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HOW TO AVOID THE USE OF GREEN'S THEOREM IN THE CIARLET-RAVIART THEORY OF VARIATIONAL CRIMES (*)

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Abstract. — The paper generalizes the theory developed in [1] and [2, Section 4.4] to the case that the solution $u$ of the given variational problem belongs to $H^1(\Omega)$ only. Mixed boundary conditions, approximation of a curved boundary and numerical integration are taken into account. The considerations are restricted to the two-dimensional case.


The foundations of the theory mentioned in the title of this paper are given in Ciarlet, Raviart [1] and Ciarlet [2, Section 4.4]. Some extensions of this theory (which will be briefly denoted as the CR-theory) to the case of boundary value problems with various stable and unstable boundary conditions were done in Ženíšek [9], [10]. In all these papers the maximum rate of convergence is proved; thus the assumed smoothness of the exact solution $u$ is unrealistic in the majority of problems appearing in applications. The smoothness of $u$ allows us to use the Green's theorem in estimating the third term on the right-hand side of [2, (4.4.21)] — see also the first term on the right-hand side of (35). This simplifies very much considerations.

In this paper we consider the variational problem corresponding to a general elliptic boundary value problem with combined Dirichlet's and Neumann's boundary conditions. We assume only that the solution $u$ of the variational

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problem exists, i.e. \( u \in H^1(\Omega) \). Thus we cannot transform the term \( \partial_m(\tilde{u}, w) \) (defined by (21)) to the form (50) by means of Green's theorem. Instead of it our main tool becomes Zlámal's ideal curved triangular element (see Zlámal [7]) which is considered simultaneously with the associate curved triangular element used in applications. As \( u \in H^1(\Omega) \) the complete result of this paper will be only the proof of convergence (without any rate of convergence). The considerations of this paper are based on some results from [9]; thus we use some notions and symbols introduced in [9] without any deeper explanation and with reference to [9] only.

The notation of Sobolev spaces, their norms and seminorms is the same as in the book [2] and other references of this paper.

Let \( \Omega \) be a bounded domain in \( E_2 \) with a Lipschitz-continuous boundary \( \Gamma \). Let \( a(v, w) : H^1(\Omega) \times H^1(\Omega) \to R \) be a bilinear form which is bounded and \( V \)-elliptic,

\[
| a(v, w) | \leq M \| v \|_1 \| w \|_1 \quad \forall v, w \in H^1(\Omega),
\]

(1)

\[
\alpha \| v \|_1^2 \leq a(v, v) \quad \forall v \in V,
\]

(2)

where \( \alpha, M \) are positive constants and

\[
V = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1, \operatorname{mes}_1 \Gamma_1 > 0, \Gamma_1 \subset \Gamma \},
\]

(3)

and let \( L(v) : H^1(\Omega) \to R \) be a bounded linear form,

\[
| L(v) | \leq K \| v \|_1 \quad \forall v \in H^1(\Omega),
\]

(4)

where \( K \) is a positive constant. (In (1)-(4) and in what follows we write for a greater simplicity \( \| \cdot \|_1 \) instead of \( \| \cdot \|_{1, \alpha} \).

Remark 1: If \( \operatorname{mes}_1 \Gamma_1 < \operatorname{mes}_1 \Gamma \) and \( \Omega \) is a simply connected domain we consider only the case that \( \Gamma_1 \) consists of a finite number of disjoint arcs. The end-points of these arcs belong (by definition) to \( \Gamma_1 \). Thus \( \Gamma_2 = \Gamma - \Gamma_1 \) consists of a finite number of arcs without end-points. In the case of a multiply connected domain \( \Omega \) the situation is similar.

Problem P: Let

\[
W = \{ v \in H^1(\Omega) : v = \tilde{u} \text{ on } \Gamma_1 \},
\]

(5)

where \( \tilde{u} \in H^{1/2}(\Gamma_1) \) is a given function. Find a function \( u \in W \) such that

\[
a(u, v) = L(v) \quad \forall v \in V.
\]

(6)

The Lax-Milgram lemma implies that Problem P has just one solution \( u \in W \).
In what follows we shall consider $a(v, w)$ and $L(v)$ of the forms

$$a(v, w) = \int_{\Omega} k_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx$$

and

$$L(v) = L^\Omega(v) + L^\Gamma(v) \equiv \int_{\Omega} vf \, dx + \int_{\Gamma_2} qv \, ds ,$$

respectively, where $\Gamma_2 = \Gamma - \Gamma_1$. In (7) and in what follows a summation convection over repeated subscripts is adopted.

We assume that the following sufficient conditions for the validity of (1), (2), (4) hold:

- $k_{ij}$ are measurable and bounded functions on $\Omega$,
- $k_{ij}(x) \xi_i \xi_j \geq \mu_0 \xi_i \xi_i \quad \forall \xi_i, \xi_j \in R \quad \forall x \in \Omega ,$

where $\mu_0$ is a positive constant,

$$f \in L_2(\Omega) , \quad q \in L_2(\Gamma_2) .$$

In the case of the use of numerical integration we shall have additional requirements concerning the smoothness of the functions $k_{ij}$, $f$ and $q$.

Similarly as in [1], [2], [10] we shall consider three following variational crimes (the notion « variational crime » is due to Strang (see [4], [5])):

1. Approximation of the space $V$ and the manifold $W$ by a finite dimensional space $V_h$ and manifold $W_h$ respectively.
2. Approximation of the domain $\Omega$ by a domain $\Omega_h$ with a boundary $\Gamma_h$ which is simpler than $\Gamma$.
3. Approximation of the forms $a(v, w)$, $L(v)$ by forms $a_h(v, w)$, $L_h(v)$ which are obtained by means of numerical integration.

Combining these three basic variational crimes we can obtain various situations; we shall consider the most general case.

Let us choose a sequence $\{ h_m \}$ of real numbers with the following properties :

$$1 > h_m > 0 , \quad h_m > h_{m+1} , \quad \lim_{m \to \infty} h_m = 0 .$$

For every $m$ let us construct an ideal triangulation $\tau^i_m$ of the domain $\Omega$ and its approximation $\tau^n_m$ (where $n$ is a given integer) in the following way: Let us
choose a finite number of nodal points on \( \Gamma \); each corner of \( \Gamma \) (if any) and each end-point of arcs forming \( \Gamma_1 \) (if \( \Gamma_1 \neq \Gamma \) — see Remark 1) belong to these points; the distance between two neighbouring nodal points is not greater than \( h_m \). The triangulation \( \tau_m^{id} \) is chosen in such a way that two different arcs, in which the boundary \( \Gamma \) is divided by the nodal points, are sides of two different boundary triangles. Further, the interior triangles of \( \tau_m^{id} \) have only straight sides. Finally,

\[
\hat{h}_m \leq h_m, \quad \overline{h}_m \geq c_0 h_m, \quad g_m \geq g_0,
\]

where \( c_0, g_0 \) are positive constants and

\[
\hat{h}_m = \max_{T \in \tau_m} h_T, \quad \overline{h} = \min_{T \in \tau_m} h_T, \quad g_m = \min_{T \in \tau_m} g_T.
\]

In (14) \( h_T \) and \( g_T \) are the length of the greatest side of \( T \) and the magnitude of the smallest angle of \( T \), respectively, and \( \tau_m \) is the triangulation, which arises from \( \tau_m^{id} \) if we substitute triangles with one curved side by triangles with straight sides. (If \( \Omega \) has a polygonal boundary then \( \tau_m = \tau_m^{id} \).)

If \( \Omega \) has not a polygonal boundary we obtain the triangulation \( \tau_m^w \) associated with \( \tau_m^{id} \) in the following way: Let us choose an integer \( n \geq 1 \) and on each curved side of \( \tau_m^{id} \) let us choose \( n - 1 \) nodal points with the coordinates

\[
[x(i/n), y(i/n)] \quad (i = 1, \ldots, n - 1),
\]

where

\[
x_1 = \varphi(t), \quad x_2 = \psi(t) \quad (0 \leq t \leq 1)
\]

is the local parametric representation of the considered curved side (in more detail see [9, eqs. (6)]), where the symbols \( \varphi, \psi \) are used instead of \( \varphi, \psi \). The side (15) is then approximated by the arc

\[
x_1 = \varphi^*(t), \quad x_2 = \psi^*(t), \quad 0 \leq t \leq 1,
\]

where \( \varphi^*(t) \) and \( \psi^*(t) \) are the Lagrange interpolation polynomials of degree \( n \) of the functions \( \varphi(t) \) and \( \psi(t) \), respectively, uniquely determined by the relations

\[
\varphi^*(i/n) = \varphi(i/n) \quad (i = 0, 1, \ldots, n),
\]
\[
\psi^*(i/n) = \psi(i/n) \quad (i = 0, 1, \ldots, n).
\]

The arcs of the type (16) form curved sides of the boundary triangles of the triangulation \( \tau_m^w \) and the union of closed triangles \( \bar{T} \in \tau_m^w \) forms the approximation \( \Omega_m^w \) of \( \Omega \).
Now we choose the remaining nodal points of \( \tau_n^m \) and \( \tau_{\text{id}}^m \). If \( n = 1 \) then they are formed by the vertices of the triangles of \( \tau_n^m \) or \( \tau_{\text{id}}^m \). If \( n = 2 \) then they are formed by the vertices of the triangles and by the mid-points of the straight sides. If \( n = 3 \) then they are formed by the vertices of the triangles, by the points dividing the straight sides of the triangles into thirds and by the « centres of gravity » \( P_T^0 \) of all triangles \( T \in \tau_n^m \) (or \( T \in \tau_{\text{id}}^m \)). (In the case of a triangle \( T \) with straight sides the point \( P_T^0 \) is the center of gravity of \( T \), in the case of a curved triangle \( T \) the point \( P_T^0 \) is the image of the point \((1/3, 1/3)\) in the transformation mapping the well-known standard triangle \( T_0 \) (see [6]-[10]) onto \( T \).

On every triangle with straight sides function values prescribed at the nodal points determine uniquely a polynomial of degree \( n \). On every curved triangle (both an ideal one and an approximating one) function values prescribed at the nodal points determine uniquely a function which is on both straight sides a polynomial of degree \( n \) in one variable. (Details are omitted; they can be found in [2], [6]-[8].)

Piecing together just mentioned finite elements we obtain \( N \)-dimensional spaces \( X_m^n \) and \( Y_m^n \) of continuous functions on \( \tau_n^m \) and \( \tau_{\text{id}}^m \), respectively, where \( N \) is the number of nodal points in both triangulations \( \tau_n^m \) and \( \tau_{\text{id}}^m \).

Let \( \Gamma_{m1} \) be the approximation of \( \Gamma_1 \) defined by the triangulation \( \tau_n^m \); we set

\[
V_m = \{ v \in X_m^n : v = 0 \text{ on } \Gamma_{m1} \}
\]

\[
= \{ v \in X_m^n : v(P_k) = 0, P_k \in \Gamma_{m1} \}
\]

where \( P_k \) are the nodal points. In order to define suitably the finite element approximation \( W_m \) of \( W \) we shall assume that the function \( \bar{u} \) is so smooth that there exists a function \( z \in H^2(\Omega) \) such that \( z = \bar{u} \) on \( \Gamma_1 \). Then we can set

\[
W_m = \{ v \in X_m^n : v(P_k) = \bar{u}(P_k), P_k \in \Gamma_{m1} \}
\]

Remark 2 : In the definitions of \( V_m \) and \( W_m \) we need the space \( X_m^n \) only. The space \( Y_m^n \) will be used in (52)-(54).

In what follows we assume (similarly as in the CR-theory) that there exists a bounded domain \( \tilde{\Omega} \) such that

\[
\tilde{\Omega} \supset (\Omega \cup \Omega_m) \quad \forall m
\]

and that \( k_{ijp}, f \) are continuous and bounded functions on \( \Omega \) having continuous and bounded extensions \( \tilde{k}_{ijp}, \tilde{f} \) onto \( \tilde{\Omega} \). As to the functions \( \tilde{k}_{ij} \) we further assume that there exists a constant \( \tilde{\mu}_0 \) such that

\[
\tilde{k}_{ij}(x) \xi_i \xi_j \geq \tilde{\mu}_0 \xi_i \xi_i \quad \forall \xi_0, \xi_j \in R \quad \forall x \in \tilde{\Omega}.
\]
Thus every bilinear form

\[ \tilde{a}_m(v, w) = \int_{\Omega_m} \hat{k}_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx \]  

has the property

\[ \tilde{a}_m(v, v) \geq \mu_0 |v|_{1, \Omega_m}^2 \quad \forall v \in H^1(\Omega_m). \]  

Further we define

\[ \tilde{L}_m(v) = \tilde{L}_m^\Omega(v) + \tilde{L}_m^F(v) \]  

where

\[ \tilde{L}_m^\Omega(v) = \int_{\Omega_m} \tilde{f} \, dx, \quad \tilde{L}_m^F(v) = \int_{\Gamma_m} q_m \, v \, ds \]  

with \( \Gamma_m = \Gamma - \Gamma_{m1} \). The symbol \( q_m \) denotes the function which is obtained by « transferring » the function \( q \) from \( \Gamma_2 \) onto \( \Gamma_{m2} \) (see [10]), i.e. if \( c \) is a part of \( \Gamma_2 \) with parametric representation (15) and \( c_m \) its approximation with parametric representation (16) then

\[ \int_c qv \, ds = \int_0^1 q(\varphi(t), \psi(t)) \, \nu(\varphi(t), \psi(t)) \, \rho(t) \, dt, \]  

\[ \int_{c_m} q_m \, v \, ds = \int_0^1 q(\varphi^*(t), \psi^*(t)) \, \nu^*(t) \, \rho^*(t) \, dt \]  

where

\[ \rho(t) = \left[ (\dot{\varphi}(t))^2 + (\dot{\psi}(t))^2 \right]^{1/2}, \]  

\[ \rho^*(t) = \left[ (\dot{\varphi}^*(t))^2 + (\dot{\psi}^*(t))^2 \right]^{1/2}. \]

Using quadrature formulas on the triangles with integration points lying in \( \overline{\Omega} \cap \overline{\Omega}_m \) we replace the forms \( \tilde{a}_m(v, w) \) and \( \tilde{L}_m^\Omega(v) \) by the forms \( a_m(v, w) \) and \( L_m^\Omega(v) \), respectively. (Details can be found in [1], [2] or [8].) Further, computing numerically the integral on the right-hand side of (26) for each \( c_m \subseteq \Gamma_{m2} \) (see [10]) we obtain a linear form \( L_m^F(v) \). Denoting

\[ L_m(v) = L_m^\Omega(v) + L_m^F(v) \]  

we can formulate the following discrete problem:
Problem $P_m$: Find a function $u_m \in W_m$ such that
\[ a_m(u_m, v) = L_m(v) \quad \forall v \in V_m. \] (30)

First we must prove the existence and uniqueness of the solution $u_m$ of Problem $P_m$. This is solved (besides other problems) in Theorems 1-3.

**Theorem 1**: Let the boundary $\Gamma$ of the domain $Q$ be piecewise of class $C^{n+1}$. Then
\[ \| v \|_{1, \Omega_m}^2 \leq K \| v \|_{1, \Omega_m}^2 \quad \forall v \in V_m \quad \forall h_m < \tilde{h} \] (31)
where $\tilde{h}$ is sufficiently small fixed number and $K$ a positive constant independent of $v$ and $h_m$.

Theorem 1 is proved in [9] in a more general form.

**Remark 3**: If $F$ is piecewise of class $C^{n+1}$ then it has a finite number of points of $C^{n+1}$-discontinuity. These points are nodal points of all triangulations $\tau^n_m$ and $\tau^d_m$ ($m = 1, 2, ...$).

**Theorem 2**: Let $k_{ij} \in W^{(n)}(\bar{Q})$ ($i, j = 1, 2$) and let the quadrature formula on the standard triangle $T_0$ used for calculation of $a_m(v, w)$ be of degree of precision $d = \max (1, 2n - 2)$. Then for all $v, w \in V_m$ we have
\[ |\tilde{a}_m(v, w) - a_m(v, w)| \leq C\tilde{B}_n h_m \| v \|_{1, \Omega_m} \| w \|_{1, \Omega_m} \] (32)
where $C$ is a positive constant independent of $k_{ij}, v, w$ and $h_m$ and the constant $\tilde{B}_n$ is defined by
\[ \tilde{B}_n = \sum_{i,j=1}^2 \| k_{ij} \|_{n, \infty, \bar{Q}}. \] (33)

Theorem 2 follows from [8, Theorem 7] (see also [2, Chapter 4]).

**Theorem 3**: Let the assumptions of Theorems 1 and 2 be satisfied. Then for $h_m \leq \tilde{h}$ the bilinear forms $a_m(v, w)$ are uniformly $V_m$-elliptic, i.e. there exists a positive constant $\beta$ independent of $V_m$ such that
\[ \beta \| v \|_{1, \Omega_m}^2 \leq a_m(v, v) \quad \forall v \in V_m \quad \forall h_m \leq \tilde{h}, \] (34)
and Problem $P_m$ has just one solution $u_m$.

**Proof**: Relations (22) and (31) imply
\[ \tilde{a}_m(v, v) \geq \tilde{\mu}_0 K^{-1} \| v \|_{1, \Omega_m}^2 \quad \forall v \in V_m. \]
Theorem 2 gives

\[ a_m(v, v) - \bar{a}_m(v, v) \geq -CB_n h_m \| v \| \Omega_m^2 \quad \forall v \in V_m. \]

Adding both inequalities up we obtain (34) with \( \beta = \bar{\mu}_0/(2K) \) for \( h_m \leq \bar{\mu}_0/(2CB_n K) \).

Now we prove the existence and uniqueness of the solution of Problem \( P_m \).

As relation (30) represents a system of linear algebraic equations for the unknowns \( u_m(P_k) \), where \( P_k \notin \Gamma_m \), it is sufficient to prove the uniqueness, i.e. to prove that the problem "find \( u_m \in V_m \) such that \( a_m(u_m, v) = 0 \ \forall v \in V_m \)" has only the trivial solution. This follows immediately from (34) if we set \( v = u_m \).

Theorem 3 is proved.

Now we are ready to formulate an abstract error theorem which is the starting point of the CR-theory and all its modifications.

**Theorem 4:** Let the assumptions of Theorem 1 and 2 be satisfied. Then there exists a positive constant \( C \) independent of \( V_m \) and \( W_m \) such that for all \( h_m < \tilde{h} \) we have

\[
\| \bar{u} - u_m \|_{1, \Omega_m} \leq C \left\{ \sup_{w \in V_m} \frac{|L_m(w) - \bar{a}_m(\bar{u}, w)|}{\| w \|_{1, \Omega_m}} + \right. \\
+ \left. \inf_{v \in W_m} \left[ \| \bar{u} - v \|_{1, \Omega_m} + \sup_{w \in V_m} \frac{\| \bar{a}_m(v, w) - a_m(v, w) \|}{\| w \|_{1, \Omega_m}} \right] \right\} \tag{35}
\]

where \( \bar{u} \in H^1(\tilde{\Omega}) \) is the Calderon's extension of the solution \( u \in H^1(\Omega) \) of Problem \( P \) from the domain \( \Omega \) onto the domain \( \tilde{\Omega} \).

The proof of Theorem 4 follows the same lines as the proof of [2, Theorem 4.4.1] and thus it is omitted. (Of course, if \( u \in H^k(\Omega) \), \( k > 1 \), then \( \bar{u} \in H^k(\tilde{\Omega}) \) in Theorem 4.)

Before introducing the first application of Theorem 4 we remind two theorems from the theory of numerical integration in the finite element method and prove a theorem on approximations of \( \bar{u} \) in the sets \( W_m \).

**Theorem 5:** Let \( 1 \leq r \leq n \). Let \( \tilde{f} \in W^{0, r}(\tilde{\Omega}) \) and let the quadrature formula on the standard triangle \( T_0 \) used for calculation of \( L_m^\alpha(v) \) be of degree of precision \( d = \max(1, r + n - 2) \). Then for all \( v \in V_m \) we have

\[
| L_m^\alpha(v) - \tilde{L}_m^\alpha(v) | \leq C h_m \| \tilde{f} \|_{r, \omega(\tilde{T})} \| v \|_{1, \Omega_m} \tag{36}
\]

where the constant \( C \) is independent of \( h_m, v \) and \( \tilde{f} \).

The proof of Theorem 5 is very similar to the proofs of [2, Theorems 4.1.5 and 4.4.5].
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THEOREM 6: Let the part $\Gamma_2$ of the boundary $\Gamma$ be piecewise of class $C^{n+1}$ and let the function $q(x_1, x_2)$ belong to the space $C^n(U)$, where $U$ is a domain containing $\Gamma_2$. Let the quadrature formula used on the segment $[0, 1]$ for calculation of $L_m^T(v)$ be of degree of precision $d = 2n - 1$. Then for sufficiently small $h_m$ and for all $v \in V_m$ we have

$$|L_m^T(v) - \hat{L}_m^T(v)| \leq C h_m^m \|v\|_{1, \Omega_m},$$

where the constant $C$ does not depend on $h_m$ and $v$.

Theorem 6 is proved in the proof of [10, Theorem 5].

THEOREM 7: Let $\overline{u}$ be so smooth that there exists a function $z \in H^2(\Omega)$ such that $z = \overline{u}$ on $\Gamma_1$. Then there exists a sequence $\{v_m\}$, where $v_m \in W_{m}$, such that

$$\lim_{m \to \infty} \|\overline{u} - v_m\|_{1, \Omega_m} = 0.$$ \hspace{1cm} (38)

Proof: According to [3], $C^\infty(\Omega) \cap V$ is dense in $V$. Let $\{\varepsilon_k\}$ be an arbitrary sequence of real numbers with properties

$$\varepsilon_k > 0, \hspace{0.5cm} \varepsilon_k > \varepsilon_{k+1}, \hspace{0.5cm} \lim_{k \to \infty} \varepsilon_k = 0.$$ \hspace{1cm} (39)

Let us set

$$w = u - z.$$ \hspace{1cm} (40)

Then $w \in V$ and for every $k$ there exists a function $w_{\varepsilon_k} \in C^\infty(\Omega) \cap V$ such that

$$\|w - w_{\varepsilon_k}\|_{1, \Omega} \leq \varepsilon_k / (3 \bar{C}),$$ \hspace{1cm} (41)

where $\bar{C}$ is the constant from inequality (42).

Let $\bar{v}$ be the Calderon's extension of $v \in H^1(\Omega)$ into $H^1(E_2)$. Then we have

$$\|\bar{v}\|_{1, E_2} \leq \bar{C} \|v\|_{1, \Omega} \hspace{0.5cm} \forall v \in H^1(\Omega),$$ \hspace{1cm} (42)

where the constant $\bar{C}$ does not depend on $v$. Similarly, if $\bar{v}^*$ denotes the Calderon's extension of $v \in H^2(\Omega)$ into $H^2(E_2)$ then

$$\|\bar{v}^*\|_{2, E_2} \leq \bar{C}^* \|v\|_{2, \Omega} \hspace{0.5cm} \forall v \in H^2(\Omega),$$ \hspace{1cm} (43)

where the constant $\bar{C}^*$ does not depend on $v$.

Relations (41), (42) imply

$$\|\bar{w} - \bar{w}_{\varepsilon_k}\|_{1, \bar{\Omega}} \leq \bar{C} \|w - w_{\varepsilon_k}\|_{1, \Omega} \leq \varepsilon_k / 3.$$ \hspace{1cm} (44)
Let $I_m v \in V_m$ be the interpolate of $v \in H^2(\Omega)$ (i.e. the function from $V_m$ which has the same function values as $v$ at the nodal points of $\tau_m^n$). Owing to the definition of nodal points we have

$$I_m \tilde{v}^* = I_m v \quad \forall v \in H^2(\Omega).$$

(45)

Relations (43), (45) and the finite element interpolation theorems (see [2] or [8, Theorem 5]) imply

$$\left\| \tilde{\tilde{w}}_{ek}^* - I_m w_{ek} \right\|_{1,\Omega_m} \leq Ch_m \left\| \tilde{\tilde{w}}_{ek}^* \right\|_{2,\Omega_m} \leq \tilde{C} Ch_m \left\| w_{ek} \right\|_{2,\Omega}.$$

Thus, according to (12), there exists $m_k$ (depending on $k$) such that

$$\left\| \tilde{\tilde{w}}_{ek}^* - I_m w_{ek} \right\|_{1,\Omega_m} \leq \varepsilon_k / 3 \quad \forall m \geq m_k^1.$$  \hspace{1cm} (46)

Finally, using the relation

$$\lim_{m \to \infty} \left\{ \text{mes} (\Omega_m - \Omega) \right\} = 0$$

we find, according to the theorem on the absolute continuity of the Lebesgue integral,

$$\left\| \tilde{\tilde{w}}_{ek}^* - \tilde{\tilde{w}}_{ek}^* \right\|_{1,\Omega_m} = \left\| \tilde{\tilde{w}}_{ek} - \tilde{\tilde{w}}_{ek}^* \right\|_{1,\Omega_m - \Omega} \leq \varepsilon_k / 3 \quad \forall m \geq m_k^2.$$  \hspace{1cm} (47)

Both inequalities (46) and (47) hold for $m \geq m_k = \max (m_k^1, m_k^2)$. It can be easily arranged that $m_k < m_{k+1}$ ($k = 1, 2, \ldots$).

Now we can construct a sequence {$w_m$}, $w_m \in V_m$, such that

$$\lim_{m \to \infty} \left\| \tilde{\tilde{w}} - w_m \right\|_{1,\Omega_m} = 0.$$  \hspace{1cm} (48)

If $m_k \leq m < m_{k+1}$ then we set $w_m = I_m w_{ek} \in V_m$. The inequality

$$\left\| \tilde{\tilde{w}} - I_m w_{ek} \right\|_{1,\Omega_m} \leq \left\| \tilde{\tilde{w}} - \tilde{\tilde{w}}_{ek} \right\|_{1,\Omega_m} +$$

$$+ \left\| \tilde{\tilde{w}}_{ek} - \tilde{\tilde{w}}_{ek}^* \right\|_{1,\Omega_m} + \left\| \tilde{\tilde{w}}_{ek}^* - I_m w_{ek} \right\|_{1,\Omega_m}$$

and relations (39), (44), (46), (47) imply then relation (48).

Now let us set

$$v_m = w_m + I_m z.$$

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Then, according to (40), (45) and (48),
\[ \| \tilde{u} - v_m \|_{1, \Omega_m} \leq \| \tilde{w} - w_m \|_{1, \Omega_m} + \| \tilde{z} - \tilde{z}^* \|_{1, \Omega_m - \Omega} + \]
\[ + \| \tilde{z}^* - I_m z \|_{1, \Omega_m} \to 0 \quad \text{if} \quad m \to \infty. \]

Theorem 7 is proved.

In the case of a polygonal domain \( \Omega \) the preceding theorems imply the following general result:

**Theorem 8:** Let \( \Omega \) be a bounded domain with a polygonal boundary \( \Gamma \). Let the assumptions of Theorems 2, 5, 6 and 7 be satisfied, where \( \bar{\Omega} = \Omega \). Then

\[
\lim_{m \to \infty} \| u - u_m \|_{1, \Omega} = 0 , \tag{49}
\]

where \( u \) and \( u_m \) are the solutions of Problems \( P \) and \( P_m \), respectively.

**Proof:** As \( \bar{\Omega} = \Omega \) we have \( \tilde{u} = u \) and \( q_m = q \). Thus

\[
\tilde{a}_m(\tilde{u}, w) = a(u, w) = L^\Omega(w) + L^\Gamma(w) = \tilde{L}_m^\Omega(w) + \tilde{L}_m^\Gamma(w).
\]

This result and Theorems 5, 6 imply

\[
| L_m(w) - \tilde{a}_m(\tilde{u}, w) | \cdot \| w \|_{1, \Omega}^{-1} = O(h_m^e) + O(h_m^n).
\]

Thus the first term on the right-hand side of (35) tends to zero if \( m \to \infty \).

As to the second term we have, according to Theorem 7,

\[
\inf_{v \in W_m} \sup_{w \in V_m} \{ | \tilde{a}_m(v, w) - a_m(v, w) | \cdot \| w \|_{1, \Omega}^{-1} \} \leq \inf_{v \in W_m} \{ C\tilde{B}_n h_m \| v \|_{1, \Omega} \} \leq C\tilde{B}_n h_m \| v_m \|_{1, \Omega} = O(h_m)
\]

because the sequence \( \{ v_m \} \) is bounded, according to Theorem 7. Relation (49) follows now from Theorem 4. Theorem 8 is proved.

In the case of non-polygonal domains the situation is not so straightforward. Thus the CR-theory and its modifications assume the solution of Problem \( P \) sufficiently smooth and use the Green's theorem in order to find a more

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convenient expression for \( \tilde{a}_m(\tilde{u}, w) \):

\[
\tilde{a}_m(\tilde{u}, w) = \int_{\Gamma_m} \left( \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} v_{mj} \right) w \, ds - \int_{\Omega_m} \frac{\partial}{\partial x_j} \left( \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \right) w \, dx \tag{50}
\]

(see [2, p. 268] or [10]). The symbols \( v_{m1}, v_{m2} \) denote the components of the unit outward normal to \( \Gamma_m \). The solution \( u \) is so smooth that it satisfies the equation

\[
- \frac{\partial}{\partial x_j} \left( k_{ij} \frac{\partial u}{\partial x_i} \right) = f \quad \text{in} \quad \Omega.
\]

If \( \Gamma_1 = \Gamma \) then (50) can be written in the form

\[
\tilde{a}_m(\tilde{u}, w) = \iint_{\Omega_m} \tilde{f} w \, dx \equiv \tilde{L}_m(w)
\]

where the extension \( \tilde{f} \) of \( f \) is defined by the relation

\[
\tilde{f} = - \frac{\partial}{\partial x_j} \left( k_{ij} \frac{\partial u}{\partial x_i} \right).
\]

In this case the estimate of the first term on the right-hand side of (35) follows immediately from Theorem 5. (As to the case \( \Gamma_1 \neq \Gamma \) see [10].)

**Remark 4**: It should be noted that the sufficient smoothness of \( u \) enables the CR-theory to use finite element interpolation theorems instead of Theorem 7 and to obtain the optimum error estimates.

Our assumptions guarantee only \( u \in H^1(\Omega) \) and the use of Green's theorem is forbidden for us. In order to estimate the first term on the right-hand side of (35) in the case of \( u \in H^1(\Omega) \) let us define first some notions and notation.

The symbols \( \omega_+ \) and \( \omega_- \) have the following meaning :

\[
\omega_+ = \Omega_m - \Omega, \quad \omega_- = \Omega - \Omega_m.
\tag{51}
\]

The symbols \( \omega_{1+} \) and \( \omega_{2+} \) denote the parts of \( \omega_+ \) which lie along \( \Gamma_1 \) and \( \Gamma_2 \), respectively. (In other words, the boundary of \( \omega_{1+} \) is formed by parts of \( \Gamma_1 \) and \( \Gamma_{m1} \); similarly the boundary of \( \omega_{2+} \) is formed by parts of \( \Gamma_2 \) and \( \Gamma_{m2} \).) The symbols \( \omega_{1-} \) and \( \omega_{2-} \) denote the parts of \( \omega_- \) which lie along \( \Gamma_1 \) and \( \Gamma_2 \), respectively.

The symbol \( T^* \) will denote a triangle belonging to \( \tau_m^n \) and approximating a corresponding ideal curved triangle \( T^{id} \in \tau_m^{id} \). (In [9] the ideal curved triangles are denoted simply by \( T \).)
The symbol $T^id : \Gamma_1$ denotes an ideal triangle $T^id \in \tau^id$ whose curved side lies on $\Gamma_1$. The symbol $T* : \Gamma_1$ denotes a triangle $T* \in \tau^n_1$ whose one side approximates a curved part of $\Gamma_1$.

Similarly as in [9], the symbol $\overline{w}$ denotes the natural extension of a function $w \in V_m$ from the domain $\Omega_m$ to the domain $\Omega_m \cup \omega_-$. (The definition of « natural extension » is given in [9, p. 271].)

Finally, the symbol $\hat{w}$ denotes a continuous function belonging to $V$ which corresponds to a function $w \in V_m \subset X^n_m$ and is defined by the following relations:

$$\hat{w} = \overline{w} \quad \text{on} \quad \Omega - \Lambda, \quad \hat{w} = w^* \quad \text{on} \quad \Lambda,$$

where

$$\Lambda = \bigcup_{T^id, \Gamma_1} T^id$$

and where $w^*$ is the function from $Y^n_m$ which is uniquely determined by the values

$$w^*(P_k) = w(P_k) \quad (k = 1, \ldots, N),$$

$P_1, \ldots, P_N$ being the nodal points of the triangulation $\tau^id_m$. (The definition of the space $Y^n_m$ is introduced in the text between relations (16) and (17).)

As $\hat{w} \in V$ we can write, according to (6) and (24),

$$L_m(w) - \tilde{a}_m(\tilde{u}, w) = \int_{\Gamma_m^2} q_m w \, ds - \tilde{L}_m^r(w) + \{ L_m(w) - L(\hat{w}) \} + \{ a(u, \hat{w}) - \tilde{a}_m(\tilde{u}, w) \}. \quad (55)$$

This relation is the starting point for estimating the first term on the right-hand side of (35) without using the Green's theorem. Now we express the terms in brackets in a suitable way. We have

$$L^r(\hat{w}) = \int_{\Gamma_2} q \hat{w} \, ds, \quad (56)$$

$$L^q(\hat{w}) = \int_{\Omega} f \hat{w} \, dx = \int_{\Omega_m} f \hat{w} \, dx + \int_{\omega^2} f \hat{w} \, dx - \int_{\omega^2} \tilde{f}\hat{w} \, dx + \sum_{T^*} \left\{ \int_{T^id} f \hat{w} \, dx - \int_{T*} \tilde{f}\hat{w} \, dx \right\}. \quad (57)$$
\[ \int_{T^d} f\dot{w} \, dx - \int_{T^*} \tilde{f}w \, dx = \]
\[ = \int_{T^d} f\dot{w} \, dx - \int_{T^*} \tilde{f}w \, dx - \int_{T^d-T^*} f\dot{w} \, dx + \int_{T^d-T^*} \tilde{f}w \, dx \]
\[ = \int_{T^d} (\dot{w} - \tilde{w}) f \, dx - \int_{T^*} \tilde{f}w \, dx + \int_{T^d-T^*} \tilde{f}w \, dx. \tag{58} \]

Similarly
\[
 a(u, \dot{w}) - \tilde{a}_m(\tilde{u}, w) = \int_{\omega^+} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \, dx -
\]
\[ - \int_{\omega^+} \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} \, dx + \sum_{T^* \cap T_1} \left\{ \int_{T^d} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial (\tilde{w} - \tilde{w})}{\partial x_j} \, dx \right\} + \int_{T^d-T^*} \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} \, dx + \int_{T^d-T^*} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} \, dx. \tag{59} \]

Relations (55)-(59) together with (24) and with the identities
\[
 \omega^+ = \bigcup_{T^* \cap T_1} (T^* - T^d), \quad \omega^- = \bigcup_{T^* \cap T_1} (T^d - T^*) \tag{60} \]

imply the following lemma:

**Lemma 1:** We have
\[
 |L_m(w) - \tilde{a}_m(\tilde{u}, w)| \leq |L^R_m(w) - \tilde{L}^R_m(w)| +
\]
\[ + |L^T_m(w) - \tilde{L}^T_m(w)| + \left| \int_{\Gamma_m^2} q_m \, w \, ds - \int_{\Gamma_3} q\tilde{w} \, ds \right| \]
\[ + \left| \sum_{T^d \cap T_1} \int_{T^d} \left[ - (\dot{w} - \tilde{w}) f + k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial (\tilde{w} - \tilde{w})}{\partial x_j} \right] \, dx \right| \]
\[ + \left| \int_{\omega^+} \left\{ \tilde{f}w - \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} \right\} \, dx \right| \]
\[ + \left| \int_{\omega^-} \left\{ - f\tilde{w} + k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} \right\} \, dx \right|. \tag{61} \]
Now we can prove the main result of this paper:

**Theorem 9:** Let the assumptions of Theorems 1, 2, 5, 6 and 7 be satisfied. Then

$$\lim_{m \to \infty} \| \tilde{u} - u_m \|_{1, \Omega_m} = 0 \quad (62)$$

where \( \tilde{u} \in H^1(\bar{\Omega}) \) is the Calderon’s extension of the solution \( u \in H^1(\Omega) \) of Problem \( P \) and \( u_m \) is the solution of Problem \( P_m \).

**Proof:** Using Theorem 7 we find

$$\inf_{v \in W_m} \| \tilde{u} - v \|_{1, \Omega_m} \leq \| \tilde{u} - u_m \|_{1, \Omega_m} \to 0 \quad (63)$$

Similarly as in the proof of Theorem 8 we have

$$\inf \sup \{ | a_m(v, w) - a_m(v, w) | \cdot \| w \|_{1, \Omega_m}^{-1} \} = O(h_m). \quad (64)$$

It remains to prove

$$\sup \{ | L_m(w) - L_m(\tilde{u}, w) | \cdot \| w \|_{1, \Omega_m}^{-1} \} \to 0 \quad (65)$$

Assertion (62) follows then from (63)-(65) and Theorem 4. The proof of (65) is divided into five parts A)-E):

A) Using Theorems 5 and 6 we obtain

$$| L_m^r(w) - \tilde{L}_m^r(w) | + | L_m(w) - \tilde{L}_m(w) | \leq \leq C h_m \| w \|_{1, \Omega_m} (1 \leq r \leq n) \quad \forall w \in V_m, \quad (66)$$

where the constant \( C \) does not depend on \( h_m \) and \( w \).

B) Let us denote for the sake of brevity

$$D_1 = \sum_{T \in \mathcal{T}_1} \int_{T \in \mathcal{T}_1} \left( k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial (\tilde{w} - \bar{w})}{\partial x_j} - (\tilde{w} - \bar{w}) f \right) dx. \quad (67)$$

Using the assumptions \( f \in W_n(\Omega), \bar{k}_{ij} \in W_n(\bar{\Omega}) \) and the Cauchy inequalities we easily find

$$| D_1 | \leq \left( \| f \|_{0, \Omega} + \bar{B}_n \| u \|_{1, \Omega} \left\{ \sum_{T \in \mathcal{T}_1} \| \tilde{w} - \bar{w} \|_{1, T}^2 \right\}^{1/2}. \quad (68)$$

According to (52)-(54), the function \( \tilde{w} \) is an ideal interpolate of the function \( \bar{w} \).
on $T^{id}$ (because $\bar{w}(P_1) = w(P_1)$). Thus using the result proved in the proof of [7, Theorem 2] we obtain

$$\| \hat{w} - \bar{w} \|_{1, T^{id}} \leq C h_m^n \| \bar{w} \|_{n+1, T^{id}}. \quad (69)$$

Inequalities (68) and (69) imply

$$| D_1 | \leq C h_m^n \left\{ \sum_{T^* : Y_1} \left( \| w \|_{n+1, T^*}^2 + \| \bar{w} \|_{n+1, T^{id} - T^*}^2 \right) \right\}^{1/2}. \quad (70)$$

(In (69), (70) and in what follows the symbol $C$ denotes a generic constant, not necessarily the same in any two places.) Let

$$x_1 = x_1^* (\xi_1, \xi_2), \quad x_2 = x_2^* (\xi_1, \xi_2) \quad (71)$$

be a mapping which maps one-to-one the curved triangle $T^*$ onto the standard triangle $T_0$ lying in the $\xi_1, \xi_2$-plane and having the vertices (0, 0), (1, 0), (0, 1). According to the definition of the function $w \in X^n_0$ we have (see also [9, p. 269])

$$w \big|_{T^*} (x_1^*(\xi_1, \xi_2), x_2^*(\xi_1, \xi_2)) = p(\xi_1, \xi_2), \quad (72)$$

where $p(\xi_1, \xi_2)$ is a polynomial of degree $n$. Using the theorem on transformation of multiple integrals and the properties of the mapping (71) (see [9, Lemma 1]) we find (because $| p \|_{n+1, T_0} = 0$):

$$\sum_{k=2}^{n+1} | w \|_{k, T^*}^2 \leq C h_m^{2 - 2n} \sum_{k=1}^{n} | p \|_{k, T_0}^2. \quad (73)$$

Using [8, Lemma 5] and the transformation from $T_0$ on $T^*$ we obtain

$$| p \|_{k, T_0}^2 \leq C | p \|_{1, T_0}^2 \leq C \| w \|_{1, T^*}^2 \quad (k \geq 1). \quad (74)$$

Relations (73), (74) imply

$$h_m^{2n} \| w \|_{n+1, T^*}^2 \leq C h_m^2 \| w \|_{1, T^*}^2. \quad (75)$$

The second term on the right-hand side of (70) can be estimated by the technique developed in [9]. Thus the proof is only sketched. Let $N_1$ be the number of curved triangles along $\Gamma_1$. Let us denote them by the symbols $T_1^*, T_2^*, ..., T_{N_1}^*$ and the corresponding ideal curved triangles by the symbols $T_1^{id}, T_2^{id}, ..., T_{N_1}^{id}$. According to the properties of transformations (71) (see [9, Lemma 1]), we have

$$| \bar{w} \|_{k, T^{id} - T_j} \leq C \sum_{r=1}^{k} | p_j \|_{r, a} h_m^{2 - 2r} \quad (k \geq 1),$$

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where in accordance with (72) (see also [9, (27)])

\[ p_j(x_1, x_2) = w \mid_{T_j^*} \left( x_1^*(x_1, x_2), x_2^*(x_1, x_2) \right) \]

and where \( \sigma \) is a quadrilateral lying in the \( \xi_1, \xi_2 \)-plane and having vertices \( A_1(1 - \delta, 0), A_2(1 + \delta, 0), A_3(0, 1 + \delta), A_4(0, 1 - \delta); \delta \) is so small that (see [9, p. 275])

\[ \text{mes } \sigma = O(h^m_n). \]  

(76)

As \( p_j \) is a polynomial of degree \( n \) the last inequality gives

\[ \| \mathbf{w} \|_{n+1,T_{j^*}}^{2} \leq C \left\{ | p_j |_{0,\sigma}^2 h_m^2 + \sum_{k=1}^{n} \left| p_j \right|_{k,\sigma}^2 h_m^{2-2k} \right\}. \]

(77)

Each polynomial \( p_j(x_1, x_2) \) can be written in the form

\[ p_j(x_1, x_2) = \sum_{i=1}^{d} \alpha^i b_i(x_1, x_2), \]

(78)

where \( d = (n + 1)(n + 2)/2, b_i(x_1, x_2) \) are fixed basis functions and \( \alpha^i = w(P^i_j), P^i_j (i = 1, \ldots, d) \) being the nodal points of \( T_j^* \) in the local notation. Similarly as [9, (39)] we can prove

\[ | w |_{0,\Omega_m}^2 \geq C h_m^2 A(\alpha^i), \]

\[ | w |_{1,\Omega_m}^2 \geq C B(\alpha^i), \]

(79)

where

\[ A(\alpha^i) = \sum_{j=1}^{N^*} \sum_{i=1}^{d} (\alpha^i_j)^2, \quad B(\alpha^i) = \sum_{j=1}^{N^*} \sum_{i=1}^{d} (\alpha^i_j - \alpha^0_j)^2, \]

\[ \alpha^0_j = (\alpha^1_j + \alpha^2_j + \cdots + \alpha^d_j)/d; \]

(80)

\[ N^* (\geq N_1) \text{ denotes the total number of curved boundary triangles. (If } n = 1 \text{ then } N^* \text{ is the number of boundary triangles lying along the curved part of } \Gamma. \]

As \( | p_j |_{k,\sigma} = | p_j - \alpha^0 \mid_{k,\sigma} (k \geq 1) \) we have, according to (77) and (79),

\[ \sum_{T \in \Gamma_1} \| \mathbf{w} \|_{n+1,T_{j^*}}^{2} \leq C \left( \sum_{j=1}^{N_1} \left| p_j \right|_{0,\sigma}^2 \right) A(\alpha^i_j) + \]

\[ \quad + C \left( \sum_{j=1}^{N_1} \sum_{k=1}^{n} \left| p_j - \alpha^0_j \right|_{k,\sigma}^2 h_m^{2-2k} \right) B(\alpha^i_j). \]

(82)

As \( b_1 + \cdots + b_d = 1 \) we can write

\[ p_j - \alpha^0_j = \sum_{i=1}^{d} (\alpha^i - \alpha^0_j) b_i. \]

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Thus
\[
| p_j |_{0, \sigma}^2 \leq C \max_{i=1, \ldots, d} | \alpha_i^j |^2 \text{mes } \sigma ,
\]
(83)
\[
| p_j - \alpha_0^j |_{k, \sigma}^2 \leq C \max_{i=1, \ldots, d} | \alpha_i^j - \alpha_0^j |^2 \text{mes } \sigma .
\]
(84)

We see from (76), (80), (83) and (84) that the right-hand side of (82) is bounded by \( Ch_m^{2-n} \). Using this result together with (75) and
\[
\sum_{T^* \in \Gamma_1} \| w \|_{1, T^*}^2 \leq \| w \|_{1, \Omega_m}^2
\]
we obtain from (70) that
\[
| D_1 | \| w \|_{1, \Omega_m}^{-1} \leq Ch_m .
\]
(85)

C) Similarly as in part B) we have
\[
\left| \int \int_{\omega_m} \left( \tilde{f} w - \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial w}{\partial x_j} \right) dx \right| \leq
\]
\[
\leq \left( \| \tilde{f} \|_{0, \Omega} + \tilde{B}_n \| \tilde{u} \|_{1, \Omega} \right) \left\{ \sum_{T^* \in \Gamma} \| w \|_{1, T^* - T_i d}^2 \right\}^{1/2},
\]
(86)
\[
\left| \int \int_{\omega_m} \left( k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} - \tilde{f} \tilde{w} \right) dx \right| \leq
\]
\[
\leq \left( \| f \|_{0, \Omega} + \tilde{B}_n \| u \|_{1, \Omega} \right) \left\{ \sum_{T^* \in \Gamma} \| \tilde{w} \|_{1, T^* - T_i d}^2 \right\}^{1/2},
\]
(87)
\[
\| w \|_{1, T^* - T_i d}^2 \leq Ch_m^2 | p_j |_{0, \sigma}^2 + C | p_j - \alpha_0^j |_{1, \sigma}^2 ,
\]
(88)
\[
\| \tilde{w} \|_{1, T_i d - T}^2 \leq Ch_m^2 | p_j |_{0, \sigma}^2 + C | p_j - \alpha_0^j |_{1, \sigma}^2 .
\]
(89)

Relations (76), (79)-(84), (86)-(89) imply
\[
\left| \int \int_{\omega_m} \left( \tilde{f} w - \tilde{k}_{ij} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial w}{\partial x_j} \right) dx \right| \cdot \| w \|_{1, \Omega_m}^{-1} \leq Ch_m^{n/2}
\]
(90)
\[
\left| \int \int_{\omega_m} \left( k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \tilde{w}}{\partial x_j} - \tilde{f} \tilde{w} \right) dx \right| \cdot \| w \|_{1, \Omega_m}^{-1} \leq Ch_m^{n/2}
\]
(91)
D) As \( \bar{w} = w \) on \( \bar{T}_j \) we have, according to (25), (26),

\[
\int_{\Gamma_2} q(x_1, x_2) \bar{w}(x_1, x_2) \, ds - \int_{\Gamma_{m2}} q_m(x_1, x_2) w(x_1, x_2) \, ds =
\]

\[
= \sum_{j=1}^{N_2} \left\{ \int_0^{n_1} q(\phi_j(t), \psi_j(t)) \bar{w}(\phi_j(t), \psi_j(t)) \, \rho_j(t) \, dt - \int_0^{n_1} q(\phi_j(t), \psi_j(t)) \bar{w}(\phi_j^*(t), \psi_j^*(t)) \, \rho_j^*(t) \, dt \right\},
\]

where \( N_2 \) is the number of boundary triangles lying along the curved part of \( \Gamma_2 \).

Let us set for the sake of brevity

\[
\Delta_{j1} = \phi_j(t) - \phi_j^*(t), \quad \Delta_{j2} = \psi_j(t) - \psi_j^*(t).
\]

We have, according to [9, Lemma 2]:

\[
\Delta_{j1} = O(h_m^{n+1}), \quad \Delta_{j2} = O(h_m^{n+1}),
\]

\[
\rho_j(t) = \rho_j^*(t) \left[ 1 + O(h_m^n) \right], \quad \rho_j^*(t) = O(h_m).
\]

Using Taylor's theorem we can write

\[
\bar{w}(\phi_j(t), \psi_j(t)) = \bar{w}(\phi_j^*(t), \psi_j^*(t)) + \frac{\partial \bar{w}}{\partial x_1}(S_j) \Delta_{j1} + \frac{\partial \bar{w}}{\partial x_2}(S_j) \Delta_{j2},
\]

where

\[
S_j = (\phi_j^*(t) + \delta_j \Delta_{j1}, \psi_j^*(t) + \delta_j \Delta_{j2}), \quad 0 < \delta_j < 1.
\]

Thus \( S_j \in \bar{T}_j \cup \bar{T}_{j\#} \). Using (72) and (78) we can find

\[\max_{t \in [0,1]} \left| \bar{w}(\phi_j^*(t), \psi_j^*(t)) \right| \leq C m_j \]

where

\[m_j = \max_{i=1,\ldots,d} |\alpha_i^j| .\]

Finally, relations \( \frac{\partial \bar{w}}{\partial x_k} = (\partial p_j / \partial \xi_i)(\partial \xi_i / \partial x_k) \) and (13)\(_2\) together with [9, Lemma 1] give

\[\max_{t \in [0,1]} \left| \frac{\partial \bar{w}}{\partial x_i}(S_j) \right| \leq C h_m^{-1} m_j .\]
Combining all relations introduced here with (79), (80) and taking into account that \( N_2 = O(h_m^{-1}) \) we obtain

\[
\int_{\Gamma_2} q \overline{w} \, ds - \int_{\Gamma_m^2} q_m w \, ds \leq \|w\|_{1,\Omega_m}^{1/2} \leq C h_m^n \sum_{j=1}^{N_2} m_j \left\{ A(\alpha_j^k) \right\}^{-1/2} \leq C h_m^n \left\{ N_2 \sum_{j=1}^{N_2} m_j^2 \left/ \sum_{j=1}^{N^*} \sum_{i=1}^{d} (\alpha_j^i)^2 \right. \right\}^{1/2} \leq C h_m^{-1/2}. \tag{92}
\]

E) Relations (66), (67), (85), (90), (91) and (92) together with Lemma 1 imply relation (65). Theorem 9 is proved.

Remark 5: We proved more than relation (65): Under the assumptions of Theorem 9 the rate of convergence of the first term on the right-hand side of (35) is \( O(h_m^{n/2}) \) in the case \( n = 1 \) and \( O(h_m) \) in the case \( n \geq 2 \).

Remark 6: For a greater simplicity we restricted our considerations to the case of triangular finite elements of the Lagrange type. Using results of [9] we can prove theorems analogous to Theorems 7 and 9 also in the case of triangular finite \( C^0 \)-elements of the Hermite type. The proofs follows the same lines as the proofs of Theorems 7 and 9.

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