M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique

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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 22, nº 1 (1988), p. 159-176

<http://www.numdam.org/item?id=M2AN_1988__22_1_159_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol 22, n° 1, 1988, p 159 à 176)

NUMERICAL METHODS FOR A MODEL FOR WAVE PROPAGATION IN COMPOSITE ANISOTROPIC MEDIA (*)

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Communicated by J DOUGLAS

Abstract — *The problem of wave propagation through an anisotropic system consisting of an elastic solid Ω_s containing a fluid-saturated porous medium Ω_p is considered. The partial differential equations chosen to describe the propagation are the usual elastic wave equation in Ω_s and Biot's low-frequency dynamic equations for Ω_p, stated in terms of the global solid displacement vector u_1 for $\Omega = \Omega_s \cup \Omega_p$ and the relative vector movement u_2 between fluid and solid in Ω_p*

Energy flux preserving boundary conditions are used at the interface between Ω_s and Ω_p, while absorbing boundary conditions, derived in the last Section of this work, are imposed at the artificial boundaries of Ω

Since the solution $u = (u_1, u_2)$ of this wave problem is expected to lie in $[H^1(\Omega)]^3 \times H_0(\operatorname{div}, \Omega_p)$, standard finite element spaces are used to approximate the solid displacement vector u_1, while the relative movement vector u_2 is approximated using mixed finite element subspaces of $H_0(\operatorname{div}, \Omega_p)$

Existence and uniqueness results as well as continuous and discrete-time finite element methods for the approximate solution of this wave problem are given and analyzed

Also, the case of particular interest in exploration geophysics in which Ω represents a two-dimensional transverse anisotropic elastic system is analyzed in detail

Resume — *On considère ici le problème de la propagation d'onde dans un système anisotrope composé d'un solide élastique Ω_s et d'un milieu poreux saturé de fluide Ω_p Les équations aux dérivées partielles décrivant la propagation de l'onde sont l'équation d'onde de l'élasticité linéaire dans Ω_s et les equations de BIOT de dynamique à basse fréquence pour Ω_p , elles sont exprimées en fonction du vecteur déplacement u_1 pour $\Omega = \Omega_s \cup \Omega_p$ et du vecteur de déplacement relatif u_r du fluide par rapport au solide dans Ω_p*

Les conditions aux limites utilisées sont la conservation du flot à l'interface entre Ω_s et Ω_p, tandis que des conditions de frontière absorbante, obtenues dans la dernière section de ce travail, sont imposées sur les frontières artificielles de Ω

(*) Received in September 1986

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Comme la solution $u = (u_1, u_2)$ de ce problème d'onde appartient à $[H^1(\Omega)^3] \times H_0(\mathrm{div}, \Omega_p)$, des éléments finis standards sont utilisés pour l'approximation du vecteur déplacement dans le solide u_1, tandis que le vecteur de déplacement relatif u_2 est approximé dans des sous-espaces d'éléments finis mixtes de $H_0(\mathrm{div}, \Omega_p)$.

1. INTRODUCTION

We will consider the problem of wave propagation through a composite anisotropic system Ω consisting of an elastic solid Ω_s containing a porous medium Ω_p saturated by a compressible viscous fluid. The standard elastic wave equation and Biot's low-frequency dynamic equations are used to describe the propagation of waves in Ω_s and Ω_p, respectively.

The validity of Biot's low-frequency dynamic equations in Ω_p implies that several physical assumptions are made. First, wave lengths have to be appreciably greater than the diameter of the pores. Also the fluid may flow relative to the solid causing friction to arise. Dissipation is assumed to depend only on such a relative movement between fluid and solid, which is supposed to take place according to Darcy's law of fluid flow through porous anisotropic media. Finally, the form of Biot's equations chosen in this work, which were presented in [5], allows us to consider domains Ω_p having non-uniform porosity.

Appropriate energy-preserving boundary conditions are imposed at the contact surface between Ω_s and Ω_p. Also, absorbing boundary conditions for the artificial boundaries of the model are derived. These conditions are obtained from the conservation of momentum equations for anisotropic elastic solids and generalize those given in [12] for the isotropic case.

In a previous work [19], the problem considered here was treated for the isotropic case but using another form of Biot's equations which is valid only for uniform porosity [3]. Also, several earlier papers are related to the subject. Biot's theory of propagation of elastic waves in fluid-saturated anisotropic porous media will be strongly referred [4, 5]. The problem of existence and uniqueness of the solution of Biot's equations for isotropic media with uniform porosity as well as numerical methods for the approximate solution of such equations were considered in [17, 18]. Finite element methods for solving the elastic wave equation in isotropic media were presented in [16]. Boundary conditions allowing conservation of energy flux through the interface between Ω_s and Ω_p were first given in [11]. The organization of the paper is as follows. In Section 2 we describe the composite anisotropic model and state the partial differential equations as well as the initial and boundary conditions describing it. In Section 3 we present some notation and results to be used. In Section 4 we first derive the weak form of the problem and state the existence and uniqueness theorem for the solution of the composite system. Then we present the finite element spaces to be used for the spatial discretization and formulate the continuous and discrete-time Galerkin procedures. Since the existence and uniqueness results and the error bounds for the numerical methods can be obtained by repeating the arguments given for the model treated in [19], all the theorems of this Section are stated without proof. In Section 5 we apply the results previously derived to the case of particular interest in exploration geophysics in which Ω is a two-dimensional transverse anisotropic composite system. Finally, in Section 6 we present the derivation of the absorbing boundary conditions for anisotropic elastic solids used in our model.

2. THE COMPOSITE ANISOTROPIC MODEL

Let us identify the composite anisotropic elastic system with an open bounded domain $\Omega \subset R^3$ and assume that its boundary, denoted by $\partial\Omega$, is Lipschitz continuous. Let $\Gamma_0 \subset \partial\Omega$ and $\Gamma_1 = \partial\Omega \backslash \Gamma_0$ be the stress-free and artificial boundaries of the model, respectively. It will be assumed that both Γ_0 and Γ_1 have strictly positive $d\sigma$-measure, $d\sigma$ being the surface measure on $\partial\Omega$. Let $\Omega_p \subset\subset \Omega$ and $\Omega_s = \Omega \backslash \bar{\Omega}_p$ be the fluid-saturated porous medium and elastic solid parts of Ω, respectively, and set $\Gamma_2 = \partial\Omega_p$, that will also be assumed to be Lipschitz continuous.

Let $u_1(x,t) = (u_{11}(x,t), u_{12}(x,t), u_{13}(x,t))$ be the vector representing the displacement in the solid part of Ω (i.e. both in Ω_s and Ω_p) and let $\tilde{u}_2(x,t) = (\tilde{u}_{21}(x,t),\ \tilde{u}_{22}(x,t),\ \tilde{u}_{23}(x,t))$ be the locally averaged fluid displacement in Ω_p. Here u_{1i} and $\tilde{u}_{2i}$ are the displacement in the x_i-direction for $1 \leqslant i \leqslant 3$. The $\tilde{u}_{2i}$'s are defined so that the volume of fluid displaced through a unit area S normal to the x_i-direction is $\int_S \phi \tilde{u}_{2i}\, d\sigma$, $\phi = \phi(x)$ being the effective porosity.

Let $u_2(x,t) = \phi(x)(\tilde{u}_2(x,t) - u_1(x,t)) = (u_{21}, u_{22}, u_{23})$ and set $u(x,t) = (u_1(x,t),\ u_2(x,t))$. Note that u_2 represents the flow of the fluid relative to the solid but measured in terms of volume per unit area of the bulk medium Ω_p.

Next, the strain tensor in the solid is given by

$$\varepsilon_{ij}(u_1) = \frac{1}{2}\left(\frac{\partial u_{1i}}{\partial x_j} + \frac{\partial u_{1j}}{\partial x_i}\right), \quad 1 \leqslant i, j \leqslant 3 .$$

According to Hooke's law for anisotropic elastic solids, the stress tensor $\sigma_{ij}(u_1)$ is related to $\varepsilon_{ij}(u_1)$ according to

$$\sigma_{ij}(u_1) = \sum_{k,\ell} A_{ijk\ell}(x)\, \varepsilon_{k\ell}(u_1), \quad 1 \leqslant i, j \leqslant 3, \tag{2.1}$$

where, because of symmetry conditions, the tensor $A_{ijk\ell}(x)$ is such that

$$A_{ijk\ell} = A_{k\ell ij} = A_{ij\ell k} .$$

Also, the strain energy density $W_s = W_s(\varepsilon_{ij}(u_1))$ in Ω_s is a quadratic function of $\varepsilon_{ij}(u_1)$ given by

$$W_s(\varepsilon_{ij}(u_1)) = \frac{1}{2} \sum_{i,j} \sigma_{ij}(u_1)\, \varepsilon_{ij}(u_1) . \tag{2.2}$$

Next, on the set Ω_p the stress-strain relations can be described using a total stress tensor $\tau_{ij}(u)$ for the bulk material and the fluid pressure $p(u)$. They are given by [5],

$$\begin{aligned} \tau_{ij}(u) &= \sum_{k,\ell} A_{ijk\ell}(x)\, \varepsilon_{k\ell}(u_1) - Q_{ij}(x) \nabla \cdot u_2 , \quad 1 \leqslant i, j \leqslant 3 , \\ p(u) &= \sum_{k,\ell} Q_{k\ell}(x)\, \varepsilon_{k\ell}(u_1) - H(x) \nabla \cdot u_2 . \end{aligned} \tag{2.3}$$

Here $H = H(x)$ is a strictly positive coefficient and the Q_{ij}'s are such that $Q_{ij} = Q_{ji}$. Methods of measurement of the elastic coefficients $A_{ijk\ell}$, Q_{ij} and H in (2.3) are discussed in [4]. Now the strain energy density $W_p = W_p(\varepsilon_{ij}(u_1), \nabla \cdot u_2)$ in Ω_p can be written in the form [5],

$$W_p(\varepsilon_{ij}(u_1), \nabla \cdot u_2) = \frac{1}{2} \left[\sum_{i,j} \tau_{ij}(u)\, \varepsilon_{ij}(u_1) - p(u) \nabla \cdot u_2 \right] . \tag{2.4}$$

It will be assumed that the elastic coefficients $A_{ijk\ell}$, Q_{ij} and H in (2.1) and (2.3) are bounded in modulus by some positive constant and also that they satisfy certain conditions in order that the quadratic forms W_s and W_p be positive definite. Let $\lambda_s(x)$ and $\lambda_p(x)$ be the minimum eigenvalues of the symmetric matrices $E_s(x) \in R^{6 \times 6}$ and $E_p(x) \in R^{7 \times 7}$ associated with the quadratic forms W_s and W_p respectively and let $\lambda_i^* = \inf_{x \in \bar{\Omega}_i} \lambda_i(x)$, $i = s, p$.

Physical considerations allow us to assume that $\lambda_s^* > 0$ and $\lambda_p^* > 0$. Also, set

$$\begin{aligned} z_1(u_1) &= (\varepsilon_{11}(u_1), \varepsilon_{22}(u_1), \varepsilon_{33}(u_1), \varepsilon_{12}(u_1), \varepsilon_{13}(u_1), \varepsilon_{23}(u_1))^t , \\ z(u) &= (z_1^t(u_1), -\nabla \cdot u_2)^t . \end{aligned}$$

Then we have

$$\begin{aligned} W_s(\varepsilon_{ij}(u_1)) &= [E_s\, z_1(u_1), z_1(u_1)]_e \\ &\geqslant \lambda_s^* \| z_1(u_1) \|_e^2 \\ &\geqslant \frac{\lambda_s^*}{2} \sum_{i,j} (\varepsilon_{ij}(u_1))^2 \end{aligned} \tag{2.5}$$

and

$$W_p(\varepsilon_{ij}(u_1), \nabla \cdot u_2) = [E_p\, z(u), z(u)]_e \geq \frac{\lambda_p^*}{2} \left(\sum_{i,j} (\varepsilon_{ij}(u_1))^2 + (\nabla \cdot u_2)^2 \right) . \tag{2.6}$$

Here $[\cdot\,,\cdot\,]_e$ and $\|\cdot\|_e$ denote the usual euclidean inner product and norm in R^n. In Section 5, necessary and sufficient conditions for the validity of (2.5) and (2.6) will be shown for the case in which Ω is a two-dimensional transverse anisotropic system.

Next, let $\rho = \rho(x)$ be the total mass density of bulk material in Ω and let $\rho_f = \rho_f(x)$ be the mass density of fluid in Ω_p and assume that

$$0 < \rho_* \leq \rho(x) \leq \rho^* < \infty , \quad x \in \bar{\Omega} ,$$
$$0 < \rho_{f*} \leq \rho_f(x) \leq \rho_f^* < \infty , \quad x \in \bar{\Omega}_p .$$

Also let $G(x) = (g_{ij}(x)) \in R^{3\times 3}$ be a symmetric positive define matrix which depends on the coordinates of the pores and the pore geometry. Then let the mass matrix $\mathscr{A}(x) \in R^{6\times 6}$ be defined by

$$\mathscr{A}(x) = \begin{bmatrix} \rho(x)\, I & \rho_f(x)\, I \\ \rho_f(x)\, I & G(x) \end{bmatrix} ,$$

where $I \in R^{3\times 3}$ denotes the identity matrix. Let $\lambda_G(x)$ be the minimum eigenvalue of $G(x)$. Then it will be assumed that

$$\rho_f^2(x) < \lambda_G(x)\, \rho(x) , \quad x \in \bar{\Omega}_p ,$$

which is a necessary and sufficient condition for the matrix $\mathscr{A}$ to be positive definite.

Let the nonnegative dissipation matrix $\mathscr{C}(x) \in R^{6\times 6}$ be given by

$$\mathscr{C}(x) = \mu(x) \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{K}^{-1}(x) \end{bmatrix} ,$$

where $\mu(x)$ is the fluid viscosity and $\mathbb{K}(x) \in R^{3\times 3}$ is the symmetric positive definite permeability matrix.

Let $\sigma_i(u_i) = \sum_j \sigma_{ij}(u_1)\, e_j$ and $\tau_i(u) = \sum_j \tau_{ij}(u)\, e_j$, $1 \leq i, j \leq 3$, where e_j denotes the versor in the x_j-direction and let $\mathscr{L}_1(u_1)$ and $\mathscr{L}(u)$ be the differential operators given by

$$\mathscr{L}_1(u_1) = (\nabla \cdot \sigma_1(u_1), \nabla \cdot \sigma_2(u_2), \nabla \cdot \sigma_3(u_1)) ,$$
$$\mathscr{L}(u) = (\nabla \cdot \tau_1(u), \nabla \cdot \tau_2(u), \nabla \cdot \tau_3(u), -\nabla p(u)) .$$

Let $u_1^0(x)$, $v_1^0(x)$ and $f_1(x,t)$ be given for $x \in \Omega$ and $u_2^0(x)$, $v_2^0(x)$ and $f_2(x,t)$ on Ω_p. Set $u^0 = (u_1^0, u_2^0)$, $v^0 = (v_1^0, v_2^0)$, $f = (f_1, f_2)$. Then we consider the following problem : find $u(x,t) = (u_1(x,t), u_2(x,t))$ such that

$$\begin{aligned} &\text{i) } \rho \frac{\partial^2 u_1}{\partial t^2} - \mathscr{L}_1(u_1) = f_1(x,t), x \in \Omega_s, \quad t \in J = (0,T), \\ &\text{ii) } \mathscr{A} \frac{\partial^2 u}{\partial t^2} + \mathscr{C} \frac{\partial u}{\partial t} - \mathscr{L}(u) = f(x,t), \quad (x,t) \in \Omega_p \times J, \end{aligned} \tag{2.7}$$

with initial conditions

$$\begin{aligned} &\text{i) } u_1(x,0) = u_1^0, \quad x \in \Omega, \quad t = 0, \\ &\text{ii) } u_2(x,0) = u_2^0, \quad x \in \Omega_p, \quad t = 0, \\ &\text{iii) } \frac{\partial u_1}{\partial t}(x,0) = v_1^0, \quad x \in \Omega, \quad t = 0, \\ &\text{iv) } \frac{\partial u_2}{\partial t}(x,0) = v_2^0, \quad x \in \Omega_p, \quad t = 0, \end{aligned} \tag{2.8}$$

and boundary conditions

$$\begin{aligned} &\text{i) } \sigma(u_1) \cdot \nu_s = 0, \quad (x,t) \in \Gamma_0 \times J, \\ &\text{ii) } \sigma(u_1) \cdot \nu_s = -\rho^{1/2} \mathbb{D}^{1/2} \frac{\partial u_1}{\partial t}, \quad (x,t) \in \Gamma_1 \times J, \\ &\text{iii) } \sigma(u_1) \cdot \nu_s + \tau(u) \cdot \nu_p = 0, \quad (x,t) \in \Gamma_2 \times J, \\ &\text{iv) } u_2 \cdot \nu_p = 0, \quad (x,t) \in \Gamma_2 \times J. \end{aligned} \tag{2.9}$$

In the above $\nu_i = (\nu_{i1}, \nu_{i2}, \nu_{i3})$ denotes the unit outward normal along $\partial\Omega_i$, $i = s, p$, and $\sigma \cdot \nu_s = (\sigma_1 \cdot \nu_s, \sigma_2 \cdot \nu_s, \sigma_3 \cdot \nu_s)$, $\tau \cdot \nu_p = (\tau_1 \cdot \nu_p, \tau_2 \cdot \nu_p, \tau_3 \cdot \nu_p)$.

Equation (2.7.i)) is the standard elastic wave equation for Ω_s, while (2.7.ii)) is Biot's low-frequency dynamic equation describing the propagation of elastic waves in anisotropic fluid-saturated porous media in the context of nonuniform porosity [5].

Equation (2.9.i)) represents the stress-free condition on Γ_0 and (2.9.ii)) is an absorbing boundary condition for the artificial boundary Γ_1 which is derived in Section 6, the factor $\mathbb{D}^{1/2}$ in the right-hand side of (2.9.ii)) being the square root of the symmetric positive definite matrix $\mathbb{D}$ of (6.2).

Next, (2.9.iii)) states the continuity of the stress tensor on Γ_2 and (2.9.iv)) imposes the vanishing of the normal component of the relative movement between fluid and solid along Γ_2. The last two boundary conditions are used in order to have conservation of energy flux through Γ_2 [11].

For the uniform porosity isotropic case, Biot's dynamic equation (2.7.ii)) is equivalent to that given in [3] and the problem just stated above in (2.7)-(2.9) reduces to that treated in [19].

3. NOTATIONS AND PRELIMINARIES

For an integer $n \geqslant 1$, denote by $(\cdot , \cdot)_i$ and $\| \cdot \|_{0, \Omega_i}$, $i = s, p$, the inner product and norm in $[L^2(\Omega_i)]^n$ and by $(\cdot , \cdot)$ and $\| \cdot \|_0$ the inner product and norm in $[L^2(\Omega)]^n$. For $\tilde{\Omega} = \Omega, \Omega_s, \Omega_p$ and m a nonnegative integer let $H^m(\tilde{\Omega}) = W^{m,2}(\tilde{\Omega})$ be the usual Sobolev space, the norm of an element $v = (v^1, \ldots, v_n)$ in $[H^m(\tilde{\Omega})]^n$ being given by

$$\| v \|_{m, \tilde{\Omega}} = \left[\sum_{j=1}^{n} \sum_{|\alpha| \leqslant m} \int_{\tilde{\Omega}} | D^{\alpha} v_j(x) \, dx |^2 \right]^{1/2} .$$

We will omit the symbol $\tilde{\Omega}$ in the notation above in the case $\tilde{\Omega} = \Omega$.

Next, the inner product of $v, w \in [L^2(\Gamma)]^n$, $\Gamma \subset \partial\Omega_i$, will be denoted by

$$\langle v, w \rangle_{\Gamma} = \sum_{j=1}^{n} \int_{\Gamma} v_j \, w_j \, d\sigma ,$$

where $d\sigma$ is the surface measure on Γ.

Let $H(\mathrm{div}, \Omega_i) = \{ q \in [L^2(\Omega_i)]^3 : \nabla \cdot q \in L^2(\Omega_i) \}$ provided with the norm

$$\| q \|_{H(\mathrm{div}, \Omega_i)} = [\| q \|_{0, \Omega_i}^2 + \| \nabla \cdot q \|_{0, \Omega_i}^2]^{1/2} .$$

Let $H_0(\mathrm{div}, \Omega_i)$ be the closure of $[C_0^{\infty}(\Omega_i)]^3$ in the norm $\| \cdot \|_{H(\mathrm{div}, \Omega_i)}$. Then, it can be shown that [9],

$$H_0(\mathrm{div}, \Omega_i) = \{ q \in H(\mathrm{div}, \Omega_i) : q \cdot \nu_i = 0 \quad \text{on} \quad \partial\Omega_i \} .$$

Also, recall that the following formula of integration by parts holds [9] :

$$(\nabla \cdot q, v)_i + (q, \nabla v)_i = \langle q \cdot \nu_i, v \rangle_{\partial\Omega_i} , \tag{3.1}$$

for any $q \in H(\mathrm{div}, \Omega_i)$ and any $v \in H^1(\Omega_i)$.

Next let $V = [H^1(\Omega)]^3 \times H(\mathrm{div}, \Omega_p)$, the norm of an element $v = (v_1, v_2) \in V$ being given by

$$\|v\|_V = [\|v_1\|_1^2 + \|v_2\|_{H(\mathrm{div}, \Omega_p)}^2]^{1/2} .$$

Finally, if $T > 0$, $J = (0, T)$ and $\mathbb{B}$ is a Banach space with norm $\|\cdot\|_{\mathbb{B}}$, recall that

$$L^2(J, \mathbb{B}) = \left\{ v : J \to \mathbb{B} : \|v\|_{L^2(J, \mathbb{B})} = \left[\int_0^T \|v(t)\|_{\mathbb{B}}^2 \, dt \right]^{1/2} < \infty \right\}$$

and

$$L^\infty(J, \mathbb{B}) = \left\{ v : J \to \mathbb{B} : \|v\|_{L^\infty(J, \mathbb{B})} = \operatorname*{ess\,sup}_{t \in J} \|v(t)\|_{\mathbb{B}} < \infty \right\} .$$

4. EXISTENCE AND UNIQUENESS RESULTS AND NUMERICAL PROCEDURES

First we will formulate problem (2.7)-(2.9) in a weak form. For $v = (v_1, v_2) \in V$, $w = (w_1, w_2) \in V$ let

$$B_s(v_1, w_1) = \sum_{i,j} (\sigma_{ij}(v_1), \varepsilon_{ij}(w_1))_s ,$$

$$B_p(v, w) = \sum_{i,j} (\tau_{ij}(v), \varepsilon_{ij}(w_1))_p - (p(v), \nabla \cdot w_2)_p ,$$

$$B(v, w) = B_s(v_1, w_1) + B_p(v, w) .$$

Also set

$$\tilde{V} = [H^1(\Omega)]^3 \times H_0(\mathrm{div}, \Omega_p)$$
$$= \{ v = (v_1, v_2) \in V : v_2 \cdot \nu_p = 0 \quad \text{on} \quad \Gamma_2 \} .$$

Note that $\tilde{V}$ is a closed separable subspace of V and that $[C^\infty(\bar{\Omega})]^3 \times [C_0^\infty(\Omega_p)]^3$ is dense in $\tilde{V}$.

Let $v = (v_1, v_2) \in \tilde{V}$. Multiply (2.7.i)) by v_1 and integrate over Ω_s. After applying the formula of integration by parts (3.1) to the $(\mathscr{L}_1(u_1), v_1)_s$-term and the boundary conditions (2.9.i))-(2.9.ii)) we obtain

$$\left(\rho \frac{\partial^2 u_1}{\partial t^2}, v_1 \right)_s + B_s(u_1, v_1) + \left\langle \rho^{1/2} \mathbb{D}^{1/2} \frac{\partial u_1}{\partial t}, v_1 \right\rangle_{\Gamma_1}$$
$$- \langle \sigma(u_1) \cdot \nu_s, v_1 \rangle_{\Gamma_2} = (f_1, v_1)_s , \qquad t \in J . \qquad (4.1)$$

Next, multiply (2.7.ii)) by v and integrate over Ω_p. After integration by parts of the $(\mathscr{L}(u), v)_p$-term, since $v \in \tilde{V}$ we get

$$\left(\mathscr{A}\frac{\partial^2 u}{\partial t^2}, v\right)_p + \left(\mathscr{C}\frac{\partial u}{\partial t}, v\right)_p + B_p(u, v) - \langle \tau(u) \cdot \nu_p, v_1\rangle_{\Gamma_2} = (f, v)_p, \quad t \in J. \tag{4.2}$$

Thus, adding (4.1) and (4.2) and using (2.9.iii)) we conclude that

$$\begin{aligned}\left(\rho\frac{\partial^2 u_1}{\partial t^2}, v_1\right)_s + \left(\mathscr{A}\frac{\partial^2 u}{\partial t^2}, v\right)_p + \left(\mathscr{C}\frac{\partial u}{\partial t}, v\right)_p \\ + B(u, v) + \left\langle\rho^{1/2}\,\mathbb{D}^{1/2}\frac{\partial u_1}{\partial t}, v_1\right\rangle_{\Gamma_1} \\ = (f_1, v_1) + (f_2, v_2)_p, \quad v \in \tilde{V}, \quad t \in J.\end{aligned} \tag{4.3}$$

Now we will analyze the properties of the bilinear form $B(\cdot\,,\cdot)$. First note that $B(\cdot\,,\cdot)$ is symmetric and V-continuous. For analyzing the V-coercivity of the form $B(\cdot\,,\cdot)$, we recall Korn's second inequality which states that [7], [8], [14],

$$\sum_{i,j}(\varepsilon_{ij}(z), \varepsilon_{ij}(z))_k \geq C_1\|z\|^2_{1,\Omega_k} - \|z\|^2_{0,\Omega_k},$$

for any $z \in [H^1(\Omega_k)]^3$, $k = s, p$. Thus combining (2.2), (2.4), (2.5) and (2.6) we see that, for any $v \in V$,

$$\begin{aligned}B(v, v) &\geq \lambda_s^* \sum_{i,j}(\varepsilon_{ij}(v_1), \varepsilon_{ij}(v_1))_s \\ &\quad + \lambda_p^*\left[\sum_{i,j}(\varepsilon_{ij}(v_1), \varepsilon_{ij}(v_1))_p + (\nabla\cdot v_2, \nabla\cdot v_2)_p\right] \\ &\geq C_3\|v\|_V^2 - C_2(\|v_1\|_0^2 + \|v_2\|^2_{0,\Omega_p}),\end{aligned} \tag{4.4}$$

where

$$C_2 = \max(\lambda_s^*, \lambda_p^*), \quad C_3 = \min(\lambda_s^*, \lambda_p^*)\min(1, C_1).$$

Inequality (4.4) is the analogue of the Gärding inequality for a second order elliptic operator.

Now we will state the main theorem on the existence and uniqueness of the solution of problem (2.7)-(2.9). The proof is identical to that given in

[19] for the uniform porosity isotropic case and will be omitted. Set

$$P_i^2 = \left\| \frac{\partial^i f_1}{\partial t^i} \right\|^2_{L^2(J, [L^2(\Omega)]^3)} + \left\| \frac{\partial^i f_2}{\partial t^i} \right\|^2_{L^2(J, [L^2(\Omega_p)]^3)},$$

$$M_0^2 = \|u_1^0\|_2^2 + \|u_2^0\|^2_{2,\Omega_p} + \|v_1^0\|_1^2 + \|v_2^0\|^2_{1,\Omega_p} + $$
$$+ \|f_1(0)\|_0^2 + \|f_2(0)\|^2_{0,\Omega_p} + 1 .$$

THEOREM 4.1 : *Let $f = (f_1, f_2)$, $u^0 = (u_1^0, u_2^0)$ and $v^0 = (v_1^0, v_2^0)$ be given and such that $M_0 < \infty$ and $P_i < \infty$, $i = 0, 1$. Also assume that*

$$\begin{aligned} &\textit{support } (u_1^0) \cap \Omega_i \subset\subset \Omega_i \\ &\textit{support } (v_1^0) \cap \Omega_i \subset\subset \Omega_i , \qquad i = s, p , \\ &\textit{support } (u_2^0) \subset\subset \Omega_p , \\ &\textit{support } (v_2^0) \subset\subset \Omega_p . \end{aligned}$$

Then there exists a unique solution $u(x, t)$ of problem (2.7)-(2.9) *such that* $u, \dfrac{\partial u}{\partial t} \in L^\infty(J, V)$, $\dfrac{\partial^2 u_1}{\partial t^2} \in L^\infty(J, [L^2(\Omega)]^3)$ *and* $\dfrac{\partial^2 u_2}{\partial t^2} \in L^\infty(J, [L^2(\Omega_p)]^3)$.

Now we proceed to describe the numerical procedures. Let $k \geq 1$ be an integer and let $0 < h < 1$. Let $\Upsilon_h^s = \Upsilon_h^s(\Omega)$ and $\Upsilon_h^f = \Upsilon_h^f(\Omega_p)$ be quasiregular partitions of Ω and Ω_p respectively into simplices or cubes of diameter bounded by h. Let $m_h \subset [H^1(\Omega)]^3$ be a standard finite element space associated with Υ_h^s such that

$$\inf_{\chi \in m_h} \{ \|v - \chi\|_0 + h\|v - \chi\|_1 \} \leq Ch^r \|v\|_r , \quad 1 \leq r \leq k+1 . \tag{4.5}$$

Next, let W_h^f be a finite dimensional subspace of $H(\text{div}, \Omega_p)$ associated with Υ_h^f such that

$$\text{i)} \ \inf_{\chi \in W_h^f} \|w - \chi\|_{0,\Omega_p} \leq Ch^r \|w\|_{r,\Omega_p} , \quad 1 \leq r \leq k , \tag{4.6}$$

$$\text{ii)} \ \inf_{\chi \in W_h^f} \|w - \chi\|_{H(\text{div},\Omega_p)} \leq Ch^k [\|w\|_{k,\Omega_p} + \|\nabla \cdot w\|_{k,\Omega_p}] .$$

Let $\mathring{W}_h^f = W_h^f \cap H_0(\text{div}, \Omega_p)$ and set $\tilde{V}_h = m_h \times \mathring{W}_h^f$. Then $\tilde{V}_h \subset \tilde{V}$ and it follows from (4.5) and (4.6) that

$$\text{i)} \ \inf_{\chi = (\chi_1, \chi_2) \in \tilde{V}_h} (\|v_1 - \chi_1\|_0 + \|v_2 - \chi_2\|_{0,\Omega_p}) \leq Ch^r (\|v_1\|_r + \|v_2\|_{r,\Omega_p}) ,$$

for any $v = (v_1, v_2) \in ([H^r(\Omega)]^3 \times [H^r(\Omega_p)]^3) \cap \tilde{V}$, $1 \leqslant r \leqslant k$, (4.7)

ii) $$\inf_{\chi = (\chi_1, \chi_2) \in \tilde{V}_h} \|v - \chi\|_V \leqslant C h^k (\|v_1\|_{k+1} + \|v_2\|_{k, \Omega_p} + \|\nabla \cdot v_2\|_{k, \Omega_p}),$$

for any $v = (v_1, v_2) \in ([H^{k+1}(\Omega)]^3 \times [H^k(\Omega_p)]^3) \cap \tilde{V}$

such that $\nabla \cdot v_2 \in H^k(\Omega_p)$.

Let us denote by $\underset{\sim}{\mathrm{BDDF}}_k$ (respectively, $\underset{\sim}{\mathrm{RTN}}_k$) the vector part of the Brezzi-Douglas-Durán-Fortin [2], (respectively, Raviart-Thomas-Nedelec [6, 13]) space of index k associated with Υ_h^f and set

$$\begin{aligned} \underset{\sim}{\mathrm{BD\mathring{D}F}}_k &= \underset{\sim}{\mathrm{BDDF}}_k \cap H_0(\mathrm{div}, \Omega_p), \\ \underset{\sim}{\mathrm{R\mathring{T}N}}_k &= \underset{\sim}{\mathrm{RTN}}_k \cap H_0(\mathrm{div}, \Omega_p). \end{aligned}$$

Since $\underset{\sim}{\mathrm{BDDF}}_k$ and $\underset{\sim}{\mathrm{RTN}}_{k-1}$ satisfy the approximating hypotheses (4.6) [2], [6], [13], $\tilde{V}_h$ should be taken to be either $m_h \times \underset{\sim}{\mathrm{B\mathring{D}DF}}_k$ or $m_h \times \underset{\sim}{\mathrm{R\mathring{T}N}}_{k-1}$.

Now we will define the numerical procedures. The continuous-time Galerkin approximation to the solution u of problem (2.7)-(2.9) is defined as the twice-differentiable map $U = (U_1, U_2) : J \to \tilde{V}_h$ such that

$$\begin{aligned} &\left(\rho \frac{\partial^2 U_1}{\partial t^2}, v_1\right)_s + \left(\mathscr{A} \frac{\partial^2 U}{\partial t^2}, v\right)_p + \left(\mathscr{C} \frac{\partial U}{\partial t}, v\right)_p \\ &\qquad + B(U, v) + \left\langle \rho^{1/2} \mathbb{D}^{1/2} \frac{\partial u_1}{\partial t}, v_1 \right\rangle_{\Gamma_1} \\ &\qquad = (f_1, v_1) + (f_2, v_2)_p, \quad v = (v_1, v_2) \in \tilde{V}_h, \quad t \in J. \end{aligned} \tag{4.8}$$

Also, $U(0)$ and $\dfrac{\partial U}{\partial t}(0)$ must be specified in (4.8) as approximations to u^0 and v^0, respectively.

The error analysis performed in [19] can be repeated here to show that

$$\left\| \frac{\partial (u_1 - U_1)}{\partial t} \right\|_{L^\infty(J, [L^2(\Omega)]^3)} + \left\| \frac{\partial (u_2 - U_2)}{\partial t} \right\|_{L^\infty(J, [L^2(\Omega_p)]^3)} + \\ + \|u - U\|_{L^\infty(J, V)} \leqslant C_4(u^0, v^0, u) h^k.$$

The explicit form of the constant C_4 above can also be found in [19].

Now we proceed to define the discrete-time Galerkin method. Let L be a positive integer, $\Delta t = T/L$, $f^n = f(n\,\Delta t)$, $n = 0, \dots, L$. Set

$$U^{n+1/2} = \frac{1}{2}(U^n + U^{n+1}),$$

$$U^{n,1/4} = \frac{1}{2}(U^{n-1/2} + U^{n+1/2}),$$

$$d_t U^n = \frac{U^{n+1} - U^n}{\Delta t},$$

$$\partial U^n = \frac{U^{n+1} - U^{n-1}}{2\,\Delta t},$$

$$\partial^2 U^n = \frac{U^{n+1} - 2\,U^n + U^{n-1}}{(\Delta t)^2}.$$

Then the discrete-time Galerkin method for obtaining the approximate solution of problem (2.7)-(2.9) is defined as follows: find $(U^n = (U_1^n, U_2^n))_{0 \leq n \leq L} \subset \tilde{V}_h$ such that

$$\begin{aligned}(\rho\, \partial^2 U_1^n, v_1)_s + (\mathscr{A}\, \partial^2 U^n, v)_p + (\mathscr{C}\, \partial U^n, v)_p \\ + B(U^{n,1/4}, v) + \langle \rho^{1/2}\, \mathbb{D}^{1/2}\, \partial U_1^n, v_1 \rangle_{\Gamma_1} \\ = (f_1^{n,1/4}, v_1) + (f_2^{n,1/4}, v_2)_p\,, \\ v = (v_1, v_2) \in \tilde{V}_h\,, \quad 1 \leq n \leq L-1\,.\end{aligned} \tag{4.9}$$

Again the error analysis given in [19] shows that

$$\max_{1 \leq n \leq L-1} \left(\|d_t(u_1 - U_1)^n\|_0 + \|d_t(u_2 - U_2)^n\|_{0,\Omega_p} + \|(u - U)^n\|_V \right)$$

$$\leq C_5(u^0, v^0, u)[\|d_t(u_1 - U_1)^0\|_0 + \|d_t(u_2 - U_2)^0\|_{0,\Omega_p} + \|(u - U)^{1/2}\|_V + (\Delta t)^2 + h^k]\,.$$

If the initial values U^0 and U^1 needed to initialize the procedure (4.9) are chosen as indicated in [19], then the optimal order of convergence is preserved. This completes the definition and analysis of the numerical methods related to the problem (2.7)-(2.9).

5. THE TWO-DIMENSIONAL TRANSVERSE ANISOTROPIC CASE

Here the results obtained in the previous Sections will be applied to the case in which Ω is a two-dimensional domain and the bulk material is symmetric with respect to the x_2-axis in the following sense. Let $\bar{A}_{ijk\ell}$ and $\bar{Q}_{ij}$ be the tensors in the stress-strain relations (2.1) and (2.3) after a change of coordinates of the form $\bar{x}_1 = -x_1$, $\bar{x}_2 = x_2$. Then we will assume that

$$\bar{A}_{ijk\ell} = A_{ijk\ell}\,, \qquad \bar{Q}_{ij} = Q_{ij}\,. \tag{5.1}$$

The conditions above imply that plane waves travelling in the x_i-directions show the same behaviour as the longitudinal and shear waves in isotropic media. Also, it follows from (5.1) that there are only seven independent coefficients in the stress-strain relations (2.1) and (2.3), which can be written as

$$\begin{aligned}
\sigma_{11}(u_1) &= R_1(x)\,\varepsilon_{11}(u_1) + A(x)\,\varepsilon_{22}(u_1)\,,\\
\sigma_{22}(u_1) &= A(x)\,\varepsilon_{11}(u_1) + R_2(x)\,\varepsilon_{22}(u_1)\,,\\
\sigma_{12}(u_1) &= 2\,N(x)\,\varepsilon_{12}(u_1)\,,
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
\tau_{11}(u) &= R_1(x)\,\varepsilon_{11}(u_1) + A(x)\,\varepsilon_{22}(u_1) - Q_1(x)\,\nabla \cdot u_2\,,\\
\tau_{22}(u) &= A(x)\,\varepsilon_{11}(u_1) + R_2(x)\,\varepsilon_{22}(u_1) - Q_2(x)\,\nabla \cdot u_2\,,\\
\tau_{12}(u) &= 2\,N(x)\,\varepsilon_{12}(u_1)\,,\\
p(u) &= Q_1(x)\,\varepsilon_{11}(u_1) + Q_2(x)\,\varepsilon_{22}(u_1) - H(x)\,\nabla \cdot u_2\,.
\end{aligned} \tag{5.3}$$

In the above $u_1 = (u_{11}, u_{12})$ is the vector displacement in the solid part of Ω and $u_2 = \phi(\tilde{u}_2 - u_1) = (u_{21}, u_{22})$, where $\tilde{u}_2 = (\tilde{u}_{21}, \tilde{u}_{22})$ represents the average fluid displacement in Ω_p. From physical considerations it can be seen that $R_1(x)$, $R_2(x)$, $A(x)$, $N(x)$ and $H(x)$ are bounded above and below by positive constants. Also, necessary and sufficient conditions for the strain energy densities W_s and W_p to be positive definite are given by

$$\begin{aligned}
&R_1\,R_2 - A^2 > 0\,, \quad x \in \bar{\Omega}_s\,,\\
&R_1\,R_2\,H + 2\,AQ_1\,Q_2 - R_1\,Q_2^2 - R_2\,Q_1^2 - HA^2 > 0\,, \quad x \in \bar{\Omega}_p\,,\\
&R_1\,R_2 + H(R_1 + R_2) - Q_1^2 - Q_2^2 - A^2 > 0\,, \quad x \in \bar{\Omega}_p\,.
\end{aligned}$$

The inequalities above are easily obtained analyzing the characteristic polynomials of the quadratic forms $W_s(\varepsilon_{ij}(u_1))$ and $W_p(\varepsilon_{ij}(u_1), \nabla \cdot u_2)$.

Next, the symmetric properties of the material imply that matrices G and $\mathbb{K}$ in the definition of the mass matrix $\mathcal{A}$ and the dissipation matrix $\mathcal{C}$ are given by

$$G = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \qquad \mathbb{K} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

In order to define this two-dimensional model completely we have to construct the matrix $\mathbb{D}$ of the absorbing boundary condition (2.9.ii)). For that purpose, and as is indicated in Section 6, we obtain the strain energy π on the artificial boundary Γ_1 by combining (2.2) and (5.2) to write W_s as a quadratic function of $(\varepsilon_{ij}(u_1))_{1 \leq i, j \leq 2}$ and then use (6.1) to substitute $\varepsilon_{ij}(u_1)$ on Γ_1 in terms of ν_s, $\frac{\partial u_1}{\partial t}$ and the wave velocity c. In this way we get a quadratic function π of $\left(\frac{1}{c}\frac{\partial u_{1i}}{\partial t}\right)_{1 \leq i \leq 2}$ which has the following expression :

$$\pi = \frac{1}{2}\left[((\nu_{s1})^2 R_1 + (\nu_{s2})^2 N)\left(\frac{1}{c}\frac{\partial u_{11}}{\partial t}\right)^2 + 2\,\nu_{s1}\,\nu_{s2}(A+N)\frac{1}{c}\frac{\partial u_{11}}{\partial t}\frac{1}{c}\frac{\partial u_{12}}{\partial t} + ((\nu_{s1})^2 N + (\nu_{s2})^2 R_2)\left(\frac{1}{c}\frac{\partial u_{12}}{\partial t}\right)^2\right].$$

Thus, it follows from (6.2) that

$$\mathbb{D} = \begin{bmatrix} (\nu_{s1})^2 R_1 + (\nu_{s2})^2 N & \nu_{s1}\,\nu_{s2}(A+N) \\ \nu_{s1}\,\nu_{s2}(A+N) & (\nu_{s1})^2 N + (\nu_{s2})^2 R_2 \end{bmatrix}.$$

Note that when Ω is a rectangle the matrix $\mathbb{D}$ is diagonal and consequently the boundary conditions (2.9.ii)) have a simple expression in terms of the elastic coefficients R_1 or R_2 and N.

Finally, since we are dealing with a two-dimensional domain, the finite element spaces used for the spatial discretization have to be changed as follows. Let $\Upsilon_h = \Upsilon_h(\Omega)$ and $\Upsilon_h^f = \Upsilon_h^f(\Omega_p)$ be quasiregular partitions of Ω and Ω_p respectively into triangles or rectangles of diameter bounded by h and let $m_h \subset [H^1(\Omega)]^2$ be a standard finite element space associated with Υ_h satisfying the approximating hypotheses (4.5). Let $\underset{\sim}{\mathrm{BDM}}_k$ (respectively, $\underset{\sim}{\mathrm{RT}}_k$) denote the vector part of the Brezzi-Douglas-Marini [1], (respectively, Raviart-Thomas [6], [15], [20]) two-dimensional space of index k associated with Υ_h^f and set $\overset{\circ}{\underset{\sim}{\mathrm{BDM}}}_k = \{q \in \underset{\sim}{\mathrm{BDM}}_k : q \cdot \nu_p = 0 \text{ on } \Gamma_2\}$, $\overset{\circ}{\underset{\sim}{\mathrm{RT}}}_k = \{q \in \underset{\sim}{\mathrm{RT}}_k : q \cdot \nu_p = 0 \text{ on } \Gamma_2\}$. Then $\tilde{V}_h$ should be taken to be either $m_h \times \overset{\circ}{\underset{\sim}{\mathrm{BDM}}}_k$ or $m_h \times \overset{\circ}{\underset{\sim}{\mathrm{RT}}}_{k-1}$. Now the two-dimensional transverse anisotropic model is totally defined and the continuous and discrete-time Galerkin procedures (4.8) and (4.9) can be implemented.

6. AN ABSORBING BOUNDARY CONDITION FOR ANISOTROPIC ELASTIC SOLIDS

In any realistic seismic model we are led to consider wave propagation in infinite solids. One way of representing an infinite system by a finite one is to introduce non-reflecting artificial boundaries into the finite system.

In this Section we will derive boundary conditions ensuring that most of the energy arriving at the artificial boundary Γ_1 will be absorbed. These boundary conditions will be obtained from the momentum equations of wave fronts arriving normally to Γ_1, thus making it transparent to that kind of waves. For the isotropic case, the equations derived here reduce to those presented in [12] by Lysmer and Kuhlemeyer.

First we will summarize some results given in [10] about the velocity of waves in anisotropic solids. Let us consider a wave arriving normally to Γ_1 with a velocity c. The strain tensor on Γ_1 can be written in the form

$$\varepsilon_{ij}(u_1^c) = -\frac{1}{2}\left(\nu_{si}\frac{1}{c}\frac{\partial u_{1j}^c}{\partial t} + \nu_{sj}\frac{1}{c}\frac{\partial u_{1i}^c}{\partial t}\right), \ 1 \leqslant i, j \leqslant 3, \ (x, t) \in \Gamma_1 \times J . \tag{6.1}$$

Next, let $v^c = \frac{1}{c}\frac{\partial u_1^c}{\partial t} = (v_1^c, v_2^c, v_3^c)$. Then combining Hooke's law (2.1) and (6.1), the strain energy W_s in (2.2) can be written on Γ_1 as a quadratic function $\pi(v^c) = W_s(\varepsilon_{ij}(v^c))$ in the form

$$\cancel{\pi = \frac{1}{2}[\mathbb{D}v^c, v^c]_e , \qquad (x, t) \in \Gamma_1 \times J , \qquad (6.2)}$$

where $\mathbb{D}(x) \in R^{3\times 3}$ is a symmetric, positive definite matrix depending on ν_s and the elastic coefficients $A_{ijk\ell}(x)$ in (2.1).

Next, the momentum equation on Γ_1 are

$$-\rho c\frac{\partial u_{1k}^c}{\partial t} = \sum_j \frac{\partial W_s}{\partial \varepsilon_{kj}}\nu_{sj} , \quad 1 \leqslant k \leqslant 3 , \quad (x, t) \in \Gamma_1 \times J .$$

Also, combining (2.1)-(2.2) and (6.1),

$$\begin{aligned} \frac{\partial \pi}{\partial v_k^c} &= \sum_{i,j}\frac{\partial W_s}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial v_k^c} \\ &= -\sum_j \frac{\partial W_s}{\partial \varepsilon_{kj}}\nu_{sj} , \quad 1 \leqslant k \leqslant 3 . \end{aligned} \tag{6.3}$$

Thus,

$$\frac{\partial \pi}{\partial v_k^c} = \rho c^2 v_k^c , \quad 1 \leqslant k \leqslant 3 , \quad (x, t) \in \Gamma_1 \times J , \tag{6.4}$$

which in turn implies that c^2 satisfies the equation

$$\det(\mathbb{D} - \rho c^2 I) = 0 . \tag{6.5}$$

Equation (6.5) yields three real positive values for c^2, so that there are three real wave-velocities, say $(c_k)_{1 \leqslant k \leqslant 3}$, corresponding to any direction of propagation of waves. Equations (6.1)-(6.5) are derived in [10, pp. 295-298].

Now consider a velocity $\dfrac{\partial u_1}{\partial t}$ on the surface Γ_1 due to the simultaneous normal arrival of waves of possible different velocities $(c_k)_{1 \leqslant k \leqslant 3}$. Let M be the matrix containing, as rows, the eigenvectors M_k of D and let Λ be the diagonal matrix containing the eigenvalues $(\rho c_k^2)_{1 \leqslant k \leqslant 3}$ of D, so that $D = M^t \Lambda M$. Also set $v^{c_k} = \dfrac{1}{c_k}\left[M_k, \dfrac{\partial u_1}{\partial t}\right]_e M_k$ and let $\mathscr{F}_k$ be the force on Γ_1 corresponding to v^{c_k}. Then according to (6.4) $\mathscr{F}_k$ satisfies the relations

$$\begin{aligned} \mathscr{F}_k &= \frac{\partial \pi}{\partial v^{c_k}} \\ &= \rho c_k^2 v^{c_k} \\ &= \rho c_k \left[M_k, \frac{\partial u_1}{\partial t}\right]_e M_k . \end{aligned} \tag{6.6}$$

Next we observe that the total energy of waves arriving normally to Γ_1 is the sum of the partial energies $\pi(v^{c_k})$ and the total force $-\sigma(u_1) \cdot \nu_s$ on Γ_1 is equal to the sum of the partial forces $\mathscr{F}_k$. Then, according to (6.6),

$$\begin{aligned} -\sigma(u_1) \cdot \nu_s &= -\sum_{k=1}^{3} [M_k, \sigma(u_1) \cdot \nu_s]_e M_k \\ &= \rho \sum_k c_k \left[M_k, \frac{\partial u_1}{\partial t}\right]_e M_k , \end{aligned}$$

so that

$$-[M_k, \sigma(u_1) \cdot \nu_s]_e = \rho c_k \left[M_k, \frac{\partial u_1}{\partial t}\right]_e , \quad 1 \leqslant k \leqslant 3 .$$

In matrix form the equations above can be written

$$-M(\sigma(u_1) \cdot \nu_s) = \rho^{1/2} \Lambda^{1/2} M \frac{\partial u_1}{\partial t} ,$$

so that multiplying by M^t we finally get

$$-\sigma(u_1) \cdot \nu_s = \rho^{1/2} \mathbb{D}^{1/2} \frac{\partial u_1}{\partial t} , \quad (x, t) \in \Gamma_1 \times J . \tag{6.7}$$

Using (6.7) as boundary conditions on Γ_1, the conservation of momentum implies that the energy of wave fronts arriving normally to Γ_1 will be absorbed. Note also that when the medium Ω_s is isotropic the diagonal matrix $\Lambda = (\Lambda_{ij})$ is given by $\Lambda_{11} = \rho\alpha^2$, $\Lambda_{22} = \Lambda_{33} = \rho\beta^2$, α and β being the velocities of the compressional and shear waves, respectively, and that the rows of M contain the unit outward normal and two unit tangent vectors to Γ_1 in that order. Thus, in this case conditions (6.7) reduce to those given in [12] by Lysmer and Kuhlemeyer.

7. ACKNOWLEDGEMENTS

The authors are indebted to Professor Jim Douglas, Jr., for his suggestions and advice in the preparation of this paper. They are also thankful to Yacimientos Petrolíferos Fiscales (Argentina's National Oil Company) where this work has been done at the Departamento de Estudios Especiales, Gerencia General de Exploración. Besides, to Adriana Schollaert for helping us in the English version of the article.

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