PHILIPPE LE FLOCH
J. C. NEDEREC

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ASYMPTOTIC TIME-BEHAVIOR FOR
WEIGHTED SCALAR CONSERVATION LAWS (*)

Philippe Le Floch (¹), J. C. Neudelec (¹)

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Abstract. — For weighted scalar conservation laws introduced as model equations for gas flows in axisymmetric coordinates or in a nozzle, the asymptotic time-behavior of an entropy weak solution is obtained according to the behaviors of the weight and flux functions, thanks to the explicit formula previously derived in [10].

Résumé. — Considérant une loi de conservation scalaire avec poids qui modélise l'évolution d'un gaz en géométrie axisymétrique ou dans une tuyère, on utilise la formule explicite obtenue dans [10] pour préciser le comportement asymptotique de la solution faible entropique suivant le comportement de la fonction-flux et de la fonction-poids de l'équation.

1. INTRODUCTION : WEIGHTED SCALAR CONSERVATION LAWS

We are interested in weighted scalar nonlinear hyperbolic conservation laws in the half space

(1.1) \[ \frac{\partial}{\partial t} (r(x) u(x, t)) + \frac{\partial}{\partial x} (r(x) f(u(x, t))) = 0, \quad x > 0, \quad t > 0 \]

with a \( C^2 \) convex flux-function \( f : \mathbb{R} \to \mathbb{R} \) and a \( C^2 \) positive weight-function \( r : ]0, \infty[ \to ]0, \infty[ \). Such equations are considered as model equations for gas dynamics in axisymmetric coordinates where the weight-function satisfies

(H.1) \[ r \in L^\infty(0, 1) \quad \text{and} \quad \frac{1}{r} \in L^\infty(1, \infty) \]

(for example take \( r(x) = x^\alpha, \, \alpha = 0, 1, 2 \), or as model equations for gas...
flows in a nozzle where there exists two positive constant \( r_- \) and \( r_+ \) such that

\[
0 < r_- \leq r(\cdot) \leq r_+ < + \infty.
\]

For equations of the form (1.1), an uniqueness and existence result was proved in Le Floch-Nedelec [10] thanks to an explicit formula generalizing the one of Lax [7]. Moreover, a suitable formulation of boundary condition (at \( x = 0 \)) was proposed. Refer also to previous works of Whitham [17] and Schonbek [15]. In this paper, we look for the asymptotic time-behavior of the solution of (1.1). We generalize well-known results of time-behavior for equations (1.1) without weight-function (that is \( r(\cdot) = 1 \)). The rate of convergence is specified according to the properties of the weight-function and the flux-function. For classical results on time-behavior of conservation laws, we refer to Lax [7], Dafermos [2], Conway [1], Liu-Pierre [13]... We present our result of time-behavior in the following section 2. Before, we detail some important properties of the mixed problem associated to the weighted scalar conservation law (1.1). The proofs can be found in [8]-[11].

We look for weak solutions (that is in the sense of distributions) of (1.1) satisfying an initial condition

\[
(1.2) \quad u(x, 0) = u_0(x), \quad x > 0
\]

with an initial data \( u_0 : \mathbb{R}_+ \to \mathbb{R} \). And for the sake of uniqueness, one adds an entropy condition (Lax [7], Oleinik [14], Kruskov [6])

\[
(1.3) \quad u(x - 0, t) \geq u(x + 0, t), \quad x > 0, t > 0.
\]

We impose a zero boundary condition at \( x = 0 \), since we are only interested in this paper in the rate of convergence and not in the nonlinear aspect of the boundary condition. Concerning boundary conditions, see [10] and also Le Floch [8, 9], Dubois-Le Floch [3, 4].

Our assumptions concerning the initial data are as follows

\[
(H.3) \quad r \cdot u_0 \in L^1(\mathbb{R}_+), \quad r \cdot f(u_0) \in L^\infty(\mathbb{R}_+).
\]

And, the flux-function satisfies for simplifications

\[
(H.4) \quad f'' > 0, \quad f(0) = f'(0) = 0.
\]

(for example, take \( f(u) = \frac{u^2}{2} \)). Then, the problem (1.1)-(1.3) admits one and only one solution \( u(\cdot) \) such that \( r \cdot u \) belongs to \( L^\infty(\mathbb{R}_+^+); L^1(\mathbb{R}_+^+)) \) and \( rf(u) \) to \( L^\infty(\mathbb{R}_+^+ \times \mathbb{R}_+^+) \). It satisfies also the following stability properties for almost every \( t \geq 0 \) : the \( L^\infty \)-stability

\[
(1.4) \quad \| r(\cdot) f(u(\cdot, t)) \|_{L^\infty(\mathbb{R}_+^+)} \leq \| r(\cdot) f(u_0(\cdot)) \|_{L^\infty(\mathbb{R}_+^+)}
\]
and the $L^1$-semi-group property

\begin{equation}
\| (u(., t) - v(., t)) \cdot r(.) \|_{L^1(\mathbb{R}_+)} \leq \| (u_0(.) - v_0(.)) \cdot r(.) \|_{L^1(\mathbb{R}_+)}
\end{equation}

for two solutions $u$ and $v$ corresponding to two initial data $u_0$ and $v_0$ respectively. The property (1.5) generalizes a previous result of Keyfitz [4].

Moreover, take $a = f'$ and let $f_+^{-1}: \mathbb{R}_+ \to \mathbb{R}_+$ and $f_-^{-1}: \mathbb{R}_- \to \mathbb{R}_-$ be the two inverse functions of the convex function $f$. For each $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$, we can define — thanks to the hypotheses (H.1) or (H.2) (see [10]) — the function :

\[
\mathbb{R}_+ \times \{-1, +1\} \ni (c, \varepsilon) \mapsto y(c, \varepsilon) \in \mathbb{R}_+
\]

by the algebraic relation

\begin{equation}
t = \int_{y(c, \varepsilon)}^{x} \frac{d\xi}{a\left(f_+^{-1}\left(\frac{c}{r(\xi)}\right)\right)}
\end{equation}

and the function $G: \mathbb{R}_+ \times \{-1, +1\} \ni (c, \varepsilon) \mapsto G(c, \varepsilon)$ as follows :

\begin{equation}
G(c, \varepsilon) = \int_{0}^{y(c, \varepsilon)} u_0(\xi) \cdot r(\xi) \, d\xi - c \cdot t + \int_{y(c, \varepsilon)}^{x} f_+^{-1}\left(\frac{c}{r(\xi)}\right) \cdot r(\xi) \, d\xi.
\end{equation}

Then the value $u(x, t)$ of the solution $u$ of (1.1)-(1.3) is provided by the following explicit formula

\begin{equation}
u(x, t) = f_+^{-1}\left(\frac{c(x, t)}{r(x)}\right)
\end{equation}

where $(c(x, t), \varepsilon(x, t))$ minimizes the function $G$.

2. ASYMPTOTIC TIME-BEHAVIOR

Using the explicit formula (1.6)-(1.7), we are able as in Lax [7] to get an uniform decay in power of $t$ for the solution $u$ of problem (1.1)-(1.3), according to the behavior of the function $f$. Assuming that

\begin{equation}
k_- \cdot |v|^{p-2} \leq f''(v) \leq k_+ \cdot |v|^{p-2}, \quad \forall v \in \mathbb{R}; \ p \geq 2, \ 0 < k_- \leq k_+
\end{equation}

we have :

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THEOREM Under hypotheses (H1) or (H2) and (H3)-(H4), the solution $u(.,.)$ of (1.1)-(1.3), when the time $t$ tends to infinity, decreases with the following rate

$$|u(x,t)| \leq \left( \frac{kM}{r(x)} \right)^{1/p} \cdot \frac{1}{t^{1/p}} \quad t > 0, x > 0$$

where the constants $k$ and $M$ depend only on the flux-function and the initial data respectively

$$k = 2 \frac{p}{k_+}$$

and

$$M = \|r \cdot u_0\|_{L^1(\mathbb{R}^+)}$$

In the case of flows in axisymmetric coordinates

$$r(x) = x^\alpha, \quad \alpha \geq 0$$

the estimation (2.2) becomes

$$|u(x,t)| \leq \frac{c}{x^{\alpha/p}} \cdot \frac{1}{t^{1/p}} \quad (c > 0)$$

Note that, when $\alpha > 0$, this inequality (2.3) provides also the behavior of $u$ when $x \to 0^+$ and $x \to +\infty$

Proof of theorem Multiplying (2.1) by $v \in \mathbb{R}$ and integrating lead to

$$k_- \cdot \frac{|v|^p}{p} \leq a(v)v - f(v) \leq k_+ \cdot \frac{|v|^p}{p}$$

because of

$$\frac{d}{dv} (a(v)v - f(v)) = a'(v)v = f''(v)v$$

Moreover, by two successive integrations of (2.1), we have also

$$k_- \cdot \frac{|v|^p}{p(p-1)} \leq f(v) \leq k_+ \cdot \frac{|v|^p}{p(p-1)}$$

Then from (2.4)-(2.5) it results that

$$\frac{k_-}{k_+} (p - 1) f(v) \leq a(v)v - f(v) \leq \frac{k_+}{k_-} (p - 1) f(v)$$
or, for \( v > 0 \):

\[
(2.6a) \quad \frac{k_-}{k_+} (p - 1) \frac{f(v)}{a(v)} \leq v - \frac{f(v)}{a(v)} \leq -\frac{k_+}{k_-} (p - 1) \frac{f(v)}{a(v)}
\]

and for \( v < 0 \):

\[
(2.6b) \quad \frac{k_+}{k_-} (p - 1) \frac{f(v)}{a(v)} \leq v - \frac{f(v)}{a(v)} \leq -\frac{k_-}{k_+} (p - 1) \frac{f(v)}{a(v)}
\]

Henceforth, the following minoration of the function \( G \) defined by (1.6) is an immediate consequence of (2.6) (used with the value \( v = f_{\varepsilon}^{-1} \left( \frac{c}{r(\xi)} \right) \))

\[
G(c, \varepsilon) = \int_0^{y(c, \varepsilon)} u_0 r d\xi + \int_{y(c, \varepsilon)}^x \left\{ f_{\varepsilon}^{-1} \left( \frac{c}{r(\xi)} \right) - \frac{c}{a f_{\varepsilon}^{-1} \left( \frac{c}{r(\xi)} \right)} \right\} r(\xi) d\xi
\]

\[
= -M + \int_{y(c, \varepsilon)}^x \frac{k_-}{k_+} (p - 1) \frac{c}{a f_{\varepsilon}^{-1} \left( \frac{c}{r(\xi)} \right)} \cdot r(\xi) d\xi
\]

where \( M = \| r \cdot u_0 \|_{L^1(\mathbb{R}^+)} \). Thus, the inequality

\[
(2.7) \quad G(c, \varepsilon) \geq -M + \frac{k_-}{k_+} (p - 1) c \cdot t
\]

holds for all \( c = 0, \varepsilon = \pm 1, t > 0 \).

Furthermore, for each \( (x, t) \) in \( \mathbb{R}_+ \times \mathbb{R}_+ \), the minimum value of the function \( G \) — which is by notation attained at \( (c(x, t), \varepsilon(x, t)) \) — may be majorized by the value of \( G \) at \( c = 0 \):

\[
(2.8) \quad G(c(x, t), \varepsilon(x, t)) \leq \int_0^x r u_0 d\xi \leq M.
\]

Now comparing (2.7) and (2.8), it results the estimation

\[
(p - 1) t \frac{k_-}{k_+} c(x, t) \leq 2 M
\]

or

\[
0 \leq c(x, t) \leq \frac{k_+}{k_-} \frac{2 M}{(p - 1) t}.
\]
Thus in virtue of (1.7), we have proved that \( r_f(u) \) decreases uniformly in \( x \in \mathbb{R}_+ \) when \( t \to \infty \):

\[
\sup_{x \in \mathbb{R}_+} |r(x) f(u(x, t))| \leq \frac{k_+ 2M}{k_- (p - 1) t}.
\]

It remains to use again (2.5):

\[
k_- \cdot \frac{|u(x, t)|^p}{p(p - 1)} \leq f(u(x, t)) \leq \frac{2Mk_+}{k_-} \frac{1}{r(x)} t
\]

which gives (2.2). ■

When the solution \( u \) is bounded — say the weight-function \( r \) satisfies the properties (H.2) — it suffices to assume that the inequalities (2.1) hold in the neighborhood of \( v = 0 \) with one \( p_0 \geq 2 \) and also that \( f'' \) is uniformly bounded:

\[
\text{Cte} \leq f'' \leq \text{Cte}'.
\]

Namely, thanks to the theorem applied with \( p = 2 \), we know that the solution \( u \) tends to zero uniformly in \( x \in \mathbb{R}_+ \) when the time goes to infinity

\[
\sup_{x \in \mathbb{R}_+} |u(x, t)| \leq \left( \frac{kM}{r_-} \right)^\frac{1}{2} \cdot \frac{1}{\sqrt{t}} \quad t > 0.
\]

So for \( t \) sufficiently large, say \( t > T_0 \), \( u(x, t) \) is for each \( x \) in \( \mathbb{R}_+ \) in the neighborhood where (2.1) holds. And the same proof as previously gives

\[
|u(x, t)| \leq \left( \frac{kM}{r_-} \right)^\frac{1}{p_0} \cdot \frac{1}{t^{1/p_0}}, \quad \forall x \in \mathbb{R}_+, \forall t > T_0.
\]

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