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M2AN. Mathematical modelling and numerical analysis - Modélisation mathématique et analyse numérique, tome 20, nº 1 (1986), p. 113-128

<http://www.numdam.org/item?id=M2AN_1986__20_1_113_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(Vol. 20, n° 1, 1986, p. 113 à 128)

ELASTIC WAVE PROPAGATION IN FLUID-SATURATED POROUS MEDIA. PART I. THE EXISTENCE AND UNIQUENESS THEOREMS (*)

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Communicated by J. DOUGLAS Jr.

Abstract. — *The propagation of elastic waves in a porous solid saturated by a compressible viscous fluid can be described by a set of partial differential equations given by Biot. In the paper denoted here as Part I the problem of the existence and uniqueness of the solution of Biot's equations is analyzed. Numerical methods for the approximate solution of such equations shall be derived in Part II of this work.*

Résumé. — *La propagation d'ondes élastiques dans un solide poreux saturé par un fluide visqueux compressible peut être décrit par un ensemble d'équations aux dérivées partielles donné par Biot. Dans le présent article* (1re *partie*) *le problème de l'existence et l'unicité de solution des équations de Biot est analysé. Des Méthodes Numériques pour l'approximation de telles équations seront décrites dans la* 2e *partie de ce travail.*

1. INTRODUCTION

We shall analyze the problem of the existence and uniqueness of Biot's dynamic equations describing elastic wave propagation in a system composed of a porous solid saturated by a compressible viscous fluid. The numerical solution of such equations shall also be considered.

On a macroscopic scale the fluid-solid aggregate shall be assumed isotropic and elastic. For the validity of Biot's equations the wave length has to be appreciably larger than the diameter of the pores and relative movement between fluid and solid has to take place according to Darcy's law of fluid flow through porous media. The effect of dissipation is taken into account under the assumption that it depends only on such relative movement between fluid and solid. For simplicity only the two-dimensional case shall be analyzed.

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The organization of the paper denoted here as Part I is as follows. In Section 2 we introduce the Biot model and write the associated partial differential equations in a compact form more appropriate for the analysis. In Section 3 we present some notation and known results to be used. Then in Section 4 we derive the weak form of the problem and prove the results on existence and uniqueness. Finally in the paper denoted as Part II we define the continuous and discrete-time Galerkin procedures and perform the corresponding error analysis.

2. THE BIOT MODEL

We shall consider wave propagation in a fluid-solid system identified with an open bounded domain $\Omega \subset \mathbb{R}^2$ such that its boundary, denoted $\partial\Omega$, is piecewise smooth. Let $x = (x_1, x_2)$ denote a generic point in $\mathbb{R}^2$. Let $u(x, t) = (u_1(x, t), u_2(x, t))$ be the displacement vector on Ω, where $u_1 = (u_{11}, u_{12})$ is the displacement vector of the solid and $u_2 = (u_{21}, u_{22})$ is the average fluid displacement, u_{ji} being the displacement in the x_i-direction for $1 \leqslant i, j \leqslant 2$. Let

$$\varepsilon_{ij}(u_1) = \frac{1}{2}\left(\frac{\partial u_{1i}}{\partial x_j} + \frac{\partial u_{1j}}{\partial x_i}\right), \qquad 1 \leqslant i, j \leqslant 2,$$

$$\sigma_{ij}(u_1) = A\delta_{ij} \sum_{k=1}^{2} \varepsilon_{kk}(u_1) + 2\, N\varepsilon_{ij}(u_1), \quad 1 \leqslant i, j \leqslant 2,$$

be the strain and stress tensors in the solid. Here δ_{ij} denotes the Kronecker symbol and $A = A(x)$ and $N = N(x)$ are the Lamé coefficients of the solid frame. It shall be assumed that

$$\begin{aligned} &\text{(i) } 0 < A_* \leqslant A(x) \leqslant A^* < \infty, \quad x \in \bar{\Omega} = \Omega \cup \partial\Omega, \\ &\text{(ii) } 0 < N_* \leqslant N(x) \leqslant N^* < \infty, \quad x \in \bar{\Omega}. \end{aligned} \tag{2.1}$$

Next let the tensor $\theta_{ij}(u)$ and the scalar $s(u)$ be defined by

$$\theta_{ij}(u) = \sigma_{ij}(u_1) + Q\, \delta_{ij}\, \nabla . u_2 ,$$

$$s(u) = Q\, \nabla . u_1 + R\, \nabla . u_2 .$$

Also set

$$\theta_i(u) = (\theta_{i1}(u), \theta_{i2}(u)), \quad i = 1, 2 .$$

The coefficient $Q = Q(x)$ is a measure of the coupling between the volume change of the solid and that of the fluid. The coefficient $R = R(x)$ is a measure of the pressure required on the fluid to force a certain volume of the fluid into the aggregate while the total volume remains constant [1], [2].

No difficulty appears in the evaluation of the coefficient N since it is the shear modulus of the bulk material. The coefficients A, Q and R can not be measured directly but they can be expressed in terms of another set of physical constants which can be evaluated in laboratory tests. Such expressions are

$$A = \frac{\dfrac{\omega}{\psi} + \beta^2 + (1 - 2\beta)\left(1 - \dfrac{p}{\psi}\right)}{\omega + p - \dfrac{p^2}{\psi}} - \frac{2}{3} N ,$$

$$Q = \frac{\beta\left(1 - \beta - \dfrac{p}{\psi}\right)}{\omega + p - \dfrac{p^2}{\psi}} ,$$

$$R = \frac{\beta^2}{\omega + p - \dfrac{p^2}{\psi}} ,$$

where β is the effective porosity, ψ and p are respectively the jacketed and unjacketed compressibilities and ω is the fluid content, which is the ratio between the volume of fluid entering the pores of a solid sample and the pressure applied during an unjacketed compressibility test. For a description of the jacketed and unjacketed tests and the physical interpretation of the coefficients ω, p and ψ we refer to [3]. Some measurements of these coefficients can be found in [5].

The physical properties of the fluid-solid system allow us to assume that

$$\begin{aligned} &\text{(i) } 0 \leqslant Q_* \leqslant Q(x) \leqslant Q^* < \infty , && x \in \bar{\Omega} , \\ &\text{(ii) } 0 < R_* \leqslant R(x) \leqslant R^* < \infty , && x \in \bar{\Omega} , \\ &\text{(iii) } R(A + N) - Q^2 > 0 , && x \in \bar{\Omega} , \end{aligned} \tag{2.2}$$

Let $\rho_1 = \rho_1(x)$ (respectively $\rho_2 = \rho_2(x)$) be the mass of solid (respectively fluid) per unit volume of the aggregate and let $\rho_{12} = \rho_{12}(x)$ be a mass coupling parameter between fluid and solid. Set

$$\rho_{11} = \rho_1 - \rho_{12} ,$$
$$\rho_{22} = \rho_2 - \rho_{12} ,$$

and assume that

$$\begin{aligned}
&\text{(i)} && 0 < \rho_{11}^m \leqslant \rho_{11}(x) \leqslant \rho_{11}^M < \infty\,, && x \in \bar{\Omega}\,,\\
&\text{(ii)} && 0 < \rho_{22}^m \leqslant \rho_{22}(x) \leqslant \rho_{22}^M < \infty\,, && x \in \bar{\Omega}\,,\\
&\text{(iii)} && -\infty < \rho_{12}^m \leqslant \rho_{12}(x) \leqslant \rho_{12}^M \leqslant 0\,, && x \in \bar{\Omega}\,,\\
&\text{(iv)} && \rho_{11}(x)\,\rho_{22}(x) - (\rho_{12}(x))^2 > 0\,, && x \in \bar{\Omega}\ \text{(redundant)}\,.
\end{aligned} \tag{2.3}$$

Next let $b = b(x)$ be a dissipation coefficient defined by

$$b = \frac{\mu\beta^2}{\mathbb{K}}\,,$$

where $\mathbb{K} = \mathbb{K}(x)$ is the permeability and $\mu = \mu(x)$ is the fluid viscosity and assume that

$$0 < b_* \leqslant b(x) \leqslant b^* < \infty\,, \quad x \in \bar{\Omega}\,. \tag{2.4}$$

Then the propagation of elastic waves on Ω can be described by the equations [2],

$$\begin{aligned}
&\text{(i)}\ \frac{\partial^2}{\partial t^2}(\rho_{11}\,u_{1i} + \rho_{12}\,u_{2i}) + b\frac{\partial}{\partial t}(u_{1i} - u_{2i}) - \nabla.\theta_i(u) = f_i(x, t)\,,\\
&\text{(ii)}\ \frac{\partial^2}{\partial t^2}(\rho_{12}\,u_{1i} + \rho_{22}\,u_{2i}) - b\frac{\partial}{\partial t}(u_{1i} - u_{2i}) - \frac{\partial}{\partial x_i}s(u) = g_i(x, t)\,,
\end{aligned} \tag{2.5}$$

for $x \in \Omega$, $t \in J = (0, T)$ and $i = 1, 2$.

Let the matrices $\mathscr{A}$, $\mathscr{C} \in R^{4\times 4}$ and the vector $F \in R^4$ be given by

$$\mathscr{A} = \begin{bmatrix} \rho_{11} & 0 & \rho_{12} & 0\\ 0 & \rho_{11} & 0 & \rho_{12}\\ \rho_{12} & 0 & \rho_{22} & 0\\ 0 & \rho_{12} & 0 & \rho_{22} \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} b & 0 & -b & 0\\ 0 & b & 0 & -b\\ -b & 0 & b & 0\\ 0 & -b & 0 & b \end{bmatrix},$$

$$F = (f_1, f_2, g_1, g_2)^t\,.$$

Note that it follows from (2.3 (iv)) and (2.4) that $\mathscr{A}$ is positive definite and $\mathscr{C}$ is nonnegative.

Let $\mathscr{L}(u)$ be the differential operator defined by

$$\mathscr{L}(u) = (\nabla.\theta_1(u), \nabla.\theta_2(u), \nabla s(u))\,.$$

Then the set of equations (2.5) can be written in the equivalent form

$$\mathscr{A}\frac{\partial^2 u}{\partial t^2} + \mathscr{C}\frac{\partial u}{\partial t} - \mathscr{L}(u) = F(x, t), \quad (x, t) \in \Omega \times J. \qquad (2.6)$$

We shall impose initial conditions

$$\begin{aligned} &\text{(i) } u(x, 0) = u^0 = (u_1^0, u_2^0), \quad x \in \Omega, \\ &\text{(ii) } \frac{\partial u}{\partial t}(x, 0) = v^0 = (v_1^0, v_2^0), \quad x \in \Omega, \end{aligned} \qquad (2.7)$$

and boundary conditions

$$\begin{aligned} &\text{(i) } (\theta_1(u).\nu, \theta_2(u).\nu) = \phi(x, t), \quad (x, t) \in \partial\Omega \times J, \\ &\text{(ii) } s(u) = \eta(x, t), \quad (x, t) \in \partial\Omega \times J, \end{aligned} \qquad (2.8)$$

$\nu = \nu(x)$ being the outer unit exterior normal along $\partial\Omega$.

3. NOTATION AND PRELIMINARIES

For m a nonnegative integer let $H^m(\Omega) = W^{m,2}(\Omega)$ be the usual Sobolev space with the norm

$$\| v \|_m = \left[\sum_{|\alpha| \leqslant m} \int_\Omega | D^\alpha v(x) |^2 \, dx \right]^{1/2}.$$

For $n \geqslant 1$ the norm of $v = (v_1, \ldots, v_n)$ in $[H^m(\Omega)]^n$ will be given by

$$\| v \|_m = \left[\sum_{i=1}^n \| v_i \|_m^2 \right]^{1/2}.$$

The inner product and norm in $[L^2(\Omega)]^n$ will be denoted by

$$(v, w) = \sum_{i=1}^n \int_\Omega v_i w_i \, dx, \quad \| v \|_0^2 = (v, v).$$

Also we shall denote the inner product and norm in $[L^2(\partial\Omega)]^n$ by

$$\langle v, w \rangle = \sum_{i=1}^n \int_{\partial\Omega} v_i w_i \, d\sigma, \quad | v |_0^2 = \langle v, v \rangle,$$

$d\sigma$ being the arc length on $\partial\Omega$.

Denote by X' the dual space of X. Then by definition set $[H^{-m}(\Omega)]^n = [(H^m(\Omega))']^n$ with the norm

$$\| \zeta \|_{-m} = \sup_{0 \neq v \in [H^m(\Omega)]^n} \frac{(\zeta, v)}{\| v \|_m},$$

where $(\, , \,)$ denotes the duality between $[H^{-m}(\Omega)]^n$ and $[H^m(\Omega)]^n$.

Next recall that $\mathcal{C}_v = v/\partial\Omega \in [L^2(\partial\Omega)]^n$ for any $v \in [H^1(\Omega)]^n$ and that

$$| \mathcal{C}_v |_0 \leqslant C \| v \|_0^{1/2} \| v \|_1^{1/2} .$$

Here and later C is a generic constant that may be different at different places.

Then $[H^{1/2}(\partial\Omega)]^n$ is defined as the image of $[H^1(\Omega)]^n$ under $\mathcal{C}$ [7]. The norm of $\zeta \in [H^{1/2}(\partial\Omega)]^n$ will be given by

$$| \zeta |_{1/2} = \inf_{\substack{v \in [H^1(\Omega)]^n \\ \mathcal{C}_v = \zeta}} \| v \|_1 .$$

Let $[H^{-1/2}(\partial\Omega)]^n$ be the dual space of $[H^{1/2}(\partial\Omega)]^n$, the norm of an element $g \in [H^{-1/2}(\partial\Omega)]^n$ being given by

$$| g |_{-1/2} = \sup_{0 \neq \zeta \in [H^{1/2}(\partial\Omega)]^n} \frac{\langle g, \zeta \rangle}{| \zeta |_{1/2}} .$$

Here $\langle \, , \, \rangle$ denotes the duality between $[H^{-1/2}(\partial\Omega)]^n$ and $[H^{1/2}(\partial\Omega)]^n$.

Let $H(\mathrm{div}, \Omega) = \{ q \in [L^2(\Omega)]^2 : \nabla . q \in L^2(\Omega) \}$, provided with the norm

$$\| q \|_{H(\mathrm{div}, \Omega)} = (\| q \|_0^2 + \| \nabla . q \|_0^2)^{1/2} .$$

Denote by $\nu = \nu(x)$ the outer unit exterior normal along $\partial\Omega$. Then it is known that $q . \nu \in H^{-1/2}(\partial\Omega)$ for any $q \in H(\mathrm{div}, \Omega)$ and

$$| q . \nu |_{-1/2} \leqslant C \| q \|_{H(\mathrm{div}, \Omega)} . \tag{3.1}$$

Also it can be seen that for any $q \in H(\mathrm{div}, \Omega)$ and any $v \in H^1(\Omega)$ the following formula of integration by parts holds [7] :

$$(\nabla . q, v) + (q, \nabla v) = \langle q . \nu, v \rangle . \tag{3.2}$$

Set $V = [H^1(\Omega)]^2 \times H(\mathrm{div}, \Omega)$, the norm of an element $v = (v_1, v_2) \in V$ being defined by

$$\| v \|_V = (\| v_1 \|_1^2 + \| v_2 \|_{H(\mathrm{div}, \Omega)}^2)^{1/2} .$$

Note that since $[H(\text{div}, \Omega)]'$ can be identified with a closed subspace of $[L^2(\Omega)]^3$, any element in V' can be represented by a quintuple $(z_1, z_2, z_3, z_4, z_5)$, where $z_1, z_2 \in H^{-1}(\Omega)$ and $z_3, z_4, z_5 \in L^2(\Omega)$.

Let $w = (w_1, w_2) = ((w_{11}, w_{12}), (w_{21}, w_{22})) \in V$ and $z = (z_1, z_2, z_3, z_4, z_5) \in V'$ and denote by $[\ ,\ ]$ the duality between V' and V, i.e.,

$$[z, w] = ((z_1, z_2), w_1) + \int_\Omega ((z_3, z_4).w_2 + z_5 \nabla.w_2)\, dx .$$

Then,

$$\begin{aligned} |[z, w]| &\leqslant \| (z_1, z_2) \|_{-1} \| w_1 \|_1 + \| (z_3, z_4, z_5) \|_0 \| w_2 \|_{H(\text{div},\Omega)} \\ &\leqslant \| z \|_{V'} \| w \|_V . \end{aligned}$$

Denote by $\mathscr{D}(\Omega)$ the space of C^∞-functions on Ω which have compact support in Ω, and by $\mathscr{D}'(\Omega)$ the space of distributions on Ω. Also if X is a Banach space with norm $\| \ \|_X$ let $\mathscr{D}'(J, X)$ denote the set of distributions on J with values on X, i.e., $\mathscr{D}'(J, X) = \mathscr{L}(\mathscr{D}(J), X)$ [8]. If $f \in \mathscr{D}'(J, X)$, its derivative in the distributional sense is defined by [8],

$$\frac{\partial f}{\partial t}(\phi) = - f\left(\frac{\partial \phi}{\partial t}\right) \forall \phi \in \mathscr{D}(J) .$$

Finally, recall that

$$L^2(J, X) = \left\{ v : J \to X : \| v \|_{L^2(J,X)} = \left[\int_0^T \| v(t) \|_X^2 \, dt \right]^{1/2} < \infty \right\}$$

and

$$L^\infty(J, X) = \{ v : J \to X : \| v \|_{L^\infty(J,X)} = \sup_{t \in J} \| v(t) \|_X < \infty \} .$$

4. THE EXISTENCE AND UNIQUENESS THEOREMS

First we shall obtain the weak form of the problem (2.6)-(2.8). For $v = (v_1, v_2) \in V$, $w = (w_1, w_2) \in V$ set

$$M(v_1, w_1) = \int_\Omega \left[A\nabla.v_1 \nabla.w_1 + 2N \sum_{i,j=1}^{2} \varepsilon_{ij}(v_1)\, \varepsilon_{ij}(w_1) \right] dx ,$$

$$B(v, w) = M(v_1, w_1) + (Q\nabla.v_2, \nabla.w_1) + (Q\nabla.v_1 + R\nabla.v_2, \nabla.w_2) .$$

Then we multiply (2.6) by a test function $v \in V$ and integrate over Ω. Applying the formula of integration by parts (3.2) to the $(\mathscr{L}(u), v)$ — term and taking into account the boundary conditions (2.8) we conclude that

$$\left(\mathscr{A}\frac{\partial^2 u}{\partial t^2}, v\right) + \left(\mathscr{C}\frac{\partial u}{\partial t}, v\right) + B(u, v) = (F, v) + \langle \phi, v_1 \rangle + \langle v_2.\nu, \eta \rangle, \quad v \in V, t \in \dot{J}. \quad (4.1)$$

Next we shall analyze the properties of the bilinear form B. First note that B is symmetric and

$$| B(v, w) | \leqslant C(\| v_1 \|_1 + \| \nabla . v_2 \|_0)(\| w_1 \|_1 + \| \nabla . w_2 \|_0) \leqslant C \| v \|_V \| w \|_V, \quad v, w \in V. \quad (4.2)$$

Also recall Korn's second inequality, which states that, [4], [6], [9],

$$\int_\Omega \left[\sum_{i,j=1}^{2} (\varepsilon_{ij}(v))^2 \right] dx + \| v \|_0^2 \geqslant C_1 \| v \|_1^2, \quad (4.3)$$

for all $v \in [H^1(\Omega)]^2$.

Next let the symmetric matrix $E = (e_{ij}) \in R^{4 \times 4}$ be defined by $e_{11} = e_{22} = A + 2N$, $e_{33} = 4N$, $e_{44} = R$, $e_{12} = A$, $e_{14} = e_{24} = Q$, $e_{13} = e_{23} = e_{34} = 0$, and let $z(v) = (\varepsilon_{11}(v_1), \varepsilon_{22}(v_1), \varepsilon_{12}(v_1), \nabla . v_2)^t$. Then note that

$$B(v, v) = \int [Ez(v), z(v)]_l(x) \, dx,$$

$[,]_l$ denoting the usual scalar product in R^4.

The assumptions (2.1)-(2.2) on A, N, Q and R imply that E is positive definite. Thus if λ_* denotes the minimum eigenvalue of E and $C_2 = \min\left(\frac{\lambda_* C_1}{2}, \lambda_*\right)$, using (4.3) we conclude that

$$B(v, v) \geqslant C_2 \| v \|_V^2 - \lambda_* \| v \|_0^2, \quad v \in V. \quad (4.4)$$

It will be convenient for the analysis that follows to define the bilinear form B_γ by the rule

$$B_\gamma(v, w) = B(v, w) + \gamma(v, w), \quad v, w \in V,$$

where γ is a fixed constant such that $\gamma \geqslant \lambda_*$. Then B_γ is symmetric and

$$\text{(i)}\ \left| B_\gamma(v, w) \right| \leqslant C \| v \|_V \| w \|_V, \quad v, w \in V, \tag{4.5}$$
$$\text{(ii)}\ B_\gamma(v, v) \geqslant C_2 \| v \|_V^2, \quad v \in V.$$

Next, set

$$\text{(i)}\ D_s^2 = \left\| \frac{\partial^s \phi}{\partial t^s} \right\|^2_{L^\infty(J,[H^{-1/2}(\partial\Omega)]^2)} + \left\| \frac{\partial^{s+1} \phi}{\partial t^{s+1}} \right\|^2_{L^2(J,[H^{-1/2}(\partial\Omega)]^2)} + \left\| \frac{\partial^s \eta}{\partial t^s} \right\|^2_{L^\infty(J,H^{1/2}(\partial\Omega))} + \left\| \frac{\partial^{s+1} \eta}{\partial t^{s+1}} \right\|^2_{L^2(J,H^{1/2}(\partial\Omega))} + \left\| \frac{\partial^s F}{\partial t^s} \right\|^2_{L^2(J,[L^2(\Omega)]^4)}, \tag{4.6}$$

$$\text{(ii)}\ G_0^2 = \| u^0 \|_2^2 + \| v^0 \|_1^2 + \| F(0) \|_0^2 + 1 .$$

Now we can state the theorem of existence.

THEOREM 4.1 : *Let F, ϕ, η, u^0, v^0 be given and such that $G_0 < \infty$, $D_0 < \infty$ and $D_1 < \infty$. Then there exists a solution $u(x, t)$ of* (2.6)-(2.8) *such that u, $\frac{\partial u}{\partial t} \in L^\infty(J, V)$ and $\frac{\partial^2 u}{\partial t^2} \in L^\infty(J, [L^2(\Omega)]^4)$.*

Proof: Following [8], let $(v_n)_{n \geqslant 1}$ be a sequence of functions in $[H^2(\Omega)]^4$ such that for all m $v_1, ..., v_m$ are linearly independent and the finite linear combinations of the v_i's are denses in $[H^2(\Omega)]^4$. Let $S_m = \text{Span}\,(v_1, ..., v_m)$ and let $u_m = (u_{1m}, u_{2m}) \in S_m$ be determined by the relations

$$\left(\mathscr{A} \frac{\partial^2 u_m}{\partial t^2}, v\right) + \left(\mathscr{C} \frac{\partial u_m}{\partial t}, v\right) + B(u_m, v) = (F, v) + \langle \phi, v_1 \rangle + \langle v_2 . \mathrm{v}, \eta \rangle,$$
$$v = (v_1, v_2) \in S_m,\ t \in J. \tag{4.7}$$

$$\text{(i)}\ u_m(0) \in S_m,\ u_m(0) \xrightarrow[m \to \infty]{} u^0 \quad \text{in} \quad [H^2(\Omega)]^4,$$
$$\text{(ii)}\ \frac{\partial u_m}{\partial t}(0) \in S_m,\ \frac{\partial u_m}{\partial t}(0) \xrightarrow[m \to \infty]{} v^0 \ \text{in} \ [H^1(\Omega)]^4. \tag{4.8}$$

Note that

$$\frac{d}{dt} \| u_m \|_0^2 \leqslant \| u_m \|_0^2 + \left\| \frac{\partial u_m}{\partial t} \right\|_0^2 .$$

Then the choice $v = \frac{\partial u_m}{\partial t}$ in (4.7) implies that

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\mathscr{A}^{1/2}\frac{\partial u_m}{\partial t}\right\|_0^2 + B_\gamma(u_m, u_m)\right] + \left(\mathscr{C}\frac{\partial u_m}{\partial t}, \frac{\partial u_m}{\partial t}\right) \leqslant$$

$$\leqslant C\left[\| F \|_0^2 + \| u_m \|_0^2 + \left\|\frac{\partial u_m}{\partial t}\right\|_0^2\right] + \left\langle \phi, \frac{\partial u_{1m}}{\partial t}\right\rangle + \left\langle \frac{\partial u_{2m}}{\partial t} \cdot \nu, \eta \right\rangle. \quad (4.9)$$

Next we shall obtain bounds for the integrals in time of the last two terms in the right-hand side above. Using integration by parts with respect to time we have

$$| I_1 | = \left| \int_0^t \left\langle \phi, \frac{\partial u_{1m}}{\partial t}\right\rangle (s)\, ds \right|$$

$$= \left| \langle \phi, u_{1m} \rangle (t) - \langle \phi, u_{1m} \rangle (0) - \int_0^t \left\langle \frac{\partial \phi}{\partial t}, u_{1m} \right\rangle (s)\, ds \right|$$

$$\leqslant \varepsilon \| u_{1m}(t) \|_1^2 + C\left[D_0^2 + \| u_{1m}(0) \|_1^2 + \int_0^t \| u_{1m}(s) \|_1^2\, ds\right],$$

$$| I_2 | = \left| \int_0^t \left\langle \frac{\partial u_{2m}}{\partial t} \cdot \nu, \eta \right\rangle (s)\, ds \right|$$

$$= \left| \langle u_{2m} \cdot \nu, \eta \rangle (t) - \langle u_{2m} \cdot \nu, \eta \rangle (0) - \int_0^t \left\langle u_{2m} \cdot \nu, \frac{\partial n}{\partial t} \right\rangle (s)\, ds \right|$$

$$\leqslant \varepsilon \| u_{2m}(t) \|_{H(\mathrm{div},\Omega)}^2 + C\left[D_0^2 + \| u_{2m}(0) \|_{H(\mathrm{div},\Omega)}^2 + \int_0^t \| u_{2m}(s) \|_{H(\mathrm{div},\Omega)}^2\, ds\right].$$

Thus if we integrate (4.9) in time from to 0 to t, the bounds for $I_1 - I_2$ and (4.5) imply that

$$\left\| \mathscr{A}^{1/2}\frac{\partial u_m}{\partial t}(t)\right\|_0^2 + C_2 \| u_m(t) \|_V^2 + \int_0^t \left(\mathscr{C}\frac{\partial u_m}{\partial t}, \frac{\partial u_m}{\partial t}\right)(s)\, ds$$

$$\leqslant \varepsilon \| u_m(t) \|_V^2 + C\left[D_0^2 + \left\|\mathscr{A}^{1/2}\frac{\partial u_m}{\partial t}(0)\right\|_0^2 + \| u_m(0) \|_V^2 + \int_0^t \left(\left\|\frac{\partial u_m}{\partial t}(s)\right\|_0^2 + \| u_m(s) \|_V^2\right) ds\right]. \quad (4.10)$$

Next note that $\left\| \mathscr{A}^{1/2} \frac{\partial u_m}{\partial t}(t) \right\|_0$ is equivalent to $\left\| \frac{\partial u_m}{\partial t}(t) \right\|_0$ and that from (4.8) we have

$$\left\| \mathscr{A}^{1/2} \frac{\partial u_m}{\partial t}(0) \right\|_0^2 + \| u_m(0) \|_V^2 \leqslant C G_0^2 .$$

Then since $\mathscr{C}$ is nonnegative the choice $\varepsilon = \frac{1}{2} C_2$ in (4.10) and a Gronwall argument show that

$$\left\| \frac{\partial u_m}{\partial t} \right\|_{L^\infty(J,[L^2(\Omega)]^4)} + \| u_m \|_{L^\infty(J,V)} \leqslant C(D_0 + G_0) . \qquad (4.11)$$

Now taking derivative in time in (4.7) we obtain

$$\left(\mathscr{A} \frac{\partial^3 u_m}{\partial t^3}, v \right) + \left(\mathscr{C} \frac{\partial^2 u_m}{\partial t^2}, v \right) + B\left(\frac{\partial u_m}{\partial t}, v \right) =$$

$$= \left(\frac{\partial F}{\partial t}, v \right) + \left\langle \frac{\partial \phi}{\partial t}, v_1 \right\rangle + \left\langle v_2 . \mathrm{v}, \frac{\partial \eta}{\partial t} \right\rangle , \quad v \in S_m , \quad t \in J .$$

The choice $v = \frac{\partial^2 u_m}{\partial t^2}$ in the relation above and a repetition of the argument leading to (4.10) imply that

$$\left\| \frac{\partial^2 u_m}{\partial t^2}(t) \right\|_0^2 + \left\| \frac{\partial u_m}{\partial t}(t) \right\|_V^2 \leqslant C\left[D_1^2 + G_0^2 + \left\| \frac{\partial^2 u_m}{\partial t^2}(0) \right\|_0^2 + \right.$$

$$\left. + \int_0^t \left(\left\| \frac{\partial^2 u_m}{\partial t^2}(s) \right\|_0^2 + \left\| \frac{\partial u_m}{\partial t}(s) \right\|_V^2 \right) ds \right] . \qquad (4.12)$$

Next we shall obtain a bound for the term $\left\| \frac{\partial^2 u_m}{\partial t^2}(0) \right\|_0$. Using (4.7) and the formula of integration by parts (3.2) we can deduce that

$$\left(\mathscr{A} \frac{\partial^2 u_m}{\partial t^2}(0), v \right) = (F(0), v) - \left(\mathscr{C} \frac{\partial u_m}{\partial t}(0), v \right) + (\mathscr{L}(u_m(0)), v) , \quad v \in S_m .$$

Then choosing $v = \frac{\partial^2 u_m}{\partial t^2}(0)$ in the equation above we obtain

$$\left\| \frac{\partial^2 u_m}{\partial t^2}(0) \right\|_0^2 \leqslant C\left[\| F(0) \|_0^2 + \| u_m(0) \|_2^2 + \left\| \frac{\partial u_m}{\partial t}(0) \right\|_0^2 \right] \leqslant G_0^2 . \tag{4.13}$$

From (4.12)-(4.13) and Gronwall's lemma we finally get the estimate

$$\left\| \frac{\partial^2 u_m}{\partial t^2} \right\|_{L^\infty(J,[L^2(\Omega)]^4)} + \left\| \frac{\partial u_m}{\partial t} \right\|_{L^\infty(J,V)} \leqslant C[D_1 + G_0] . \tag{4.14}$$

But note that

$$L^\infty(J, V) = [L^1(J, V')]'$$

and

$$L^\infty(J, [L^2(\Omega)]^4) = [L^1(J, [L^2(\Omega)]^4)]' .$$

Thus (4.11) and (4.14) imply that there exists a subsequence of $(u_m)_{m \geqslant 1}$, which for simplicity in the notation we call again $(u_m)_{m \geqslant 1}$, such that

$$\begin{aligned} &\text{(i) } u_m \xrightarrow[m \to \infty]{} u \quad \text{in } L^\infty(J, V) \text{ weak-*} , \\ &\text{(ii) } \frac{\partial u_m}{\partial t} \xrightarrow[m \to \infty]{} \frac{\partial u}{\partial t} \quad \text{in } L^\infty(J, V) \text{ weak-*} , \\ &\text{(iii) } \frac{\partial^2 u_m}{\partial t^2} \xrightarrow[m \to \infty]{} \frac{\partial^2 u}{\partial t^2} \text{ in } L^\infty(J, [L^2(\Omega)]^4) \text{ weak-*} , \end{aligned} \tag{4.15}$$

i.e.,

$$\int_0^T [v, u_m](t)\, dt \xrightarrow[m \to \infty]{} \int_0^T [v, u](t)\, dt ,$$

for all $v \in L^1(J, V')$ and similarly for (4.15 (ii)) and (4.15 (iii)).

Now we observe that (4.15 (ii)) and (4.15 (iii)) imply that, for any $v \in S_m$,

$$\begin{aligned} &\text{(i) } \left(\mathscr{A} \frac{\partial^2 u_m}{\partial t^2}, v \right) \xrightarrow[m \to \infty]{} \left(\mathscr{A} \frac{\partial^2 u}{\partial t^2}, v \right) \text{ in } L^\infty(J) \text{ weak-*} , \\ &\text{(ii) } \left(\mathscr{C} \frac{\partial u_m}{\partial t}, v \right) \xrightarrow[m \to \infty]{} \left(\mathscr{C} \frac{\partial u}{\partial t}, v \right) \text{ in } L^\infty(J) \text{ weak-*} . \end{aligned} \tag{4.16}$$

Next let $g(t) \in L^1(J)$ and $v \in S_m$. Using (4.15 (i)) and applying twice the formula of integration by parts (3.2) we obtain

$$\lim_{m \to \infty} \int_0^T B(u_m, v)(t)\, g(t)\, dt =$$

$$= \lim_{m \to \infty} \int_0^T [-(\mathscr{L}(v), u_m) + \langle (\theta_1(v).\nu, \theta_2(v).\nu), u_{1m} \rangle + \\ + \langle u_{2m}.\nu, s(v) \rangle](t)\, g(t)\, dt$$

$$= \int_0^T [-(\mathscr{L}(v), u) + \langle (\theta_1(v).\nu, \theta_2(v).\nu), u_1 \rangle + \\ + \langle u_2.\nu, s(v) \rangle](t)\, g(t)\, dt$$

$$= \int_0^T B(u, v)(t)\, g(t)\, dt\,,$$

so that

$$B(u_m, v) \xrightarrow[m \to \infty]{} B(u, v) \quad \text{in } L^\infty(J) \text{ weak-*}\,, \tag{4.17}$$

for any $v \in S_m$.

Then taking limit in m in (4.7) and using (4.16) and (4.17) we conclude that

$$\left(\mathscr{A} \frac{\partial^2 u}{\partial t^2}, v\right) + \left(\mathscr{C} \frac{\partial u}{\partial t}, v\right) + B(u, v) = \\ = (F, v) + \langle \phi, v_1 \rangle + \langle v_2.\nu, \eta \rangle\,, \quad v \in S_m\,, \quad \text{a.e. in } J\,. \tag{4.18}$$

The density of the sequence $(v_m)_{m \geq 1}$ in $[H^2(\Omega)]^4$ implies that (4.18) also holds for any $v \in [H^2(\Omega)]^4$.

Next note that $B(u, v) = -(\mathscr{L}(u), v)$ for any $v \in \mathscr{D}(\Omega)$, where the application of the differential operator $\mathscr{L}$ to u is in the distributional sense in $\mathscr{D}'(\Omega)$. Thus it follows from (4.18) that

$$\mathscr{A} \frac{\partial^2 u}{\partial t^2} + \mathscr{C} \frac{\partial u}{\partial t} - \mathscr{L}(u) = F \quad \text{in } [\mathscr{D}'(\Omega)]^4\,, \quad \text{a.e. in } J\,. \tag{4.19}$$

But since $\frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}$ and F are all functions in $L^\infty(J, [L^2(\Omega)]^4)$, (4.19) is also an equality as functions in $L^\infty(J, [L^2(\Omega)]^4)$ and so (2.6) holds.

Now we shall show that the boundary conditions (2.8) are satisfied. First using (4.18) and the density of $[C^{\infty}(\overline{\Omega})]^4$ in $[H^2(\Omega)]^4$ and in V we have

$$\left(\mathscr{A}\frac{\partial^2 u}{\partial t^2}, v\right) + \left(\mathscr{C}\frac{\partial u}{\partial t}, v\right) + B(u, v) =$$
$$= (F, v) + \langle \phi, v_1 \rangle + \langle v_2 . \nu, \eta \rangle , \quad \text{for any } v = (v_1, v_2) \in V . \quad (4.20)$$

and a.e. in J.

Now the fact that $u \in L^{\infty}(J, V)$ and $\mathscr{L}(u) \in L^{\infty}(J, [L^2(\Omega)]^4)$ allows us to apply again the formula of integration by parts (3.2) and obtain

$$B(u, v) = -(\mathscr{L}(u), v) + \langle (\theta_1(u).\nu, \theta_2(u).\nu), v_1 \rangle +$$
$$+ \langle v_2 . \nu, s(u) \rangle , \quad \text{for any } v = (v_1, v_2) \in V \quad (4.21)$$

and a.e. in J.

Thus combining (4.19) and the remark immediately below with (4.20)-(4.21) we conclude that

$$\begin{aligned} &\text{(i)} \ \langle (\theta_1(u).\nu, \theta_2(u).\nu), v_1 \rangle = \langle \phi, v_1 \rangle , \quad v_1 \in [H^1(\Omega)]^2 , \quad \text{a.e. in } J , \\ &\text{(ii)} \ \langle v_2 . \nu, s(u) \rangle = \langle v_2 . \nu, \eta \rangle , \quad v_2 \in H(\text{div}, \Omega) , \quad \text{a.e. in } J . \end{aligned} \quad (4.22)$$

It follows from (4.22 (i)) that (2.8 (i)) holds. Since for any $z \in H^{-1/2}(\partial\Omega)$ there exists $q \in H(\text{div}, \Omega)$ such that $q.\nu = z$, (4.22 (ii)) implies that (2.8 (ii)) holds. Finally the argument given in [8] can be used here to show that the initial conditions (2.7) are satisfied. This completes the proof.

Now we shall give an uniqueness result for the problem (2.6)-(2.8).

Theorem 4.2 : *Under the hypothesis of theorem* 4.1, *the solution obtained in theorem* 4.1 *is unique.*

Proof : Let u_1 and u_2 be solutions of the problem (2.6)-(2.8) in the sense of theorem 4.1 and let $u = u_1 - u_2$. Then $u, \frac{\partial u}{\partial t} \in L^{\infty}(J, V), \frac{\partial^2 u}{\partial t^2} \in L^2(J, [L^2(\Omega)]^4)$ and u satisfies the relations

$$\begin{aligned} &\text{(i)} \ \mathscr{A}\frac{\partial^2 u}{\partial t^2} + \mathscr{C}\frac{\partial u}{\partial t} - \mathscr{L}(u) = 0 , \quad (x, t) \in \Omega \times J , \\ &\text{(ii)} \ u(x, 0) = 0 , \qquad x \in \Omega , \\ &\text{(iii)} \ \frac{\partial u}{\partial t}(x, 0) = 0 , \qquad x \in \Omega , \end{aligned} \quad (4.23)$$

(iv) $(\nu.\theta_1(u), \nu.\theta_2(u)) = 0$, $\quad (x, t) \in \partial\Omega \times J$,

(v) $s(u) = 0$, $\quad (x, t) \in \partial\Omega \times J$.

The argument given in the derivation of (4.1) can be repeated here to see that

$$\left(\mathscr{A}\frac{\partial^2 u}{\partial t^2}, v\right) + \left(\mathscr{C}\frac{\partial u}{\partial t}, v\right) + B(u, v) = 0\,, \quad v \in V\,, \quad t \in J\,.$$

The choice $v = \frac{\partial u}{\partial t}$ in the relation above implies that

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\mathscr{A}^{1/2}\frac{\partial u}{\partial t}\right\|_0^2 + B(u, u)\right] + \left(\mathscr{C}\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right) = 0\,.$$

Next, add the inequality

$$\frac{\gamma}{2}\frac{d}{dt}\| u \|_0^2 \leqslant \frac{\gamma}{2}\left[\| u \|_0^2 + \left\|\frac{\partial u}{\partial t}\right\|_0^2\right]$$

to the equation above and integrate the result from 0 to t. Then,

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_0^2 + \| u(t) \|_V^2 \leqslant C\int_0^t\left(\left\|\frac{\partial u}{\partial t}(s)\right\|_0^2 + \| u(s) \|_V^2\right) ds\,,$$

so that $u(t) \equiv 0$ and the theorem is proved.

5. ACKNOWLEDGMENTS

I wish to thank Professor Jim Douglas Jr. for his support in pursuing this research and also Mr. O. Lovera for his useful remarks about the physics of the problem.

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