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EVERTED EQUILIBRIA OF A SPHERICAL CAP : A SINGULAR PERTURBATION METHOD (*)

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Abstract. — *A singular perturbation method is used to construct an explicit branch of solutions of a system of nonlinear differential equations governing the axially symmetric equilibria of a thin spherical cap.*

The physical interest lies in the construction of a branch of solutions depending on a parameter ε *(to be thought of as the « thickness » of the cap) which approaches the « everted » configuration as* $\varepsilon \to 0$.

The existence of such a branch is proved in [1] *via topological methods ; the main result of this paper, namely theorem* 4.1, *provides an explicit « Taylor » expansion of the solution.*

Résumé. — *Une méthode de perturbation singulière est utilisée pour construire une branche de solutions explicites d'un système d'équations différentielles non linéaires gouvernant l'équilibre axial symétrique d'une calotte sphérique mince.*

L'intérêt physique est dans la construction d'une branche de solutions dépendant d'un paramètre ε *(qui est supposé mesurer la « minceur » de la calotte) qui approche la configuration « renversée » lorsque* $\varepsilon \to 0$.

L'existence d'une telle branche est prouvée dans [1] *via des méthodes topologiques ; le résultat principal de cet article, le théorème* 4.1, *nous donne un développement de Taylor explicite de la solution.*

1. INTRODUCTION

This paper deals with the axially symmetric equilibria of a thin elastic spherical cap. The reference configuration is given, in spherical coordinates, by the inequalities :

$$R - s \leq \rho \leq R + s \,, \quad 0 \leq \theta \leq \theta_0 \,, \quad 0 \leq \phi \leq 2\pi \,,$$

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where the radius R of the middle sphere, the half-thickness s ($s \ll R$) and the opening angle θ_0 are given. Throughout the present work the cap is supposed to be free from loads and a linear constitutive assumption as well as an appropriate version of the Kirchoff-Love hypothesis are supposed to hold.

The equations governing the system are deduced in [1] together with the appropriate functional setting which is shortly described in § 2.

In [1] a topological argument is used to prove the existence of a branch of solutions (with respect to the parameter k defined below) approaching the « everted » configuration (i.e. the configuration obtained turning inside-out the cap) as the « thickness » of the cap goes to zero.

More precisely, the « thickness » parameter beeing :

$$k := (R/s)\sqrt{3(1-\nu)/2}\ , \quad \nu = \text{Poisson ratio}\ ,$$

the equilibria of the cap are described by the following system of two second order ordinary differential equations :

$$(1.1)\qquad \begin{cases} g''(\theta) + (3/\theta)\, g'(\theta) = -\, kh(\theta)[h(\theta) - 2]\,; \\ h''(\theta) + (3/\theta)\, h'(\theta) = kg(\theta)[h(\theta) - 1] \end{cases} \quad 0 < \theta < \theta_0$$

where the functions g and h are required to satisfy the boundary conditions :

$$(1.2)\qquad \begin{cases} \qquad\qquad g'(0) = h'(0) = 0\ , \\ g(\theta_0) = 0\ , \quad \theta_0\, h'(\theta_0) + (1+\nu)\, h(\theta_0) = 2(1+\nu)\ . \end{cases}$$

Remark : Actually in [1] the unknowns are g and $f := -2 + h$; here we think of the unknown h as a perturbation of the « everted » configuration : $f_0 \equiv -2$.

Because only axially simmetric deformations are considered and because of the thickness of the cap, the displacement is described by the components in the meridional and normal directions to the cap's middle surface ; let's denote them with $v(\theta)$ and $w(\theta)$, where θ is the polar angle. Here (see [1]) we have :

$$h(\theta) = 2 + \frac{v + w'}{R\theta} \qquad g(\theta) = \frac{2\,kS}{E}$$

where E is the Young modulus and S is the meridional stress.

The boundary conditions at $\theta = 0$ take in account the symmetry of the cap ; on the other hand the boundary conditions at $\theta = \theta_0$ express the requirement on the displacement to have null vertical component at the rim of the cap.

We define the parameter :

$$(1.3)\qquad \varepsilon = \left(\frac{1}{2}\,\theta_0^3\sqrt{k}\right)^{-1}$$

and multiply both equations of (1.1) by k^{-1} ; we can now think of (1.1)-(1.2) as a singular perturbation problem (in [2] a similar idea is applied to the theory of nonlinear plates).

We refer to [3] and [4] for a systematic survey of the singular perturbation theory together with many applications and to [5] for an interesting abstract approach to the theory.

The aim of the present paper is to construct a branch of solutions to (1.1)-(1.2) of the form (here $m \geq 1$ is an integer) :

$$\begin{cases} g(\theta, \varepsilon) = \sum_{j=1}^{m} \varepsilon^j g_j(t) + \varepsilon^{m+1} \bar{g}(\theta, \varepsilon) \\ h(\theta, \varepsilon) = \sum_{j=1}^{m} \varepsilon^j h_j(t) + \varepsilon^{m+1} \bar{h}(\theta, \varepsilon) \end{cases} \tag{1.4}$$

where the independent variable t is defined by :

$$\theta^{-2} = \theta_0^{-2} + \varepsilon t \quad 0 < t < +\infty . \tag{1.5}$$

The functions g_j and h_j $(j > 0)$ are thought of as the coefficients of an appropriate Taylor expansion in ε and are defined by means of some linear inhomogeneous systems which are deduced and studied in § 3. The functions $\bar{g}$ and $\bar{h}$ are now the very unknowns of the problem ; their existence is deduced in § 4 by using the contracting mapping principle.

2. THE FUNCTIONAL SETTING

In order to study problem (1.1)-(1.2) we first observe that :

$$\frac{d^2}{d\theta^2} + (3/\theta)\frac{d}{d\theta} = \theta^{-3}\frac{d}{d\theta}\left(\theta^3 \frac{d}{d\theta}\right),$$

so to define the weak solutions of (1.1)-(1.2) it is quite natural to introduce the following weighted Hilbert spaces [1] :

K = space of the square integrable functions in $(0, \theta_0)$ with respect to the weight θ^3,

K^1 = space of the functions whose first derivative belongs to K,

K_0^1 = subspace of K^1 of the functions vanishing in θ_0.

The scalar products and the norms are :

$$\text{in } K \quad (f_1, f_2)_0 = \int_0^{\theta_0} d\theta\, \theta^3 f_1(\theta) f_2(\theta), \qquad \|\cdot\|_0$$

$$\text{in } K^1 \quad (f_1, f_2)_1 = (f_1', f_2')_0 + \theta_0^2(1+\nu) f_1(\theta_0) f_2(\theta_0), \quad \|\cdot\|_1 .$$

In [1] the following inequality is proved to hold :

$$\int_0^{\theta_0} d\theta\, \theta^3 f(\theta)^4 \leq \text{const} \|f\|_1^4 , \quad f \in K^1 ;$$

accordingly, the right sides of both equations of (1.1) belong to K if both g and h belong to K^1.

In [1] it is also proved that K^1 is compactly imbedded in K ; hence for any $q \in K$ unique weak solutions $G_0\, q$ and $G_1\, q$ are easily proved to exist for the problems :

$$\text{(P0)} \qquad \begin{cases} -\theta^{-3}(\theta^3 f'(\theta))' = q(\theta) \\ f'(0) = f(\theta_0) = 0 , \end{cases}$$

$$\text{(P1)} \qquad \begin{cases} -\theta^{-3}(\theta^3 f'(\theta))' = q(\theta) \\ f'(0) = \theta_0 f'(\theta_0) + (1+\nu) f(\theta_0) = 0 . \end{cases}$$

As a consequence, both G_0 and G_1 are linear and compact operators from K into K_0^1 (K^1, respectively) ; therefore problem (1.1)-(1.2) takes the functional form :

$$\text{(2.1)} \qquad \begin{cases} g = -kG_0(h^2 - 2h) \\ h = 2 + kG_1[g(h-1)] \end{cases}$$

or, equivalently,

$$\text{(2.2)} \qquad 0 = F(h) := h - 2 + k^2 G_1[(h-1)\, G_0(h^2 - 2h)]$$

where F maps K^1 into itself.

Simple algebraic manipulations prove that F is the Frechét derivative of the functional :

$$\text{(2.3)} \qquad J(h) := \frac{1}{2} \|h - 2\|_1^2 + \frac{k^2}{4} \|G_0(h^2 - 2h)\|_1^2 ;$$

moreover the Frechét derivative of F at $h \in K^1$ is associated to the bilinear symmetric form :

$$\text{(2.4)} \quad F'(h, \phi) = \|\phi\|_1^2 + k^2 (G_0[h^2 - 2h], \phi^2)_0 + 2k^2 \|G_0[\phi(h-1)]\|_1^2 .$$

It is clear that J attains, for every $k > 0$, its absolute minimum in $h_0 \equiv 2$ (that is the reference configuration) ; in [1] a topological argument provides, for large values of k, the existence of a second branch of solutions approaching 0 as $k \to \infty$. In the rest of the present paper we prove such solution to be stable and to have the form (1.4), which is also usefull from the numerical point of view.

3. THE TAYLOR COEFFICIENTS

We can think of g and h as the components for a two-dimensional vector $\underset{\sim}{u}$; therefore we define the matrices :

$$A \equiv \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \qquad B \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and we write system (2.1) in vector form ; actually it is convenient to use the variable t defined in (1.5) to write (1.1) in the equivalent form :

$$(3.1) \qquad (1 + \theta_0^2 \, \varepsilon t)^3 \frac{d^2 \underset{\sim}{u}}{dt^2} + A \underset{\sim}{u} = hB\underset{\sim}{u} \quad 0 < t < +\infty .$$

Remark : In this section we shall use the symbol « ′ » for the derivative with respect to t.

The boundary conditions are :

$$(3.2) \qquad \begin{cases} g'(+\infty) , \quad h'(+\infty) = o(t^{-3/2}) \quad t \to +\infty \\ g(0) = 0 \quad 2\,h'(0) - \theta_0^2 \, \varepsilon(1+\nu)\,h(0) = -2\,\theta_0^2 \, \varepsilon(1+\nu) . \end{cases}$$

We try to solve the problem (3.1)-(3.2) by means of the formal series :

$$(3.3) \qquad \underset{\sim}{u}(t, \varepsilon) \sim \sum_{j=1}^{\infty} \varepsilon^j \, \underset{\sim}{u}_j(t) \quad \underset{\sim}{u}_j \equiv (g_j, h_j) .$$

Formal algebraic manipulations provide the equations for the $\underset{\sim}{u}_j$'s :

$$(3.4)_1 \qquad \underset{\sim}{u}_1'' + A\underset{\sim}{u}_1 = 0$$

$$(3.4)_2 \qquad \underset{\sim}{u}_2'' + A\underset{\sim}{u}_2 = h_1 \, B\underset{\sim}{u}_1 - 3\,\theta_0^2 \, t\underset{\sim}{u}_1''$$

$$(3.4)_3 \qquad \underset{\sim}{u}_3'' + A\underset{\sim}{u}_3 = h_1 \, B\underset{\sim}{u}_2 + h_2 \, B\underset{\sim}{u}_1 - 3\,\theta_0^4 \, t^2 \underset{\sim}{u}_1'' - 3\,\theta_0^2 \, t\underset{\sim}{u}_2''$$

and, for $j > 3$:

$$(3.4)_j \quad \underset{\sim}{u}_j'' + A\underset{\sim}{u}_j = \sum_{1 \leq l < j} h_{j-l} \, B\underset{\sim}{u}_l - 3\,\theta_0^2 \, t\underset{\sim}{u}_{j-1}'' - 3\,\theta_0^4 \, t^2 \underset{\sim}{u}_{j-2}'' - \theta_0^6 \, t^3 \underset{\sim}{u}_{j-3}'' .$$

We impose the boundary conditions :

$$(3.5)_j \qquad g_j'(+\infty) = 0 , \quad h_j'(+\infty) = 0 , \qquad j \geq 1$$

$$(3.6)_1 \qquad g_1(0) = 0 , \qquad h_1'(0) = -\theta_0^2(1+\nu)$$

$$(3.6)_j \qquad g_j(0) = 0 , \quad 2\,h_j'(0) = \theta_0^2(1+\nu)\,h_{j-1}(0) \qquad j \geq 2 .$$

Problems $(3.4)_j$-$(3.5)_j$-$(3.6)_j$ have the form :

$$(3.7)\qquad \begin{cases} \underset{\sim}{v}'' + A\underset{\sim}{v} = \underset{\sim}{q}(t) & 0 < t < +\infty \\ \underset{\sim}{v}'(+\infty) = 0 \\ v_1(0) = 0\,, \quad v_2'(0) = a \end{cases}$$

where the vector-valued function $\underset{\sim}{q}(t)$ and the real number a are given.

The identity $A^2 = -2I$ (I = unit matrix) suggests to consider the matrix-valued function :

$$(3.8)\qquad G(t) = e^{-rt}\begin{bmatrix} \cos rt & -\sqrt{2}\sin rt \\ \sqrt{1/2}\sin rt & \cos rt \end{bmatrix} \quad r = 2^{-1/4}$$

which satisfies :

$$(3.9)\qquad \begin{cases} G''(t) + AG(t) = 0 \\ G(0) = I\,,\ G(t)\,,\ G'(t) = 0(e^{-rt}) \\ G(t)^{-1} = 0(e^{rt})\,,\quad \det G(t) = e^{-rt} > 0\,. \end{cases}$$

Further algebric manipulations yield the following formula :

$$(3.10)\qquad \underset{\sim}{v}(t) = G(t)\,\underset{\sim}{v}(0) - G(t)\int_0^t d\tau\, G(\tau)^{-2}\int_\tau^{+\infty} ds\, G(s)\,\underset{\sim}{q}(s)$$

for all solutions $\underset{\sim}{v}$ of (3.7), the initial value $\underset{\sim}{v}(0)$ beeing :

(3.11)

$$\begin{cases} v_1(0) = 0 \\ v_2(0) = -r^{-1}a - r^{-1}\displaystyle\int_0^{+\infty} ds\, e^{-rs}\{2^{-1/2}q_1(s)\sin rs + q_2(s)\cos rs\}\,. \end{cases}$$

Formulae (3.10) and (3.11) show that the family of problems (3.4)-(3.5)-(3.6) has a unique solution $\{\underset{\sim}{u}_j, j \in N\}$, each $\underset{\sim}{u}_j$ vanishing exponentially as $t \to +\infty$. In next section we need the explicit form of $g_1(t)$:

$$(3.12)\qquad g_1(t) = -\theta_0^2(1+\nu)\,2^{3/4}\,e^{-rt}\sin rt\,.$$

4. THE BRANCH OF SOLUTIONS

The series (3.3) is not expected to converge even for small values of ε ; nevertheless, for given $m \geq 1$, the « Taylor polinomia » :

$$(4.1)\qquad g^*(t,\varepsilon) := \sum_{j=1}^{m}\varepsilon^j g_j(t),\quad h^*(t,\varepsilon) := \sum_{j=1}^{m}\varepsilon^j h_j(t) - \varepsilon^m h_m(0)$$

will be of use.

Remark : The term $-\varepsilon^m h_m(0)$ in the right side of the second equation in (4.1) takes in account the boundary condition on h at $t = 0$.

Actually the « coefficients » g_j and h_j are functions of both θ and ε because of (1.5) ; throughout this section we shall think of g^* and h^* as function of the variable θ ; they don't satisfy (2.1) but :

$$\left\{\begin{aligned} g^* &= -kG_0(h^{*2} - 2h^*) + \varepsilon^{m-2} S(\theta, \varepsilon) \\ h^* &= 2 + kG_1(g^* - g^* h^*) + \varepsilon^{m-1} T(\theta, \varepsilon) \end{aligned}\right. \tag{4.2}$$

where both S and T are uniformly bounded ; accordingly h^* doesn't satisfy (2.2) but

$$F(h^*) = \varepsilon^{m-4} Z(\theta, \varepsilon) \tag{4.3}$$

where Z is also uniformly bounded. Lemma (4.2) below implies :

$$\|F'[h^*(.,\varepsilon)]^{-1}\| \leqslant \text{const} \qquad \varepsilon \ll 1 . \tag{4.4}$$

Remark : The norm in (4.4) denotes the norm in the space of bounded linear operators in K^1.

Pick $m + 5$ instead of m, (4.3) takes the form :

$$F(h^*) = \varepsilon^{m+1} Z . \tag{4.5}$$

Let's look at solutions of the form :

$$h(\theta, \varepsilon) = h^*(\theta, \varepsilon) + \varepsilon^{m+1} \bar{h}(\theta, \varepsilon) \quad \bar{h} \text{ uniformly bounded} . \tag{4.6}$$

The equation in $\bar{h}$ is :

$$0 = F(h) = F(h^*) + \varepsilon^{m+1} F'(h^*) \bar{h} + \varepsilon^{2m+2} \Gamma(\varepsilon, \bar{h}) , \tag{4.7}$$

where :

$$\|\Gamma(\varepsilon, \bar{h})\| \leqslant \text{const} \|\bar{h}\|_1^2 \tag{4.8}$$

in a fixed sphere of K^1.

Comparing (4.5) and (4.7) we get :

$$0 = Z(., \varepsilon) + F'(h^*) \bar{h} + \varepsilon^{m+1} \Gamma(\varepsilon, \bar{h}) . \tag{4.9}$$

It is now easily seen that, if $\varepsilon \ll 1$, the contracting mapping principle applies to a ball whose radius is independent of ε. We have just proved :

THEOREM 4.1 : *Problem* (1.1)-(1.2) *has a branch of solutions of the form* (1.4) *where* $\bar{g}$ *and* $\bar{h}$ *are bounded independently of* $\varepsilon \ll 1$.

Next lemma proves (4.4) :

LEMMA 4.2 : *The least eigenvalue of* $F'(h^*(\varepsilon, .))$ *goes to* 1 *as* ε *goes to zero.*

Proof: From (2.4) we deduce :

$$F'(h^*, \phi) \geqslant \|\phi\|_1^2 + k(g^*, \phi^2)_0 . \tag{4.10}$$

We observe :

$$g^* = \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g^{**} \tag{4.11}$$

where $\|g^{**}(., \varepsilon)\|_0$ is uniformly bounded. Let's remark

$$|(\gamma, \phi^2)_0| \leqslant \text{const} \|\gamma\|_0 \|\phi\|_1^2 ; \quad \gamma \in K, \phi \in K^1 . \tag{4.12}$$

From $(3.4)_2$ we deduce that $\|g_2(., \varepsilon)\|_0 \to 0$; $\varepsilon^2 k$ beeing constant the K-norm of the last two terms of the right side of (4.11) vanish with ε ; the term εg_1 is more difficult to handle. For, let's first consider the case :

$$|\phi(\theta_0)| \leqslant \delta \|\phi'\|_0 , \tag{4.13}$$

$\delta > 0$ to be choosen later.

As ϕ is (1/2)-hölder-continuous we have :

$$|\phi(\theta)| \leqslant \text{const } (\delta + (\theta_0 - \theta)^{1/2}) \|\phi\|_1 \tag{4.14}$$

in any subinterval of $(0, \theta_0)$ bounded away from zero.

From (3.12) we have :

$$\int_0^{\theta_0} d\theta\, \theta^3 |g_1(\theta, \varepsilon)| = 0(\varepsilon) . \tag{4.15}$$

Let $0 < \beta$ and choose θ_ε such that :

$$(2^{1/4}/\varepsilon)(\theta_0^{-2} - \theta_\varepsilon^{-2}) = \ln\,(\beta\varepsilon) ;$$

clearly $\theta_0 - \theta_\varepsilon \sim \varepsilon \ln \varepsilon$ as $\varepsilon \to 0$.

From (1.3), (3.12), (4.14) and (4.15) we have :

$$\begin{aligned} |(k\varepsilon g_1, \phi^2)_0| &\leqslant \text{const } \varepsilon^{-1} \int_0^{\theta_0} d\theta\, \theta^3 |g_1(\theta, \varepsilon)|\, |\phi(\theta)|^2 \leqslant \\ &\leqslant \text{const} \int_0^{\theta_\varepsilon} d\theta\, \theta^3\, \beta\, |\phi(\theta)|^2 + \text{const max } \{\phi(\theta)^2 ; \theta_\varepsilon \leqslant \theta \leqslant \theta_0\} \leqslant \\ &\leqslant \text{const } \beta\, \|\phi\|_0^2 + \text{const } [\delta^2 + \varepsilon \ln \varepsilon^{-1}] \|\phi\|_1^2 . \end{aligned} \tag{4.16}$$

Hence, for every $\sigma > 0$, we can choose $\beta = \beta(\sigma)$ and $\delta = \delta(\sigma)$ in such a way that :

$$F'(h(.,\varepsilon);\phi) \geqslant (1-\sigma)\|\phi\|_1^2$$

if $\varepsilon \ll 1$ and ϕ satisfies (4.13).

Finally we consider the case :

$$|\phi(\theta_0)| \geqslant \delta(\sigma)\|\phi'\|_0 . \tag{4.17}$$

Accordingly :

$$\phi(\theta) = \phi(\theta_0)\,\gamma(\theta) \quad \|\gamma\|_1 \leqslant \sqrt{\theta_0^2(1+\nu)+\delta^{-2}} .$$

From (2.4) we have :

$$F'(h(.,\varepsilon);\phi) \geqslant -\operatorname{const} \varepsilon^{-1}\|\phi\|_1^2 + \mu(\delta)\,\varepsilon^{-4}\|\phi\|_1^2$$

where $\mu(\delta)$ is positive because $\|G_0\gamma\|_1$ attains a minimum in the set :

$$\gamma(\theta_0) = 1, \quad \|\gamma\|_1 \leqslant \sqrt{\theta_0^2(1+\nu)+\delta^{-2}} .$$

This completes the proof.

REFERENCES

[1] P. PODIO-GUIDUGLI, M. ROSATI, A. SCHIAFFINO and V. VALENTE, Equilibrium of an elastic spherical cap pulled at the rim (1987), to appear on S.I.A.M. *J. of Appl. Math.*

[2] L. S. SRUBSHCHIK, On the asymptotic integration of a system of nonlinear equations of plate theory, *Appl. Math. Mech., Trans. PMM 27,* 335-349 (1964).

[3] W. ECKHAUS, *Asymptotic Analysis of Singular Perturbations,* North-Holland, Amsterdam (1979).

[4] K. W. CHANG, F. A. HOWES, Nonlinear Singular Perturbation Phenomena : Theory and Application, *Applied Math. Sciences* n. 56, Springer-Verlag (1984).

[5] M. S. BERGER, L. E. FRAENKEL, On singular perturbations of nonlinear operator equations, *Indiana Univ. Math. Journ.* 20, 623-631 (1971).