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ATTRACTOR FOR A NAVIER-STOKES FLOW IN AN UNBOUNDED DOMAIN

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Abstract. — *We present an existence result for the global attractor associated to the Navier-Stokes Equations in an infinite strip in $\mathbb{R}^2$, and provide an estimate for its fractal dimension in terms of the Reynolds number*

0. INTRODUCTION

It is commonly agreed upon that a thorough understanding of the long-time behaviour of Navier-Stokes fluids is essential in many respects. Starting with the results of C. Foias and R. Temam [F-T] is a series of papers [C-F] [C-F-M-T] [C-F-T (1)] [C-F-T (2)] leading to the following conclusion : when the domain enclosing the fluid is a smooth bounded open set of $\mathbb{R}^2$, the dynamical system associated to the Navier-Stokes equations possesses a global attractor of finite fractal dimension. In other words, the asymptotic behavior of such a system is determined by a finite-dimensional object. Unfortunately, there are numerous physically important situations that are not covered by this result : the three dimensional case is still a partly open problem, and even in two dimensions, the case of an unbounded domain is still unsolved. In this paper, we want to present a result that extends the 2-D theory to some particular unbounded domains of $\mathbb{R}^2$. Specifically, we consider the flow of a viscous fluid in an infinite strip $\Omega = \mathbb{R} \times (0, \ell)$ of $\mathbb{R}^2$; such a flow is classically modelled by the following system of equations :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in} \quad \Omega , \\ \nabla \cdot u = 0 \quad \text{in} \quad \Omega , \\ u(x, 0 ; t) = u(x, \ell ; t) = 0 \quad \text{for all} \quad t\text{'s} , \\ u(. , . ; 0) = u_0 \quad \text{in} \quad \Omega . \end{cases}$$

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The viscosity ν is > 0, u_0 is in $(L^2(\Omega))^2$, $\nabla \cdot u_0 = 0$, and the forcing term is in $(L^2(\Omega))^2$. Due to the particular geometry of Ω, the classical results of existence and regularity on a bounded domain [T2, chap. III] can be easily extended. However, this is not so for the existence of a global attractor, mainly because of the non-compactness of Sobolev's imbeddings.

Some recent results [A] are used to overcome this difficulty, and lead to a satisfactory conclusion : if the forcing term f is small enough for large x's, we prove the existence of a finite dimensional global attractor for (0.1), and give an estimate of its fractal dimension. These results hinge upon some time-dependent weighted estimates for the solution u of (0.1), which require a careful treatment of the pressure p.

The paper is organized as follows : in section I, we recall the mathematical setting adapted to (0.1), as well as the existence and regularity results ; section II consists in an exposition of the results of [A] to be used for this particular problem ; finally, in section III, we state and prove our main result, the existence and finite dimensionality of the global attractor associated to (0.1). Our notations are those commonly used in the theory of Navier-Stokes equations [T1], and we may use the letter C rather carelessly to denote a strictly positive constant.

I. NAVIER-STOKES EQUATIONS IN AN INFINITE STRIP

We let Ω be the strip $\mathbb{R} \times (0, \ell)$ in $\mathbb{R}^2$; the classical formulation of the Navier-Stokes equations in Ω is :

(N-S) To find a vector-valued function u and a scalar function p, defined in Ω, and meeting the following requirements :

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in} \quad \Omega, \tag{I.1}$$

$$u \text{ is divergence free : } \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \tag{I.2}$$

$$u(\,.\,, 0\,; t) = u(\,.\,, \ell\,; t) = 0 \quad \text{for all} \quad t\text{'s}, \tag{I.3}$$

$$u(\,.\,,\,.\,; 0) = u_0 \quad \text{in} \quad \Omega. \tag{I.4}$$

The right-hand side f of (I.1) is in $(L^2(\Omega))^2$, and so is the initial datum u_0 ; the viscosity ν is > 0. The suitable framework for this problem is by now classical [T1] : we set

$$\mathscr{V} = \{v \in (\mathscr{D}(\Omega))^2, \nabla \cdot v = 0 \text{ in } \Omega\},$$

$V =$ the closure of $\mathscr{V}$ in $(H^1(\Omega))^2$, $H =$ the closure of $\mathscr{V}$ in $(L^2(\Omega))^2$. We then define the operator $A = -\nu P \Delta$ (the Stokes operator), where P is the projector from $(L^2(\Omega))^2$ onto H, and the nonlinear operator $B(u) \equiv \mathscr{B}(u, u)$, where

$$\langle \mathscr{B}(u, v), w \rangle_{V' \times V} = \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx . \tag{I.5}$$

It is well known that determining the velocity field for the flow in the strip amounts to solving the abstract evolution equation :

$$\begin{cases} \dfrac{du}{dt} + Au + B(u) = Pf \quad \text{in} \quad H , \\ u(0) = u_0 \quad \text{in} \quad H , \quad u \in V . \end{cases} \tag{I.6}$$

We now state the existence and regularity result for (I.1)-(I.4) :

PROPOSITION I.1 : *For every f in $(L^2(\Omega))^2$, u_0 in H, there exists a unique u in $C_b([0, +\infty) ; H) \cap L^2_{\mathrm{loc}}(0, +\infty ; V)$, and a function p defined up to a constant, such that $(u ; p)$ solves* (I.1)-(I.4) ; *furthermore, for every $t_0 > 0$, u is in $L^\infty(t_0, +\infty ; (H^2(\Omega))^2)$, and ∇p is in $L^\infty(t_0, +\infty ; L^2(\Omega))^2)$.*

Sketch of the proof: the existence of a solution u in $L^\infty(0, +\infty ; H) \cap L^2_{\mathrm{loc}}(0, +\infty ; V)$ is standard ; to prove further regularity results on u and p, we use the solution u_n of N.S.E. in the truncated strip $\Omega_n = (-n, n) \times (0, \ell)$. u_n converges to u as $n \to +\infty$, and satisfies the uniform (in n) estimate, for a given $t_0 > 0$:

$$\sup_{t_0 \le t < +\infty} \int_{\Omega_n} \left| \frac{\partial u_n}{\partial t} \right|^2 + |\Delta u_n|^2 \, dx \, dy < C .$$

This estimate is a consequence of the analyticity results of [F-T], and a careful examination of the analysis in the latter paper shows that the constant in the estimate above can be chosen to depend on ℓ, ν, $\int_\Omega |u_0|^2 \, dx \, dy$, $\int_\Omega |f|^2 \, dx \, dy$, but *not* on n. This is sufficient to prove the required regularity of u and ∇p, thanks to the equivalence of norms proven in the appendix.

To conclude this introductory section, we give a further estimate on the pressure :

LEMMA I.3 : *Let $p(x, y ; t)$ be the pressure field associated to* (I.1)-(I.4) ; *p can be chosen so as to satisfy the following inequality :*

$$\sup_{t \ge t_0} \left(\int_\Omega \frac{|p^2(x, y ; t)|}{(1 + x^2)} \, dx \, dy \right) < C_{t_0} , \tag{I.7}$$

on every interval $[t_0, +\infty]$, $t_0 > 0$.

Proof: Lemma I.3 is a straightforward consequence of Hardy's inequality [HLP] :

$$\int_{-\infty}^{+\infty} \left| \frac{q(x) - q(0)}{x} \right|^2 dx \leqslant 4 \int_{-\infty}^{+\infty} q'^2(x)\, dx ,$$

as we now show. We first extend the inequality above to smooth functions in Ω, and obtain :

$$\int_\Omega \frac{|q(x,y) - q(0,y)|^2}{x^2}\, dx\, dy \leqslant 4 \int_\Omega |\nabla q(x,y)|^2\, dx\, dy ,$$

from which we can derive :

$$\int_\Omega \frac{|q(x,y)|^2}{1+x^2}\, dx\, dy \leqslant 4 \int_\Omega |\nabla q|^2\, dx\, dy + \int_\Omega \frac{|q(0,y)|^2}{1+x^2}\, dx\, dy ,$$

which yields :

$$\int_\Omega \frac{|q(x,y)|^2}{1+x^2}\, dx\, dy \leqslant 4 \int_\Omega |\nabla q|^2\, dx\, dy + \pi \int_0^\ell |q(0,y)|^2\, dy .$$

We then extend the last inequality to functions such that $\nabla q \in (L^2(\Omega))^2$, by regularization ; note that $\int_0^1 q(0,y)^2\, dy$ is well-defined, for $q(0,.) \in H^{1/2}(0,\ell)$ if $\nabla q \in (L^2(\Omega))^2$.

Eventually, we choose a fixed subdomain $\Omega_1 = (-1,1) \times (0,\ell)$ of Ω, say, and determine the constant in p so that the equality $\iint_{\Omega_1} p\, dx\, dy = 0$ holds true for every time $t > 0$. This, together with Poincaré's inequality in Ω_1, implies :

$$p \in L^\infty(t_0, +\infty\,; H^1(\Omega_1)) \tag{I.8}$$

for every $t > 0$, and in particular :

$$p(0,.\,;t) \in L^\infty(t_0, +\infty\,; H^{1/2}(0,\ell))\,; \tag{I.9}$$

this completes the proof of Lemma I.3 □

Remark I.4 : Due to the existence of an absorbing set in V, the constant C_{t_0} can be chosen independently of t_0 and the initial condition u_0, for u_0 in a bounded set in H, and t_0 large enough.

II. EXISTENCE OF THE GLOBAL ATTRACTOR

We start with a statement that stems from the results of [A], and is the key tool for the existence of the global attractor :

LEMMA II.1 : *Let $\Psi(x, t)$ be a smooth weight function meeting the following requirements :*

(II.i) $\Psi(x, t) > 0$ *for* $t > 0$, $\Psi(x, 0) = 0$,
(II.ii) *each derivative of order* ≥ 1 *of* Ψ *is a bounded function,*
(II.iii) $\Psi(x, t) \to +\infty$ *as* $(|x|, t) \to +\infty$,

and let us assume that the velocity field u associated to (I.1)-(I.4) *satisfies the following assumption :*

$$\sup_{t \geq t_0} \int_\Omega |u(x, y, t)|^2 \Psi(x, t)\, dx\, dy < +\infty \tag{II.1}$$

for some $t_0 > 0$; *then, the dynamical system defined by equation* (I.6) *possesses a global attractor* $\mathscr{A}$, *i.e. a compact invariant set in H, which attracts every bounded set of H, and is maximal with respect to these properties.*

Sketch of the proof: Condition (II.1), together with the results in Theorem 1, imply that the ω-limit set of a bounded set of H is compact in H ; the existence of an absorbing set for equation (I.6) in V, proven as in [T2, chap. III], therefore leads to the existence of the global attractor (see [A] for more details) □

We now proceed to prove (II.1), with a specific choice of $\psi(x, t)$, and under some assumptions on the right-hand side f of (I.1).

We set

$$\varphi(x) = \operatorname{Log}\,(1 + x^2) \tag{II.2}$$

and :

$$\Psi(x, t) = \varphi(x)\left(1 - \exp\left(\frac{-t}{\varphi(x)}\right)\right). \tag{II.3}$$

It is straightforward to verify that (II.i)-(II.iii) hold true for this choice of Ψ, and that we have the following bound for the (space) gradient $\nabla\Psi$ of Ψ :

$$\sup_{t \geq 0} |\nabla\Psi(x, t)| \leq \frac{C}{(1 + x^2)^{1/2}}. \tag{II.4}$$

We now state the main result of this section :

THEOREM II.2 : *Let f (in (I.1)) be such that*

$$\int_\Omega |f(x,y)|^2 \varphi(x)\, dx\, dy < +\infty . \tag{II.5}$$

Then, the dynamical system S defined by the solution operator of Equation (I.6) *in H :*

$$S(t) \, . \, u_0 = u(\, . \,, . \,; t)$$

possesses a global attractor $\mathcal{A}$ in H.

Proof: According to Lemma II.1, we need to check assumption (II. 1) ; to do so, we start from (I.1) :

$$\frac{\partial u}{\partial t} - \nu \, \Delta u + (u \, . \, \nabla) u + \nabla p = f ,$$

take the inner product with $(u \, . \, \Psi)$, and integrate on Ω :

$$\frac{1}{2} \frac{d}{dt} \left(\int_\Omega |u|^2 \, \Psi \, dx\, dy \right) - \nu \int_\Omega u \, . \, \Delta u \, \Psi \, dx\, dy + \int_\Omega (u \, . \, \nabla) u \, . \, u \Psi \, dx\, dy$$
$$+ \int_\Omega \nabla p \, . \, u\Psi \, dx\, dy = \int_\Omega f \, . \, u \Psi \, dx\, dy + \frac{1}{2} \int_\Omega |u|^2 \frac{\partial \Psi}{\partial t} \, dx\, dy .$$

Using some straightforward integrations by parts, we thus obtain :

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 \, \Psi \, dx\, dy + \nu \int_\Omega |\nabla u|^2 \, \Psi \, dx\, dy + \int_\Omega \nabla p \, . \, u \Psi \, dx\, dy =$$
$$\int_\Omega (f \, . \, u) \, \Psi \, dx\, dy + \frac{1}{2} \int |u|^2 \left(\frac{\partial \Psi}{\partial t} + \nu \, \Delta \Psi \right) dx\, dy$$
$$+ \frac{1}{2} \int_\Omega |u|^2 \, (u \, . \, \nabla \Psi) \, dx\, dy . \tag{II.6}$$

In the right-hand side of (II.7), the first term can be bounded from above by

$$\left(\frac{1}{2\,C} \int f^2 \, \Psi \, dx\, dy + \frac{C}{2} \int |u|^2 \, \Psi \, dx\, dy \right) ,$$

and the other two are bounded (see Proposition I.1), so that we get, using Poincaré's inequality :

$$\frac{d}{dt} \int_\Omega |u|^2 \, \Psi \, dx\, dy + K \int_\Omega |u|^2 \, \Psi \, dx\, dy \leq C + \left| \int_\Omega (\nabla p \, . \, u) \, \Psi \, dx\, dy \right| ,$$

for every $t_0 > 0$, where K is a strictly positive constant. In order to deal with the term involving the pressure, we integrate by parts and use Lemma I.3 :

$$\begin{aligned}
\left| \int_\Omega \nabla p \,.\, u\Psi \, dx\, dy \right| &\leqslant \left| \int_\Omega p \,\nabla \,.\, (u\Psi)\, dx\, dy \right| \\
&\leqslant (\text{for } \nabla \,.\, u = 0) \\
&\leqslant \left| \int_\Omega p\, \nabla\Psi \,.\, u\, dx\, dy \right| \\
&\leqslant \left(\int_\Omega |u|^2\, dx\, dy \right)^{1/2} \left(\int_\Omega p^2 |\nabla\Psi|^2\, dx\, dy \right)^{1/2} \\
&\leqslant C \left(\int_\Omega |u|^2\, dx\, dy \right)^{1/2} \left(\int_\Omega p^2 \frac{dx\, dy}{(1+x^2)} \right)^{1/2} \\
&\leqslant (\text{using Theorem I.1 and Lemma I.3 }) \\
&\leqslant C\, .
\end{aligned}$$

Eventually, we have :

$$\frac{d}{dt}\left(\int_\Omega |u|^2\, \Psi\, dx\, dy \right) + K \left(\int_\Omega |u|^2\, \Psi\, dx\, dy \right) \leqslant C\ ,$$

and this implies, after integration between $t_0 > 0$ and t :

$$\sup_{t \geqslant t_0} \int_\Omega |u|^2\, \Psi\, dx\, dy < +\infty\ ,$$

or, more precisely :

$$\sup_{t \geqslant t_0} \int_\Omega |u|^2\, \Psi\, dx\, dy \leqslant C\,(f, u_0)\ ,$$

where $C\,(f, u_0)$ depends boundedly on $\left(\int_\Omega |u_0|^2\, dx\, dy \right)$ and $\left(\int_\Omega |f|^2\, \varphi\, dx\, dy \right)$.

This concludes the proof of Theorem II.2 □

Remark II.3 : (i) The above integrations by parts are legitimate, because $\Psi(x, t)$ is always bounded in x, for every finite time.

(ii) The key condition (II.1) is similar to the asymptotic smoothing property in the terminology of [H].

Remark II.4 : In the three dimensional case, if one is willing to assume that $\frac{\partial u}{\partial t}$, $\Delta u \in L^\infty((t_0, +\infty)\ ;\ H)$, then Theorem II.2 still holds true.

III. ESTIMATES FOR THE FRACTAL DIMENSION OF $\mathscr{A}$

In this last section, we prove the finite dimensionality of $\mathscr{A}$ (in the sense of the fractal dimension), and provide some estimates for its dimension $d_F(\mathscr{A})$. We shall use some fairly general results, for which we refer to [T2], and now just recall briefly the

PROPOSITION III.1 : *Let* Γ *be the nonlinear operator (see section* I*)*

$$\Gamma(u) = Au + B(u)$$

and $\Gamma'(u)$ *be its derivative at* u *; let furthermore* Q *be an arbitrary time-independent projector in* H. *We define a sequence* q_n *by :*

$$q_n = \lim_{t \to +\infty} \left(\inf_{u_0 \in \mathscr{A}} \left(\frac{1}{t} \int_0^t \inf_{\operatorname{rank}(Q) = n} \operatorname{Tr}\left(\Gamma'(u(s)) \circ Q\right) ds \right) \right),$$

where $u(t)$ *is the solution of* (I.6) *with initial datum* u_0. *If there exists an integer* n *such that* $q_n > 0$, *then we have :*

$$d_F(\mathscr{A}) \leqslant n \left(1 + \max_{1 \leqslant i \leqslant n} \frac{-q_i}{q_n} \right).$$

In the situation we consider, the following result is valid :

THEOREM III.2 . *The fractal dimension of the global attractor* $\mathscr{A}$ *is bounded from above by* $(1 + 2C_1^2 \mathrm{Re}^4)^2$, *where*

$$\mathrm{Re} = \frac{\left(\int_\Omega f^2(x, y)\, dx\, dy \right)^{1/2} . \ell}{\nu}$$

is the Reynolds number, and C_1 *is an absolute constant.*

Proof : The proof is very similar to that in the bounded case [C-F-T], and proceeds along the following lines : we first choose an arbitrary family $(\zeta_i)_{1 \leqslant i \leqslant m}$ in $(H^2(\Omega)^2 \cap V)^m$, orthonormal in H, and denote by Q_m the orthogonal projector on Span $\{\zeta_i\}_{1 \leqslant i \leqslant m}$; we then write :

$$\begin{aligned} \operatorname{Tr}(\Gamma'(u) \circ Q_m) &= \sum_{i=1}^m (\Gamma'(u)\zeta_i, \zeta_i) \\ &= \sum_{i=1}^m (-\nu \Delta\zeta_i + (u . \nabla)\zeta_i + (\zeta_i . \nabla) u, \zeta_i) \\ &= \nu \int_\Omega \left(\sum_{i=1}^m |\nabla\zeta_i|^2 \right) dx\, dy + \int_\Omega \sum_{i=1}^m (\zeta_i . \nabla) u . \zeta_i \, dx\, dy \end{aligned}$$

(for $\nabla \cdot u = 0$ implies that $\int_\Omega (u \cdot \nabla)\, \zeta_i \cdot \zeta_i \, dx\, dy = 0$).

Let us now set $\rho = \sum_{i=1}^{m} \zeta_i^2$; we have :

$$\operatorname{Tr}\,(\Gamma'(u) \circ Q_m) \geq$$

$$\geq \nu \int_\Omega \left(\sum_{i=1}^{m} |\nabla \zeta_i|^2 \right) dx\, dy - \left(\int_\Omega \rho^2 \, dx\, dy \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \, dx\, dy \right)^{1/2}$$

$\geq$ (using Lieb-Thirring inequality)

$$\geq \nu \int_\Omega \sum_{i=1}^{m} |\nabla \zeta_i|^2 \, dx\, dy - C_1 \left(\int_\Omega |\nabla u|^2 \, dx\, dy \right)^{1/2} \left(\int_\Omega \sum_{i=1}^{m} |\nabla \zeta_i|^2 \, dx\, dy \right)^{1/2},$$

where C_1 is an absolute constant, see [L-T], [G-M-T] ; from here on, we obtain by means of Young's inequality :

$$\operatorname{Tr}\,(\Gamma'(u) \circ Q_m) \geq \frac{\nu}{2} \int_\Omega \sum_{i=1}^{m} |\nabla \zeta_i|^2 \, dx\, dy - \frac{C_1^2}{2\,\nu} \int_\Omega |\nabla u|^2 \, dx\, dy .$$

As $\{\zeta_i\}_{1 \leq i \leq m}$ is orthonormal in H, Poincaré's inequality finally yields (the width of the strip is ℓ, and the Poincaré constant is $\leq \ell$) :

$$\operatorname{Tr}\,(\Gamma'(u) \circ Q_m) \geq \frac{\nu m}{2\,\ell^2} - \frac{C_1^2}{2\,\nu} \left(\int_\Omega |\nabla u|^2 \, dx\, dy \right) . \qquad \text{(III.1)}$$

From (III.1), we carry on the calculations as follows :

$$\frac{1}{t} \int_0^t \operatorname{tr}\,(\Gamma'(u(s)) \circ Q_m)\, ds \geq \frac{\nu m}{2\,\ell^2} - \frac{C_1^2}{2\,\nu} \frac{1}{t} \int_0^t \left(\int_\Omega |\nabla u|^2 \, dx\, dy \right),$$

and, setting $\beta^2 = \overline{\lim_{t \to +\infty}} \frac{1}{t} \int_0^t \left(\int_\Omega |\nabla u|^2 \, dx\, dy \right) ds$, we obtain :

$$\frac{1}{t} \int_0^t \operatorname{Tr}\,(\Gamma'(u(s)) \circ Q_m)\, ds \geq \frac{\nu m}{2\,\ell^2} - \frac{C_1^2}{2\,\nu} (2\,\beta^2)$$

for t large enough. In particular, when m satisfies

$$m > \frac{2\, C_1^2\, \beta^2\, \ell^2}{\nu^2},$$

q_m is > 0, and so we have proven that $d_F(\mathscr{A})$ is finite, provided that β is finite Actually, it is fairly easy to derive the following estimate for β

$$\beta \leqslant \frac{\ell\left(\int_\Omega |f^2|\, dx\, dy\right)^{1/2}}{\nu} \equiv \frac{|f|\,\ell}{\nu},$$

see for instance [A, Theorem III 1 3] , therefore, q_m is strictly greater than zero for $m > \frac{2\, C_1^2 |f|^2 \ell^4}{\nu^4}$, and some rather simple estimates lead to the following upper bound for $d_F(\mathscr{A})$

$$d_F(\mathscr{A}) \leqslant \left(1 + \frac{2\, C_1^2 \ell^2 \beta^2}{\nu^2}\right)^2$$

If we introduce the Reynolds number $\mathrm{Re} = \frac{|f|^{1/2}\,\ell}{\nu}$, we then have .

$$d_F(\mathscr{A}) \leqslant (1 + 2\, C_1^2 \,\mathrm{Re}^4)^2 \qquad \square$$

Remark III 3 (i) It is interesting to point out that the estimate for the dimension $d_F(\mathscr{A})$ of the global attractor does not involve the weighted norm of f, any weight function leading to the existence of a global attractor in H would provide the same estimate for $d_F(\mathscr{A})$

(ii) It is obvious that the estimate for $d_F(\mathscr{A})$ is not as good as that in the bounded case , the main (technical) reason is the fact that the volume of Ω is infinite, which prevents us from using Lieb-Thirring inequality in an optimal manner

APPENDIX TWO EQUIVALENT NORMS IN $(H^2(\Omega))^2 \cap V$

We want to prove that the canonical norm in $(H^2(\Omega))^2 \cap V$ is equivalent to the norm of Δu in $(L^2(\Omega))^2$ Thanks to the Closed Graph Theorem, this amounts to proving the following regularity result

$$\begin{cases} \text{the solution } u \text{ of the Stokes system} \\ \quad \begin{cases} -\Delta u + \nabla p = f & \text{in} \quad (L^2(\Omega))^2 , \\ \nabla \cdot u = 0 \quad \text{in} \quad \Omega , \\ u \in V , \end{cases} \\ \text{belongs to} \quad (H^2(\Omega))^2 \end{cases} \tag{A 1}$$

The proof is quite straightforward, and not new, but we prefer to give it for the sake of completeness We first recall that a variational formulation of the Stokes system yields a solution $u \in V$, $p \in L^2_{\mathrm{loc}}(\Omega)$, $\nabla p \in V'$. To prove higher regularity, we use the classical method of differential quotients in the longitudinal (infinite) direction.

We write $u_h = u(x+h, y)$, $p_h = p(x+h, y)$, $f_h = f(x+h, y)$; u_h, p_h is the solution of

$$\begin{cases} -\Delta u_h + \nabla p_h = f_h & \text{in} \quad \Omega\,, \\ \nabla \cdot u_h = 0 \quad \text{in} \quad \Omega\,, \\ u_h \in V\,, \end{cases}$$

and therefore, $u_h \in V$, and $|u_h|_V$ is bounded above by $C\,|f_h|_{V'}$; as f is in $(L^2(\Omega))^2$, we conclude that u_h is bounded in V, independently of h. Upon passing to the limit as $h \to 0$, we obtain that $\frac{\partial u}{\partial x}$ belongs to V. Let us now write $u = (u_1, u_2)$, $f = (f_1, f_2)$; we have

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial p}{\partial x} = f_1\,, \\ -\dfrac{\partial^2 u_2}{\partial x^2} - \dfrac{\partial^2 u_2}{\partial y^2} + \dfrac{\partial p}{\partial y} = f_2\,, \\ \dfrac{\partial u_1}{\partial x} + \dfrac{\partial u_2}{\partial y} = 0\,. \end{cases} \tag{A.2}$$

From the last equation in (A.2), we infer that $\frac{\partial u_2}{\partial y}\left(= -\frac{\partial u_1}{\partial x}\right)$ is in $H^1_0(\Omega)$, and therefore, that u_2 is in $H^2(\Omega) \cap H^1_0(\Omega)$. As for u_1, we already know that u_1, $\frac{\partial u_1}{\partial x}$, $\frac{\partial u_1}{\partial y}$, $\frac{\partial^2 u_1}{\partial x^2}$, $\frac{\partial^2 u_1}{\partial x\,\partial y}$ are in $L^2(\Omega)$, and that $\frac{\partial^2 u_1}{\partial y^2}$, $\frac{\partial^3 u_1}{\partial x^3}$, $\frac{\partial^3 u_1}{\partial x\,\partial y^2}$, $\frac{\partial^3 u_1}{\partial x^2\,\partial y}$ are in $H^{-1}(\Omega)$; differentiating the first equation of (A.2) with respect to y, and remarking that $\frac{\partial p}{\partial y}$ is in $L^2(\Omega)$, we obtain that $\frac{\partial^3 u_1}{\partial y^3}$ is also in $H^{-1}(\Omega)$. A classical regularity result, see e.g. [L-M, chap. I, Lemma 12.3], allows us to assert that u_1 is in $H^2(\Omega)$, and the proof is complete.

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