullet = L^2(I_\bullet) \times \cdots \times$ (N or M times) $\times \cdots \times L^2(I_\bullet)$. Space S is endowed with a norm by simply replacing H^2 by L^2 in the definition of the norm in the working space V. Furthermore, S is strictly normed (or strictly convex or rotund), which implies that L^2 is a strictly normed space (see, e.g., Singer (1970, p. 111).

Fourth, we introduce the operator $T: V \to S$ as follows: $Th = h''$ with $h'' = (h_x'', h_y'')$, $h_\bullet'' = (h_{\bullet_1}'', ..., h_{\bullet_L}'')$. Recalling the definition of $\|\bullet\|_V$ and $\|\bullet\|_S$, it is readily inferred that T is a bounded linear operator from V into S. As regards the question on the closedness of $T(V)$ in S, the answer is in the affirmative since the equation $d^2h/dx^2 = g$ is solvable in $H^2(0, 1)$ for any $g \in L^2(0, 1)$, which in its turn implies that d^2/dx^2 is surjective on $L^2(0, 1)$.

Fifth, we introduce the linear operator $A: V \to \mathbf{R}^q$, $q = 4(N + M)$, defined as follows:

$$Ah = \{h_{xi}(x), h_{xi}'(x)\,(\text{or } 0)\,, \quad x = x_1, x_M\,, \quad i = 1, ..., N\,,$$

$$h_{yj}(y), h_{yj}'(y)\,(\text{or } 0)\,, \quad y = y_1, y_N, \quad j = 1, ..., M\ \}\,,$$

according as type-I (or -II') boundary constraints are imposed. Exploiting the imbedding relation $H^2(I_\bullet) \subset C^1(I_\bullet)$, we can easily prove that A is a bounded linear map from V into $\mathbf{R}^q$. Furthermore, the problem: given an $r \in \mathbf{R}^q$ find an $h \in V : Ah = r$, is solvable by the curve mesh of cubic polynomials which satisfy the type-I (or -II') boundary conditions specified by the vector r. Thus A is also onto $\mathbf{R}^q$.

In conclusion, the previously defined spaces V, S, the set K and the operators T and A fulfill the requirements cited in Lemma 2.2.1, and, on the other hand, the minimization problem (5) resulting from them is but the curve mesh fairing problem in the space V. Accordingly, we can state.

THEOREM 2.2.1 : *The curve mesh fairing problem possesses a unique solution in the space V.*

2.3. Construction of the Solution

In this section a process is developed for constructing the unique solution of the curve mesh fairing problem (see Eqs. (1)-(4)). To start with, we note that I_{fair} and I_{acc} are quadratic functionals, whereas the compatibility and the boundary constraints are linear ones. The curve mesh fairing problem may then be considered as a convex-programming problem (see, e.g., Ioffe and Tihomirov (1979, chap. 1.1.2)). In other to treat this problem by means of the Kuhn-Tucker theorem, we first note that $I_{\text{fair}}(h)$ and $I_{\text{acc}}(h)$ are Fréchet differentiable everywhere in V, which can be easily proved by exploiting the definition of $\|\bullet\|_V$, the imbedding relation $H^2 \hookrightarrow C^1$, and the Cauchy-Schwarz inequality. Furthermore, the Fréchet derivatives of $I_{\text{fair}}(h)$ and $I_{\text{acc}}(h)$ are given by the formulae

$$\Lambda_{\text{fair}}(h)\ \delta h = 2 \sum_{i=1}^{N} \int_{x_1}^{x_M} \delta h''_{xi}(x)\ h''_{xi}(x)\ dx + \\ + 2 \sum_{j=1}^{M} \int_{y_1}^{y_N} \delta h''_{yj}(y)\ h''_{yj}(y)\ dy\,, \quad (10a)$$

and

$$\Lambda_{\text{acc}}(h)\ \delta h = 2 \sum_{i} \sum_{j} \left(h_{xi}(x_j) - z_{ij}\right) \delta h_{xi}(x_j)\,, \quad (10b)$$

respectively. Finally, the so-called Slater condition is satisfied by the cubic curve-mesh which interpolates the noisy data and satisfies the boundary conditions (4). Then, recalling a stronger version of the Kuhn-Tucker theorem we are led to.

THEOREM 2.3.1 : *If $h \in X = \{h \in V :$ constraints (3), (4) are fulfilled$\}$ is a solution of the curve mesh fairing problem, then it is necessary and sufficient that there exists a Lagrange multiplier $\lambda \geqslant 0$ such that, for any $\delta h \in V$ with $h + \delta h \in X$, the following inequalities hold*

$$\Lambda_{\text{fair}}(h)\ \delta h + \lambda\, \Lambda_{\text{acc}}(h)\ \delta h = 0\,, \quad (11a)$$

$$I_{\text{acc}}(h) \leqslant 0\,, \qquad \lambda I_{\text{acc}}(h) = 0\,. \quad (11b)$$

Let us now restrict ourselves to a C^2 spline subspace V^* of V. Performing then twice integration by parts in (11a) and taking into account the Lagrange lemma, as well as that that δh_* satisfies the constraints (3), (4), and that $\delta h_{*xi}(x_j)$ are in general linearly independent, we finally get that (11a) is equivalent to the following set of equalities

$$(h'''_{*xi}(x_j+) - h'''_{*xi}(x_j-)) + (h'''_{*yj}(y_i+) - h'''_{yj}(*y_i-)) + \\ + \lambda\,(h_{*xi}(x_j) - z_{ij}) = 0\,. \quad (12)$$

Summarizing the hitherto obtained results, we state

COROLLARY 2.3.1 : *If* $h_* \in X^* = \{h \in V^* :$ *constraints* (3), (4) *are fulfilled*$\}$ is a solution of the curve mesh fairing problem, then it is necessary and sufficient that h_{*xi}, $i = 1, ..., N$ and h_{*yj}, $j = 1, ..., M$, are C^2 cubic splines and a Lagrange multiplier $\lambda \geqslant 0$ exists so that (11b) and (12) are fulfilled.

The boundary cubic splines h_{*x1}, h_{*xN} and h_{*y1}, h_{*yM} are uniquely determined by the C^2 continuity requirements and the boundary conditions (4). From now on we shall confine our attention to determining the interior cubic splines. For this purpose, we first adopt the following cubic spline representation :

$$g(x) = g_j(1-q_x) + g_{j+1}\,q_x + \Delta_{xj}^2 g_j''\,F(1-q_x) + \Delta_{xj}^2 g_{j+1}''\,F(q_x)\,, \quad (13)$$

where $x \in (x_j, x_{j+1})$, $\Delta x_j = x_{j+1} - x_j$, $q_x = (x_{j+1} - x_j)/\Delta x_j$ and $F(q) = (q^3 - q)/6$. It can easily be seen from (13) that $h'_{*xi}(x_j-) = h'_{*xi}(x_j+)$ are sufficient conditions ensuring that $h_{xi} \in C^2[x_1, x_M]$. Combining these conditions with equations (12) and taking into account the boundary constraints (4), we get after some straightforward but elaborate algebra the following matrix equation :

$$M_{x2}\,H_x''^T + M_{x0}\,Z_f^T = B_x^T\,, \quad (14)$$

where Z_f is an $(N-2) \times (M-2)$ matrix containing the faired z-components of the interior noisy data, H_x'' is an $(N-2) \times (M-2)$ matrix containing the second order x-derivatives of the faired curve mesh h_* at the interior grid points and M_{x1}, M_{x2} are two $(M-2) \times (M-2)$ symmetric and tridiagonal matrices, the non-zero elements of which are given by the following formulae :

$$(M_{x0})_{j,j-1} = -1/\Delta_{x,j-1}\,, \quad (M_{x0})_{jj} = 1/\Delta_{x,j-1} + 1/\Delta_{xj}\,, \\ (M_{x0})_{j,j+1} = -1/\Delta_{xj}\,, \quad (15a)$$

$$(M_{x0})_{22} = 3/2\,\Delta_{x1} + 1/\Delta_{x2}\,, \quad (M_{x0})_{M-1,M-1} = 3/2\,\Delta_{x,M-1} + 1/\Delta_{x,M-2}\,, \quad (15b)$$

$$(M_{x2})_{j,j-1} = \Delta_{x,j-1}/6\,, \quad (M_{x2})_{jj} = \Delta_{x,j-1}/3 + \Delta_{xj}/3\,, \quad (M_{x2})_{j,j+1} = \Delta_{xj}/6\,, \tag{16a}$$

$$(M_{x2})_{22} = \Delta_{x1}/4 + \Delta_{x2}/3\,, \ (M_{x2})_{M-1,M-1} = \Delta_{x,M-1}/4 + \Delta_{x,M-2}/3\,, \tag{16b}$$

with $j = 3, ..., M - 2$. Finally, B_x is an $(N - 2) \times (M - 2)$ sparse matrix expressing the influence of the boundary conditions (4). Choosing the cubic spline representation (13) for $h_{*yj}(y)$, $j = 1, ..., M$, and working similarly, we arrive at the following matrix equation

$$M_{y2}\, H_y''^T + M_{y0}\, Z_f^T = B_y^T\,, \tag{17}$$

the matrices M_{y0}, M_{y2} and B_y being defined with direct analogy to M_{x0}, M_{x2} and B_x, respectively. Rewriting now equations (12) in the context of the cubic spline representation (13) and combining the resulting set of equations with the matrix equations (14) and (17), we get the following matrix equation for Z_f:

$$Z_f\, M_R + M_L\, Z_f + \lambda Z_f = \lambda Z + B\,, \tag{18}$$

where Z is an $(N - 2) \times (M - 2)$ matrix containing the z-components of the interior noisy data and

$$M_R = M_{x0}\, M_{x2}^{-1}\, M_{x0} + B_{Gx}\,, \quad M_L = M_{y0}\, M_{y2}^{-1}\, M_{y0} + B_{Gy}\,. \tag{19}$$

Finally, B, B_{Gx}, B_{Gy} are sparse matrices expressing the influence of the boundary conditions (4). More specifically, in the case of type-I boundary conditions B_{Gx} is an $(M - 2) \times (M - 2)$ diagonal matrix whose non-zero elements are given by

$$(B_{Gx})_{22} = 3/\Delta_{x1}^3\,, \quad (B_{Gx})_{M-1,M-1} = 3/\Delta_{x,M-1}^3\,, \tag{20}$$

whereas in the case of type-II′ boundary conditions B_{Gx} degenerates to the null matrix. Obviously, a directly analogous statement can be made for B_{Gy}.

Let us now investigate the well-posedness of the matrix equation (18). At first, relations (15) and (16) readily imply that M_{x2} is positive definite and M_{x0} is non-negative definite. To examine whether zero is an eigenvalue of M_{x0} we appeal to a well known result of Taussky (1948) stating that, if a matrix is irreducible and an eigenvalue lies on the boundary of one of the associated Gerschgorin circles, then it should lie on the common boundary of all Gerschgorin circles. By virtue of relations (16) it can easily be seen that $0 + \iota 0$ is not a common boundary point of all the Gerschgorin circles of M_{x0}. On the other hand, M_{x0} is irreducible for it can be obtained by replacing zero entries of an irreducible matrix, namely the matrix

$$(M_{x2})_{j,j-1} = \Delta_{x,j-1}/6\,, \quad (M_{x2})_{jj} = 0\,, \quad (M_{x2})_{j,j+1} = \Delta_{xj}/6\,, \tag{21}$$

by non-zero ones (see, e.g., Lankaster and Tismenetsky (1985, p. 530)). Thus M_{x0} is eventually positive definite. The same statement can be clearly made for M_{y2} and M_{y0}. Finally, by virtue of the remarks made above, B_{Gx} and B_{Gy} are non-negative definite. Recalling now that, given two matrices, A, B with A positive definite and B non-singular, then BAB^T is also positive definite (see, e.g., Voyevodin (1983, chap. 9, § 77)), that the matrices M_R and M_L are positive definite. Then, standard matrix-equation theory (see, e.g., Gantmacher (1977, vol. 1, chap. VIII, § 3)) yields.

LEMMA 2.3.1 : *For every* $\lambda \geqslant 0$ *matrix equation* (18) *possesses a unique solution.*

Next, we investigate the existence of λ for which conditions (11*b*) are fulfilled. For this purpose, it will suffice to study the properties of $I_{\text{acc}}(Z_f(\lambda))$ as a function of λ. To start with, we set I_{acc} in the form

$$I_{\text{acc}}(Z_f(\lambda)) = \sum_{j=1}^{M-2} (Z_d(\lambda)\, e_j, Z_d(\lambda)\, e_j)\,, \quad Z_d(\lambda) = Z_f(\lambda) - Z\,, \tag{22}$$

where $(.,.)$ denotes the inner product in R^{M-2} and e_j, $j = 1, \ldots, M-2$, is an orthonormal basis in R^{M-2} given by $e_1 = \{1, 0, \ldots, 0\}^T$, $e_2 = \{0, 1, \ldots, 0\}^T$ and so on. Differentiating (22) with respect to λ, we find after some straightforward matrix algebra

$$\begin{aligned} dI_{\text{acc}}(Z_f(\lambda))/d\lambda &= -2 \sum_{j=1}^{M-2} (\lambda\,(\zeta_d e_j, \zeta_d e_j) + \\ &\quad + (\zeta_d e_j, M_L \zeta_d e_j) + e_j^T M_R(\zeta_d^T \zeta_d)\, e_j)\,, \end{aligned} \tag{23}$$

where $\zeta_d = (\mathcal{M} + \lambda I_\kappa)^{-1} Z_d$, with $\mathcal{M}$ being a linear operator acting on the space of $(N-2) \times (M-2)$ rectangular matrices according to the formula $\mathcal{M}\bullet = \bullet M_R + M_L\bullet$, and I the identity matrix of dimension $\kappa \times \kappa$. Recalling that M_L and M_R are positive definite matrices, we readily conclude that the right-hand side of equation (23) is negative for $\lambda \geqslant 0$. We thus have that I_{acc} is a strictly decreasing function of λ on $[0, \infty]$, which in turn leads to

LEMMA 2.3.2 : *There exists a unique* λ *in* $[0, \infty]$ *satisfying inequalities* (11*b*). *If* $I_{\text{acc}}(Z_f(0)) \leqslant 0$ *then* $\lambda = 0$. *Otherwise,* λ *is found by applying a standard Newton-Raphson method to the system of equations* $I_{\text{acc}}(Z_f(\lambda)) = 0$ *and* (18).

Lemmas 2.3.1 and 2.3.2 yield.

THEOREM 2.3.2 : *The curve mesh fairing problem admits of a unique cubic spline solution h_* in the C^2 spline subspace V^* of V.*

The question naturally arises whether h_* coincides with the solution h of the curve mesh fairing problem in the space V, whose existence and uniqueness has been established in Section 2.2. To investigate this question we appeal to the following identity

$$I_{\text{fair}}(h_* - g) + \lambda I_{\text{acc}}(h_* - g) + I_{\text{fair}}(h_*) + \lambda I_{\text{acc}}(h_*) = \\ = I_{\text{fair}}(g) + \lambda I_{\text{acc}}(g) , \quad (24)$$

which is valid for any curve-mesh $g \in X$. The above identity is but the first integral relation for the fairing cubic-spline curve-meshes (see Kaklis (1989, App. 2)). From (24) it is readily inferred that h_* minimizes $I_{\text{fair}}(g) + \lambda I_{\text{inf}}(g)$ in V. On the other hand, h_* satisfies by construction the inequality constraints (11*b*). Theorem 2.2.1 then implies that $h_* = h$.

2.4. Computational Experiments

The fairing method developed in Sections 2.2 and 2.3 has been submitted to extensive Monte Carlo experimentation. These experiments have been carried out along the following lines. First, an orthogonal curve mesh h_{id}, henceforth referred to as the ideal curve mesh, is drawn from an ideal (= noise free) and smooth surface s_{id} defined on a rectangular domain in the (x, y)-plane. Second, using a Gaussian pseudo-random deviate generator we insert noise of specified standard deviation σ to the z-components of the interior data. The so obtained noisy data are then interpolated by the so-called noisy curve mesh h_n, which consists of C^2 cubic splines satisfying the type-I (or-II') boundary constraints induced by the ideal surface. Finally, the noisy data are faired by constructing the faired curve mesh h, as described in the preceding Section 2.3.

Figure 1 contains the graphical output of a Monte Carlo experiment with the ship-like C^1 surface $z = s_{\text{id, ship}}(x, y) = f(x)\, g(y)$, $0 \leq x \leq 12$, $0 \leq y \leq 2$, with $f(x) = 5$ for $0 \leq x_1 = x/12 \leq 0.4$, $f(x) \approx 2.5 - 23.15(x_1 - 1.3)(x_1 - 0.4)^2$ for $0.4 \leq x_1 \leq 1$ and $g(y) = 1 - (1 - y_1)^{10}$ for $0 \leq y_1 = y/2 \leq 1$ (see Kuo (1971, chap. 3)). The chosen ideal curve mesh consists of 40 curves (20 uniformly spaced curves along the x- and the y-direction). Furthermore, $\sigma = 0.01$, $\varepsilon = \sigma^2(\kappa - \sqrt{2\,\kappa})$ with $\kappa = 18 \times 18$ and boundary constraints of type I have been imposed, the required first-order boundary derivatives being calculated using $z = s_{\text{id, ship}}(x, y)$. Finally, $\lambda = 1\,706$.

The fairing criterion formulated in Section 2.2 can be easily extended to less structured (= scattered) data sets whose spatial distribution is, however,

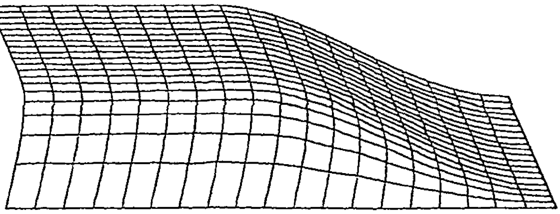

Figure 1a. — The noisy curve mesh h_n.

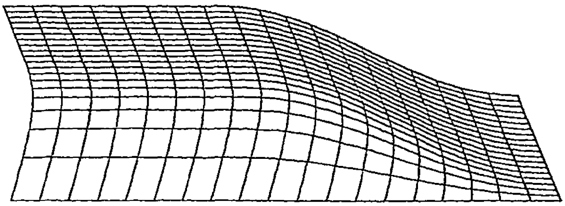

Figure 1b. — The faired curve mesh h.

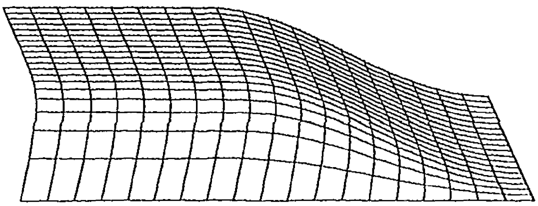

Figure 1c. — The ideal (= noise free) curve mesh h_{id}.

Figure 1. — Monte Carlo experimentation with a ship-like surface.

adequately regular so that one can adopt the hypothesis that the parameter knots of all curves of the curve mesh lie on a rectangular grid in the (u, v)-plane. Under these hypotheses, a curve mesh fairing criterion can be formulated as follows :

The curve mesh fairing problem (uniform parameter case). Given a set of data $\{(x_{ij}, y_{ij}, z_{ij}),\ i = 1, ..., N, j = 1, ..., M\}$, with the interior data being noisy, the noise having normal density distribution, zero mean and equal variances σ_x^2, σ_y^2, σ_z^2 along the x-, y- and z-direction, respectively. Find a curve mesh $h = \{\vec{h}_{ui} = (h_{xui},\ h_{yui},\ h_{zui})^T \in C^2[u_1, u_M],\ i = 1,\ ...,\ N,$ $\vec{h}_{vj} = (h_{xvj}, h_{yvj}, h_{zvj})^T \in C^2[v_1, v_N], j = 1, ..., M\}$, with the parameter knots $\{u_1 < u_2 < ...\ u_M\}$, $\{v_1 < v_2 < ...\ v_N\}$ being given, which minimizes the fairness functional $I_{\text{fair}}(h_x) + I_{\text{fair}}(h_y) + I_{\text{fair}}(h_z)$, with $h_\bullet = \{(u, v_i, h_{\bullet ui}(u)), i = 1, ..., N, (u_j, v, h_{\bullet vj}(v)), j = 1, ..., M, \bullet = x, y, z\}$ and, furthermore, satisfies the accuracy constraints $I_{\text{acc}}(h_\bullet, \varepsilon_\bullet) \leq 0$, with $\sigma_\bullet^2(\kappa - \sqrt{2\kappa}) \leq \varepsilon_\bullet \leq \sigma_\bullet^2(\kappa + \sqrt{2\kappa})$, the interior compatibility constraints $\vec{h}_{ui}(u_j) = \vec{h}_{vj}(v_i)$ and type-I (or-II') boundary constraints.

The numerical performance of the above criterion is illustrated in figure 2 by giving the results of a Monte Carlo experiment with the glass-like surface $r = s_{\text{id, glass}}(z, \theta) = f(z)$, $0 \leq z \leq 13$, $0 \leq \theta \leq \pi$, with $f(z)$ being a C^1 cubic spline interpolating the data set $\{(0, 7), (3, 1), (11, 1), (13, 5)\}$ and satisfying the constraints $f(z) = 1$, for $3 \leq z \leq 11$ and $f'(0) = 0$, $f'(13) = 0.35$. The chosen ideal curve mesh consists of 30 curves (10 « meridians » and 20 « parallels », uniformly spaced). The rectangular grid in the parameter (u, v)-plane, which is a rather poor approximation in the case of « parallels », is constructed by using the arithmetic means of chordlengths between adjacent noisy data. Furthermore, $\sigma_x/13 = 0.010$, $\sigma_y/7 = \sigma_z/7 = 0.005$, and $\varepsilon_\bullet = \sigma_\bullet^2(\kappa - \sqrt{2\kappa})$ with $\kappa = 18 \times 8$. Finally, type-I boundary-constraints have been imposed, the required first-order boundary parametric derivatives being approximated using the noisy data lying in the vicinity of the boundary, and $\lambda_x = 9.62$, $\lambda_y = 3.69$, $\lambda_z = 1.27$.

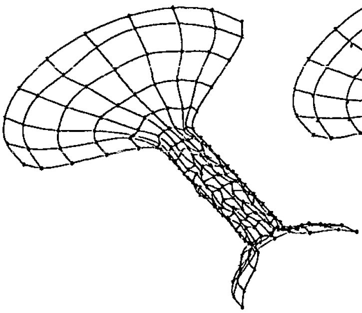

Figure 2a. — The noisy curve mesh h_n.

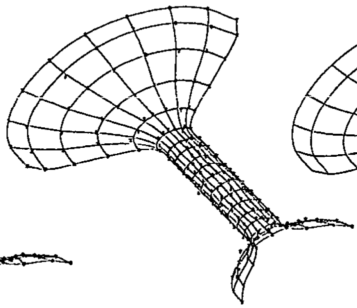

Figure 2b. — The faired curve mesh h.

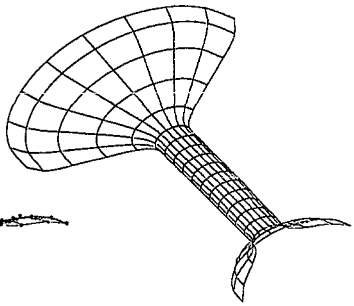

Figure 2c. — The ideal (= noise free) curve mesh h_{id}.

Figure 2. — Monte Carlo experimentation with a glass-like surface.

3. GC^2 SURFACE INTERPOLATION

3.1. State of the Art

A great variety of surface interpolations schemes has been developed in recent years for purposes of surface definition in CAD systems based on given data points (discrete case) or curves (transfinite case). These methods can be classified by the data set topologies (regular mesh/irregular mesh) and the level of continuity achieved (parametric/geometric continuity, first/second order). Hoschek and Lasser (1989) and Farin (1990) give good overviews of the underlying concepts and available options.

In the majority of cases the GC^1, regular mesh case with tensor product interpolants has been of interest. Many original thoughts go back to Coons (1961). More recently GC^1 regular mesh methods have been presented by Chiyokura and Kimura (1984) and Liu and Hoschek (1989). The latter authors have formulated necessary and sufficient conditions for GC^1 continuity between Bézier patches. Nowacki, Liu and Lü (1988) have applied this method to the construction of a global patchwork of Bézier elements of a ship surface.

The case of GC^1 continuity for irregular mesh topologies has been well covered by Sarraga (1987, 1990) and Reuding (1989). These solutions are based on triangulations of the original data sets.

For the GC^2 regular mesh case discrete interpolation schemes were proposed by Kahmann (1983) and Johansson (1989). Jones (1988) and Hahn (1989) provided interpolation procedures of GC^2 continuity level for filling n-sided holes, which includes the case of irregular mesh topologies.

The method presented here, based on Weber's work (1990), deals with the transfinite (Boolean sum) interpolation of a curve mesh. The method was developed for regular and irregular meshes, although only the regular case is implemented and presented here.

3.2. Problem Formulation

The approach makes use of the assumption that each surface element, e.g., Coons patch, can be parametrized independently. Thus at boundary lines between continuous patches the original parametrizations do not normally match so that appropriate transformations must be taken into account when stating the continuity conditions. The advantage lies in the additional freedom of choosing the transformation function rather arbitrarily which can be exploited to develop a local method for GC^2 surface construction.

The transition from one parametrization to another is made by a transformation function and the chain rule. Let $T(t)$ be such a transformation function for a curve $\underline{f}(t)$ and $\underline{F} = \underline{f}(T(t))$. Then, according to the chain rule

$$\underline{F}_t = \underline{f}_T T_t, \quad \underline{F}_{tt} = \underline{f}_{TT} T_t^2 + \underline{f}_T T_{tt}. \tag{25}$$

For a surface $\underline{f}(R, S)$ which possesses an alternative parametrization, say

$(R(r,s), S(r,s))$, the chain rule yields the following formulae for the partial derivatives of the reparametrized surface $\underline{F}(r,s) = \underline{f}(R(r,s), S(r,s))$:

$$\underline{F}_r = \underline{f}_R R_r + \underline{f}_S S_r , \tag{26}$$

$$\underline{F}_{rr} = \underline{f}_{RR} R_r^2 + \underline{f}_{SS} S_r^2 + 2\,\underline{f}_{RS} R_r S_r + \underline{f}_R R_{rr} + \underline{f}_S S_{rr} , \tag{27}$$

and analogously for $\underline{F}_s$, $\underline{F}_{ss}$ and $\underline{F}_{sr}$.

Let us now consider a quadrilateral surface element (*fig.* 3) represented as a Coons Boolean sum patch with the quintic Hermite interpolants $H_0(t), \ldots, H_5(t)$ with $t = r$ or s. In Coons-like notation this element is described by

$$\begin{aligned} \underline{\underline{Q}}(r,s) = {} & \underline{R0}\, H_0(s) + \underline{R1}\, H_5(s) + \underline{R0}_s\, H_1(s) + \underline{R1}_s\, H_4(s) + \\ & + \underline{R0}_{ss}\, H_2(s) + \underline{R1}_{ss}\, H_3(s) + \underline{0S}_s\, H_0(r) + \underline{1S}_s\, H_5(r) \\ & + \underline{0S}_r\, H_1(r) + \underline{1S}_r\, H_4(r) + \underline{0S}_{rr}\, H_2(r) + \underline{1S}_{rr}\, H_3(r) \\ & - \sum_{i=0}^{5} \sum_{j=0}^{5} \underline{B}_{ij}\, H_i(r)\, H_j(s) , \end{aligned} \tag{28}$$

with

$$\underline{B}_{ij} = \begin{pmatrix} \underline{00} & \underline{00}_r & \underline{00}_{rr} & \underline{01}_{rr} & \underline{01}_r & \underline{01} \\ \underline{00}_s & \underline{00}_{rs} & \underline{00}_{rrs} & \underline{01}_{rrs} & \underline{01}_{rs} & \underline{01}_s \\ \underline{00}_{ss} & \underline{00}_{rss} & \underline{00}_{rrss} & \underline{01}_{rrss} & \underline{01}_{rss} & \underline{01}_{ss} \\ \underline{10}_{ss} & \underline{10}_{rss} & \underline{10}_{rrss} & & & \\ \underline{10}_{ss} & \underline{10}_{rss} & \underline{10}_{rrss} & \underline{11}_{rrss} & \underline{11}_{rss} & \underline{11}_{ss} \\ \underline{10}_s & \underline{10}_{rs} & \underline{10}_{rrs} & \underline{11}_{rrs} & \underline{11}_{rs} & \underline{11}_s \\ \underline{10} & \underline{10}_r & \underline{10}_{rr} & \underline{11}_{rr} & \underline{11}_r & \underline{11} \end{pmatrix} \tag{29}$$

The symbols $\underline{R0}$, $\underline{R0}_s$, $\underline{R0}_{ss}$ etc. denote boundary curves (*fig.* 3) and cross partial derivatives of the surface at the boundaries. The matrix B_{ij} contains information about offsets and partial derivatives at the four patch corners, including mixed partial derivatives.

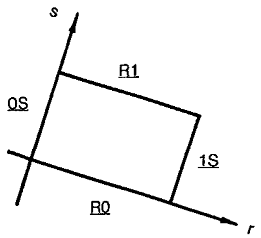

Figure 3. — Coons Boolean-sum patch, notation for patch boundaries.

We now introduce a quintic Hermite interpolant as a univariate parameter transformation function of the type

$$T(a, b, t) = aH_0(t) + bH_5(t)\,, \tag{30}$$

where t may be r or s in the surface. $T(a, b, t)$ interpolates the end values a and b and has zero first and second order end derivatives. The fact that T is itself constructed to be of continuity C^2 for a sequence of consecutive curve segments ensures that any given GC^2 continuity of a curve or its derivatives before transformation is retained under the transformation (Weber (1990)). Replacing the partial derivatives R_r, R_{rr}, S_r, S_{rr} in (26) and (27) by type-T interpolants and using the Coons patch notation, we obtain the following expressions for the cross partial derivatives at the patch boundaries after reparametrization:

$$\underline{iS}_r = \overline{\underline{iS}}_r\, T(R_{r_{i0}}, R_{r_{i1}}, s) + \overline{\underline{iS}}_s\, T(S_{r_{i0}}, S_{r_{i1}}, s)\,, \tag{31}$$

$$\begin{aligned}\underline{iS}_{rr} &= \overline{\underline{iS}}_{rr}\, T^2(R_{r_{i0}}, R_{r_{i1}}, s) + \overline{\underline{iS}}_{ss}\, T^2(S_{r_{i0}}, S_{r_{i1}}, s) + \\ &\quad + \overline{\underline{iS}}_{rs}\, 2\, T(S_{r_{i0}}, S_{r_{i1}}, s)\, T(S_{r_{i0}}, S_{r_{i1}}, s) \\ &\quad + \overline{\underline{iS}}_r\, T(R_{rr_{i0}}, R_{rr_{i1}}, s) + \overline{\underline{iS}}_s\, T(S_{rr_{i0}}, S_{rr_{i1}}, s)\,,\end{aligned} \tag{32}$$

and similarly for $\underline{Rj}_s$, $\underline{Rj}_{ss}$, where $i, j = 0$ or 1. The right-hand side partial derivatives with overbar and underbar are interpreted as partial derivatives of a neighbouring patch in its own parametrization, whereas the left-hand side singly barred quantities belong to the patch under investigation. In the surface construction these partial derivatives must be matched. Note that some right-hand side doubly barred quantities are boundary curve derivatives which are known when the curve mesh is given. The remaining ones are cross and mixed partial derivatives which must be found during the surface construction.

In Weber's method a polynomial representation is used for the cross partial derivatives. The polynomial degree is chosen so as to achieve GC^2 continuity and at the same time arrive at a local problem formulation for each knot in the mesh. It turns out (Weber (1990)) that, for irregular mesh topologies, Hermite interpolants of degree seven are necessary to meet these requirements, whereas for regular meshes, with only two lines intersecting at a knot (*fig.* 4), polynomials of degree five are sufficient. In this case a cross partial derivative at the patch boundary can be expressed in terms of corner point properties, e.g., as follows ($i = 0$ or 1):

$$\begin{aligned}\overline{\underline{iS}}_r &= \overline{\underline{i0}}_r\, H_0(s) + \overline{\underline{i0}}_{rs}\, H_1(s) + \\ &\quad + \overline{\underline{i0}}_{rss}\, H_2(s) + \overline{\underline{i1}}_r\, H_3(s) \\ &\quad + \overline{\underline{i1}}_{rs}\, H_4(s) + \overline{\underline{i1}}_{rss}\, H_5(s)\,.\end{aligned} \tag{33}$$

3.3. Surface Construction

An interpolating GC^2 surface is constructed through a given curve mesh on the basis of the following conditions :

— Curve continuity at knots (GC^2),
— Matching cross partial derivatives at patch corners,
— Compatibility of mixed partial derivatives.

Only the regular mesh case is treated here. (See Weber (1990) for the irregular case.) In the implemented examples it was assumed that the curve segments in the mesh consisted of cubic Bézier curves.

STEP 1 : Curve Matching

Second order geometric continuity (GC^2) for two adjoining curve segments $\underline{Q}_{i-1}$ and $\underline{Q}_{i+1}$ (*fig.* 4) requires (for $i = 2, 3$)

$$\partial \underline{Q}_{i-1}/\partial t = a_i \, \partial \underline{Q}_{i+1}/\partial t \,, \tag{34}$$

$$\partial^2 \underline{Q}_{i-1}/\partial t^2 = a_i^2 \, \partial^2 \underline{Q}_{i+1}/\partial t^2 + b_i \, \partial \underline{Q}_{i+1}/\partial t \,. \tag{35}$$

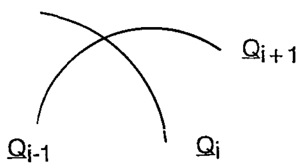

Figure 4. — Intersecting curves in regular mesh.

This result permits to simplify the form of the curve parametrizations, equations (31) and (32), further :

$$i\underline{S}_r = \overline{i\underline{S}}_r \, T(R_{r_{i0}}, R_{r_{i1}}, s) \,, \tag{36}$$

$$i\underline{S}_{rr} = \overline{i\underline{S}}_{rr} \, T^2(R_{r_{i0}}, R_{r_{i1}}, s) + \overline{i\underline{S}}_r \, T(R_{rr_{i0}}, R_{rr_{i1}}, s) \,, \tag{37}$$

etc.

Matching curve segments at the knots therefore yields conditions of the form

$$\partial \underline{Q}_{i+1}/\partial t = \frac{1}{a_i} \partial \underline{Q}_{i-1}/\partial t = \frac{\alpha_i}{a_i} \underline{E}_{i-1} \,, \tag{38}$$

$$\partial^2 \underline{Q}_{i+1}/\partial t^2 = \left(\frac{\alpha_i}{a_i} \right)^2 \underline{Z}_{i-1} + \left[\left(\beta_i - \frac{b_i \, \alpha_i}{a_i} \right) / a_i \right] \underline{E}_{i-1} \,, \tag{39}$$

where $\partial^2 \underline{Q}_{i-1}/\partial t^2 = \alpha_i^2 \underline{Z}_{i-1} + \beta_i \underline{E}_{i-1}$. For some given curve segment $\underline{Q}_{i-1}(t)$, the terms $\underline{E}_{i-1}$ and $\underline{Z}_{i-1}$ can be evaluated. The first and second derivatives of the adjoining curve segment are then found from the above conditions. In this process the parameters α_i and β_i are free and can be chosen in some suitable way, say, so as to arrive at a reasonably uniform parametrization.

STEP 2 : Cross partial derivatives (CPD's)

The CPD's along some patch boundary are represented by Hermite interpolants in accordance with equation (35), but similarly also for $\overline{\underline{iS}}_{rr}$, $\overline{\underline{Rj}}_s$, $\overline{\underline{Rj}}_{ss}$.

The solution procedure assumes that locally, at some knot, these CPD's are known for two adjoining patches, say A and B in figure 5, and must be matched up continuously with the other two elements, C and D, bordering on this knot. For the CPD's this yields the following types of conditions :

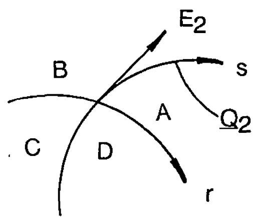

Figure 5. — Adjoining patches *A*, *B*, *C*, *D*.

Matching partial derivatives along the boundary lines (*fig.* 5) :

$$\overline{\underline{00}}_s^{rA} = \overline{\underline{10}}_s^{rB} = \overline{\underline{11}}_s^{rC} = \overline{\underline{01}}_s^{rD} = \underline{E}_2 , \tag{40}$$

and so on for the partial derivatives with respect to r, ss, rr.

Matching mixed partial derivatives :

16 conditions for the mixed partial derivatives from second to fourth order (from rs to $rrss$) of the type :

$$\overline{\underline{00}}_{rs}^{sA} = \overline{\underline{10}}_{rs}^{sB} = \underline{G}_2 . \tag{41}$$

Note that the scalar unknowns which occur inside the transformation functions of equations (33), (34), e.g., R_{ri0}, R_{rri1}, can be determined by means of eqs. (38) and (39).

STEP 3 : Compatibility of mixed partial derivatives (MPD's) or « twist » vectors

It is a well known requirement for surfaces that the result of evaluating an MDP, in particular at a corner point, should be independent of the sequence of differentiations followed. As a special case the familiar twist compatibility requirement can be stated as

$$\partial \underline{iS}_r/\partial s\,|_{s=j} = \underline{ij}_{rs} = \partial \underline{Rj}_s/\partial r\,|_{r=i} = \underline{ij}_{sr}\,. \tag{42}$$

Similar conditions are obtained for higher partial derivatives.

From the consideration that these surface properties must meet certain continuity requirements at the knots of the mesh, Weber (1990) derived a set of 16 further vector equations. The resulting matrix equations have certain rank deficiencies so that four vector unknowns (MDP's) in each patch can be chosen arbitrarily. The remaining unknowns are then fully defined by the set of vector equations.

3.4. Examples and Results

Weber's method was implemented for the case of a regular mesh with cubic Bézier boundary curves. In accordance with equation (32), the surface element would thus be of polynomial degrees 15 by 15 in general. This would require 256 Bézier points per patch, in practice a prohibitive number.

To reduce this effort a simplified scheme was implemented by Weber as follows :

1. The regular mesh area was organized in a chessboard pattern (*fig.* 6) into « black » and « white » elements (0 and 1).

2. The black patches are then represented by polynomials of degree 5 by 15, in order to give all CPD's in one direction and to adjust them to the neighbour patch only in the other direction.

3. Conversely the white patches have a representation of degree 15 by 5 so that their given and free boundaries are reversed.

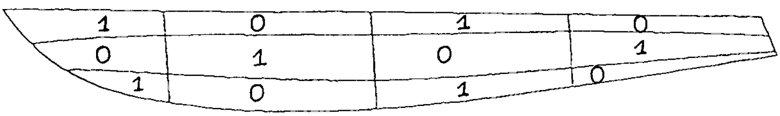

Figure 6. — Chessboard pattern on the regular curve mesh.

As a result every boundary line has a neighbouring patch with given and one with freely adjustable CPD's. This reduces the number of Bézier points to 96 per patch, still a considerable number. As a further simplification the mesh was parametrized in a uniform (integer) way.

Under these assumptions several examples were computed to explore the potential of the surface interpolation method. Figures 7 to 10 show the results for a sailing yacht hull form. More specifically, figure 7 shows the lines of the given curve mesh, whereas figures 8 to 10 present isolines of normal vector components and of the principal curvatures κ_1 and κ_2 of the interpolating surface.

In order to test the efficiency of the two-stage fairing method developed in Sections 2.2, 2.3, 3.2 and 3.3, a colour raster visualisation scheme has been implemented for the display of any desired state variables of a surface, in particular scalar values of the first and second partial derivative components, normal vector components, principal curvatures, the mean and Gaussian curvatures, as well as components of the fundamental forms of differential geometry. Comparison of these colour pictures, before and

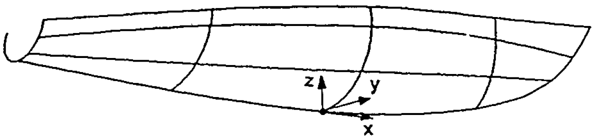

Figure 7. — Regular curve mesh for a sailing yacht hull form.

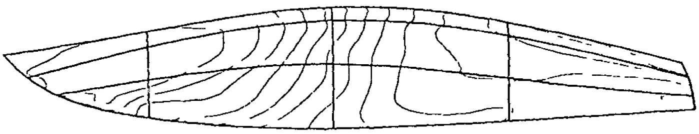

Figure 8. — Isolines of the *z*-components of the normal vector.

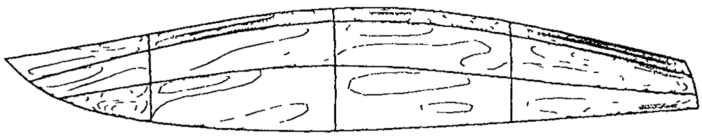

Figure 9. — Isolines of the first principal curvature κ_1.

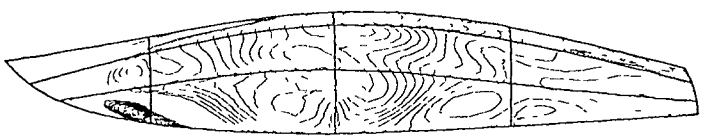

Figure 10. — Isolines of the second principal curvature κ_2.

after fairing, gave evidence of the fairness and continuity properties of the surface up to a very high degree of resolution. Unfortunately, colour pictures cannot be reproduced here. Nevertheless, the following conclusions can be derived from this evidence as well as figures 1, 2 and figures 7 to 10 :

— The curve mesh prefairing stage results in curvature continuous mesh lines of high fairness, i.e., with gradual changes in curvature, and a close correspondence in the shape character of neighbouring curves (see *figs.* 1, 2).

— The surface interpolation method ensures curvature continuity (GC^2) at cross patch boundaries. This is demonstrated, e.g., by the smooth character of the principal curvature patterns shown in figures 9 and 10. This kind of geometric continuity is achieved although the local patches are parametrized independently and hence discontinuously so that the parametric partial derivatives at patch boundaries in general do not match.

— The fairness quality of the resulting surface is certainly adequate for manufacturing purposes. Nevertheless in some cases minor imperfections in curvature patterns become visible at high resolution which seem to stem from systematic geometric rather than random numerical influences. The arbitrary assignment of mixed partial derivatives at the knots in the surface interpolation method is probably the main cause of these small disturbances. Further work on this matter, especially with regard to the best chose of twist vectors, is in progress.

— In conclusion, the available evidence suggests that the combination of curve mesh fairing and GC^2 interpolation results in fair surfaces of a high quality when compared to the original data and does ensure curvature continuity in the resulting surface.

4. SUMMARY

This article presents a method for achieving a fair surface from noisy data, consisting of two modules, namely a curve mesh fairing module and a surface interpolation module. The curve mesh fairing module is formulated as the minimisation of a fairness functional subject to accuracy (= inequality) constraints steming from statistical considerations. The surface interpolation module is a transfinite (Boolean sum) interpolation scheme for constructing curvature continuous (GC^2) surfaces interpolating the mesh provided by the curve mesh fairing module. The resulting faired surfaces retain the fairness character of the mesh lines but are not yet totally free of local shape imperfections, probably due to the arbitrary assignment of MDP's at the knots. Both modules of this two-stage fairing procedure are under further development.

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