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ON THE APPLICATION OF MIXED FINITE ELEMENT METHODS TO THE WAVE EQUATIONS (*)

by Tunc Geveci (1)

Abstract — The convergence of certain semidiscrete approximation schemes based on the « velocity-stress » formulation of the wave equation and spaces such as those introduced by Raviart and Thomas is discussed. The discussion also applies to similar schemes for the equations of elasticity.

Resumé. — La convergence de certains schémas d'approximation semi-discrète basés sur la formulation « vitesse-contrainte » de l'équation d'onde et d'espace tel que ceux introduits par Raviart et Thomas est discuté. La discussion s'applique également pour les schémas similaires aux équations d'élasticité.

1. THE « VELOCITY-STRESS » FORMULATION OF THE WAVE EQUATION AND A SEMIDISCRETE VERSION

Let us consider the following initial-boundary value problem for the wave equation:

\[ D_t^2 u(t, x) - \Delta u(t, x) = f(t, x), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^2, \]

\[ u(t, x) = 0, \quad t > 0, \quad x \in \Gamma, \]

\[ u(0, x) = u_0(x), \quad D_t u(0, x) = v_0(x), \quad x \in \Omega, \]

where \( \Omega \) is a bounded domain with boundary \( \Gamma \), and \( f, u_0, v_0 \) are given functions. Introducing the « stress » \( \sigma = \nabla u \), (1.1) may be reformulated as

\[ D_t^2 u(t, x) - \text{div} \sigma(t, x) = f(t, x), \quad t > 0, \quad x \in \Omega, \]

\[ \sigma(t, x) = \nabla u(t, x), \quad t > 0, \quad x \in \Omega, \]

\[ u(t, x) = 0, \quad t > 0, \quad x \in \Gamma, \]

\[ u(0, x) = u_0(x), \quad D_t u(0, x) = v_0(x), \quad x \in \Omega. \]

(*) Received in October 1986.
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We use the notation of Johnson and Thomée [12]:

\[ V = L^2(\Omega), \quad H = \{ \chi \in L^2(\Omega)^2 : \text{div} \chi \in L^2(\Omega) \} . \]

Using Green's formula

\[ \int_{\Omega} u \text{div} \chi \, dx = \int_{\Gamma} u \chi \cdot n \, ds - \int_{\Omega} \nabla u \cdot \chi \, dx \]

where \( n \) is the unit exterior normal to \( \Gamma \), a Galerkin version of (1.2) is to seek \( u(t) \in V, \sigma(t) \in H, t > 0 \), satisfying

\[ (D_t^2 u(t), w) - (\text{div} \sigma(t), w) = (f(t), w), \quad w \in V, \]

\[ (\sigma(t) \chi) + (u(t), \text{div} \chi) = 0, \quad \chi \in H, \]

\[ u(0) = u_0, \quad D_t u(0) = v_0, \]

where the parentheses denote the appropriate inner products (\( L^2 \)-inner product in \( V \), \( L^2(\Omega)^2 \)-inner product in \( H \)). If \( V_h \subset V \) and \( H_h \subset H \) are finite dimensional subspaces, such as the spaces introduced by Raviart and Thomas [13], and by Brezzi, Douglas, Jr. and Marini [6], a semidiscrete version of (1.3) seeks \( u_h(t) \in V_h, \sigma_h(t) \in H_h, t > 0 \), satisfying

\[ (D_t^2 u_h(t), w_h) - (\text{div} \sigma_h(t), w_h) = (f(t), w_h), \quad w_h \in V_h, \]

\[ (\sigma_h(t) \chi_h) + (u_h(t), \text{div} \chi_h) = 0, \quad \chi_h \in H_h, \]

\[ u_h(0) = u_{0,h}, \quad D_t u_h(0) = v_{0,h}, \]

where \( u_{0,h}, v_{0,h} \in V_h \) are approximations to \( u_0 \) and \( v_0 \), respectively.

Johnson and Thomée [12] have discussed the parabolic counterpart of (1.3). The analysis of convergence of (1.3) can be carried out along similar lines, parallel to Baker and Bramble [4], for example. (1.3) is treated essentially as a non-conforming « displacement » model for the wave equation (1.1). The purpose of this note is to discuss the convergence of the « velocity-stress » models based on pairs of spaces \( (V_h, H_h) \) such as those in [6], [12], [13]. Thus, defining \( v = D_t^2 u, v_h = D_t u_h, (1.3) \) and (1.3) are transformed, respectively, to

\[ (D_t v(t), w) - (\text{div} \sigma(t), w) = (f(t), w), \quad w \in V, \]

\[ (D_t \sigma(t), \chi) + (v(t), \text{div} \chi) = 0, \quad \chi \in H, \]

\[ v(0) = v_0, \quad \sigma(0) = \sigma_0 = \nabla u_0, \]

where \( v(t) \in V, \sigma(t) \in H, t \geq 0 \), and

\[ (D_t v_h(t), w_h) - (\text{div} \sigma_h(t), w_h) = (f(t), w_h), w_h \in V_h, \]
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(1.4_h) \[ (D, \sigma_h(t), \chi_h) + (v_h(t), \text{div} \chi_h) = 0, \quad \chi_h \in H_h, \]
\[ v_h(0) = v_{0,h}, \quad \sigma_h(0) = \sigma_{0,h}, \]
where \( v_h(t) \in V_h, \sigma_h(t) \in H_h, t \geq 0. \)

We now list the basic features of the space \( V_h, H_h \) which lead to a straightforward analysis of the convergence of \( v_h \) to \( v \) and \( \sigma_h \) to \( \sigma \):

(H.1) There exists a linear operator \( \Pi_h : H \rightarrow H_h \) such that

\[ (\text{div} \Pi_h \chi, w_h) = (\text{div} \chi, w_h) \quad \forall w_h \in V_h, \quad \chi \in H, \]
\[ \| \Pi_h \chi - \chi \| \leq C h^s \| \chi \| \quad \text{s for } 1 \leq s \leq r, \quad r \geq 2 \]

(\( \| \| \) is the \( L_2(\Omega) \)-norm, and \( \| \|_s \) is the \( H^s(\Omega) \)-norm).

(H.2) There exists a linear operator \( P_h : V \rightarrow V_h \) such that

\[ (P_h v, \text{div} \chi_h) = (v, \text{div} \chi_h) \quad \forall \chi_h \in H_h, \quad v \in V, \]
\[ \| P_h v - v \| \leq C h^s \| v \|_s, \quad 1 \leq s \leq r, \quad r \geq 2 \]

(\( \| \| \) is the \( L_2(\Omega) \)-norm, and \( \| \|_s \) is the \( H^s(\Omega) \)-norm, and, as usual, \( C \) denotes a generic constant which depends only on the data and on the particular discretization scheme).

If \( \Omega \) is a polygonal domain and \( V_h, H_h \) are the Raviart-Thomas spaces [12], [13], or if these spaces are the pairs introduced in the paper by Brezzi, Douglas, Jr., and Marini [6], \( \text{div} \chi_h \in V_h, \) and \( P_h \) can be taken to be the \( L_2 \)-projection. For an example of a pair \( (V_h, H_h) \) satisfying the above hypotheses (with \( r = 2 \)), where \( P_h \) is not the \( L_2 \)-projection, we refer the reader to the paper by Johnson and Thomée [12]. We would also like to point out that (H.1) and (H.2) are valid for the mixed method that has been introduced by Arnold, Douglas, Jr., and Gupta [3] to approximate solution of plane elasticity problems. Our analysis is readily adapted to the corresponding (genuine) velocity-stress formulation of the time-dependent problem.

We can now state and prove our convergence result:

**Theorem:** If \( u \) is the solution of (1.1), \( v = Du, \sigma = \nabla u, \) and if the pair \( \{ v_h, \sigma_h \} \) is the solution of (1.4_h), under the hypotheses (H.1) and (H.2) we have, for \( 1 \leq s \leq r, \quad r \geq 2, \)

\[ \| v_h(t) - v(t) \| + \| \sigma_h(t) - \sigma(t) \| \leq C \left( \| v_0 - v_{0,h} \| + \| \sigma_0 - \sigma_{0,h} \| \right) + \]
\[ + C h^s \left( \| v_0 \|_s + \| \sigma_0 \|_s + \int_0^t (\| D \tau v(\tau) \|_s + \| D \tau \sigma(\tau) \|_s) \, d\tau \right). \]
Proof: Let us denote by $X$ the space $V \times H$, the elements of which will be designated as $\xi = \{v, \sigma\}$ or $\zeta = \{w, \chi\}$ and set

$$((\xi, \zeta)) = (v, w) + (\sigma, \chi),$$

$$\|\xi\| = \sqrt{((\xi, \xi))}.$$ 

Let $X_h = V_h \times H_h$ be equipped with $((., .))$ and the induced norm $\|\cdot\|$. Elements of $X_h$ will be designated as $\xi_h = \{v_h, \sigma_h\}$ or $\zeta_h = \{w_h, \chi_h\}$. We define the bilinear form $a(., .)$ on $X$ by

$$(1.10) \quad a(\xi, \zeta) = -(\text{div} \sigma, w) + (v, \text{div} \chi)$$

for $\xi = \{v, \sigma\}$, $\zeta = \{w, \chi\}$.

We can now express (1.4) as

$$(1.11) \quad ((D_t \xi(t), \zeta)) + a(\xi(t), \zeta) = (f(t), w), \quad \zeta \in X$$

$$(\xi(t) = \{v(t), \sigma(t)\}, \zeta = \{w, \chi\}),$$

and we can express (1.4) as

$$(1.11_h) \quad ((D_t \xi_h(t), \zeta_h)) + a(\xi_h(t), \zeta_h) = (f(t), w_h), \quad \zeta_h \in X_h$$

$$(\xi_h(t) = \{v_h(t), \sigma_h(t)\}, \zeta_h = \{w_h, \chi_h\}).$$

Let us define $P_h \xi = \{P_h v, \Pi_h \sigma\}$ for $\xi = \{v, \sigma\} \in X$, and observe that

$$(1.12) \quad a(P_h \xi, \zeta_h) = a(\xi, \zeta_h), \quad \zeta_h \in X_h$$

by (H.1) ((1.5)) and (H.2) ((1.7)). Therefore we obtain from (1.11)

$$(1.13) \quad ((D_t P_h \xi(t), \zeta_h)) + a(P_h \xi(t), \zeta_h) = (f(t), w_h) + ((P_h D_t \xi(t) - D_t \xi(t), \zeta_h)), \quad \zeta_h \in X_h.$$ 

Setting $\varepsilon_h(t) = P_h \xi(t) - \xi_h(t)$, (1.11) and (1.13) yield

$$(1.14) \quad ((D_t \varepsilon_h(t), \zeta_h)) + a(\varepsilon_h(t), \zeta_h) = (P_h D_t \xi(t) - D_t \xi(t), \zeta_h)), \quad \zeta_h \in X_h.$$ 

Let us define $\Lambda_h : X_h \rightarrow X_h$ by

$$(1.15) \quad ((\Lambda_h \xi_h, \zeta_h)) = a(\xi_h, \zeta_h), \quad \xi_h, \zeta_h \in X_h.$$ 

Since

$$(1.16) \quad a(\xi_h, \zeta_h) = -a(\xi_h, \zeta_h), \quad \xi_h, \zeta_h \in X_h.$$
as is readily seen (cf. (1.10)), $\Lambda_h$ is shew-adjoint,

$$((\Lambda_h \xi_h, \zeta_h)) = -((\xi_h, \Lambda_h \zeta_h)), \quad \xi_h, \zeta_h \in X_h,$$

and $-\Lambda_h$ generates the unitary group $e^{-t\Lambda_h}$. In particular,

$$\| e^{-t\Lambda_h} \xi_h(0) \| = \| \xi_h(0) \|, \quad t \in \mathbb{R}. \quad (1.18)$$

Let us denote by $P_h^0 : X \to X_h$ the projection with respect $((., .))$.

We can now express (1.14) as

$$D_t \varepsilon_h(t) + \Lambda_h \varepsilon_h(t) = P_h^0 (P_h D_t \xi(t) - D_t \xi(t)) \quad (1.19)$$

so that

$$\varepsilon_h(t) = e^{-t\Lambda_h} \varepsilon_h(0) + \int_0^t e^{-(t-\tau)\Lambda_h} P_h^0 (P_h D_\tau \xi(\tau) - D_\tau \xi(\tau)) d\tau. \quad (1.20)$$

(1.18) and (1.20) yield the estimate

$$\| \varepsilon_h(t) \| \leq \| \varepsilon_h(0) \| + \int_0^t \| P_h D_\tau \xi(\tau) - D_\tau \xi(\tau) \| d\tau \quad (1.21)$$

($P_h^0$ is the $((., .))$-projection).

(1.21) is readily translated to

$$\| P_h v(t) - v_h(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \|$$

$$\leq C (\| P_h v_0 - v_{0,h} \| + \| \Pi_h \sigma_0 - \sigma_{0,h} \| + \int_0^t (\| P_h D_\tau v(\tau) - D_\tau v(\tau) \| + \| \Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau) \| ) d\tau$$

$$\leq C (\| v_0 - v_{0,h} \| + \| \sigma_0 - \sigma_{0,h} \| + \| P_h v_0 - v_0 \| + \| \Pi_h \sigma_0 - \sigma_0 \| + \int_0^t (\| P_h D_\tau v(\tau) - D_\tau v(\tau) \| + \| \Pi_h D_\tau \sigma(\tau) - D_\tau \sigma(\tau) \| ) d\tau),$$

and this, together with (1.6) and (1.8), yields

$$\| P_h v(t) - v_h(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \|$$

$$\leq C (\| v_0 - v_{0,h} \| + \| \sigma_0 - \sigma_{0,h} \| + h^s (\| v_0 \|_s + \| \sigma_0 \|_s)$$

$$+ Ch^s \int_0^t (\| D_\tau v(\tau) \|_s + \| D_\tau \sigma(\tau) \|_s) d\tau. \quad (1.22)$$
Since
\[ \| v(t) - v_h(t) \| + \| \sigma(t) - \sigma_h(t) \| \]
\[ \leq \| v(t) - P_h v(t) \| + \| P_h v(t) - v_h(t) \| \]
\[ + \| \sigma(t) - \Pi_h \sigma(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \| \]
\[ \leq C h^t (\| v(t) \|_s + \| \sigma(t) \|_s ) + \| P_h v(t) - v_h(t) \| + \| \Pi_h \sigma(t) - \sigma_h(t) \| , \]
by (1.6) and (1.8), and
\[ \| v(t) \|_s \leq \| v_0 \|_s + \int_0^t \| D \tau v(\tau) \|_s \, d\tau , \]
\[ \| \sigma(t) \|_s \leq \| \sigma_0 \|_s + \int_0^t \| D \tau \sigma(\tau) \|_s \, d\tau , \]
(1.22) leads to (1.9), the assertion of the theorem.

2. SOME OBSERVATIONS IN REGARD TO THE TIME-DIFFERENCING OF THE SEMIDISCRETE MODEL

(1.4) leads to a system of ordinary differential equations in the form
\[ M_0 D_t W - D \Sigma = F , \]
\[ M_1 D_t \Sigma + D^T W = 0 , \]
where \( W \) corresponds to \( v_h \), \( \Sigma \) corresponds to \( \sigma_h \), \( M_0, M_1 \) are symmetric, positive-definite matrices, and \( D^T \) denotes the transpose of \( D \). The application of implicit Euler time-differencing
\[ M_0 \frac{W^{n+1} - W^n}{k} - D \Sigma^{n+1} = F^{n+1} , \]
\[ M_1 \frac{\Sigma^{n+1} - \Sigma^n}{k} + D^T W^{n+1} = 0 , \]
(\( k \) denotes the time step), necessitates the solution of
\[ M_0 W^{n+1} - k D \Sigma^{n+1} = k F^{n+1} + M_0 W^n , \]
\[ M_1 \Sigma^{n+1} + k D^T W^{n+1} = M_1 \Sigma^n . \]
\( M_0 \) is in block-diagonal form if \( V_h \) consists of functions with no continuity requirement across inter-element boundaries, as is the case in [6], [12], [13], and the elimination of \( W^{n+1} \) in (2.3) is efficiently implementable. This leads to a system in the form
\[ (M_1 + k^2 D^T M_0^{-1} D) \Sigma^{n+1} = G , \]
where $M_1$ is symmetric, positive definite and $D^T M_0^{-1} D$ is symmetric, positive-semidefinite, for the determination of $\Sigma^{n+1}$.

On the other hand, (1.3*) leads to

\begin{equation}
M_0 D_t^2 U - D \Sigma = F,
\end{equation}

\begin{equation}
M_1 \Sigma + D^T U = 0,
\end{equation}

where $U$ corresponds to $u_h$. (2.5) can be expressed as

\begin{equation}
M_0 D_t^2 U + D M_1^{-1} D^T U = F,
\end{equation}

where $M_0$, $D M_1^{-1} D^T$ are symmetric, positive-definite [12]. If (2.6) is expressed as a system in $\{U, W\}$,

\begin{equation}
D_t U - W = 0
\end{equation}

\begin{equation}
M_0 D_t W + D M_1^{-1} D^T U = F,
\end{equation}

and implicit Euler time-differencing is applied to (2.7),

\begin{equation}
\frac{U^{n+1} - U^n}{k} - W^{n+1} = 0
\end{equation}

\begin{equation}
M_0 \frac{W^{n+1} - W^n}{k} + D M_1^{-1} D^T U^{n+1} = F^{n+1},
\end{equation}

elimination of $W^{n+1}$ leads to a system in the form

\begin{equation}
(M_0 + k^2 D M_1^{-1} D^T) U^{n+1} = \tilde{G}.
\end{equation}

The matrix in (2.9) is symmetric, positive-definite, so that (2.9) is solvable. But $M_1$ is not block-diagonal, unlike $M_0$, so that deriving the reduced system (2.9), which includes inverting $M_1$, is more expensive than forming the reduced system (2.4). The time-independent counterpart of (2.5),

\begin{equation}
- D \Sigma = F,
\end{equation}

\begin{equation}
M_1 \Sigma + D^T U = 0,
\end{equation}

led Arnold and Brezzi [2] to relax the requirement that $\text{div } \sigma_h \in L_2(\Omega)$ in order to have a block-diagonal matrix instead of $M_1$ and be able to eliminate $\Sigma$ efficiently. This approach has to introduce a multiplier corresponding to the relaxation of the requirement $\text{div } \sigma_h \in L_2(\Omega)$.

The above considerations suggest that the « velocity-stress » formulation (1.4$_h$) may be preferable to (1.3$_h$) if the approximation of the « stress » $\sigma$ is of primary concern.

The application of diagonally implicit Runge-Kutta methods (see, for example, Crouzeix [8], Crouzeix and Raviart [9], Alexander [1], Burrage vol. 22, n° 2, 1988
[7], Dougalis and Serbin [10]) to (2.1) leads to systems similar to (2.4) so that our discussion is relevant to higher-order time differencing as well. We will not prove error estimates for such full-descrete approximation schemes based on \((1.4_h)\). Such estimates should be obtainable by employing techniques that have been utilized in [5] or [11], for example.

**REFERENCES**


