M2AN. MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
- MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 25, nº 1 (1991), p. 111-128

<http://www.numdam.org/item?id=M2AN_1991__25_1_111_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 25, n° 1, 1991, p. 111 à 128)

ERROR ESTIMATES AND STEP-SIZE CONTROL FOR THE APPROXIMATE SOLUTION OF A FIRST ORDER EVOLUTION EQUATION (*)

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Communicated by V. THOMÉE

Abstract. — *Rigorous and computable error bounds are derived for the approximate solution of a first order evolution equation by means of the implicit Euler method. All effects resulting from space discretization, approximation of coefficients or truncation of iterative methods for the nonlinear difference equations, respectively, are controlled step by step in a very simple manner. Hence time and space discretization may be treated separately. The paper is completed by a pilot investigation of a step-size control for a linear equation of parabolic type.*

Résumé. — *On déduit des majorations rigoureuses et computables de l'erreur pour la résolution approchée d'une équation d'évolution du premier ordre par la méthode implicite d'Euler. On contrôle pas-à-pas les effets résultant de la discrétisation en espace, de l'approximation des coefficients et de la troncation des méthodes itératives pour les équations aux différences non linéaires dans une manière très simple. Alors on peut traiter séparément les discrétisations en temps et en espace. Ce papier est complété par une investigation pilote d'un contrôle des pas pour une équation parabolique linéaire.*

1. INTRODUCTION

It is the aim of this paper to derive rigorous and computable error bounds for the approximate solution of a first order evolution equation by means of the implicit Euler method.

The problem under investigation is specified as follows. Assume $V \hookrightarrow H \hookrightarrow V'$ is a triple of real separable Hilbert spaces, V is dense in H and the duality pairing $\langle .,. \rangle$ of V and its dual V' is a continuous extension of the scalar product $(.,.)$ in H. $|\cdot|_V$, $|\cdot|_H$, $|\cdot|_{V'}$ are the norms in V, H and V', respectively. This setting includes in particular the finite dimensional

(*) Received in June 1989.

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case where all spaces are identical sets, but equipped with different scalar products and corresponding norms.

For $I = [0, 1]$ let $\mathcal{V} = L_2(I, V)$, $\mathcal{V}' = L_2(I, V')$, $\mathcal{X} = \mathcal{V} \cap C_0(I, H)$ and $\mathcal{W} = \{x \in \mathcal{V} \mid x' \in \mathcal{V}'\}$. Provided with any canonical norm, $\mathcal{W}$ is a Banach space and $\mathcal{W} \hookrightarrow C_0(I, H)$, hence $\mathcal{W} \hookrightarrow \mathcal{X}$ (*cf.* Lions and Magenes [9] or Gajewski, Gröger and Zacharias [5]).

Let A be a continuous mapping of V onto V' which is strongly monotone, i.e.

$$\langle Au - Av, u - v \rangle \geq |u - v|_V^2 \quad \text{for all} \quad u, v \in V . \tag{1.1}$$

If $f_0 \in \mathcal{V}'$ and $z_0 \in H$ are fixed data, then there exists a unique solution $x_0 \in \mathcal{W}$ of the initial value problem

$$\left.\begin{array}{l} x' + Ax = f_0, \quad x \in \mathcal{W}, \\ x(0) = z_0 \end{array}\right\} \tag{1.2}$$

(*cf.* [5] or [12]).

Since the work of Rothe [15], the implicit Euler method has been used to prove existence and regularity of solutions of evolution equations (*cf.* Raviart [13], Nečas [12], Gröger [6], Kačur [8] and the references quoted there). Combined with some space discretization, it is the most popular method to approximate the solutions of such equations.

The method may be treated simultaneously as a Galerkin method with piecewise constant test and trial functions in $\mathcal{V}$ and as a collocation method with piecewise linear trial functions in $\mathcal{W}$. The first approach was generalized by Ericsson, Johnson and Thomée [4] who have pointed out that some methods which are based on subdiagonal Padé approximations of higher order may be formulated in an equivalent way as Galerkin methods with piecewise polynomial but discontinuous test and trial functions. Axelsson [1] investigated the convergence of the θ-method which might be regarded as a modification of the collocation approach.

Any reasonable space discretization of an evolution equation in infinite dimensional spaces results in a system of ordinary differential equations which then should be investigated within the same functional analytic framework. For such a system the spaces of the triple $V \hookrightarrow H \hookrightarrow V'$ are most conveniently chosen as finite dimensional subspaces of the corresponding spaces for the original equation. In contrast to the usual treatment of ordinary differential equations, this introduces different norms on these spaces. However, this is the appropriate way to derive results which are valid uniformly with respect to the family of space discretizations. (For the same reasons that require enhanced notions of stability and convergence for the investigation of arbitrarily stiff systems, the justification of step-size procedures for evolution equations cannot be derived from the results for standard ordinary differential equations, *cf.* Sanz-Serna and Verwer [16] for an instructive discussion of the problems which result from a straightforward application of one-step methods in this field.)

In contrast to the fact, that the convergence of the implicit Euler method has been extensively investigated even for rather general types of nonlinear equations, there are only a few theoretical results for linear equations which provide a sound theoretical basis for error estimation and adaptive choice of step-sizes. Assuming some weak but non-trivial regularity concerning the data and the solution of the problem, Johnson, Nie and Thomée [7] proved a priori and a posteriori estimates in $L_\infty(I, H)$ for the discontinuous Galerkin variant of the implicit Euler method (*cf.* [4]). In connection with an adaptive space discretization of a highly regular problem, Reiher [14] investigated a step-size control which is based on an estimate of the local truncation errors by means of a comparison with the trapezoidal rule. In both papers some rather restrictive uniformity conditions for the step-sizes of the meshes had to be imposed.

In this paper the time discretization of equation (1.2) is treated by a modified collocation method which covers a variety of approximations of f_0 by means of piecewise constants (section 2). This collocation approach has the advantage that x_0 is approximated by functions from some subspace of $\mathscr{W}$ and all effects resulting from space discretization, approximation of coefficients or truncation of iterative methods for the nonlinear difference equations, respectively, are controlled in a very simple manner. Above all, the time and space discretization of (1.2), including the corresponding error estimates, can be treated separately. The investigation is restricted to time independent operators only for simplicity of presentation.

The stability of the solution of (1.2) relative to variations of the data yields a posteriori estimates of the approximation errors in $\mathscr{X}$ and $\mathscr{W}$ which depend only on the amount by which the approximate solution fails to satisfy (1.2) (section 3). Consequently, these estimates are comparatively easy to compute. The reliability of the error estimates and the convergence of the method are established by means of some a priori bounds for the derivatives of the approximations for meshes with bounded step-size and for rather general data. In some cases such bounds have already been investigated by Gröger [6]. Under fairly mild hypotheses on the data the convergence of the method is of order $h_\Delta^{1/2}$ where h_Δ is the maximum step-size of a mesh Δ. For sufficiently regular data the approximations converge linearly.

The paper is completed by a pilot investigation of a step-size control for a linear equation of parabolic type (section 4). The consequences of some hypotheses are discussed by means of a constant coefficient diffusion equation which is considered in various function space settings. In this way some control procedures which have been used for a long time are put on a sound basis. Effectivity in the sense of not using an excessive number of time steps will depend upon additional regularity of the data and the solution of the problem and is not investigated in this paper.

2. THE IMPLICITE EULER METHOD

To define discrete-time approximations of equation (1.2) for given data $f_0 \in \mathscr{V}'$ and $z_0 \in H$, let Θ be the set of all meshes

$$\Delta = \{q_i \mid q_i = [t_{i-1}, t_i], 1 \leq i \leq n_\Delta, 0 = t_0 < \cdots < t_{n_\Delta} = 1\} \quad (2.1)$$

on I. For each $\Delta \in \Theta$ let

$$h_i = t_i - t_{i-1} \quad \text{for} \quad 1 \leq i \leq n_\Delta \quad \text{and} \quad h_\Delta = \max_{1 \leq i \leq n_\Delta} h_i \quad (2.2)$$

(the dependence of the intervals and their characteristics upon Δ is often suppressed in the notation below). The set Θ is partially ordered by refinement and so it is a directed index sequence with the minimal element $\{I\}$ and $\lim_{\Delta \in \Theta} h_\Delta = 0$.

Each $\Delta \in \Theta$ determines the spaces

$$\left.\begin{aligned} \mathscr{X}_\Delta &= \{x \in \mathscr{X} \mid x \text{ linear on } q_i \text{ for } 1 \leq i \leq n_\Delta\}, \\ \mathscr{V}'_\Delta &= \{y \in \mathscr{V}' \mid y \text{ constant on int } (q_i) \text{ for } 1 \leq i \leq n_\Delta\} \end{aligned}\right\} \quad (2.3)$$

and an interpolation mapping p_Δ of $C_0(I, V')$ onto $\mathscr{V}'_\Delta$ such that

$$p_\Delta y|_{\text{int}(q_i)} \equiv y(t_i) \text{ for } 1 \leq i \leq n_\Delta \text{ and all } y \in C_0(I, V'). \quad (2.4)$$

For $\Delta \in \Theta$, $y \in \mathscr{V}'_\Delta$ and $1 \leq i \leq n_\Delta$ let $y^i = y(t_i - 0)$.

If Δ is a mesh on I, then with any $z_\Delta \in V$ and $f_\Delta \in \mathscr{V}'_\Delta$ the original equation (1.2) is accompanied by the equation

$$\left.\begin{aligned} x' + p_\Delta Ax &= f_\Delta, \quad x \in \mathscr{X}_\Delta, \\ x(0) &= z_\Delta. \end{aligned}\right\} \quad (2.5)$$

The functions $x \in \mathscr{X}_\Delta$ are uniquely determined by their nodal values $x(t_i) \in V$, $1 \leq i \leq n_\Delta$. Therefore equation (2.5) is equivalent with the uniquely solvable system of difference equations

$$\left.\begin{aligned} (x(t_i) - x(t_{i-1}))/h_i + Ax(t_i) &= f^i_\Delta, \quad 1 \leq i \leq n_\Delta, \\ x(t_0) &= z_\Delta. \end{aligned}\right\} \quad (2.6)$$

Hence (2.5) admits a unique solution $x_\Delta \in \mathscr{X}_\Delta$.

In the general case, the initial value z_0 will not be an element of V. Even if $z_0 \in V$, then, for practical reasons, z_0 has to be approximated very often by some element z_Δ of a finite dimensional subspace of V.

Special variants of the method result from special choices of f_Δ. If $f_0 \in C_0(I, V')$ and $f_\Delta = p_\Delta f_0$, then (2.5) is the original implicit Euler method. It turns out that this choice is the most convenient one to deal with arbitrary meshes $\Delta \in \Theta$. For obvious theoretical and practical reasons it is also of interest to investigate equations (2.5) with slightly perturbed f_Δ such that

$$f_\Delta = p_\Delta f_0 + \varepsilon_\Delta \quad \text{and} \quad |\varepsilon_\Delta^i|_{V'} \leq \gamma h_\Delta \quad \text{for} \quad 1 \leq i \leq n_\Delta, \tag{2.7}$$

where $\gamma \geq 0$ is an external parameter which may be chosen depending upon or independent of $\Delta \in \Theta$, respectively.

If $f_0 \in W_2^1(I, V')$, then (2.7) implies, with $C_\gamma = \|f_0'\|_{\mathscr{V}'}/\sqrt{2} + \gamma$,

$$\begin{aligned} \|f_\Delta - f_0\|_{\mathscr{V}'} &\leq \|(I - p_\Delta) f_0\|_{\mathscr{V}'} + \|\varepsilon_\Delta\|_{\mathscr{V}'} \\ &\leq \left(\sum_{i=1}^{n_\Delta} h_i^2 \int_{q_i} |f_0'|_{V'}^2\, ds/2\right)^{1/2} + \|\varepsilon_\Delta\|_{\mathscr{V}'} \\ &\leq h_\Delta C_\gamma . \end{aligned} \tag{2.8}$$

For a sufficiently smooth forcing function f_0, (2.7) includes the variant $f_\Delta^i = f(t_{i-1} + \theta h_i)$ for $1 \leq i \leq n_\Delta$ and some $\theta \in [0, 1]$. Moreover, f_Δ may be the orthogonal projection of f_0 onto $\mathscr{V}_\Delta'$. This modification of the method is the most general one with respect to the assumptions on f_0 and therefore it is the most preferred one in the literature.

There is an alternative interpretation of the fact that a fixed $x_\Delta \in \mathscr{X}_\Delta$ solves equation (2.5) for some $f_\Delta \in \mathscr{V}_\Delta'$ which satisfies (2.7). If ε_Δ is not a priori fixed but implicitely defined by means of x_Δ, then combining (2.5) and (2.7) for $\gamma > 0$ results in the system of difference inequalities

$$\left.\begin{aligned} |(x(t_i) - x(t_{i-1}))/h_i + Ax(t_i) - f(t_i)|_{V'} \leq \gamma h_\Delta, \quad 1 \leq i \leq n_\Delta \\ x(t_0) = z_\Delta \end{aligned}\right\} \tag{2.9}$$

These inequalities determine a set of solutions. Any solution algorithm for (2.9) will fix a unique $x_\Delta \in \mathscr{X}_\Delta$ and the corresponding $f_\Delta \in \mathscr{V}_\Delta'$.

Especially, inequalities (2.9) cover all those algorithms which in a step by step procedure solve the difference equations (2.6) with $f_\Delta = p_\Delta f_0$ approximately within some prescribed tolerance and in this way decouple the full discretization of (1.2) with respect to time and space. The defects in the difference equations may originate from different sources such as space discretization or an approximate solution of the nonlinear equations. (2.9) may be realized in a comparatively easy way because it only involves restrictions on the defects but not on the approximation of the exact solutions of the difference equations.

Later on it will be necessary to restrict the investigation to a subsequence $\Theta_\rho \subset \Theta$ for some $\rho \geq 1$ such that

$$\Delta \in \Theta_\rho \quad \text{iff} \quad \rho^2 \geq h_i/h_{i-1} \quad \text{for} \quad 2 \leq i \leq n_\Delta . \tag{2.10}$$

Even if ρ is of moderate size this is no serious restriction in practice. Almost all realizations of a step-size control are provided with such a bound to stabilize the control procedure.

3. ERROR ESTIMATES IN $\mathscr{X}$ AND $\mathscr{W}$

It is convenient to define norms $\|.\|_{\mathscr{X}}$, $\|.\|_{\mathscr{W}}$ on $\mathscr{X}$ and $\mathscr{W}$ such that

$$\|x\|_{\mathscr{X}} = \max_{t \in I} \left(|x(t)|_H^2/2 + \int_0^t |x(s)|_V^2 \, ds \right)^{1/2} \quad \text{for all} \quad x \in \mathscr{X} \tag{3.1}$$

and

$$\|x\|_{\mathscr{W}} = (\|x' + Jx\|_{\mathscr{V}'}^2 + |x(0)|_H^2)^{1/2} \quad \text{for all} \quad x \in \mathscr{W} , \tag{3.2}$$

where J is the duality map, i.e. the canonical isomorphism from V onto V' such that $|u|_V^2 = \langle Ju, u \rangle = |Ju|_{V'}^2$ for all $u \in V$.

Evidently

$$\|x\|_{\mathscr{X}} = \max_{t \in I} \left(\int_0^t \langle x' + Jx, x \rangle \, ds + |x(0)|_H^2/2 \right)^{1/2} \quad \text{for all } x \in \mathscr{W} . \tag{3.3}$$

$\|.\|_{\mathscr{W}}$ is equivalent to any canonical norm on $\mathscr{W}$ because

$$\begin{aligned}
&\max[\|x\|_{C_0(I,H)}, (\|x\|_{\mathscr{V}}^2 + \|x'\|_{\mathscr{V}'}^2)^{1/2}] \leq \\
&\leq \max_{t \in I} \left(|x(t)|_H^2 + \int_0^t |x'(s)|_{V'}^2 \, ds + \int_0^t |x(s)|_V^2 \, ds \right)^{1/2} \qquad (3.4) \\
&= \max_{t \in I} \left(2 \int_0^t \langle x', x \rangle \, ds + |x(0)|_H^2 + \int_0^t (\langle x', J^{-1} x' \rangle + \langle Jx, x \rangle) \, ds \right)^{1/2} \\
&= \max_{t \in I} \left(\int_0^t |x' + Jx|_{V'}^2 \, ds + |x(0)|_H^2 \right)^{1/2} \\
&= \|x\|_{\mathscr{W}} \leq (\|x\|_{C_0(I,H)}^2 + \|x\|_{\mathscr{V}}^2 + \|x'\|_{\mathscr{V}'}^2)^{1/2} \quad \text{for all} \quad x \in \mathscr{W}
\end{aligned}$$

and $\mathscr{W} \hookrightarrow C_0(I, H)$.

$\|\cdot\|_{\mathscr{X}}$ and $\|\cdot\|_{\mathscr{W}}$ are specially tailored to fit the structure of equation (1.2). The strong monotonicity of A directly yields the stability of the solution of (1.2) with respect to the forcing function and the initial value.

LEMMA 3.1 : *If $f_1, f_2 \in \mathscr{V}'$, $z_1, z_2 \in H$ and $x_1, x_2 \in \mathscr{W}$ are the solutions of the equations*

$$\left.\begin{aligned} x' + Ax = f_i\,, \quad x \in \mathscr{W}, \\ x(0) = z_i \end{aligned}\right\} \quad for \quad i = 1, 2\,, \tag{3.5}$$

then

$$\|x_1 - x_2\|_{\mathscr{X}} \leq \|f_1 - f_2\|_{\mathscr{V}'} + |z_1 - z_2|_H/\sqrt{2}\,. \tag{3.6}$$

Proof: Let $w = x_1 - x_2$. (3.6) follows immediately from (3.3) and (3.5) via the estimate

$$\begin{aligned} \|w\|_{\mathscr{X}}^2 &= \max_{t \in I} \int_0^t \langle w' + Jw, w \rangle \, ds + |w(0)|_H^2/2 \\ &\leq \max_{t \in I} \int_0^t \langle x_1' - x_2' + Ax_1 - Ax_2, w \rangle \, ds + |z_1 - z_2|_H^2/2 \\ &= \max_{t \in I} \int_0^t \langle f_1 - f_2, w \rangle \, ds + |z_1 - z_2|_H^2/2 \\ &\leq \|w\|_{\mathscr{X}} \left(\|f_1 - f_2\|_{\mathscr{V}'} + |z_1 - z_2|_H/\sqrt{2}\right). \end{aligned}$$

□

LEMMA 3.2 : *If moreover A is lipschitzian and L' is a Lipschitz-constant of $A - J$, then*

$$\|x_1 - x_2\|_{\mathscr{W}} \leq (1 + L')(\|f_1 - f_2\|_{\mathscr{V}'} + |z_1 - z_2|_H)\,. \tag{3.7}$$

Proof: With the same notation as above the assumptions yield

$$\begin{aligned} \|w\|_{\mathscr{W}}^2 &\leq \int_I \langle w' + Jw, J^{-1}(x_1' - x_2' + Ax_1 - Ax_2) \rangle \, ds \\ &\quad + \int_I \langle w', J^{-1}((J - A)\, x_1 - (J - A)\, x_2) \rangle \, ds + |z_1 - z_2|_H^2 \\ &\leq \int_I \langle w' + Jw, J^{-1}(f_1 - f_2) \rangle \, ds \\ &\quad + L' \, \|w'\|_{\mathscr{V}'} \, \|x_1 - x_2\|_{\mathscr{V}} + |z_1 - z_2|_H^2 \\ &\leq \|w\|_{\mathscr{W}} (\|f_1 - f_2\|_{\mathscr{V}'} + L' \, \|x_1 - x_2\|_{\mathscr{X}} + |z_1 - z_2|_H)\,. \end{aligned}$$

Combined with Lemma 3.1 this proves (3.7). □

Due to Lemmas 3.1 and 3.2 for any $f_0 \in \mathcal{V}'$ and $z_0 \in H$ the solution x_0 of (1.2) is a priori bounded such that

$$\|x_0\|_{\mathcal{X}} \leq \|f_0 - A0\|_{\mathcal{V}'} + |z_0|_H/\sqrt{2}. \tag{3.8}$$

In addition, if A is lipschitzian, then

$$\|x_0\|_{\mathcal{W}} \leq (1 + L')(\|f_0 - A0\|_{\mathcal{V}'} + |z_0|_H). \tag{3.9}$$

THEOREM 3.1 : *Let* $x_0 \in \mathcal{W}$ *be the solution of equation* (1.2) *for some* $f_0 \in \mathcal{V}'$ *and* $z_0 \in H$.

(i) *If* Δ *is a mesh on* I *and* $x_\Delta \in \mathcal{X}_\Delta$ *is the solution of equation* (2.5) *for some* $f_\Delta \in \mathcal{V}'_\Delta$ *and* $z_\Delta \in V$, *then*

$$\|x_\Delta - x_0\|_{\mathcal{X}} \leq \|(I - p_\Delta) Ax_\Delta + (f_\Delta - f_0)\|_{\mathcal{V}'} + |z_\Delta - z_0|_H/\sqrt{2}. \tag{3.10}$$

(ii) *If moreover* A *is Lipschitz-continuous and* L' *is a Lipschitz-constant of* $A - J$, *then*

$$\|x_\Delta - x_0\|_{\mathcal{W}} \leq (1 + L')(\|(I - p_\Delta) Ax_\Delta + (f_\Delta - f_0)\|_{\mathcal{V}'} + |z_\Delta - z_0|_H). \tag{3.11}$$

Proof: Because x_Δ is the solution of the equation

$$x' + Ax = (I - p_\Delta) Ax_\Delta + f_\Delta, \quad x \in \mathcal{W},$$
$$x(0) = z_\Delta,$$

estimates (3.12-3.13) follow directly from Lemmas 3.1 and 3.2. □

Theorem 3.1 provides a convenient basis for error estimation and step-size control for arbitrary meshes Δ on I as soon as the a posteriori bounds in (3.10-3.11) can be traced back on a priori estimates for approximating the data in (1.2), i.e. on $\|f_\Delta - f_0\|_{\mathcal{V}'}$ and $|z_\Delta - z_0|_H$.

LEMMA 3.3 : *If* A *is Lipschitz-continuous with a constant* L *and* Δ *is a mesh on* I, *then*

$$\|(I - p_\Delta) Ax\|_{\mathcal{V}'} \leq L\left(\sum_{i=1}^{n_\Delta} h_i^2 \int_{q_i} |x'|_V^2\, ds/3\right)^{1/2} \leq Lh_\Delta \|x'\|_{\mathcal{V}}/\sqrt{3} \tag{3.12}$$

or, equivalently,

$$\|(I - p_\Delta) Ax\|_{\mathcal{V}'} \leq L\left(\sum_{i=1}^{n_\Delta} h_i |x(t_i) - x(t_{i-1})|_V^2/3\right)^{1/2}. \tag{3.13}$$

for all $\Delta \in \Theta$ *and* $x \in X_\Delta$.

Proof: If $x \in \mathscr{X}_\Delta$, then actually holds

$$\begin{aligned}\|(I - p_\Delta) Ax\|^2_{\mathscr{V}'} &= \sum_{i=1}^{n_\Delta} \int_{q_i} |Ax(s) - Ax(t_i)|^2_{V'} ds \\ &\leq L^2 \sum_{i=1}^{n_\Delta} \int_{q_i} |x(s) - x(t_i)|^2_V ds = L^2 \sum_{i=1}^{n_\Delta} h_i^3 |(x')^i|^2_V/3 \\ &= L^2 \sum_{i=1}^{n_\Delta} h_i^2 \int_{q_i} |x'|^2_V ds/3 .\end{aligned}$$

Bounding the step-sizes by their maximum completes the proof. □

LEMMA 3.4: *Let* $f_0 \in W_2^1(I, V')$.

(i) *If* $\Delta \in \Theta$ *and* x_Δ *is the solution of equation* (2.5) *for* $f_\Delta = p_\Delta f_0$ *and some* $z_\Delta \in V$, *then*

$$\max[\|x'_\Delta\|_{L_\infty(I,H)}, \|x'_\Delta\|_{\mathscr{V}}] \leq \|f'_0\|_{\mathscr{V}'} + |Az_\Delta - f_0(0)|_{V'}/\sqrt{h_1} . \tag{3.14}$$

(ii) *If, in addition,* $z_0 \in V$, $Az_0 - f_0(0) \in H$, $|z_\Delta - z_0|^2_V \leq h_1$ *and* L *is a Lipschitz-constant of* A, *then*

$$\max[\|x'_\Delta\|_{L_\infty(I,H)}, \|x'_\Delta\|_{\mathscr{V}}] \leq K_0 \tag{3.15}$$

for $K_0 = \|f'_0\|_{\mathscr{V}} + |Az_0 - f_0(0)|_H + L$.

Proof: (i) Let $v = x'_\Delta$. (2.5) implies

$$|v^1|^2_H + h_1 |v^1|^2_V \leq |(f_0(t_1) - f_0(t_0)) - (Az_\Delta - f_0(0))|^2_{V'}/h_1$$

and

$$|v^l|^2_H + h_l |v^l|^2_V \leq |(f_0(t_l) - f_0(t_{l-1}))|^2_{V'}/h_l + |v^{l-1}|^2_H \text{ for } 2 \leq l \leq n_\Delta .$$

Hence

$$\begin{aligned}|v^l|^2_H + \sum_{i=1}^{l} h_i |v^i|^2_V &\leq \sum_{i=2}^{l} |(f_0(t_i) - f_0(t_{i-1}))|^2_{V'}/h_i + \\ &+ |(f_0(t_1) - f_0(t_0)) - (Az_\Delta - f_0(0))|^2_{V'}/h_1 \quad \text{for} \quad 1 \leq l \leq n_\Delta\end{aligned}$$

and therefore (3.14) is valid.

(ii) The additional assumption about z_0 yields the alternative estimate

$$|v^1|^2_H + h_1 |v^1|^2_V \leq |(f_0(t_1) - f_0(t_0)) - (Az_\Delta - Az_0)|^2_{V'}/h_1 + \\ + |Az_0 - f_0(0)|^2_H$$

which results in

$$\max \left[\| x'_\Delta \|_{L_\infty(I, H)}, \ \| x'_\Delta \|_{\mathscr{V}'} \right] \leq$$
$$\leq \| f'_0 \|_{\mathscr{V}'} + | Az_0 - f_0(0) |_H + | Az_\Delta - Az_0 |_{V'} / \sqrt{h_1} \,.$$

Due to the Lipschitz-continuity of A this proves (3.15). □

Combining Lemmas 3.3, 3.4 and Theorem 3.1 results in a priori estimates of the approximation errors for arbitrary meshes and the special choice $f_\Delta = p_\Delta f_0$. As discussed in Section 2, there is a serious demand for similar results in more general situations which are governed by condition (2.7). Under some more restrictive condition than (2.7) and for subsequences of meshes Δ which are inversely regular, i.e. $\sup_{2 \leq i \leq n_\Delta} h_{i-1}/h_i$ is uniformly bounded, Gröger [6] derived an a priori bound like (3.15). The step by step realizations of (2.5) and inverse regularity assumptions are incompatible, however.

LEMMA 3.5 : *If* $\Delta \in \Theta$, $f_1, f_2 \in \mathscr{V}'_\Delta$, $z_1, z_2 \in V$ *and* $x_1, x_2 \in \mathscr{X}_\Delta$ *are the solutions of the equations*

$$\left. \begin{aligned} x' + p_\Delta A x &= f_i, \ x \in \mathscr{X}_\Delta, \\ x(0) &= z_i \end{aligned} \right\} \quad \text{for} \quad i = 1, 2 \,, \tag{3.16}$$

then

$$\max \left[\| x_1 - x_2 \|_{C^0(I, H)}, \ \| p_\Delta(x_1 - x_2) \|_{\mathscr{V}} \right] \leq \| f_1 - f_2 \|_{\mathscr{V}'} + | z_1 - z_2 |_H \,. \tag{3.17}$$

Proof: Let $w = x_1 - x_2$. (3.16) and the strong monotonicity of A imply

$$| w(t_l) |_H^2 + h_l | w(t_l) |_V^2 \leq h_l | f_1^l - f_2^l |_{V'}^2 + | w(t_{l-1}) |_H^2$$

and therefore

$$| w(t_l) |_H^2 + \sum_{j=1}^{l} h_j | w(t_j) |_V^2 \leq \sum_{j=1}^{l} h_j | f_1^j - f_2^j |_{V'}^2 + | w(t_0) |_H^2$$
$$\leq \| f_1 - f_2 \|_{\mathscr{V}'}^2 + | z_1 - z_2 |_H^2 \quad \text{for} \quad 1 \leq l \leq n_\Delta \,,$$

such that (3.17) is valid. □

LEMMA 3.6 : *Let* $f_0 \in W_2^1(I, V')$.

(i) *If* $\Delta \in \Theta_\rho$ *and* x_Δ *is the solution of equation* (2.5) *for some* $f_\Delta \in \mathscr{V}'_\Delta$ *which satisfies* (2.7) *and some* $z_\Delta \in V$, *then, with* $M_\gamma = \| f'_0 \|_{\mathscr{V}'} + (1 + \rho) \gamma$,

$$\left(\sum_{i=1}^{n_\Delta} h_i^2 \int_{q_i} |x'_\Delta|_V^2 \, ds\right)^{1/2} \leq h_\Delta(M_\gamma + |Az_\Delta - f_0(0)|_{V'}/\sqrt{h_1}) . \qquad (3.18)$$

(ii) *The additional assumptions of Lemma* 3.4 (ii) *imply that*

$$\left(\sum_{i=1}^{n_\Delta} h_i^2 \int_{q_i} |x'_\Delta|_V^2 \, ds\right)^{1/2} \leq h_\Delta K_\gamma , \qquad (3.19)$$

for $K_\gamma = K_0 + (1 + \rho)\gamma$.

Proof: Let $\bar{x}_\Delta$ be the solution of (2.5) for $\bar{f}_\Delta = p_\Delta f_0$ and let $w = x_\Delta - \bar{x}_\Delta$. Then

$$\|p_\Delta w\|_{\mathcal{V}} \leq \|f_\Delta - p_\Delta f_0\|_{\mathcal{V}'} \leq h_\Delta \gamma$$

as a consequence of Lemma 3.5 and (2.7). Because of

$$\begin{aligned}\left(\sum_{i=1}^{n_\Delta} h_i^2 \int_{q_i} |x'_\Delta|_V^2 \, ds\right)^{1/2} &\leq \left(\sum_{i=1}^{n_\Delta} h_i^2 \int_{q_i} |\bar{x}'_\Delta|_V^2 \, ds\right)^{1/2} \\ &\quad + \left(\sum_{i=1}^{n_\Delta} h_i |w(t_i) - w(t_{i-1})|_V^2\right)^{1/2} \\ &\leq h_\Delta \|\bar{x}'_\Delta\|_{\mathcal{V}} + (1+\rho)\|p_\Delta w\|_{\mathcal{V}} ,\end{aligned}$$

(3.18)-(3.19) follow immediately from Lemma 3.4. □

THEOREM 3.2: *Let A be Lipschitz-continous with a constant L. Let $x_0 \in \mathcal{W}$ be the solution of equation* (1.2) *for some $z_0 \in H$ and $f_0 \in W_2^1(I, V')$.*

(i) *If $\Delta \in \Theta_\rho$ and $x_\Delta \in \mathcal{X}_\Delta$ is the solution of equation* (2.5) *for some $f_\Delta \in \mathcal{V}'_\Delta$ which satisfies* (2.7) *and some $z_\Delta \in V$, then*

$$\begin{aligned}\|x_\Delta - x_0\|_{\mathcal{X}} &\leq h_\Delta L(M_\gamma + |Az_\Delta - f_0(0)|_{V'}/\sqrt{h_1})/\sqrt{3} \\ &\quad + h_\Delta C_\gamma + |z_\Delta - z_0|_H/\sqrt{2} . \qquad (3.20)\end{aligned}$$

(ii) *If moreover $z_0 \in V$, $Az_0 - f_0(0) \in H$ and $|z_\Delta - z_0|_V^2 \leq h_1$, then*

$$\|x_\Delta - x_0\|_{\mathcal{X}} \leq h_\Delta(LK_\gamma/\sqrt{3} + C_\gamma) + |z_\Delta - z_0|_H/\sqrt{2} . \qquad (3.21)$$

Proof: (3.20)-(3.21) result from a straightforward combination of Theorem 3.1 (i), Lemmas 3.3 and 3.6 and estimate (2.8). □

The same hypotheses support similar estimates of $\| x_\Delta - x_0 \|_{\mathscr{W}}$. For fixed values of γ and ρ, some additional assumptions concerning the initial values and the size of the initial step in Δ yield various estimates of the order of convergence relative to h_Δ.

COROLLARY 3.1 : *If, in addition to the assumptions of Theorem* 3.2 (i), $h_\Delta \leq c_1^2 h_1$, $z_0 \in V$, $|z_\Delta - z_0|_V \leq c_2$, *and* $|z_\Delta - z_0|_H^2 \leq 2 h_\Delta$, *then*

$$\| x_\Delta - x_0 \|_{\mathscr{X}} \leq \sqrt{h_\Delta}(L(M_\gamma + c_1 |Az_0 - f_0(0)|_{V'} + Lc_1 c_2)/\sqrt{3} + C_\gamma + 1) . \tag{3.22}$$

COROLLARY 3.2 : *If the assumptions of Theorem* 3.2 (ii) *hold and* $|z_\Delta - z_0|_H \leq h_1 \sqrt{2}$, *then*

$$\| x_\Delta - x_0 \|_{\mathscr{X}} \leq h_\Delta(LK_\gamma/\sqrt{3} + C_\gamma + 1) . \tag{3.23}$$

Except for (3.22), Theorem 3.2 and its corollaries hold without any serious restrictions on the underlying mesh and without any a priori regularity assumptions concerning the solution x_0 of the original equation (1.2). For $\gamma = 0$ the bounds are independent of ρ, thus they hold for arbitrary meshes $\Delta \in \Theta$.

In fact, the bounds from (3.20)-(3.23) majorize the term on the right hand side of (3.10). Hence the a posteriori estimates from Theorem 3.1 are convergent with the same rate as the approximations themselves. Because any bound involving the maximum step-size will be rather pessimistic, the estimates (3.20)-(3.23) are mainly of theoretical interest.

With respect to a single mesh $\Delta \in \Theta_\rho$, the constant γ may be regarded as a free parameter. Defining $\delta = \gamma h_\Delta$, the combination of (3.12) and (3.18) reads as

$$\begin{aligned} \| (I - p_\Delta) Ax \|_{\mathscr{V}'} \leq Lh_\Delta(\| f_0' \|_{\mathscr{V}'} + |Az_\Delta - f_0(0)|_{V'}/\sqrt{h_1})/\sqrt{3} \\ + L(1 + \rho)\,\delta/\sqrt{3} . \end{aligned} \tag{3.24}$$

Thus, with an a priori information about L, for each fixed tolerance level $\kappa > 0$ the bound in (2.9) can be chosen independently of h_Δ such that $\| x_\Delta - x_0 \|_{\mathscr{X}} \leq \kappa$ for appropriate meshes $\Delta \in \Theta_\rho$.

4. STEPSIZE CONTROL FOR A LINEAR EQUATION

The most effective way to approximate the solution x_0 of equation (1.2) within a prescribed tolerance is the simultaneous step by step construction of an appropriate mesh Δ and the solution x_Δ (defining f_Δ) under control of the estimates (3.10)-(3.11). Because the evaluation of the a posteriori bounds of Theorem 3.1 in no way is a trivial task for general nonlinear equations, the investigation is subsequently restricted to the most simple linear case.

Assume A is linear and symmetric and f_0 is a constant. Without loss of generality let $A = J$ such that the constants L and L' in the estimates of Section 3 are 1 and 0, respectively. In particular

$$\| (I - p_\Delta) Ax \|_{\mathscr{V}'} = \left(\sum_{i=1}^{n_\Delta} h_i |x(t_i) - x(t_{i-1})|_V^2 / 3 \right)^{1/2} \tag{4.1}$$

for all $\Delta \in \Theta$ and $x \in X_\Delta$.

Based on this equality, various step-size control algorithms can be derived in a straightforward way.

ALGORITHM 4.1 :

0. *Let* $\kappa > 0$ *and* $0 < h_1 \leq 1$.
1. $i = 0$. $t_0 = 0$. *Choose* $u_0 \in V$ *such that* $|u_0 - z_0|_H \leq \kappa/2$.
2. $i = i + 1$.
3. *Choose* $u_i \in V$ *such that* $|(u_i - u_{i-1})/h_i + Au_i - f_0|_{V'} \leq \kappa/2$.
4. $k = [-ld(\max[1/2, \sqrt{h_i/3}\,|u_i - u_{i-1}|_V/\kappa])]$ *(integer part)*.
5. *If* $k \leq -1$, *then* $h_i = 2^k h_i$ *and goto* 3.
6. $t_i = t_{i-1} + h_i$. *If* $t_i < 1$, *then* $h_{i+1} = \min[2^k h_i, 1 - t_i]$ *and goto* 2.
7. $n_\Delta = i$.
8. *Stop.*

The same arguments as in the proof of Lemma 3.6 combined with estimate (3.14) now prove that Algorithm 4.1 is always finite. Especially, if again $\bar{x}_\Delta$ denotes the result of the exact implicit Euler method on the same mesh $\Delta \in \Theta_{\sqrt{2}}$, then

$$\sqrt{h_i}\,|u_i - u_{i-1}|_V \leq h_i \|\bar{x}'_\Delta\|_{\mathscr{V}} + (1 + \sqrt{2})\,\kappa/2 \quad \text{for} \quad 1 \leq i \leq n_\Delta . \tag{4.2}$$

If $x_\Delta \in X_\Delta$ is defined by $x_\Delta(t_i) = u_i$ for $0 \leq i \leq n_\Delta$ then

$$\begin{aligned} \|x_\Delta - x_0\|_{\mathscr{W}} &\leq \|(I - p_\Delta) Ax_\Delta\|_{\mathscr{V}'} + \|f_\Delta - f_0\|_{\mathscr{V}'} + |u_0 - z_0|_H \\ &\leq (\sqrt{n_\Delta} + 1)\,\kappa . \end{aligned} \tag{4.3}$$

Algorithm 4.1 approximately equidistributes the terms in the sum of (4.1) (error per step control). Though this strategy is known to be optimal for more regular problems, it lacks from the fact that n_Δ cannot be determined in advance. At least under the assumptions of Lemma 3.4 the step-sizes are uniformly bounded from below by a multiple of κ, hence $\|x_\Delta - \bar{x}_0\|_{\mathscr{W}} = 0(\sqrt{\kappa})$ for κ small enough.

The alternative error per unit step control is justified only within the setting of Lemma 3.4, recalling the argumentation from the proof of (3.14) and (4.2). Some estimates like (3.14) seem to be necessary to prove the algorithm to be finite (in fact the generated approximation x_Δ must be bounded in $C_0(I, V)$).

ALGORITHM 4.2 :

0. *Let* $\kappa > 0$, *and* $0 < h_1 \leq 1$.
1. $i = 0$. $t_0 = 0$. *Choose* $u_0 \in V$ *such that* $|u_0 - z_0|_H \leq \kappa/2$.
2. $i = i + 1$.
3. *Determine* $u_i \in V$ *such that* $(u_i - u_{i-1})/h_i + Au_i = f_0$.
4. $k = [-ld(\max [1/2, |u_i - u_{i-1}|_V/(\sqrt{3}\,\kappa/2)])]$ *(integer part).*
5. *If* $k \leq -1$, *then* $h_i = 2^k h_i$ *and goto* 3.
6. $t_i = t_{i-1} + h_i$. *If* $t_i < 1$, *then* $h_{i+1} = \min [2^k h_i, 1 - t_i]$ *and goto* 2.
7. $n_\Delta = i$.
8. *Stop.*

For each tolerance level $\kappa > 0$ this algorithm stops after a finite number of steps and determines a mesh $\Delta \in \Theta_{\sqrt{2}}$ and some $x_\Delta \in X_\Delta$ with $x_\Delta(t_i) = u_i$ for $0 \leq i \leq n_\Delta$ such that

$$\|x_\Delta - x_0\|_{\mathscr{W}} \leq \|(I - p_\Delta) Ax_\Delta\|_{\mathscr{V}'} + |u_0 - z_0|_H \leq \kappa . \tag{4.4}$$

Obviously the realization of any of these algorithms depends on the availability of some algorithms which perform the determination of the $u_i \in V$ for $0 \leq i \leq n_\Delta$.

Examples :

Some specific aspects concerning a realization of the implicit Euler method, in particular of Algorithms 4.1 and 4.2, are discussed for the diffusion equation

$$\left.\begin{aligned} x_t - \operatorname{div}(a \,.\, \operatorname{grad} x) + cx &= f_0 \text{ on } \Omega \times I \\ x &= 0 \text{ on } \partial\Omega \times I \\ x(.,0) &= z_0 \text{ on } \Omega, \end{aligned}\right\} \tag{4.5}$$

where $\Omega \subseteq R^m$, $1 \leq m \leq 3$, is a bounded, simply connected and polyhedral domain, $a, c \in L_\infty(\Omega)$, $a(\omega) \geq a_0 > 0$, $c(\omega) \geq 0$ for all $\omega \in \Omega$, $z_0 \in L_2(\Omega)$ and $f_0 \in (W_2^1(\Omega))'$.

1. Let $H_0 = L_2(\Omega)$, $H_1 = \mathring{W}_2^1(\Omega)$ and $H_{-1} = H_1'$. The scalar products $(.,.)_0$, $(.,.)_1$ and the duality map $A \in L(H_1, H_{-1})$ are defined by

$$\left.\begin{aligned}(u, v)_0 &= \int_\Omega uv \, d\omega \quad \text{for all } u, v \in H_0, \\ (u, v)_1 &= \langle Au, v \rangle_1 = \int_\Omega (a \,.\, \operatorname{grad} u^T \operatorname{grad} v + cuv) \, d\omega \\ &\qquad \text{for all } u, v \in H_1 .\end{aligned}\right\} \tag{4.6}$$

Then $H_1 \hookrightarrow H_0 \hookrightarrow H_{-1}$ is an appropriate triple of Hilbert spaces and equation (1.2) for

$$\mathscr{W} = \{x \in L_2(I, H_1) \,|\, x' \in L_2(I, H_{-1})\} \subseteq \mathscr{X} = L_2(I, H_1) \cap C_0(I, H_0)$$

is the weak formulation of (4.5).

The implicit Euler method is determined by (2.6), thus resulting in a system of elliptic boundary value problems. With respect to Algorithm 4.1 it is advisable to solve these equations approximately by means of an adaptive conforming finite element method. That is, the selection of u_i in step 3 is carried out by a feedback algorithm which successively produces meshes on Ω and the corresponding finite element solutions of the i-th difference equation from (2.6) until the stopping criterion of step 3 is satisfied. u_0 is constructed by a similar algorithm based on information about z_0. General principles for such methods originate from Babuška and Rheinboldt [2] (*cf.* Lippold [10] and references for specific topics on adaptivity as well as Bietermann, Babuška [3] and Reiher [14] for applications in the field of parabolic equations).

On the other hand, the bounds from Theorem 3.1 can also be used within the classical approach starting from a primary space discretization. Then the resulting evolution equation in finite dimensional spaces is approximately solved by the implicit Euler method, e.g. by Algorithm 4.2. The combined effects of space and time discretization, numerical integration etc. are supervised via (2.9) and (3.10)-(3.11), applied for the original equation.

2. Let $H_2 = A^{-1} H_0 = \{u \in H_1 \,|\, a \,.\, \operatorname{grad} u \in H(\Omega, \operatorname{div})\}$ and let $\Lambda \in L(H_2, H_0)$ be the restriction of A to H_2, i.e.

$$\Lambda u = -\operatorname{div} (a \,.\, \operatorname{grad} u) + cu \quad \text{for all} \quad u \in H_2 . \tag{4.7}$$

A canonical scalar product $(.,.)_2$ on H_2 is determined by

$$(u, v)_2 = (\Lambda u, \Lambda v)_0 \quad \text{for all} \quad u, v \in H_2 . \tag{4.8}$$

Now $H_2 \hookrightarrow H_1 \hookrightarrow H_0$ is the space triple under consideration. By definition

$$\langle v, u \rangle_2 = (v, \Lambda u)_0 \quad \text{for all} \quad v \in H_0 , \quad u \in H_2 , \tag{4.9}$$

hence Λ is the duality map of H_2 onto H_0.

If $f_0 \in L_2(\Omega)$ and $z_0 \in H_1$, then the equation

$$\left.\begin{aligned} x' + \Lambda x = f_0, x \in \mathcal{W}, \\ x(0) = z_0, \end{aligned}\right\} \tag{4.10}$$

now for

$$\mathcal{W} = \{x \in L_2(I, H_2) \mid x' \in L_2(I, H_0)\} \subseteq \mathcal{X} = L_2(I, H_2) \cap C_0(I, H_1),$$

fits into the general setting of equation (1.2). Theoretically all arguments from the preceding example remain valid. From a practical point of view, however, the space discretization by means of conforming finite elements in H_2 will cause serious difficulties at least for $m > 1$. Thomée and Wahlbin investigated the primary space discretization of (4.10). Motivated by the underlying space triple the method was called an H_1 Galerkin method.

3. An alternative choice is the investigation of (4.5) in the triple $H_0 \hookrightarrow H_{-1} \hookrightarrow H_{-2}$ where H_{-2} is the dual of H_2 in the triple $H_2 \hookrightarrow H_0 \hookrightarrow H_{-2}$ too. Using the duality map Λ^* of H_0 onto H_{-2}, i.e.

$$\langle \Lambda^* u, v \rangle_0 = (u, v)_0 \quad \text{for all} \quad u, v \in H_0, \tag{4.11}$$

and $\mathcal{W} = \{x \in L_2(I, H_0) \mid x' \in L_2(I, H_{-2})\}$, equation (4.5) can be written in a very weak form as

$$\left.\begin{aligned} x' + \Lambda^* x = f_0, x \in \mathcal{W}, \\ x(0) = z_0 . \end{aligned}\right\} \tag{4.12}$$

There is a wide variety of numerical methods which attack the space discretization of (4.12), one of them is the H_{-1} Galerkin method due to Wheeler [18].

An interesting topic is the combination of the conforming finite element approximations in H_1 from the first example with the error estimates in this very weak formulation. The defects in (2.9) or step 3 of Algorithm 4.1, respectively, now are bounded in H_{-2}, i.e. the accuracy of the solutions of the difference equations is measured in H_0. At the same time the step control is based on the values $\sqrt{h_i}\, |u_i - u_{i-1}|_{H_0}$ or $|u_i - u_{i-1}|_{H_0}$, $1 \le i \le n_\Delta$, respectively, thus confirming an approach which, motivated by physical reasoning, has been used for a long time in the numerical analysis of parabolic differential equations.

Based on some more restrictive assumptions already mentioned in the introduction, Johnson, Nie and Thomée proved in [7] that bounding the terms $|u_i - u_{i-1}|_{H_0}$, $1 \le i \le n_\Delta$, results in an optimal step-size control for an error estimate in $L_\infty(I, H_0)$ (Theorem 3.1 only provides an estimate in $L_2(I, H_0)$).

In fact there ıs a sequence $(H_k)_\Gamma$ of spaces whıch are defined together wıth the correspondıng scalar products by means of the subsequent ınteger powers of A The choıce of examples 1 to 3 ıs ın correspondence wıth the needs for most applıcatıons

ACKNOWLEDGEMENT

The author ıs grateful to the edıtor end hıs referee for very helpful suggestıons ın the revısıon of thıs paper

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