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UPSTREAM WEIGHTING AND MIXED FINITE ELEMENTS IN THE SIMULATION OF MISCIBLE DISPLACEMENTS (*)

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Résumé. — *On présente et analyse une méthode d'éléments finis pour l'approximation des déplacements miscibles incompressibles dans un milieu poreux. On utilise une méthode d'éléments finis mixtes pour l'équation en pression, et pour l'équation en concentration on utilise un schéma de décentrage discontinu associé à une méthode d'éléments finis mixtes. On calcule des estimateurs d'erreur qui restent valides quand la diffusion est nulle.*

Abstract. — *A finite element method for approximating incompressible miscible displacements in porous media is presented and analysed. A mixed finite element approximation is used for the pressure equation while a discontinuous upstream weighting scheme in conjunction with a mixed finite element method is employed for the concentration equation. Error estimates, which remain valid for vanishing diffusion, are derived.*

1. INTRODUCTION

We consider the incompressible, miscible displacement of one fluid by another in a porous medium. The mathematical formulation we shall use is described in [3] and [5]. The reservoir Ω will be assumed to be of unit thickness and shall be identified with a bounded domain in $\mathbb{R}^2$, and $J = [0, T]$ will denote a fixed interval of time.

Let p denote the pressure in the fluid mixture and c the concentration of one of the component fluids in the mixture, $0 \leqslant c \leqslant 1$. The pressure equation is

$$- \operatorname{div} \{ a(x, c) (\nabla p - \gamma(x, c)) \} = q \quad \text{in} \quad \Omega \times J , \tag{1.1}$$

where $a(x, c) = (a_1(x, c), a_2(x, c))$ is the mobility of the fluid mixture, $q = q(x, t)$ an imposed external flow rate, and $\gamma(x, c)$ a function modelling the effects due

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to gravity. The concentration equation is

$$\phi(x)\frac{\partial c}{\partial t} - \operatorname{div}(D\,\nabla c) + u.\nabla c = g(x, t, c) \quad \text{in} \quad \Omega \times J\,, \tag{1.2}$$

where ϕ is the porosity of the rock, u the Darcy velocity of the fluid mixture, g a known function representing sources, and D a velocity dependent tensor diffusion. The diffusion D is given by

$$D = D(x, u) = \phi(x)\,[d_m\, I + |\,u\,|\,\{\, d_l\, E(u) + d_t\, E^{\perp}(u)\,\}]\,, \tag{1.3}$$

where d_m, d_l, and d_t are respectively the molecular, longitudinal, and tranverse diffusion constants, I the identity 2×2 matrix, $E(u)$ the matrix of projection in the direction of the flow, and $E^{\perp}(u)$ the matrix of projection in the direction orthogonal to the flow, i.e.,

$$\begin{aligned} E_{ij} &= \frac{1}{|\,u\,|^2}\, u_i\, u_j\,, \quad i, j = 1,2\,. \\ E^{\perp} &= I - E\,. \end{aligned} \tag{1.4}$$

We remark that in reality d_l is larger than d_t and we shall assume in the following that this is the case. We also make the following assumptions on the data functions. All data functions, including q are assumed to be smooth. In particular, the functions a, γ, and g are supposed to be bounded and also to be Lipschitz functions of the concentration. The porosity ϕ and the components of the mobility a_i, $i = 1, 2$, are assumed to be bounded away from zero.

Darcy's law states that

$$u = -\,a(x, c)\,(\nabla p - \gamma(x, c))\,. \tag{1.5}$$

Thus we can rewrite the pressure equation (1.1) as a first order system in p and u,

$$\operatorname{div} u = q \qquad \text{in} \quad \Omega \times J\,, \tag{1.6}$$

$$u + a(x, c)\,\nabla p = a(x, c)\,\gamma(x, c) \quad \text{in} \quad \Omega \times J\,. \tag{1.7}$$

Similarly, introducing the variable r, we can express the saturation equation as a first order system in c and r,

$$\phi(x)\frac{\partial c}{\partial t} + \operatorname{div} r + u.\nabla c = g(x, t, c) \quad \text{in} \quad \Omega \times J\,, \tag{1.8}$$

$$r + D\,\nabla c = 0\,, \qquad \text{in} \quad \Omega \times J\,. \tag{1.9}$$

We take for boundary condition that there be no fluid flow across the boundary

$$u.\nu = 0 \quad \text{on} \quad \partial\Omega \times J\,, \tag{1.10}$$

$$r.\nu = 0 \quad \text{on} \quad \partial\Omega \times J\,, \tag{1.11}$$

where ν is the exterior normal to $\partial\Omega$; and we specify the initial condition

$$c(\cdot\,, 0) = c_0 \quad \text{in} \quad \Omega\,. \tag{1.12}$$

Observe that condition (1.10) together with the incompressibility of the fluids implies that the data function q must satisfy

$$\int_\Omega q(x, t)\, dx = 0\,.$$

The purpose of this work is to define and analyse an appropriate finite element method for the problem (1.6), ..., (1.12). For the pressure equation a mixed finite element method [11] is used. This is particularly suitable since the pressure itself does not appear directly in the concentration equation and only the velocity u is present so that we are particularly interested in obtaining accurate approximations to the velocity, cf. [3]. For the concentration equation as it is transport dominated, we use a discontinuous, upstream weighted scheme [10] for the transport term in conjunction with a mixed finite element method. Only the continuous time version shall be considered here.

We point out that the ideas used in this scheme have already been successfully applied to model immiscible displacements [2].

The problem (1.6), ... (1.12) or equivalently (1.1), (1.2), (1.10) ... (1.12), has been approximated by various methods and error estimates have been obtained for these methods, cf. [3], [5], [12], and [13] among others. Note in particular that in [3] the mixed finite element method was used for the pressure equation in combination with a standard finite element method for the concentration equation, and the method was extended and analysed for the compressible case in [4]. The scheme used to approximate the concentration equation has been analysed in [7] for a linear, diffusion-convection equation, and the analysis we shall give here follows the general outline of the arguments in [3] and [7].

The organization of the paper is as follows. In section 2 the mixed weak formulation of the problem is given. In section 3 the numerical method is defined, and in section 4 the existence and uniqueness of the solution to the approximated problem is demonstrated. Finally, the error estimates are derived in section 5.

2. MIXED WEAK FORMULATION OF THE PROBLEM

Let $H(\text{div}, \Omega)$ be the set of vector functions in $L^2(\Omega)^2$ whose divergence is in $L^2(\Omega)$, and let V be the set of those functions in $H(\text{div}, \Omega)$ with normal component vanishing on $\partial\Omega$. The space V will be a space of test functions for both the equation in u (1.7) and that in r (1.9). The space of test functions for the concentration equation (1.8) will be $W_c = L^2(\Omega)$ but for the pressure equation (1.6), we shall use $W_p = L^2(\Omega)/\text{constants}$ as p is determined only up to an additive constant.

For notational convenience we introduce the following bilinear forms. For $\theta \in L^\infty(\Omega)$, define the bilinear form $A(\theta; \cdot, \cdot)$ on $V \times V$ by

$$A(\theta; \alpha, \beta) = \sum_{i=1}^{2} \int_\Omega \frac{1}{a_i(\theta)} \alpha_i \beta_i \, dx, \tag{2.1}$$

and for any v sufficiently smooth define $G(v; \cdot, \cdot)$ on $\mathrm{H}^1(\Omega) \times L^2(\Omega)$ by

$$G(v; \phi, \psi) = \int_\Omega (v . \nabla\phi) \psi \, dx . \tag{2.2}$$

Observe that if the coefficient d_m is non zero then D is uniformly positive definite, i.e.

$$\sum_{i,j=1}^{2} D_{i,j}(x, u) \xi_i \xi_j \geqslant D_* |\xi|^2 , \quad \xi \in \mathbb{R}^2 , \tag{2.3}$$

with D_* independent of both x and u. In particular, in this case D is invertible and D^{-1} takes the form

$$D^{-1}(x, u) = \frac{1}{\phi[d_m^2 + d_m(d_l + d_t)|u| + d_l d_t |u|^2]} \times$$
$$\times \left[d_m I + |u| \{ d_t E(u) + d_l E^\perp(u) \}\right] . \tag{2.4}$$

Moreover, for each u bounded in $L^\infty(\Omega)$, $D^{-1}(x, u)$ is positive definite, uniformly in x; and, in its norm as a linear map, $D^{-1}(x, u)$ is bounded independently of u and x.

We shall assume in the following that d_m is not zero.

Dividing componentwise by a in equation (1.7), multiplying in equation (1.9) by D^{-1}, and taking into account the boundary conditions on elements of V, we can express the mixed weak formulation of the problem (1.6) ... (1.11) as follows.

Find the differentiable maps $(p, u) : J \to W_p \times V$ and $(c, r) : J \to H^1(\Omega) \times V$ satisfying

$$(\operatorname{div} u, w) = (q, w), \qquad w \in W_p, \tag{2.5}$$

$$A(c; u, v) - (p, \operatorname{div} v) = (\gamma(c), v), \quad v \in V, \tag{2.6}$$

$$\left(\phi \frac{\partial c}{\partial t}, z\right) + (\operatorname{div} r, z) + G(u; c, z) = (g(c), z), \quad z \in W_c, \tag{2.7}$$

$$(D^{-1}(u)\, r, s) - (c, \operatorname{div} s) = 0, \qquad s \in V \tag{2.8}$$

We remark that the boundary conditions (1.10) and (1.11) are taken into account in this formulation as we seek u and r in V, elements of which have normal components vanishing on $\partial\Omega$. Note, however, that though u is sought as an element of V, more regularity is required for u in order to give meaning to the expressions $(D^{-1}(u)\, r, s)$ and $G(u; c, z)$. This regularity is assured by the requirements of sufficient smoothness on the coefficients.

The above formulation of the saturation equation is used to separate the treatment of the transport and diffusion terms in order to handle problems with large transport. The transport term will be approximated by discontinuous upwinding techniques, and since the concentration c will be approximated by a discontinuous function, we shall approximate the diffusion term by mixed finite elements, cf. [7].

3. THE APPROXIMATION PROCEDURE

For a domain $\mathscr{D}$ we shall denote norms in the Sobolov space $H^m(\mathscr{D})$ by $\| \cdot \|_{m,\mathscr{D}}$ omitting the subscript $\mathscr{D}$ when $\mathscr{D} = \Omega$, and for Γ the boundary of $\mathscr{D}$ (or a portion there of) the norms in $H^m(\Gamma)$ shall be indicated by $| \cdot |_{m,\Gamma}$ omitting the subscript Γ when $\Gamma = \partial\Omega$. We shall also write $\| \quad \|_\infty$ and $\| \quad \|_m$ for the norms in $L^\infty(\Omega)^2$ and $H^m(\Omega)^2$ as well as for those in $L^\infty(\Omega)$ and $H^m(\Omega)$.

Let $\mathscr{T}_h$ be a quasi regular discretisation of Ω into triangles and quadrangles of diameter not exceeding h, and let $V_h^l \times W_h^l$ be a Raviart-Thomas space of index l, $l \geqslant 0$, subordinate to $\mathscr{T}_h$. Associated with V_h^l there is the projection operator $\Pi_h^l : H(\operatorname{div}, \Omega) \to V_h^l$, cf. [9], satisfying for all $v \in H(\operatorname{div}, \Omega)$,

$$(\operatorname{div} (\Pi_h^l v - v), w) = 0, \qquad w \in W_h^l, \tag{3.1}$$

and also

$$\| \Pi_h^l v - v \|_0 \leqslant M h^j \| v \|_j, \qquad 1 \leqslant j \leqslant l + 1, \tag{3.2}$$

$$\| \operatorname{div} (\Pi_h^l v - v) \|_0 \leqslant M h^j \| \operatorname{div} v \|_j, \qquad 0 \leqslant j \leqslant l + 1, \tag{3.3}$$

whenever $\| v \|_j$ and $\| \operatorname{div} v \|_j$ are defined. Furthermore Π_h^l maps V into $V_h^l \cap V$. Associated with W_h^l we have the L^2-projection $\rho_h^l : L^2(\Omega) \to W_h^l$ satisfying, for each $w \in L^2(\Omega)$,

$$(\rho_h^l w - w, z) = 0, \quad z \in W_h^l, \tag{3.4}$$

and also,

$$\| \rho_h^l w - w \|_{m,K} \leqslant M h^{j-m} \| w \|_{j,K}, \quad 0 \leqslant m < j \leqslant l + 1, \quad K \in \mathcal{T}_h, \tag{3.5}$$

$$| \rho_h^l w - w |_{0;\partial K} \leqslant M h^{j-1/2} \| w \|_{j,K}, \quad 0 < j \leqslant l + 1, \quad K \in \mathcal{T}_h, \tag{3.6}$$

whenever w lies in $H^j(\Omega)$.

We shall also find useful the following inequalities valid for each $K \in \mathcal{T}_h$:

$$\| w \|_{1,K} \leqslant M h^{-1} \| w \|_{0,K}, \quad w \in W_h^l, \tag{3.7}$$

$$| w |_{0,\partial K} \leqslant M h^{-1/2} \| w \|_{0,K}, \quad w \in W_h^l, \tag{3.8}$$

$$\| \operatorname{div} v \|_{0,K} \leqslant M h^{-1} \| v \|_{0,K}, \quad v \in V_h^l, \tag{3.9}$$

$$| v . \nu |_{0,\partial K} \leqslant M h^{-1/2} \| v \|_{0,K}, \quad v \in V_h^l. \tag{3.10}$$

In each of the inequalities above, M represents a constant independent of h.

In the approximation procedure that we shall define, we expect some loss of accuracy in the approximation of the concentration due to the upstream weighting that we shall use. In order to balance the precision in the approximation of the concentration equation and the approximation of the pressure equation, we shall approximate c, respectively r, by polynomials of one degree greater than that of those we shall use to approximate p, respectively u. Thus, given $k \geqslant 0$, we define V_{u_h} to be $V_h^k \cap V$ and V_{r_h} to be $V_h^{k+1} \cap V$, and we put $W_{p_h} = W_h^k/\text{constants}$ and $W_{c_h} = W_h^{k+1}$. Then we shall approximate the pair (p, u) by $(p_h, u_h) \in W_{p_h} \times V_{u_h}$ and (c, r) by $(c_h, r_h) \in W_c \times V_r$.

Note that the concentration c is approximated in the space W_{c_h}, which is not included in $H^1(\Omega)$. Consequently the bilinear form $G(v; \cdot, \cdot)$ on $H^1(\Omega) \times L^2(\Omega)$ does not restrict to a forme on $W_{c_h} \times W_{c_h}$. Thus to define our approximation procedure we need to give an approximation to G. This shall be done using discontinuous upstream-weighting techniques described in [10]. Toward this end we make the following definitions.

For $K \in \mathcal{T}_h$ define the upstream boundary and the downstream boundary of K, cf. figure 1, by

$$\partial K_- = \{ x \in \partial K : u . \nu_K \leqslant 0 \} = \text{upstream boundary}, \qquad (3.11)$$
$$\partial K_+ = \{ x \in \partial K : u . \nu_K > 0 \} = \text{downstream boundary},$$

where ν_K denotes the unit outward normal on ∂K.

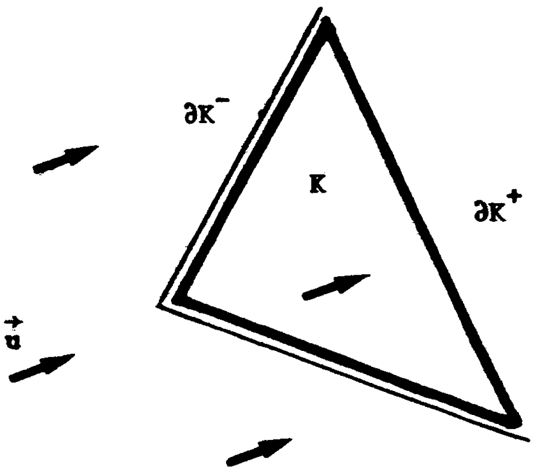

Figure 1. — Upstream and downstream boundaries of an element K.

As there is no requirement of continuity of elements of W_{c_h} across the boundaries of elements K of $\mathcal{T}_h$, we define for each $\phi \in W_{c_h}$ and for each $K \in \mathcal{T}_h$ both an upstream trace and a downstream trace of ϕ on ∂K, $K \in \mathcal{T}_h$, cf. figure 2, as follows :

$$\phi^- = \left\{ \begin{array}{l} \text{exterior trace of } \phi \text{ on } \partial K_- \\ \text{interior trace of } \phi \text{ on } \partial K_+ \end{array} \right\} = \text{upstream trace},$$
$$\phi^+ = \left\{ \begin{array}{l} \text{interior trace of } \phi \text{ on } \partial K_- \\ \text{exterior trace of } \phi \text{ on } \partial K_+ \end{array} \right\} = \text{downstream trace}, \qquad (3.12)$$

where we arbitrarily set the exterior trace of ϕ on $\partial K \cap \Gamma$ to be 0.

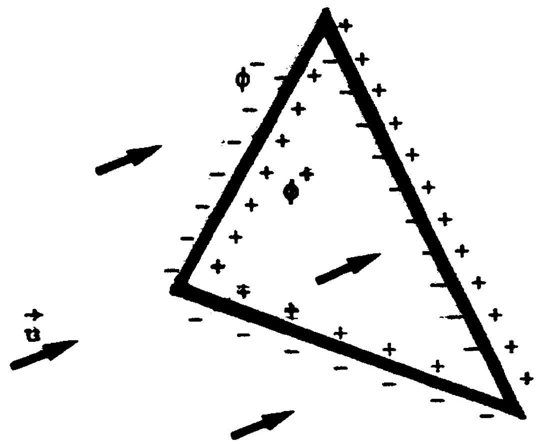

Figure 2. — Upstream and downstream traces of a function ϕ.

Now for $V \in V_{u_h}$ we define the bilinear form $G_h(v; ., .)$ on $W_{c_h} \times W_{c_h}$ by

$$G_h(v; \phi, \psi) = \sum_{K \in \mathcal{C}_h} \left\{ \int_K (v.\nabla\phi)\, \psi \, dx - \right.$$

$$\left. - \frac{1+\delta}{2} \int_{\delta K_-} v.\nu_K(\phi^+ - \phi^-)\, \psi^+ \, d\gamma - \frac{1-\delta}{2} \int_{\delta K_+} v.\nu_K(\phi^- - \phi^+)\, \psi^- \, d\gamma \right., \quad \phi, \psi \in W_{c_h}, \qquad (3.13)$$

where δ is a parameter of dissipation, $0 \leqslant \delta \leqslant 1$, determining the amount of upstream weighting. For $\delta = 1$, the upstream weighting and dissipation are maximal, and for $\delta = 0$, the derivation is centered and there is no dissipation cf. [6].

Our continuous in time approximation procedure is to find mappings $(p_h, u_h) : J \to W_{p_h} \times V_{u_h}$ and $(c_h, r_h) : J \to W_{c_h} \times V_{r_h}$ satisfying :

$$(\operatorname{div} u_h, w_h) = (q, w_h), \qquad w_h \in W_{p_h}, \qquad (3.14)$$

$$A(c_h; u_h, v_h) - (p_h, \operatorname{div} v_h) = (\gamma(c_h), v_h), \quad v_h \in V_{u_h}, \qquad (3.15)$$

$$\left(\phi \frac{\partial c_h}{\partial t}, z_h\right) + (\operatorname{div} r_h, z_h) + G_h(u_h; c_h, z_h) = (g(c_h), z_h), \quad z_h \in W_{c_h}, \qquad (3.16)$$

$$(D^{-1}(u_h)\, r_h, s_h) - (c_h, \operatorname{div} s_h) = 0, \qquad s_h \in V_{r_h}, \qquad (3.17)$$

with the initial data,

$$c_h(0) = c_{0h}\,, \tag{3.18}$$

where c_{0h} is the L^2 projection on W_{c_h} of the continuous initial data c_0.

4. EXISTENCE AND UNIQUENESS OF THE APPROXIMATE SOLUTION

First we state without proof two lemmas which will be useful in the following arguments. The first concerns an important property of the bilinear form G_h. The second gives results concerning the pressure equation.

For the proof of lemma 4.1 see [7] or [8].

LEMMA 4.1 : *Let S_h denote the set of interior edges of the mesh $\mathcal{C}_h$. Then for each $v \in V_{u_h}$, the bilinear form $G_h(v\,;\,\cdot\,,\,\cdot\,)$ defined by* (3.13) *satisfies :*

$$G_h(v\,;z,z) = -\frac{1}{2}\int_\Omega \operatorname{div} v\,|z|^2\,dx + \frac{\delta}{2}\sum_{S\in S_h}\int_S |v.\nu|\,(z^+ - z^-)^2\,d\gamma\,,$$

$$z \in W_{c_h}\,, \tag{4.1}$$

where ν is any unit normal to S and z^+ and z^- are the downstream and upstream traces of z on S with respect to the flow given by v.

The demonstration of lemma 4.2 is given by Douglas et al. in [3]. It uses the arguments of Brezzi [1] and the boundedness of the functions of the concentration $\dfrac{1}{a_i}$, $= 1, 2$. First we define, for $z \in L^\infty(\Omega)$ and $(f, g) \in (L^2(\Omega))^2 \times W_p$, the continuous and discretised problemes :

$$\left.\begin{aligned} &\text{Find } (\alpha, \beta) \in V \times W_p \quad \text{such that} \\ &\qquad (\operatorname{div}\alpha, w) = (g, w)\,, \quad w \in W_p\,, \\ &A(z\,;\alpha, v) - (\beta, \operatorname{div} v) = (f, v)\,, \quad v \in V\,, \end{aligned}\right\} \tag{4.2}$$

$$\left.\begin{aligned} &\text{Find } (\alpha_h, \beta_h) \in V_{u_h} \times W_{p_h} \text{ satisfying} \\ &\qquad (\operatorname{div}\alpha_h, w_h) = (g, w_h)\,, \quad w_h \in W_{p_h}\,, \\ &A(z\,;\alpha_h, v_h) - (\beta_h, \operatorname{div} v_h) = (f, v_h)\,, \quad v_h \in V_{u_h}\,. \end{aligned}\right\} \tag{4.3}$$

Now we may state the following lemma :

LEMMA 4.2 : *Problems* (4.2) *and* (4.3) *have unique solutions. Moreover the following inequalities are satisfied.*

$$\| \alpha \|_V + \| \beta \|_0 \leqslant M_1[\| f \|_0 + \| g \|_0], \tag{4.4}$$

$$\| \alpha_h \|_V + \| \beta_h \|_0 \leqslant M_1[\| f \|_0 + \| g \|_0], \tag{4.5}$$

$$\| \alpha - \alpha_h \|_V + \| \beta - \beta_h \|_0 \leqslant M_2 \, h^{k+1} \| \beta \|_{L^\infty(J;H^{k+3}(\Omega))}, \tag{4.6}$$

where M_1 and M_2 are constants independent of h and z, and M_1 is independent of f and g.

Now we return to the proof of the existence and uniqueness of the solution of the discretised problem (3.14), ..., (3.18).

THÉORÈM 4.1 : *The discretised problem* (3.14), ..., (3.18) *has a unique solution.*

Proof : Following an argument given in [3], from lemma 4.2, the boundedness of a and γ, and the assumed regularity of q, we obtain the following inequality :

$$\| u_h \|_V + \| p_h \|_{W_p} \leqslant M .$$

Then quasi-regularity of the mesh implies

$$\| u_h \|_\infty + \| \operatorname{div} u_h \|_\infty \leqslant M h^{-1} , \tag{4.7}$$

and it follows that for each h, $D^{-1}(u_h)$ is positive definite uniformly in x.

Setting $z_h = c_h$ in (3.16) and $s_h = r_h$ in (3.17) and adding the two equations, we obtain

$$\left(\phi \frac{\partial c_h}{\partial t}, c_h \right) + (D^{-1}(u_h)\, r_h, r_h) + \frac{\delta}{2} \sum_{S \in S_h} \int_S | u_h . \nu | (c_h^+ - c_h^-)^2 \, d\gamma =$$

$$= (g(c_h), c_h) + \frac{1}{2} \int_\Omega \operatorname{div} u_h \, | c_h |^2 \, dx .$$

Using (4.7), the nonsingularity of ϕ, and the positive-definiteness of $D^{-1}(u_h)$, we may write

$$\frac{d}{dt} \| c_h \|_0^2 \leqslant M h^{-1} \| c_h \|_0^2 ,$$

which yields

$$\| c_h(t) \|_0 \leqslant M(h) , \quad t \in J , \tag{4.8}$$

with $M(h)$ a constant dependent on h.

To bound r_h we observe that (3.9), (3.17), and (4.8) together with the quasi-regularity of the mesh and the positive definiteness of $D^{-1}(u_h)$ imply

$$\| r_h \|_\infty \leqslant M h^{-1} \| r_h \|_0 \leqslant M h^{-2} \| c_h \|_0 \leqslant \tilde{M}(h), \tag{4.9}$$

where $\tilde{M}(h)$ denotes an h dependent constant.

Now, using estimates (4.7), (4.8), (4.9), one can demonstrate the existence and uniqueness of a solution of the system of differential equations (3.14), ..., (3.18).

5. ERROR ESTIMATES

Our aim in this section is to demonstrate the error estimate stated in Theorem 5.1 (case of nonvanishing diffusion) and Theorem 5.2 (general case). For simplicity of exposition, the proof is given only in the first case, and we observe that the argument can easily be extended to cover the second case.

THEOREM 5.1 : *Let (c, p, u) be the solution of the continuous problem (1.6), ..., (1.12) and (c_h, p_h, u_h) the solution to the discretised problem (3.14), ... (3.18). Then, for h sufficiently small, the following estimates hold :*

$$\| c - c_h \|_{L^\infty(J;W_c)} + \sup_{[0,T]} \left[\sum_{S \in S_h} \int_S | u_h . \nu | (c_h^+ - c_h^-)^2 \, d\gamma \right]^{1/2} + \\ + \| p - p_h \|_{L^\infty(JW_p)} + \| u - u_h \|_{L^\infty(J,V)} \leqslant M h^{k+1} \tag{5.1}$$

where M depends on the norms $\| p \|_{L^\infty(J;H^{k+3}(\Omega))}$, $\| c \|_{L^\infty(J;H^{k+2}(\Omega))}$ and

$$\left\| \frac{\partial c}{\partial t} \right\|_{L^\infty(J;H^{k+1}(\Omega))}.$$

Before proceeding to the proof of the theorem we make several observations. The estimates of the errors in the approximations of p and u are of optimal order since these are approximated in a Raviart-Thomas space of order k. The estimate of the error in the approximation of c, as c is approximated in a Raviart-Thomas space of order $k + 1$, is of one order less than optimal due to the upstream weighting. We also obtain as a by product an estimate on the jumps in the direction of the flow across the element boundaries of the discontinuous approximation c_h. Estimates on the error in the approximation of r have not been pursued as r has no especially interesting physical significance in the problem at hand.

Proof: We shall, as usual, make use of projections of the continuous solution into the finite element spaces. Consider first the pressure equation. Following [3], we introduce the elliptic projection $(\bar{p}_h, \bar{u}_h)$ of (p, u) into $W_{p_h} \times V_{u_h}$ defined for a concentration $c : J \to H^1(\Omega)$ to be the map $(\bar{p}_h, \bar{u}_h) : J \to W_{p_h} \times V_{u_h}$ satisfying

$$(\operatorname{div} \bar{u}_h, w_h) = (q, w_h), \qquad w_h \in W_{p_h}, \tag{5.2}$$

$$A(c; \bar{u}_h, v_h) - (\bar{p}_h, \operatorname{div} v_h) = (\gamma(c), v_h), \quad v_h \in V_{u_h}. \tag{5.3}$$

From lemma 4.2 we have

$$\| u - \bar{u}_h \|_V + \| p - \bar{p}_h \|_0 \leqslant M h^{k+1} \| p \|_{L^\infty(J; H^{k+3}(\Omega))} \tag{5.4}$$

with M independent of c.

Next we need estimates for $u_h - \bar{u}_h$ and $p_h - \bar{p}_h$. Subtracting (5.2) from (3.14) and (5.3) from (3.15) we obtain the following error equations for the pressure and velocity respectively :

$$A(c_h, u_h - \bar{u}_h, v_h) - (p_h - \bar{p}_h, \operatorname{div} v_h) = A(c; \bar{u}_h, v_h) - A(c_h; \bar{u}_h, v_h) + \\ + (\gamma(c_h) - \gamma(c), v_h), \quad v_h \in V_{u_h}, \tag{5.5}$$

$$(\operatorname{div}(u_h - \bar{u}_h), w_h) = 0, \quad v_h \in V_{p_h}. \tag{5.6}$$

Again applying lemma 4.2 and using the assumptions that γ and a are Lipschitz functions of c, we have

$$\| u_h - \bar{u}_h \|_V + \| p_h - \bar{p}_h \|_0 \leqslant M \| c - c_h \|_0 [\| \bar{u}_h \|_\infty + 1].$$

Then from (5.4) with $k = 0$, the quasi-regularity of the grid, and the assumption that p is bounded in $L^\infty(J; H^3(\Omega))$ and that u is bounded in $L^\infty(J; L^\infty(\Omega))$ it follows that

$$\| u_h - \bar{u}_h \|_V + \| p_h - \bar{p}_h \|_0 \leqslant M \| p \|_{L^\infty(J; H^{k+3}(\Omega))} \| c - c_h \|_0 . \tag{5.7}$$

We turn now to the estimation of $c - c_h$. Here we shall need the L^2-projection $\bar{c}_h = \rho_h^{k+1} c$ of c defined by (3.4) and the projection $\bar{r}_h = \Pi_h^{k+1} r$ of r defined by (3.1). As estimates for $(c - \bar{c}_h)$ and $(r - \bar{r}_h)$ are given by (3.5) and (3.2) we are interested in the differences $(c_h - \bar{c}_h)$ and $(r_h - \bar{r}_h)$. Using (2.7), (2.8), (3.16), (3.17), and the definitions of the above projections, we arrive at the error equations

$$\left(\phi \frac{\partial}{\partial t}(c_h - \bar{c}_h), z\right) + (\operatorname{div}(r_h - \bar{r}_h), z) + G_h(u_h ; c_h - \bar{c}_h, z) =$$

$$= G_h(u ; c - \bar{c}_h, z) + G_h(u - u_h ; \bar{c}_h, z) + (g(c_h) - g(c), z) + \left(\phi \frac{\partial}{\partial t}(c - \bar{c}_h), z\right)$$

and $$z \in W_{c_h}, \quad (5.8)$$

$$(D^{-1}(u_h)(r_h - \bar{r}_h), s) - (c_h - \bar{c}_h, \operatorname{div} s) =$$

$$= (D^{-1}(u_h)(r - \bar{r}_h), s) + (D^{-1}(u) r, s) - (D^{-1}(u_h) r, s), \quad s \in V_{r_h}. \quad (5.9)$$

Let,

$$\begin{aligned} \xi &= c_h - \bar{c}_h & \rho &= r_h - \bar{r}_h \\ \eta &= c - \bar{c}_h & \sigma &= r - \bar{r}_h , \end{aligned} \quad (5.10)$$

and take for test functions $z = \xi$ in (5.8) and $s = \rho$ in (5.9). Then add (5.8) and (5.9) to obtain

$$\frac{1}{2}\frac{d}{dt}(\phi\xi, \xi) + (D^{-1}(u_h)\rho, \rho) + G_h(u_h ; \xi, \xi) = G_h(u ; \eta, \xi) +$$

$$+ G_h(u - u_h ; \bar{c}_h, \xi) + (g(c_h) - g(c), \xi) + (D^{-1}(u_h)\sigma, \rho)$$

$$+ (D^{-1}(u) r, \rho) - (D^{-1}(u_h) r, \rho) + \left(\phi \frac{\partial}{\partial t}\eta, \xi\right). \quad (5.11)$$

We consider first the terms of the left hand side.
Since $D^{-1}(u_h)$ is positive-definite, we may write

$$(D^{-1}(u_h)\rho, \rho) = \| D^{-1/2}(u_h)\rho \|_0^2 . \quad (5.12)$$

For the term $G_h(u_h ; \xi, \xi)$, we make use of equalities (4.1), and (2.5) to write

$$G_h(u_h ; \xi, \xi) = \frac{\delta}{2}\sum_{S \in S_h}\int_S | u_h . \nu | (\xi^+ - \xi^-)^2 \, d\gamma - \frac{1}{2}\int_\Omega q \, |\xi|^2 \, dx +$$

$$+ \frac{1}{2}\int_\Omega \operatorname{div}(u - u_h) \, |\xi|^2 \, dx .$$

As $q \in L^\infty(\Omega)$ we have

$$\left| \int_\Omega q \, |\xi|^2 \, dx \right| \leqslant M \| \xi \|_0^2 .$$

Since div u_h and div $\bar{u}_h$ lie in the test space W_{p_h}, equations (3.14) and (5.2) imply div $u_h \equiv$ div $\bar{u}_h$, so that using (5.4) and the quasi-regularity of the grid we obtain

$$\begin{aligned}\left|\int_\Omega \operatorname{div}(u - u_h)\,|\,\xi\,|^2\,dx\right| &= \left|\int_\Omega \operatorname{div}(u - \bar{u}_h)\,|\,\xi\,|^2\,dx\right| \\ &\leqslant \|\operatorname{div}(u - \bar{u}_h)\|_0\,\|\xi\|_\infty\,\|\xi\|_0 \\ &\leqslant Mh^k\,\|p\|_{L^\infty(J;H^{k+3}(\Omega))}\,\|\xi\|_0^2 \leqslant M\,\|\xi\|_0^2\,.\end{aligned}$$

Thus, it follows that

$$G_h(u_h;\xi,\xi) \geqslant \frac{\delta}{2}\sum_{S\in S_h}\int_S |\,u_h.\nu\,|\,(\xi^+ - \xi^-)^2\,d\gamma - M\,\|\xi\|_0^2\,. \quad (5.13)$$

Now we consider the terms of the right-hand side of (5.10). Assuming that u is in $L^\infty(J \times \Omega)^2$ with div $u \in L^\infty(J \times \Omega)$ we make use of (3.5), (3.6), (3.8), and (3.13), to obtain

$$G_h(u;\eta,\xi) \leqslant Mh^{k+1}\,\|c\|_{k+2}\,\|\xi\|_0^2 \leqslant M(\|\xi\|_0^2 + h^{2k+2}\,\|c\|_{k+2}^2)\,. \quad (5.14)$$

Next, from (3.13), we have

$$\begin{aligned}G_h(u - u_h;\bar{c}_h,\xi) \leqslant M \sum_{K\in\mathcal{T}_h} [\|u - u_h\|_{H(\operatorname{div};K)}\,\|\xi\|_{0,K}\,(\|\nabla\bar{c}_h\|_{\infty,K} + \\ + h^{-1}\,|\,c - \bar{c}_h\,|_{0,\partial K})]\,.\end{aligned}$$

The inequalities (3.5) and (3.6) with $j = 2$ and the assumed boundedness of c imply that $\|\nabla\bar{c}_h\|_{\infty,K} + h^{-1}\,|\,c - \bar{c}_h\,\|_{0,\partial K}$ is bounded so that :

$$G_h(u - u_h;\bar{c}_h,\xi) \leqslant M\,\|u - u_h\|_V\,\|\xi\|_0\,.$$

But, from (5.4), (5.7) and (3.5) we obtain

$$\begin{aligned}G_h(u - u_h;\bar{c}_h,\xi) &\leqslant M[h^{k+1}\,\|p\|_{L^\infty(J;H^{k+3}(\Omega))} + h^{k+2}\,\|c\|_{k+2} + \|\xi\|_0]\,\|\xi\|_0 \\ &\leqslant M[\|\xi\|_0^2 + h^{2k+2}\,\|p\|_{L^\infty(J;H^{k+3})}^2 + h^{2k+4}\,\|c\|_{k+2}]\,,\end{aligned} \quad (5.15)$$

where M depends on $\|p\|_{L^\infty(J;H^{k+3})}$.

For the third term on the right hand side of (5.11), since g is assumed to be

Lipshitz, from (3.5) we have

$$\begin{aligned}(g(c_h) - g(c), \xi) &= (g(c_h) - g(\overline{c}_h), \xi) + (g(\overline{c}_h) - g(c), \xi) \\ &\leqslant M(h^{k+2} \| c \|_{k+2} + \| \xi \|_0) \| \xi \|_0 \\ &\leqslant M(\| \xi \|_0^2 + h^{2k+4} \| c \|_{k+2}^2) . \end{aligned} \tag{5.16}$$

For the next term we recall that $D^{-1}(u_h)$ is bounded as a linear map independently of u_h and thus $D^{-1/2}(u_h)$ is also. Hence have

$$\begin{aligned}(D^{-1}(u_h) \sigma, p) &\leqslant \| D^{-1/2}(u_h) \sigma \|_0 \| D^{-1/2}(u_h) \rho \|_0 \\ &\leqslant M \| \sigma \|_0 \left\| D^{-1/2}(u_h) \sigma \right\|_0 . \end{aligned}$$

Now (3.2) implies that

$$\begin{aligned}(D^{-1}(u_h) \sigma, \rho) &\leqslant M h^{k+1} \| c \|_{k+2} \left\| D^{-1/2}(u_h) \rho \right\|_0 \\ &\leqslant \varepsilon \left\| D^{-1/2}(u_h) \rho \right\|_0^2 + M h^{2k+2} \| c \|_{k+2}^2 . \end{aligned} \tag{5.17}$$

For the next two terms in the right hand side of (5.10) one may write

$$\begin{aligned}(D^{-1}(u) r, \rho) - (D^{-1}(u_h) r, \rho) &= (D^{1/2}(u_h) (D^{-1}(u) - D^{-1}(u_h)) r, D^{-1/2}(u_h) \rho) \\ &= (D^{1/2}(u) (D^{-1}(u) - D^{-1}(u_h)) r, D^{-1/2}(u_h) \rho) \\ &\quad + ((D^{1/2}(u_h) - D^{1/2}(u)) (D^{-1}(u) - D^{-1}(u_h)) r, D^{-1/2}(u_h) \rho) . \end{aligned}$$

An argument in [3] shows that $D(u)$ is Lipshitz in u. Since $D(u)$, and consequently $D^{1/2}(u)$, has norm as a linear map bounded away from zero independently of u, it follows that $D^{1/2}(u)$ is Lipshitz in u. Also since $D^{-1}(u)$ is bounded it follows that $D^{-1}(u)$ is Lipshitz in u. As u and r are assumed to be smooth enough to be in L^{∞}, using Sobolev embedding, we have

$$\begin{aligned}(D^{-1}(u) r, \rho) - (D^{-1}(u_h) r, \rho) &\leqslant M(\| u - u_h \|_0 + \| u - u_h \|_{L^4}^2) \times \\ &\quad \times \| r \|_{\infty} \left\| D^{-1/2}(u_h) \rho \right\|_0 \leqslant \\ &\leqslant M(\| u - u_h \|_0 + \| u - u_h \|_V^2) \left\| D^{-1/2}(u_h) \rho \right\|_0 . \end{aligned}$$

From (5.4), (5.7), and (3.5) we obtain, for h small enough,

$$(D^{-1}(u)\,r,\rho)-(D^{-1}(u_h)\,r,\rho)\leqslant M[h^{k+1}\,\|p\|_{L^\infty(J;H^{k+3})}+h^{k+1}\,\|c\|_{k+1}+$$
$$+\|\xi\|_0+\|\xi\|_0^2]\,\|D^{-1/2}(u_h)\,\rho\|_0\leqslant\varepsilon\,\|D^{-1/2}(u_h)\,\rho\|_0^2+M[\|\xi\|_0^2+$$
$$+\|\xi\|_0^4+h^{2k+2}\,\|p\|_{L^\infty(J;H^{k+3})}^2+h^{2k+2}\,\|c\|_{k+1}^2]\,. \tag{5.18}$$

Finally for the last term of (5.11) we have

$$\left(\phi\frac{\partial}{\partial t}\eta,\xi\right)\leqslant M\left(h^{2k+2}\left\|\frac{\partial c}{\partial t}\right\|_{k+1}^2+\|\xi\|_0^2\right). \tag{5.19}$$

All the terms of the equality (5.11) have now been bounded. For ε sufficiently small in (5.17) and (5.18), equation (5.11) together with (5.12), ..., (5.19) gives

$$\frac{d}{dt}\|\phi^{1/2}\,\xi\|_0^2+\|D^{-1/2}(u_h)\,\rho\|_0^2+\frac{\delta}{2}\sum_{S\in S_h}\int|u_h.\nu|\,(\xi^+-\xi^-)^2\,d\gamma\leqslant$$
$$\leqslant M[\|\xi\|_0^2(1+\|\xi\|_0^2)+h^{2k+2}\,\|p\|_{L^\infty(J;H^{k+3})}^2+h^{2k+2}\,\|c\|_{k+2}^2$$
$$+h^{2k+2}\left\|\frac{\partial c}{\partial t}\right\|_{k+1}^2\Big]. \tag{5.20}$$

We now terminate the proof by the same argument as in [3].

Let us make the induction hypothesis that

$$\|\xi\|_{L^\infty(J;L(\Omega))}\leqslant 1\,. \tag{5.21}$$

Of course since $\xi(0)=0$, (5.21) holds on some interval $J=[0,T_h]$, for some $T_h>0$. Let $J_h=[0,T_h]$ denote the largest such interval. We shall show that, for h small enough, $T_h=T$ and convergence takes place at the rate $0(h^{k+1})$.

With (5.21), inequality (5.20) implies that

$$\frac{d}{dt}\|\phi^{1/2}\,\xi\|_0^2+\frac{\delta}{2}\sum_{S\in S_h}\int_S|u_h.\nu|\,(\xi^+-\xi^-)^2\,d\gamma\leqslant$$
$$\leqslant M\left[\|\xi\|_0^2+h^{2k+2}\,\|p\|_{L^\infty(J;H^{k+3})}^2+h^{2k+2}\left(\|c\|_{k+2}^2+\left\|\frac{\partial c}{\partial t}\right\|_{k+1}^2\right)\right].$$

We apply Gronwall's lemma and we obtain

$$\| \xi \|_{L^\infty(J;L^2)} \leqslant Mh^{k+1}\left[\| p \|_{L^\infty(J;H^{k+3})} + \right.$$
$$\left. + \| c \|_{L^\infty(J;H^{k+2})} + \left\| \frac{\partial c}{\partial t} \right\|_{L^\infty(J;H^{k+1})} \right]. \quad (5.22)$$

For small h, inequality (5.22) implies that $\| \xi \|_{L^\infty(J;L^2)} < 1$ on J_h. Thus $T_h = T$ and the induction hypothesis holds.

Finally, on applying (5.21), and Gronwall's lemma to (5.20) and combining the resulting equation with (5.4) and (5.7) we obtain the theorem.

Observe that in the proof of theorem 5.1, no term coming from the transport term G_h has been covered by the diffusion term $D(u_h)$. Thus the argument remains valid for vanishing diffusion.

More precisely replace d_m, d_l, d_t, by εd_m, εd_l, εd_t and denote by $(c_\varepsilon, r_\varepsilon, p_\varepsilon, u_\varepsilon)$ and $(c_{\varepsilon h}, r_{\varepsilon h}, p_{\varepsilon h}, u_{\varepsilon h})$ the solutions of the continuous and approximated problems respectively. Keeping track of ε in the calculations above, one can show the following theorem.

THEOREM 5.2 : *Set $D_\varepsilon(u) = \varepsilon D(u)$ and let (c, p, u) and $(c_\varepsilon, p_\varepsilon, u_\varepsilon)$ be the solutions of the corresponding continuous and discretized problems respectively. Then for h sufficiently small the following estimate holds :*

$$\| c_\varepsilon - c_{\varepsilon h} \|_{L^\infty(J;W_c)} + \sup_{[0,T]} \left[\sum_{S \in S_h} \int_S | u_{\varepsilon h} \cdot \nu | (c_{\varepsilon h}^+ - c_{\varepsilon h}^-)^2 \, d\gamma \right]^{1/2} +$$
$$+ \| p_\varepsilon - p_{\varepsilon h} \|_{L^\infty(J,W_p)} + \| u_\varepsilon - u_{\varepsilon h} \|_{L^\infty(J;V)} \leqslant Mh^{k+1},$$

where M depends on the norms $\| p_\varepsilon \|_{L\ (J;H^{k+3}(\Omega))}$, $\| c_\varepsilon \|_{L\ (J;H^{k+2}(\Omega))}$, $\left\| \frac{\partial c_\varepsilon}{\partial t} \right\|_{L^\infty(J;H^{k+1}(\Omega))}$ but not directly on ε.

Remark 5.1 : A slightly modified method is suggested by the observation that in the proof of theorem 5.1, inequality (3.2) has been used only up to $l = k$ (not $k + 1$). Thus one may decrease the index of V_{r_h} as in [2] and [7] and define V_{r_h} to be $V_h^k \cap V$ instead of $V_h^{k+1} \cap V$ while keeping c_h in the same approximation space $W_{c_h} = W_h^{k+1}$.

All the calculations for this modified method hold as before except there is one more term, $(\operatorname{div}(\bar{r}_h - r), z)$ in equation (5.8) as z is taken in W_h^{k+1} and $\bar{r}_h = \Pi_h^k r \in V_h^k$. Thus we would have occurring in the right hand side of (5.11) $(\operatorname{div}(r - \bar{r}_h), \xi)$ which would also have to be bounded. Using (3.3) with $l = k$, one could obtain

$$(\operatorname{div}(r - \bar{r}_h), \xi) \leqslant \| \operatorname{div}(r - \bar{r}_h) \|_0 \| \xi \|_0 \leqslant h^{k+1} \| \operatorname{div} r \|_{k+1} \| \xi \|_0$$
$$\leqslant h^{k+1} \| c \|_{k+3} \| \xi \|_0 .$$

Therefore, in this case, an estimate similar to (5.1) would hold, differing in that in the right-hand side, we would have to increase the regularity of c up to $\| c \|_{L^\infty(J;H^{k+3})}$. The same remark can be made for theorem 5.2.

RÉFÉRENCES

1. F. BREZZI, *On the existence, uniqueness and approximation of saddlepoint problems arising from Lagrangien multipliers.* R.A.I.R.O., Anal. Numér. 2 (1974), pp. 129-151.
2. G. CHAVENT, G. COHEN, M. DUPUY, J. JAFFRE, I. RIBERA, *Simulation of two dimensional water flooding using mixed finite elements,* SPEJ. 24 (1984), pp. 382-390.
3. J. DOUGLAS Jr., R. E. EWING, M. F. WHEELER, *The approximation of the pressure by a mixed method in the simulation of miscible displacement,* R.A.I.R.O., Anal. Numér. 17 (1983), pp. 17-33.
4. J. DOUGLAS Jr., J. E. ROBERTS, *Numerical methods for a model for compressible miscible displacement in porous media,* Math. Comp. 41 (1983), pp. 441-459.
5. R. E. EWING, M. F. WHEELER, *Galerkin methods for miscible displacement problems in porous media,* SIAM J. Numér. Anal. 17 (1980), pp. 351-365.
6. M. FORTIN, *Résolution numérique des équations de Navier Stokes par des éléments finis du type mixte,* Rapport INRIA n° 184, INRIA Le Chesnay (1976).
7. J. JAFFRE, *Éléments finis mixtes et décentrage pour les équations de diffusion-convection,* Calcolo 23 (1984), pp. 171-197.
8. P. JOLY, *La méthode des éléments finis mixtes appliquée au problème de diffusion-convection,* Thèse de 3e cycle, Université Pierre-et-Marie Curie, Paris (1982).
9. C. JOHNSON, V. THOMÉE, *Error estimates for some mixed finite element methods for parabolic type problems,* R.A.I.R.O., Anal. Numér., 15 (1981), pp. 41-78.
10. P. LESAINT, P. A. RAVIART, *On a finite element method for solving the neutron transport equation.* Mathematical Aspect of Finite Elements in Partial Differential Equations, Ed. Carl de Boor, Academic Press (1974), pp. 89-123.
11. P. A. RAVIART, J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems,* Mathematical Aspects of the Finite Element Method, Eds. I. Galligani and E. Magenes, Lecture Notes in Mathematics 606, Springer Verlag (1977), pp. 292-315.
12. T. F. RUSSELL, *Finite elements with characteristics for two-component incompressible miscible displacement,* 6th SPE Symposium on Reservoir Simulation, New Orleans, SPE 10500 (1982).
13. M. F. WHEELER, B. L. DARLOW, *Interior penalty Galerkin methods for miscible displacement problems in porous media,* Computational Methods in Nonlinear Mechanics, Ed. J. T. Oden, North Holland (1980).