M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique

Leopoldo P. Franca
Abimael F. D. Loula

A new mixed finite element method for the Timoshenko beam problem

M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 25, nº 5 (1991), p. 561-578

<http://www.numdam.org/item?id=M2AN_1991__25_5_561_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 25, n° 5, 1991, p. 561 à 578)

A NEW MIXED FINITE ELEMENT METHOD FOR THE TIMOSHENKO BEAM PROBLEM (*)

Leopoldo P. FRANCA (1), Abimael F. D. LOULA (1)

Communicated by J. DOUGLAS

Abstract. — *The Timoshenko beam problem, with shear force and rotation as primal variables and displacement as Lagrange multiplier, is examined under the general hypotheses of Brezzi's theorem concerning existence and uniqueness of saddle point problems. A new finite element approximation is proposed and shown to be convergent for various combinations of interpolations for the three variables.*

Résumé. — *Le problème de la poutre de Timoshenko, avec pour variables primales la force de cisaillement et la rotation, et comme multiplicateur de Lagrange le déplacement, est examiné sous les hypothèses générales du théorème de Brezzi relatives à l'existence et l'unicité des problèmes de points-selles. Une nouvelle approximation par éléments finis est proposée ici, et nous prouvons la convergence pour des combinaisons variées d'espaces d'interpolation sur les trois variables.*

1. INTRODUCTION

In [1] a mixed formulation for the Timoshenko problem was considered based upon the Hellinger-Reissner principle, with bending moment and shear force as primal variables and displacement and rotation as multipliers. Existence and uniqueness of solution was established using Brezzi's corollary [2] on the analysis of saddle-point problems which requires the satisfaction of two conditions : K-ellipticity and the Babuška-Brezzi condition. The latter condition is easily fulfilled while the former is more subtle to be satisfied in this formulation. By dropping the moment variable from this problem and admitting the shear force and rotation as primal variables, the resulting mixed formulation does not satisfy K-ellipticity but a more general inf-sup condition on K which is part of the generality of Brezzi's

(*) Received in March 1990.
(1) Laboratório Nacional de Computação Científica, Rua Lauro Muller 455, 22290 - Rio de Janeiro, RJ - Brasil.

theorem. As far as the authors are aware, this is the first time that this hypothesis has been applied to this problem. The convergence proof of the approximation considered also employs the general inf-sup condition. The resulting finite element method is valid for various combinations of interpolations of shear, rotation and displacement.

The paper is organized as follows. In Section 2 the differential equations governing this problem are presented, a mixed variational formulation in terms of nondimensional variables is derived and its analysis is shown in Proposition 2.1. In Section 3 we describe a finite element approximation and prove its convergence for a rather general family of finite element interpolations.

2. VARIATIONAL FORMULATION

According to the Timoshenko beam theory, the in-plane bending of a clamped uniform beam of length L, cross section A, moment of inertia I, Young's modulus E and shear modulus G, subjected to a distributed load $p(\bar{x})$, with $\bar{x} \in (0, L)$ representing the independent variable, is governed by the following system of differential equations :

$$-\frac{dQ}{d\bar{x}} = p\,, \tag{1}$$

$$-EI\frac{d^2\theta}{d\bar{x}^2} - Q = 0\,, \tag{2}$$

$$-\frac{Q}{\kappa GA} + \frac{dw}{d\bar{x}} - \theta = 0\,, \tag{3}$$

where $Q(\bar{x})$ is the shear force; $\theta(\bar{x})$ is the cross-sectional rotation; $w(\bar{x})$ is the transverse displacement; κ is the shear correction factor. For simplicity we consider the following boundary conditions :

$$w(0) = w(L) = 0\,, \tag{4}$$
$$\theta(0) = \theta(L) = 0\,. \tag{5}$$

To show the dependence of this problem on the small parameter

$$\varepsilon^2 = \frac{EI}{\kappa GAL^2}, \tag{6}$$

we introduce the following change of variables :

$$u_1 = \frac{w}{L}, \qquad u_2 = \theta\,, \tag{7}$$

$$\sigma_1 = \frac{QL^2}{EI}, \qquad f = \frac{pL^3}{EI}, \tag{8}$$

which reduces the original problem to finding $u_1(x)$, $u_2(x)$ and $\sigma_1(x)$, $x \in (0, 1)$ satisfying,

$$-\sigma_1' = f, \tag{9}$$

$$-u_2'' - \sigma_1 = 0, \tag{10}$$

$$-\varepsilon^2 \sigma_1 + u_1' - u_2 = 0, \tag{11}$$

with boundary conditions,

$$u_1(0) = u_1(1) = 0, \tag{12}$$

$$u_2(0) = u_2(1) = 0, \tag{13}$$

where the prime superscript denotes differentiation with respect to the dimensionless variable $x = \bar{x}/L$. We observe that the dimensionless problem above depends explicitly on a parameter ε, proportional to the ratio of thickness to length. In many applications $\varepsilon \ll 1$. In this limit the construction of finite element approximations is delicate.

We wish to consider the following variational formulation for (9)-(13): Given $f \in H^{-1}(0, 1)$, find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in W \times V$ such that

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}) = 0, \quad \forall \boldsymbol{\tau} \in W, \tag{14}$$

$$b(\boldsymbol{\sigma}, \boldsymbol{v}) = f(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in V, \tag{15}$$

where the bilinear forms $a : W \times W \to R$ and $b : W \times V \to R$ are defined by

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = -\varepsilon^2(\sigma_1, \tau_1) - (\sigma_1, v_2) - (\tau_1, u_2) + (u_2', v_2') \ \ \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in W, \tag{16}$$

$$b(\boldsymbol{\tau}, \boldsymbol{u}) = (\tau_1, u_1') \quad \forall \boldsymbol{\tau} \in W, \quad \forall \boldsymbol{u} \in V, \tag{17}$$

and

$$f(\boldsymbol{v}) = (f, v_1) \qquad \forall \boldsymbol{v} \in V, \tag{18}$$

with

$$W = L_2(0, 1) \times H_0^1(0, 1), \tag{19}$$

$$V = H_0^1(0, 1). \tag{20}$$

Herein, $\boldsymbol{\sigma} = \{\sigma_1, u_2\}^T$ is the primal variable and $\boldsymbol{u} = \{u_1\}^T$ is the Lagrange multiplier; $L_2(0, 1)$ denotes the space of square-integrable functions on the unit interval with the inner product

$$(f, g) = \int_0^1 fg \, d\xi, \quad \forall f, g \in L_2(0, 1); \tag{21}$$

$$H_0^1(0, 1) = \{g \in L_2(0, 1); g' \in L_2(0, 1); g(0) = g(1) = 0\}, \tag{22}$$

and H^{-1} is the dual of $H_0^1(0,1)$. The norms on $L_2(0,1)$, $H_0^1(0,1)$, W and V are denoted by (respectively)

$$\|f\| = (f,f)^{1/2}, \qquad \forall f \in L_2(0,1), \tag{23}$$

$$\|g\|_1 = [(g,g) + (g',g')]^{1/2}, \qquad \forall g \in H_0^1(0,1), \tag{24}$$

$$\|\boldsymbol{\tau}\|_W = [\|\tau_1\|^2 + \|u_2\|_1^2]^{1/2}, \qquad \forall \boldsymbol{\tau} \in W, \tag{25}$$

$$\|\boldsymbol{v}\|_V = \|v_1\|_1, \qquad \forall \boldsymbol{v} \in V. \tag{26}$$

PROPOSITION 2.1 : *For bounded thickness, i.e.,* $\varepsilon \leqslant C < \infty$, *problem* (16)-(17) *satisfies the general hypotheses of Brezzi's theorem given by*

(A1) : *Continuity of* $a : W \times W \to R$,

$$|a(\boldsymbol{\sigma},\boldsymbol{\tau})| \leqslant C_1 \|\boldsymbol{\sigma}\|_W \|\boldsymbol{\tau}\|_W, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in W, \tag{27}$$

(A2) : *Continuity of* $b : W \times V \to R$,

$$|b(\boldsymbol{\tau},\boldsymbol{v})| \leqslant C_2 \|\boldsymbol{\tau}\|_W \|\boldsymbol{v}\|_V, \quad \forall \boldsymbol{\tau} \in W, \quad \forall \boldsymbol{v} \in V, \tag{28}$$

(H1) : *Stability of* a,

$$\sup_{\boldsymbol{\sigma} \in K} \frac{|a(\boldsymbol{\sigma},\boldsymbol{\tau})|}{\|\boldsymbol{\sigma}\|_W} \geqslant \alpha \|\boldsymbol{\tau}\|_W, \quad \forall \boldsymbol{\tau} \in K, \tag{29}$$

where

$$K = \{\boldsymbol{\tau} \in W ; b(\boldsymbol{\tau},\boldsymbol{v}) = 0, \forall \boldsymbol{v} \in V\}, \tag{30}$$

(H2) : *Babuška-Brezzi condition,*

$$\sup_{\boldsymbol{\tau} \in W} \frac{|b(\boldsymbol{\tau},\boldsymbol{v})|}{\|\boldsymbol{\tau}\|_W} \geqslant \beta \|\boldsymbol{v}\|_V, \quad \forall \boldsymbol{v} \in V, \tag{31}$$

with constants C_1, C_2, α, $\beta > 0$, *independent of* ε.

Proof :

(A1) :

$$|a(\boldsymbol{\sigma},\boldsymbol{\tau})|^2 \leqslant \max\{1,\varepsilon^2\}(\|\sigma_1\|\|\tau_1\| + \|\sigma_1\|\|v_2\| + \\ + \|\tau_1\|\|u_2\| + \|u_2'\|\|v_2'\|)^2 .$$

Thus,

$$|a(\boldsymbol{\sigma},\boldsymbol{\tau})|^2 \leqslant \max\{1,\varepsilon^2\}\{2\|\sigma_1\|^2(\|\tau_1\|^2 + \|v_2\|^2) + \\ + \|u_2\|_1^2(\|\tau_1\|^2 + \|v_2'\|^2) + \|\sigma_1\|^2(\|\tau_1\|^2 + 2\|v_2'\|^2 + \|v_2\|^2) \\ + 2\|u_2\|^2\|\tau_1\|^2 + \|u_2'\|^2(\|\tau_1\|^2 + \|v_2\|^2)\}$$

therefore

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})|^2 \leqslant 3 \max \{1, \varepsilon^2\} (\|\sigma_1\|^2 + \|u_2\|_1^2)(\|\tau_1\|^2 + \|v_2\|_1^2) \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in W, \tag{32}$$

which proves (A1) with $C_1 = \sqrt{3} \max \{1, C\}$.

(A2) :

$$|b(\boldsymbol{\tau}, \boldsymbol{v})| \leqslant \|\tau_1\| \|v_1'\| \leqslant \|\boldsymbol{\tau}\|_W \|\boldsymbol{v}\|_V \ \forall \boldsymbol{\tau} \in W, \ \forall \boldsymbol{v} \in V. \tag{33}$$

Thus (A2) follows with $C_2 = 1$.

(H1) : For each $\boldsymbol{\tau} \in K$, $\tau_1 = \tau_1^0 = \text{constant}$. Select $\bar{\boldsymbol{\sigma}} = \{-2\tau_1^0, v_2 - \tau_1^0 \phi/12\} \in K$ with $\phi = 6x(1-x)$. Then

$$|a(\bar{\boldsymbol{\sigma}}, \boldsymbol{\tau})| = \big|2\varepsilon^2 \|\tau_1^0\|^2 + (\tau_1^0, v_2) + \|\tau_1^0\|^2 (\phi, 1)/12 + \\ + \|v_2'\|^2 - \tau_1^0(\phi', v_2')/12\big| = (2\varepsilon^2 + 1/12)\|\tau_1^0\|^2 + \|v_2'\|^2.$$

From Poincaré inequality, we have

$$|a(\bar{\boldsymbol{\sigma}}, \boldsymbol{\tau})| \geqslant \frac{1}{12} \|\tau_1^0\|^2 + \frac{2}{3} \|v_2\|_1^2 \geqslant \frac{1}{12} \|\boldsymbol{\tau}\|_W^2 \quad \forall \boldsymbol{\tau} \in K,$$

with

$$\|\bar{\boldsymbol{\sigma}}\|_W^2 = \|2\tau_1^0\|^2 + \|v_2 - \tau_1^0 \phi/12\|_1^2 \leqslant 5 \|\boldsymbol{\tau}\|_W^2 \quad \forall \boldsymbol{\tau} \in K.$$

Thus,

$$\sup_{\boldsymbol{\sigma} \in K} \frac{|a(\boldsymbol{\sigma}, \boldsymbol{\tau})|}{\|\boldsymbol{\sigma}\|_W} \geqslant \frac{|a(\bar{\boldsymbol{\sigma}}, \boldsymbol{\tau})|}{\|\bar{\boldsymbol{\sigma}}\|_W} \geqslant \frac{1}{30} \|\boldsymbol{\tau}\|_W \quad \forall \boldsymbol{\tau} \in K,$$

which yields (H1) with $\alpha = 1/30$.

(H2) : Select $\forall \boldsymbol{v} \in V$, $\bar{\boldsymbol{\tau}} = \{v_1', 0\}^T$. Then $\bar{\boldsymbol{\tau}} \in W$ and

$$\sup_{\boldsymbol{\tau} \in W} \frac{|b(\boldsymbol{\tau}, \boldsymbol{v})|}{\|\boldsymbol{\tau}\|_W} \geqslant \frac{|b(\bar{\boldsymbol{\tau}}, \boldsymbol{v})|}{\|\bar{\boldsymbol{\tau}}\|_W} \geqslant \frac{2}{3} \|\boldsymbol{v}\|_V \quad \forall \boldsymbol{v} \in V.$$

Thus (H2) follows with $\beta = 2/3$.

Remarks :

1. (H1) is a rather general condition. In most applications (H1) is replaced by a stronger condition, namely K-ellipticity

$$|a(\boldsymbol{\tau}, \boldsymbol{\tau})| \geqslant \alpha \|\boldsymbol{\tau}\|^2 \quad \forall \boldsymbol{\tau} \in K. \tag{34}$$

As far as the authors are aware, this is the first time the general hypothesis (H1) of Brezzi's theorem has been applied in the analysis of this model problem. The analysis indicates how to handle stability of the finite element approximation presented next.

2. Studying the same model problem, Arnold [3] considered $\hat{\boldsymbol{\sigma}} = \{u_1, u_2\}^T$ as primal variable and $\hat{\boldsymbol{u}} = \{\sigma_1\}$ as Lagrange multiplier. For this arrangement the abstract form (15) is replaced by

$$-\varepsilon^2(\sigma_1, \tau_1) + (\tau_1, u_1' - u_2) = 0 . \tag{35}$$

Due to the presence of the $-\varepsilon^2(\sigma_1, \tau_1)$-term in (35), in this case the analysis cannot be carried out employing Brezzi's theorem. Arnold [3] proved another result of his own to deal with this arrangement of variables. On the other hand, by insisting on the classical mixed formulation format of this problem, we were able to prove well-posedness of our model problem using Brezzi's theorem.

3. FINITE ELEMENT APPROXIMATION

Consider a partition of the unit interval $0 = x_0 < x_1 < \cdots < x_{n_{el}} = 1$ where x_e is the coordinate of the end point of the eth-element and n_{el} is the total number of elements. The domain of each element e is $\Omega^e = (x_{e-1}, x_e)$, $e = 1, 2, \ldots, n_{el}$ and the mesh parameter is denoted by $h = \max(\Omega^e)$, $e = 1, 2, \ldots, n_{el}$. Consider the set of all polynomials of degree not greater than k and denote its restriction to Ω by $P_k(\Omega)$. Let $Q_h^k(\Omega)$ be the space of C^{-1} piecewise polynomial interpolations of degree k, i.e.,

$$Q_h^k(\Omega) = \{g_h \mid g_e \in P_k(\Omega^e), e = 1, 2, \ldots, n_{el}\} \tag{36}$$

with g_e denoting the restriction of g_h to Ω^e. Let $S_h^l(\Omega)$ be the space of C^0 piecewise polynomial of degree l with zero value at $x = 0$ and $x = 1$, i.e.,

$$S_h^l(\Omega) = Q_h^l(\Omega) \cap H_0^1(0, 1) . \tag{37}$$

To simplify the convergence analysis we will assume herein quasiuniformity of each partition considered.

The finite element approximation we wish to consider is given by

Given $f \in H^{-1}(0, 1)$, and $W_h = Q_h^k \times S_h^l \subset W$, $V_h = S_h^l \subset V$, find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in W_h \times V_h$, such that

$$a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{u}_h) = g_h(\boldsymbol{\tau}_h) , \quad \forall \boldsymbol{\tau}_h \in W_h , \tag{38}$$

$$b(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) = f(\boldsymbol{v}_h) , \quad \forall \boldsymbol{v}_h \in V_h , \tag{39}$$

where,

$$a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - \delta h^2(\sigma'_{h1}, \tau'_{h1})_h , \tag{40}$$

$$g_h(\boldsymbol{\tau}_h) = \delta h^2(f, \tau'_{h1})_h , \tag{41}$$

in which δ is a dimensionless positive scalar parameter, and $(.,.)_h$ is defined by

$$(\mu_h, \eta_h)_h = \sum_{e=1}^{n_{el}} \int_0^{h_e} \mu_e \eta_e \, d\xi , \quad \forall \mu_h, \eta_h \in Q_h^k , \tag{42}$$

where μ_e and η_e are the restrictions of μ_h and η_h to element e. For $\delta = 0$, (38)-(41) reduces to the Galerkin method which, for $k > l - 1$, lacks stability of the shear variable σ_{h1} as $\varepsilon \to 0$. Addition of the δ-term enhances the stability of this variable in almost the whole space Q_h^k, even for $k > l - 1$. In general we may assert only that

$$\delta h^2 \|\tau'_{h1}\|^2 \geq 2\,\delta\, \|\tau_{h1}\|^2 \quad \forall \tau_{h1} \in Q_h^k / R^e . \tag{43}$$

In other words, the additional δ-term controls all functions in Q_h^k modulo piecewise constants. Convergence of this finite element method is proven next.

PROPOSITION 3.1 : *For* $k \geq l - 1$, $\boldsymbol{\sigma} \in H^{k+1}(0,1) \times (H^{l+1}(0,1) \cap H_0^1(0,1))$ *and* $\boldsymbol{u} \in H^{l+1}(0,1) \cap H_0^1(0,1)$ *the finite element method* (38)-(41) *has a unique solution* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in W_h \times V_h$, *and the following error estimate holds :*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{h, W} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq C(\boldsymbol{\sigma}, \boldsymbol{u})\, h^l , \tag{44}$$

in which

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{h, W} = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_W + \delta^{1/2} \sup_{\tau_h \in W_h} \frac{h\,|(\sigma'_1 - \sigma'_{h1}, \tau'_{h1})|}{\|\tau'_{h1}\|_h} . \tag{45}$$

This result holds for $\delta > 1/4$ *and* $h \leq 1/2$.

Before proving Proposition 3.1 let us establish some preliminary results.

LEMMA 3.1 : *Characterization of the Shear Component,* τ_{h1} *of* $\boldsymbol{\tau}_h = (\tau_{h1}, v_{h2}) \in K_h$ *where*

$$K_h = \{\boldsymbol{\tau}_h \in W_h ; b(\boldsymbol{\tau}_h, \boldsymbol{v}_h) = 0, \ \forall \boldsymbol{v}_h \in V_h\} . \tag{46}$$

Let τ^0_{h1} *denote the mean value of the component* τ_{h1} *of* $\boldsymbol{\tau}_h \in K_h$, i.e.,

$$\tau^0_{h1} = \int_0^1 \tau_{h1}\, d\xi \,. \tag{47}$$

Define

$$\tau^*_{h1} = \tau_{h1} - \tau^0_{h1} \,. \tag{48}$$

Then, for any $\mu_h \in Q^l_h$,

$$\int_0^{h_e} \tau^*_{e1}\, \mu'_e\, dx = 0 \,, \qquad e = 1, 2, \ldots, n_{el} \tag{49}$$

where τ^*_{e1} *is the restriction of* $\tau^*_{h1} \in Q^k_h$ *to element e, and* μ_e *is the restriction of* μ_h *to element e.*

Proof: By definition (46) for $\boldsymbol{\tau}_h \in K_h$

$$\int_0^1 \tau_{h1}\, v'_{h1}\, d\xi = 0 \,, \qquad \forall v_{h1} \in S^l_h \,. \tag{50}$$

It follows from (47) and (48) that

$$\int_0^1 \tau^*_{h1}\, v'_{h1}\, d\xi = 0 \,, \qquad \forall v_{h1} \in S^l_h \,. \tag{51}$$

Given any $\mu_h \in Q^l_h$, there exists $v_{h1} \in S^l_h$ and a constant c such that

$$v'_{e1} = \mu'_e + c \tag{52}$$

where v_{e1} is the restriction of v_{h1} to element e. Substituting (52) into (51) yields

$$\sum_{e=1}^{n_{el}} \int_0^{h_e} \tau^*_{e1}\, \mu'_e\, d\xi + c \int_0^1 \tau^*_{h1}\, d\xi = 0 \,. \tag{53}$$

By (47) and (48)

$$\int_0^1 \tau^*_{h1}\, d\xi = 0 \tag{54}$$

and therefore

$$\sum_{e=1}^{n_{el}} \int_0^{h_e} \tau^*_{e1}\, \mu'_e\, d\xi = 0 \,. \tag{55}$$

Invoking the independence of μ'_e on each element, it follows that each term in the sum vanishes, which completes the proof of the lemma.

COROLLARY 3.1 : *It follows from* (55), *by selecting* μ'_e *an arbitrary constant on each element, that*

$$\int_0^h \tau_{e1}^* \, dx = 0 \,, \qquad e = 1, 2, \ldots, n_{el} \,. \tag{56}$$

The existence of local constant modes is precluded by Corollary 3.1.

COROLLARY 3.2 (Shear Stability) : *With* τ_{h1}^* *as defined in Lemma* 3.1,

$$h^2(\tau'_{h1}, \tau'_{h1})_h \geq 2 \|\tau_{h1}\|^2 \,, \quad \forall \boldsymbol{\tau}_h \in K_h/R \,. \tag{57}$$

$$h^2(\tau'_{h1}, \tau'_{h1})_h \geq 2 \|\tau_{h1}^*\|^2 \,, \quad \forall \boldsymbol{\tau}_h \in K_h \,. \tag{58}$$

Proof: From (48) and (56) we can use the Poincaré inequality in each element to get

$$\|\tau'_{e1}\|^2 \geq \frac{2}{h_e^2} \|\tau_{e1}^*\|^2 \,, \qquad e = 1, 2, \ldots, n_{el}$$

and (58) follows by summing over elements. The only constant mode in K_h is the global constant τ_{h1}^0 to be controlled through the Galerkin terms.

We now state the theorem in which the proof of Proposition 3.1 is based upon.

THEOREM 3.1 (Generalization of Brezzi's theorem) : *Assume :*

(A1) : (*Continuity of* $a : W \times W \to R$ *and* $b : W \times V \to R$). *There exist constants* $0 < C_1, C_2 < \infty$ *such that*

$$|a(\boldsymbol{\sigma}, \boldsymbol{\tau})| \leq C_1 \|\boldsymbol{\sigma}\|_W \|\boldsymbol{\tau}\|_W \,, \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in W \,, \tag{59}$$

$$|b(\boldsymbol{\tau}, \boldsymbol{v})| \leq C_2 \|\boldsymbol{\tau}\|_W \|\boldsymbol{v}\|_V \,, \quad \forall \boldsymbol{\tau} \in W \,, \ \forall \boldsymbol{v} \in V \,, \tag{60}$$

(A2) : *(Consistency). The exact solution* $(\boldsymbol{\sigma}, \boldsymbol{u})$ *satisfies*

$$a_h(\boldsymbol{\sigma}, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{u}) = g_h(\boldsymbol{\tau}_h) \,, \quad \forall \boldsymbol{\tau}_h \in W_h \,, \tag{61}$$

$$b(\boldsymbol{\sigma}, \boldsymbol{v}_h) = f(\boldsymbol{v}_h) \,, \quad \forall \boldsymbol{v}_h \in V_h \,. \tag{62}$$

The following stability conditions must also hold :

$(\mathrm{H1})_h$: (a_h*-stability on* K_h). *There exists a constant* $\alpha_h > 0$ *such that*

$$\sup_{\boldsymbol{\sigma}_h \in K_h} \frac{|a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)|}{\|\boldsymbol{\sigma}_h\|_W} \geq \alpha_h \|\boldsymbol{\tau}_h\|_W \quad \forall \boldsymbol{\tau}_h \in K_h \,, \tag{63}$$

$(\mathrm{H2})_h$: *(Discrete Babuška-Brezzi condition). There exists a constant* $\beta_h > 0$ *such that*

$$\sup_{\boldsymbol{\tau}_h \in W_h} \frac{|b(\boldsymbol{\tau}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{\tau}_h\|_W} \geq \beta_h \|\boldsymbol{v}_h\|_V \quad \forall \boldsymbol{v}_h \in V_h, \tag{64}$$

(IE) : *There exists a constant* $\gamma > 0$ *such that*

$$h \|\tau'_{h1}\|_h \leq \gamma \|\tau_{h1}\| . \tag{65}$$

Then the finite element method (38)-(41) *has a unique solution* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in W_h \times V_h$ *and the following estimate holds :*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{h, W} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq C_h(\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{h, W} + \|\boldsymbol{u} - \boldsymbol{v}_h\|_V) \quad \forall \boldsymbol{\tau}_h \in W_h, \quad \forall \boldsymbol{v}_h \in V_h, \tag{66}$$

with $\|\cdot\|_{h, W}$ *as defined in* (45) *and* $C_h > 0$ *constant.*

(A3) : *Let* $\tilde{\boldsymbol{\sigma}}_h \in W_h$ *and* $\tilde{\boldsymbol{u}}_h \in V_h$ *be interpolants of* $\boldsymbol{\sigma}$ *and* $\boldsymbol{u}$. *Assume that the interpolation errors* $\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h$ *and* $\boldsymbol{u} - \tilde{\boldsymbol{u}}_h$ *satisfy*

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}_h\|_{h, W} \leq C(\boldsymbol{\sigma}) h^{k+1}, \tag{67}$$

$$\|\boldsymbol{u} - \tilde{\boldsymbol{u}}_h\|_V \leq C(\boldsymbol{u}) h^l, \tag{68}$$

and, if in addition α_h *and* β_h *are independent of* h, *then,*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{h, W} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_V \leq C(\boldsymbol{\sigma}, \boldsymbol{u}) h^p, \tag{69}$$

with $p = \min\{k + 1, l\}$.

Proof : See [4].

Proof of Proposition 3.1 : We show that the finite element approximation (38)-(41) satisfies the hypotheses of Theorem 3.1.

(A1) was checked in Proposition 2.1 with

$$C_1 = \sqrt{3} \max\{1, \varepsilon\}, \quad C_2 = 1 . \tag{70}$$

(A2) is implicit in the scheme.

(A3) follows from the assumed smoothness of the exact solution $(\sigma, \boldsymbol{u})$.

(IE) is simply a particular result of the inverse estimate found in Ciarlet [5],

$$h \|\tau'_{h1}\|_h \leq \gamma \|\tau_{h1}\| . \tag{71}$$

$(\mathrm{H2})_h$: For $k \geqslant l-1$, $\forall \boldsymbol{v}_h \in V_h$ select $\bar{\boldsymbol{\tau}}_h = \{v'_{h1}, 0\}^T$. Then $\bar{\boldsymbol{\tau}}_h \in W_h$ and

$$\sup_{\boldsymbol{\tau}_h \in W_h} \frac{|b(\boldsymbol{\tau}_h, \boldsymbol{v}_h)|}{\|\boldsymbol{\tau}_h\|_W} \geqslant \frac{|b(\bar{\boldsymbol{\tau}}_h, \boldsymbol{v}_h)|}{\|\bar{\boldsymbol{\tau}}_h\|_W} = \frac{\|v'_{h1}\|^2}{\|v'_{h1}\|} \geqslant \frac{2}{3}\|\boldsymbol{v}_h\|_V, \quad \forall \boldsymbol{v}_h \in V_h, \tag{72}$$

yielding $\beta_h = \beta = 2/3$.

$(\mathrm{H1})_h$: (a_h-stability on K_h). Lemma 3.1 characterizes $\boldsymbol{\tau}_h \in K_h$. Select for each $\boldsymbol{\tau}_h = \{\tau_{h1}, \boldsymbol{v}_{h2}\}^T \in K_h$

$$\bar{\boldsymbol{\sigma}}_h = \left\{-2\,\tau_{h1}, v_{h2} - \tau^0_{h1}\,\phi_h/12\right\}^T \in K_h \tag{73}$$

where $\phi_h \in S^l_h$ is such that

$$(\phi'_h, \mu'_h) = (\phi', \mu'_h) \quad \forall \mu_h \in S^l_h\,, \tag{74}$$

and satisfies

$$(\phi_h, 1) \geqslant (1-h^2)\,, \tag{75}$$

$$\|\phi'_h\|^2 \leqslant \|\phi'\|^2 = 12\,, \tag{76}$$

$$\|\phi_h\|^2 \leqslant \|\phi\|^2 = 1.2\,. \tag{77}$$

Combining the above results we get

$$\begin{aligned}
|a_h(\bar{\boldsymbol{\sigma}}_h, \boldsymbol{\tau}_h)| &= 2\,\varepsilon^2\|\tau_{h1}\|^2 + 2(\tau_{h1}, v_{h2}) - (\tau_{h1}, v_{h2} - \tau^0_{h1}\,\phi_h/12) + \\
&\quad + (v'_{h2} - \tau^0_{h1}\,\phi'_h/12, v'_{h2}) + 2\,\delta h^2\|\tau'_{h1}\|^2_h \\
&\geqslant (\tau^*_{h1}, v_{h2}) + \|\tau^0_{h1}\|^2\,(\phi_h, 1)/12 + (\tau^*_{h1}, \tau^0_{h1}\,\phi_h/12) \\
&\quad + \|v'_{h2}\|^2 + 4\,\delta\,\|\tau^*_{h1}\|^2 \\
&\geqslant \frac{1}{2}\,\|v_{h2}\|^2_1 + \frac{19-20\,h^2}{240}\,\|\tau^0_{h1}\|^2 + (4\,\delta - 1)\|\tau^*_{h1}\|^2 \\
&\geqslant \min\left\{\frac{19-20\,h^2}{240}, 4\,\delta - 1\right\}\|\boldsymbol{\tau}_h\|^2_W.
\end{aligned} \tag{78}$$

On the other hand,

$$\begin{aligned}
\|\bar{\boldsymbol{\sigma}}_h\|^2_W &= \|2\,\tau_{h1}\|^2 + \|v_{h2} - \tau^0_{h1}\,\phi_h/12\|^2_1 \leqslant \\
&\leqslant 4\|\tau_{h1}\|^2 + 4\|v_{h2}\|^2_1 + \frac{4}{3}\,\|\tau^0_{h1}\|^2\,\|\phi_h/12\|^2_1 \leqslant 5\,\|\boldsymbol{\tau}_h\|^2_W.
\end{aligned} \tag{79}$$

Combining (78) and (79) yields

$$\sup_{\sigma_h \in K_h} \frac{a_h(\sigma_h, \tau_h)}{\|\sigma_h\|_W} \geqslant \frac{a_h(\bar{\sigma}_h, \tau_h)}{\|\bar{\sigma}_h\|_W} \geqslant$$

$$\geqslant \frac{1}{\sqrt{5}} \min \left\{ \frac{19 - 20\, h^2}{240}, 4\,\delta - 1 \right\} \|\tau_h\|_W \quad \forall \tau_h \in K_h .$$

Thus $(\mathrm{H1})_h$ follows taking $\delta > 1/4$ and $h < 1/2$.

Remarks :

1. From Proposition 3.1 we observe that the finite element approximation (38)-(41) is stable and convergent for equal-order interpolation for all three variables. This combination of interpolation finite elements, based on the classical Galerkin method is well known to have poor stability properties. In the case of linear elements « locking » of the kinematic variables and spurious oscillations in the shear force are typical pathological phenomena observed in the Galerkin approximation for thin beams. Arnold [3] has shown that optimal rates of convergence for this method is obtained only when $h < \varepsilon$, which indicates the inappropriateness of Galerkin finite element approximations with equal-order interpolations.

4. NUMERICAL RESULTS

Numerical results for equal-order linear and quadratic elements (shear discontinuous on element interfaces) were obtained for the Galerkin and the PG methods (PG for Perturbation of the Galerkin method). In figure 1 the Galerkin method with 8 linear elements is plotted and compared to the exact solution of a clamped beam subjected to unit uniform load. Notice the « locking » (trivial solution) of the displacement and rotation variables, and the pathological behaviour of the shear variable with spurious oscillations throughout the domain. In figure 2 the same problem is tested employing the PG method. The erratic behaviour of the three variables is now precluded, and the comparison to the exact solution is very good.

In figure 3 the Galerkin method with 4 quadratic elements is compared to the exact solution of the same problem. The displacement and rotation variables no longer lock, but their values are far from the desirable exact solution values. Note the persistence of spurious oscillations in the shear variable. In figure 4 we observe an excellent agreement between the PG method solution and the exact solution for these quadratic elements.

In figure 5 we perform an error study for a clamped beam subjected to linear load ($f = x$). The L_2-norm and H_1-seminorm are plotted for the PG method employing equal-order two and three-node elements. Optimal rates

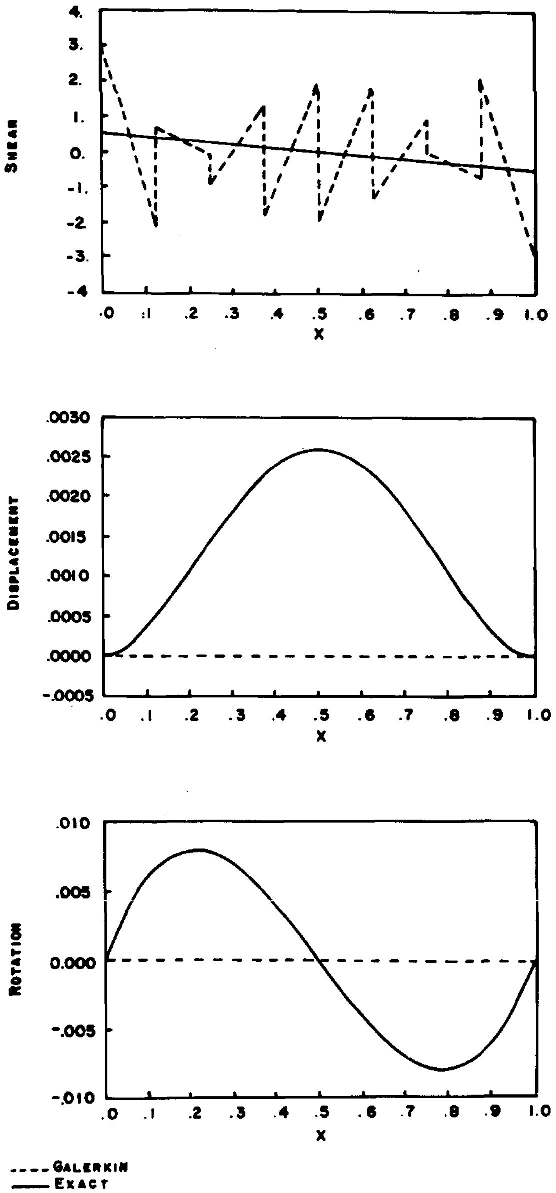

Figure 1. — Equal-order linear elements for a beam with both ends clamped subjected to unit uniform load.

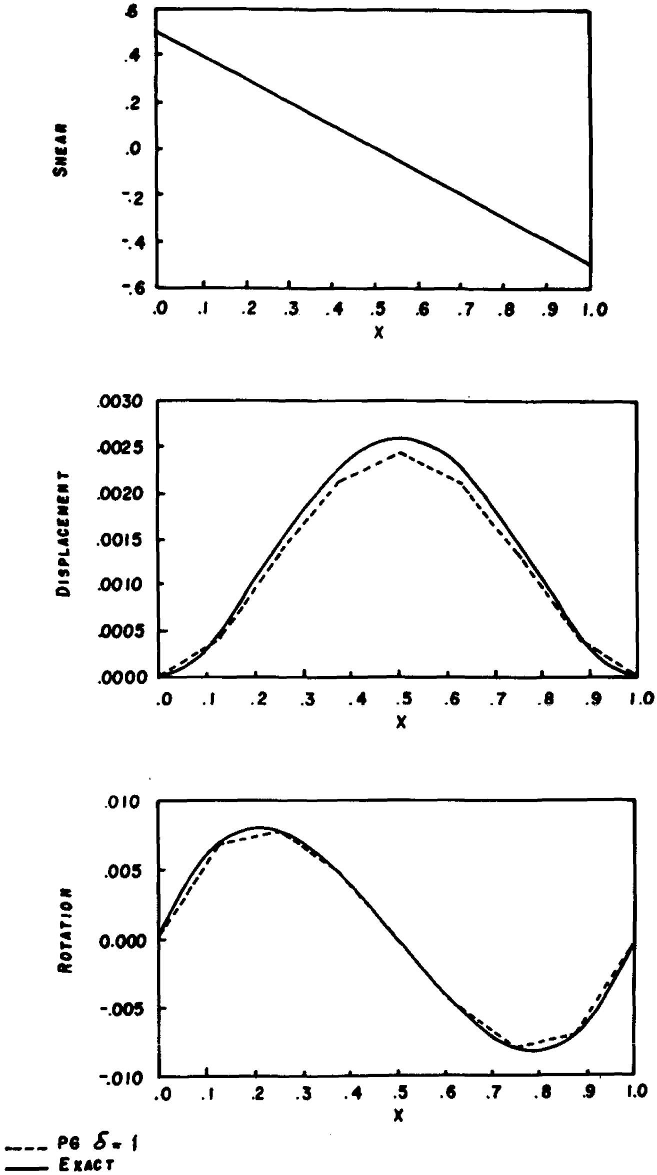

Figure 2. — Equal-order linear elements for a beam with both ends clamped subjected to unit uniform load.

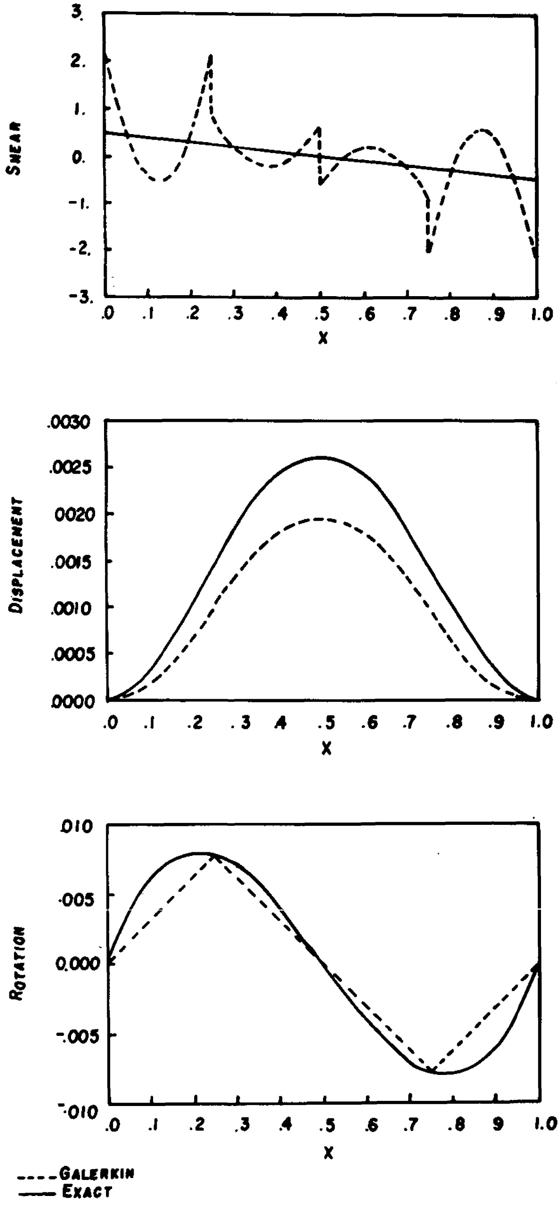

Figure 3. — Equal-order quadratic elements for a beam with both ends clamped subjected to unit uniform load.

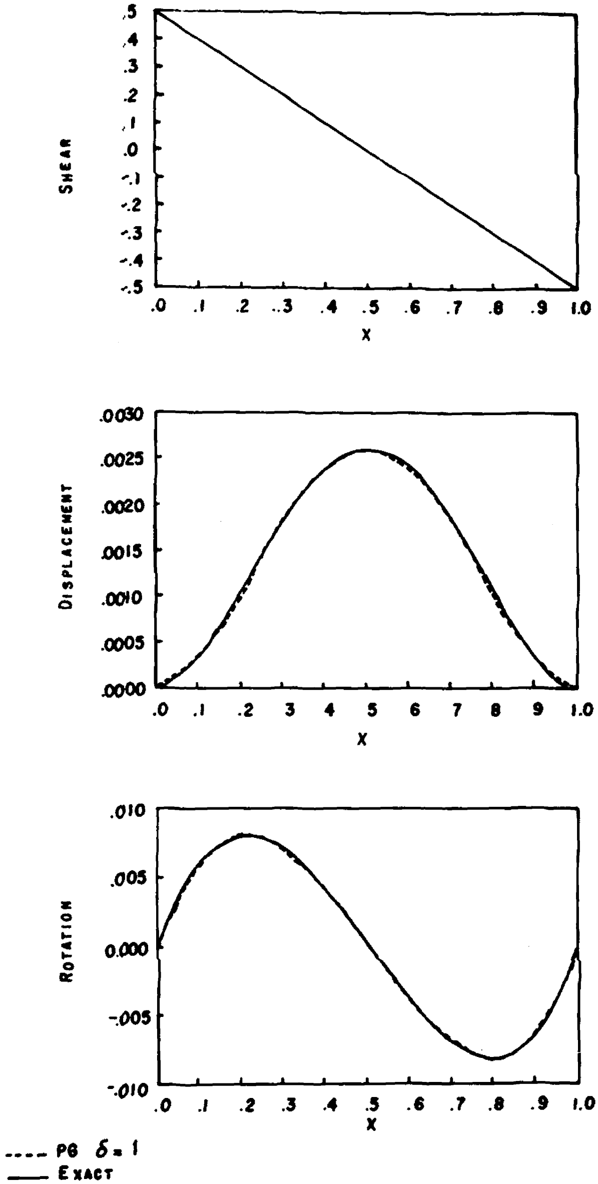

Figure 4. — Equal-order quadratic elements for a beam with both ends clamped subjected to unit uniform load.

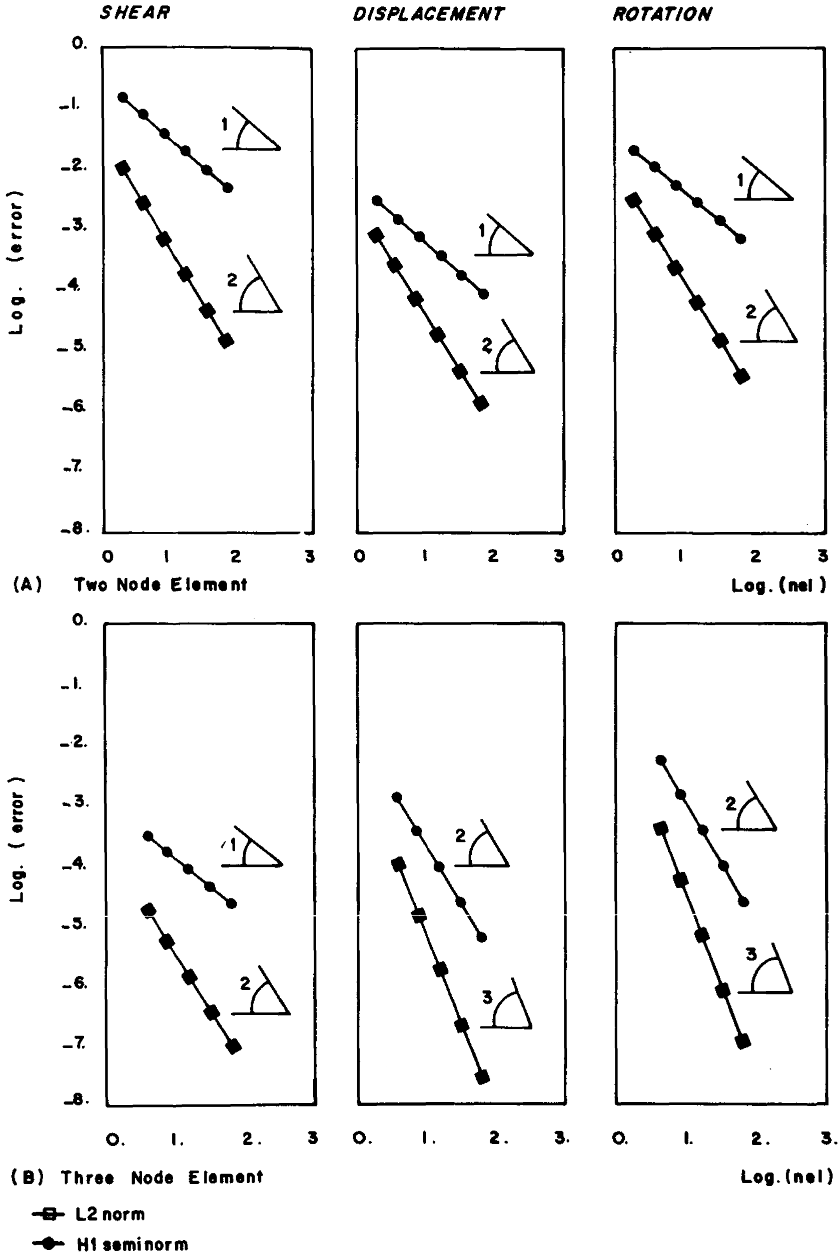

Figure 5. — Convergence study for a beam with both ends clamped subjected to linear load ($f = x$).

are obtained except for shear using the three-node element in which convergence has a gap one as predicted in the analysis.

5. CONCLUSIONS

In this paper we presented :

i) an analysis of existence and uniqueness for the Timoshenko problem employing the general theorem of Brezzi for saddle point problems ;

ii) a PG method which is convergent for a variety of combinations of finite element interpolations ;

iii) numerical results that confirm the numerical analysis of the proposed method.

We believe the analysis presented here may prove itself useful in the analysis of mixed variational formulations with more than two variables.

REFERENCES

[1] A. F. D. LOULA, T. J. R. HUGHES, L. P. FRANCA and I. MIRANDA, *Mixed Petrov-Galerkin Method for the Timoshenko Beam,* Comput. Methods Appl. Mech. Engrg., 63, 133-154, 1987.

[2] F. BREZZI, *On the Existence, Uniqueness and Approximation of Saddle-point Problems Arising From Lagrange Multipliers,* RAIRO Modél. Math. Anal. Numér., 8, R-2, 129-151, 1974.

[3] D. N. ARNOLD, *Discretization by Finite Elements of a Model Parameter Dependent Problem,* Numer. Math., 37, 405-421, 1981.

[4] L. P. FRANCA, *New Mixed Finite Element Methods,* PhD Dissertation, Stanford University, 1987.

[5] P. G. CIARLET, *The Finite Element Method for Elliptic Problems,* North-Holland Publishing Company, Amsterdam, 1978.