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## INTERIOR AND SUPERCONVERGENCE ESTIMATES FOR MIXED METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS (\*)

J. DOUGLAS, Jr. <sup>(1)</sup> and F. A. MILNER <sup>(2)</sup>

**Résumé.** — Nous établissons des estimations a priori à l'intérieur pour l'erreur dans des méthodes d'éléments finis mixtes pour des équations elliptiques linéaires ou semi-linéaires, l'erreur étant évaluée dans des espaces de Sobolev d'indice non positif. Les estimations a priori correspondantes sont prouvées pour des quotients aux différences finies de l'erreur quand l'espace des éléments finis est associé avec un réseau invariant par translation dans un sous-domaine intérieur. Dans ce cas la procédure de superconvergence de Bramble-Schatz est étendue aux méthodes d'éléments finis mixtes.

**Abstract.** — Interior estimates for the error in mixed finite element methods for linear and semi-linear elliptic equations of second order are derived in Sobolev spaces of nonpositive index. Corresponding estimates are demonstrated for difference quotients of the error when the finite element space is associated with a translation-invariant grid on an interior subdomain, and the Bramble-Schatz superconvergence procedure is extended to the mixed method.

### 0. INTRODUCTION

Let  $\Omega$  be a bounded, planar domain with smooth boundary  $\partial\Omega$ . We shall assume that for each pair of functions  $(f, g)$  in  $H^m(\Omega) \times H^{m+3/2}(\partial\Omega)$ ,  $-1 \leq m \leq 2k+2$  there exists a unique solution  $p \in H^{m+2}(\Omega)$  of the Dirichlet problem

$$\left. \begin{aligned} Np &= -\operatorname{div}(a(p) \nabla p + \underline{b}(p)) + c(p) = f \quad \text{in } \Omega, \\ p &= -g \quad \text{on } \partial\Omega, \end{aligned} \right\} \quad (0.1)$$

where  $\nabla w$  denotes the gradient of a scalar function  $w$ ,  $\operatorname{div} \underline{v} = \nabla \cdot \underline{v}$  denotes the divergence of a vector function  $\underline{v}$ , and  $a : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the requirement that  $a(x, q) \geq a_0 > 0$ .

If the coefficients  $a$ ,  $\underline{b}$ , and  $c$  have the properties that  $\frac{\partial a}{\partial p} = 0$ ,  $\underline{b}(p) = \underline{b}(x)p$ , and  $c(p) = c(x)p$ , then the quasilinear problem (0.1) becomes the linear

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Dirichlet problem

$$\left. \begin{aligned} Lp &= -\nabla \cdot (a\nabla p + bp) + cp = f \quad \text{in } \Omega, \\ p &= -g \quad \text{on } \partial\Omega, \end{aligned} \right\} \quad (0.2)$$

where we have omitted writing the variable  $x$ , as we shall do throughout the paper.

We shall denote by  $(,)$  the natural inner product in  $L^2(\Omega)$  or  $L^2(\Omega)^2$ , and by  $\langle, \rangle$  the one in  $L^2(\partial\Omega)$ . We shall use the same notations to indicate the dualities between the  $W^{r,s}(\Omega)$  and  $W^{r,s}(\Omega)'$ , and  $H^s(\partial\Omega)$  and  $H^{-s}(\partial\Omega)$ .

Let

$$\tilde{V} = \tilde{H}(\text{div}; \Omega) = \{ \tilde{v} \in L^2(\Omega)^2 \mid \text{div } \tilde{v} \in L^2(\Omega) \},$$

normed by

$$\| \tilde{v} \|_{\tilde{V}}^2 = \| \tilde{v} \|_0^2 + \| \text{div } \tilde{v} \|_0^2,$$

and

$$W = L^2(\Omega).$$

If

$$\tilde{u} = -(a(p) \nabla p + b(p)), \quad \alpha = 1/a, \quad \beta = \alpha b, \quad (0.3)$$

then  $(\tilde{u}, p) \in \tilde{V} \times W$  is a weak solution of (0.1) in the sense that it satisfies the equations

$$\left. \begin{aligned} (\alpha(p) \tilde{u}, \tilde{v}) - (\text{div } \tilde{v}, p) + (\beta(p), \tilde{v}) &= \langle g, \tilde{v} \cdot \tilde{\nu} \rangle, \quad \tilde{v} \in \tilde{V}, \\ (\text{div } \tilde{u}, w) + (c(p), w) &= (f, w), \quad w \in W, \end{aligned} \right\} \quad (0.4)$$

where  $\tilde{\nu}$  denotes the unit outward normal vector to  $\partial\Omega$ .

Let  $\mathcal{T}_h$  be a quasi-uniform polygonalization of  $\Omega$  (by triangles or rectangles) of characteristic parameter  $h \in (0, 1)$ , where boundary polygons are allowed to have one curvilinear edge. That is, if we let  $r_T$  be the radius of the circle inscribed in  $T$  and  $R_T$  the radius of the circle circumscribing  $T$ ,  $T \in \mathcal{T}_h$ , then there exist positive constants  $m$  and  $M$ , independent of  $h$ , such that

$$mh \leq r_T < R_T \leq Mh, \quad T \in \mathcal{T}_h.$$

Let

$$\tilde{V}_h \times W_h \subset \tilde{V} \times W$$

be the Raviart-Thomas-Nedelec space of index  $k \geq 0$  associated with  $\mathcal{T}_h$ , [3, 6, 8, 11], which is defined as follows.

For  $E \subset \mathbb{R}^2$  let  $P_k(E)$  denote the restrictions of polynomials of total degree  $k$  to the set  $E$  and let  $Q_k(E)$  denote the restriction of  $P_k(\mathbb{R}) \otimes P_k(\mathbb{R})$  to  $E$ . Then, let  $R_k(E) = P_k(E)$  if  $E$  is a triangle (interior or boundary) and  $R_k(E) = Q_k(E)$  if  $E$  is a rectangle (interior or boundary), and let  $\tilde{R}_k(E) = R_k(E)^2$ . For any  $E \in \mathcal{T}_h$  let

$$\tilde{V}(E) = \tilde{R}_k(E) \otimes \tilde{x}R_k(E),$$

$$W(E) = R_k(E).$$

Set

$$\begin{aligned} \tilde{V}_h &= \tilde{V}(k, \mathcal{T}_h) = \{ \tilde{v} \in \tilde{V} \mid \tilde{v}|_E \in \tilde{V}(E), E \in \mathcal{T}_h \} \\ &= \left\{ \tilde{v} \in \prod_{E \in \mathcal{T}_h} \tilde{V}(E) \mid \tilde{v}|_{E_i} \cdot \gamma_i + \tilde{v}|_{E_j} \cdot \gamma_j = 0 \text{ on } \overline{E_i} \cap \overline{E_j} \right\} \end{aligned}$$

where  $\gamma_i$  is the outer normal to  $\partial E_i$ ; also, let

$$W_h = W(k, \mathcal{T}_h) = \{ w \in W \mid w|_E \in W(E), E \in \mathcal{T}_h \}.$$

Let  $P_h : W \rightarrow W_h$  be the orthogonal  $L^2$ -projection of  $W$  into  $W_h$  defined by

$$(P_h w - w, \chi) = 0, \quad w \in W, \quad \chi \in W_h, \quad (0.5)$$

which satisfies, for  $1 \leq q \leq \infty$  and for either  $G_0 \subset \subset G_1 \subset \subset \Omega$  and  $h$  sufficiently small or  $G_0 = G_1 = \Omega$ ,

$$\| P_h w - w \|_{0,q;G_0} \leq Q \| w \|_{s,q;G_1} h^s, \quad 0 \leq s \leq k+1, \quad \text{if } w \in W^{s,q}(G_1), \quad (0.6)$$

$$\| P_h w - w \|_{-r,G_0} \leq Q \| w \|_{s,G_1} h^{r+s}, \quad 0 \leq r, s \leq k+1, \quad \text{if } w \in H^s(G_1), \quad (0.7)$$

where  $\| \cdot \|_{s,q;G}$  denotes the usual Sobolev norm in  $W^{s,q}(G)$ , with  $q = 2$  being omitted. Also, since  $\operatorname{div} \tilde{V}_h \subset W_h$ ,

$$(\operatorname{div} \tilde{v}, w - P_h w) = 0, \quad \tilde{v} \in \tilde{V}_h, \quad w \in W. \quad (0.8)$$

Let  $\pi_h : \underline{V} \rightarrow \underline{V}_h$  be the Raviart-Thomas projection, [8, 9], which satisfies

$$\operatorname{div} \pi_h = P_h \operatorname{div}, \quad (0.9)$$

$$\| \pi_h \underline{v} - \underline{v} \|_{0,q;G_0} \leq Q \| \underline{v} \|_{s,q;G_1} h^s, \quad 1/q < s \leq k+1, \quad \text{if } \underline{v} \in W^{s,q}(G_1)^2. \quad (0.10)$$

The mixed finite element method for (0.1) is the discrete form of (0.4) : Find  $(\underline{u}_h, p_h) \in \underline{V}_h \times W_h$  such that

$$\left. \begin{aligned} (\alpha(p_h) \underline{u}_h, \underline{v}) - (\operatorname{div} \underline{v}, p_h) + (\beta(p_h), \underline{v}) &= \langle g, \underline{v} \cdot \underline{\nu} \rangle, & \underline{v} \in \underline{V}_h, \\ (\operatorname{div} \underline{u}_h, w) + (c(p_h), w) &= (f, w), & w \in W_h. \end{aligned} \right\} \quad (0.11)$$

The existence and uniqueness of a solution of (0.11) and global convergence rates for the approximation  $(\underline{u}_h, p_h)$  to  $(\underline{u}, p)$  in various Sobolev spaces have been established in [3] for the linear problem (0.2). Existence, a form of uniqueness, and the convergence of the corresponding approximate solution have been demonstrated in [5] for the quasilinear problem.

Let

$$\begin{aligned} \underline{\zeta} &= \underline{u} - \underline{u}_h, & \xi &= p - p_h, & \varpi &= \pi_h \underline{u} - \underline{u}_h, & \tau &= P_h p - p_h, \\ \theta &= \tau - \xi = P_h p - p. \end{aligned}$$

Let

$$\begin{aligned} \underline{\kappa} &= \left[ \int_0^1 (1-t) \alpha_{pp}(p - t\xi) dt \right] \underline{u} + \int_0^1 (1-t) \beta_{pp}(p - t\xi) dt, \\ \lambda &= \int_0^1 \alpha_p(p - t\xi) dt, & \rho &= \int_0^1 (1-t) c_{pp}(p - t\xi) dt, \end{aligned}$$

for the nonlinear problem, and set  $\underline{\kappa} = \underline{0}$  and  $\lambda = \rho = 0$  for the linear problem. Also, let

$$\underline{\Gamma} = \alpha_p(p) \underline{u} + \beta_p(p), \quad \gamma = c_p(p)$$

for the nonlinear problem and

$$\underline{\Gamma} = \underline{\beta}, \quad \gamma = c$$

for the linear problem.

It then follows from (0.4) and (0.11) that  $\zeta$  and  $\xi$  satisfy the following error equations (see [3, 5]) :

$$\left. \begin{aligned} (\alpha(p) \zeta, v) - (\operatorname{div} v, \xi) + (\Gamma \xi, v) &= ([\kappa \xi + \lambda \zeta] \xi, v), \quad v \in V_h, \\ (\operatorname{div} \zeta, w) + (\gamma \xi, w) &= (\rho \xi^2, w), \quad w \in W_h. \end{aligned} \right\} \quad (0.12)$$

Let  $M : H^2(\Omega) \rightarrow L^2(\Omega)$  be the operator given by

$$Mw = -\nabla \cdot (a(p) \nabla w + a(p) \Gamma w) + \gamma w,$$

and let  $M^*$  be its formal adjoint; that is,

$$M^* \chi = -\nabla \cdot (a(p) \nabla \chi) + a(p) \Gamma \cdot \nabla \chi + \gamma \chi.$$

Note that  $M = L$  and  $M^* = L^*$  in the linear case.

For any open set  $E \subset \Omega$  set

$$\|z\|_{-s,E} = \sup_{\substack{y \in H_0^s(E) \\ y \neq 0}} \frac{(z, y)}{\|y\|_{s,E}}. \quad (0.13)$$

It follows from [10] that the operator  $M^*$  is coercive over  $H_0^1(G)$  for any ball  $G \subset \Omega$  of diameter not greater than some  $d_0$ . Furthermore, if the coefficients of (0.1) (or (0.2)) are sufficiently smooth, then, for any fixed  $s \geq 0$ ,  $M^*$  has a bounded inverse from  $H^s(G)$  onto  $H^{s+2}(G) \cap H_0^1(G)$ . All balls appearing in the remainder of this paper will be assumed to be of diameters less than  $d_0$ .

Let

$$\begin{aligned} V_h(G) &= \{v \in V_h : \operatorname{supp} v \subset G\}, \\ W_h(G) &= \{w \in W_h : \operatorname{supp} w \subset G\}. \end{aligned} \quad (0.14)$$

The pair  $(u_h, p_h) \in V_h \times W_h$  is said to be an interior solution of the mixed method equations (0.11) on  $G \subset \subset \Omega$  if it satisfies

$$\begin{aligned} (\alpha(p_h) u_h, v) - (\operatorname{div} v, p_h) + (\beta(p_h), v) &= 0, \quad v \in V_h(G), \\ (\operatorname{div} u_h, w) + (c(p_h), w) &= (f, w), \quad w \in W_h(G). \end{aligned} \quad (0.15)$$

Note that the boundary integral from (0.4) is missing in (0.15) as  $v \cdot \nu = 0$  on  $\partial\Omega$  if  $v \in V_h(G)$ .

The plan of the paper is as follows. In section 1 we derive a local duality lemma which will be used for both the linear and nonlinear problems. We then

derive local estimates in  $H^{-s}$ ,  $0 \leq s \leq k+1$ , for the linear problem (section 2) and for the nonlinear (section 3). In section 4 we find rates of convergence for difference quotients for  $\xi$ ,  $\zeta$ , and  $\operatorname{div} \zeta$  in  $H^{-s}$ ,  $0 \leq s \leq k+1$ . Finally, in section 5 we demonstrate local superconvergence in  $L^2$  of some particular local averages of  $p_h$ ,  $\underline{u}_h$ , and  $\operatorname{div} \underline{u}_h$  to  $p$ ,  $u$ , and  $\operatorname{div} u$ , respectively.

## 1. THE INTERIOR DUALITY LEMMA

Global error estimates for the linear problem have been derived by Douglas and Roberts [3] using an analytical approach based strongly on a duality argument, and an analogous argument has been employed by Milner [5] in treating the quasilinear problem. Lemma 1.1, which will be fundamental in obtaining our interior estimates, is a local version of Lemmas 3.1 and 3.2 of Douglas-Roberts [3] and of Lemmas 2.1 and 5.1 of Milner [5]. Let  $\delta_{ik}$  denote the Kronecker symbol below.

**LEMMA 1.1 :** *Let  $G'$  and  $G$  be concentric balls such that  $G' \subset \subset G \subset \subset \Omega$ . Assume that  $\zeta \in \underline{V}$ ,  $\underline{f}_0 \in L^2(G)^2$ ,  $f_1 \in L^2(G)$ ,  $g_0 \in L^2(G)$ . Also, let  $q \in L^{1+\delta}(G)^2$  and  $\eta \in L^{1+\delta}(G)$  for some  $\delta$ ,  $0 < \delta < 1$ . Suppose that  $z \in W_k$  satisfies the equations*

$$\left. \begin{aligned} (\alpha \zeta, \underline{v}) - (\operatorname{div} \underline{v}, z) + (\underline{\Gamma} z, \underline{v}) &= (\underline{f}_0 + q, \underline{v}) + (f_1, \operatorname{div} \underline{v}), \quad \underline{v} \in \underline{V}_h(G), \\ (\operatorname{div} \zeta, w) + (\gamma z, w) &= (g_0 + \eta, w), \quad w \in W_h(G), \end{aligned} \right\} \quad (1.1)$$

Let  $s \geq 0$  and  $l = \min(s, k)$ . Then,

$$\begin{aligned} \|z\|_{-s, G'} &\leq K [h^{l+1} \|\zeta\|_{0, G} + h^{l+2-\delta_{ik}} \|\operatorname{div} \zeta\|_{0, G} \\ &\quad + h^{l+1} \|z\|_{0, G} + h^l \{ h \|\underline{f}_0\|_{0, G} + \|f_1\|_{0, G} \\ &\quad + h^{2-\delta_{ik}} \|g\|_{0, G} \} + \|q\|_{0, 1+\delta, G} + \|\eta\|_{0, 1+\delta, G} \\ &\quad + \|\zeta\|_{-s-2, G} + \|z\|_{-s-1, G} + \|\underline{f}_0\|_{-s-1, G} \\ &\quad + \|f_1\|_{-s, G} + \|g_0\|_{-s-2, G}]. \end{aligned} \quad (1.2)$$

*Proof:* It will be convenient here and throughout the remainder of the paper to introduce a finite sequence  $\{G_i\}$  of balls concentric with  $G'$  and such that  $G' \subset \subset G_i \subset \subset G_{i+1} \subset \subset G$ . We shall use a corresponding sequence  $\{\omega_i\}$  of functions such that  $\omega_i \in C_0^\infty(G_{i+1})$  and  $\omega_i = 1$  on  $G_i$ ; normally the index  $i$  will be omitted for  $\omega_i$ .

Let  $\psi \in H_0^s(G_1)$  and determine  $\phi \in H^{s+2}(G_1) \cap H_0^1(G_1)$  as the solution of  $M^* \phi = \psi$  in  $G_1$ . Then,  $\|\phi\|_{s+2, G_1} \leq Q \|\psi\|_{s, G_1}$ , and

$$\underline{v} = \pi_h \omega_a \nabla \phi \in \underline{V}_h(G_2) (\subset \underline{V}_h(G))$$

if  $h$  is sufficiently small. Hence, we see from (1.1) that

$$\begin{aligned}
 (\omega z, \psi) &= (\omega z, M^* \phi) \\
 &= (z, -\nabla \cdot (\omega a \nabla \phi) + \Gamma \cdot \omega a \nabla \phi + \omega \gamma \phi) + (z, \nabla \omega \cdot a \nabla \phi) \\
 &= (z, -\operatorname{div} [\pi_h \omega a \nabla \phi]) + (\Gamma z, \pi_h \omega a \nabla \phi) + (\gamma z, P_h \omega \phi) \\
 &\quad + (\Gamma z, \omega a \nabla \phi - \pi_h \omega a \nabla \phi) + (\gamma z, \omega \phi - P_h \omega \phi) + (z, \nabla \omega \cdot a \nabla \phi) \\
 &= (q + f_0, \pi_h \omega a \nabla \phi) + (f_1, \operatorname{div} [\pi_h \omega a \nabla \phi]) - (\alpha \zeta, \pi_h \omega a \nabla \phi) \\
 &\quad + (g_0 + \eta, P_h \omega \phi) - (\operatorname{div} \zeta, P_h \omega \phi) + (\Gamma z, \omega a \nabla \phi - \pi_h \omega a \nabla \phi) \\
 &\quad + (\gamma z, \omega \phi - P_h \omega \phi) + (z, \nabla \omega \cdot a \nabla \phi) \\
 &= (q, \omega a \nabla \phi) + (q, \pi_h \omega a \nabla \phi - \omega a \nabla \phi) + (f_0, \omega a \nabla \phi) \\
 &\quad + (f_0, \pi_h \omega a \nabla \phi - \omega a \nabla \phi) + (f_1, \operatorname{div} \omega a \nabla \phi) \\
 &\quad + (f_1, \operatorname{div} [\pi_h \omega a \nabla \phi - \omega a \nabla \phi]) \\
 &\quad + (g_0, \omega \phi) + (g_0, P_h \omega \phi - \omega \phi) + (\eta, \omega \phi) + (\eta, P_h \omega \phi - \omega \phi) \\
 &\quad + (\alpha \zeta + \Gamma z, \omega a \nabla \phi - \pi_h \omega a \nabla \phi) + (\operatorname{div} \zeta + \gamma z, \omega \phi - P_h \omega \phi) \\
 &\quad + (\zeta, (\nabla \omega) \phi) + (z, \nabla \omega \cdot a \nabla \phi), \tag{1.3}
 \end{aligned}$$

since

$$-(\zeta, \omega \nabla \phi) - (\operatorname{div} \zeta, \omega \phi) = -(\omega, \operatorname{div} (\zeta \phi)) = (\zeta, (\nabla \omega) \phi).$$

First note that the embedding  $H^{s+1}(G) \subset L^{1+(1/\delta)}(G)$  implies that

$$\begin{aligned}
 (q, \omega a \nabla \phi) &\leq K \|q\|_{0,1+\delta;G_2} \|\omega a \nabla \phi\|_{0,1+(1/\delta);G_2} \\
 &\leq K \|q\|_{0,1+\delta;G_2} \|\nabla \phi\|_{s+1,G_2}. \tag{1.4}
 \end{aligned}$$

and that

$$\begin{aligned}
 (\eta, \omega \phi) &\leq K \|\eta\|_{0,1+\delta;G_2} \|\omega \phi\|_{0,1+(1/\delta);G_2} \\
 &\leq K \|\eta\|_{0,1+\delta;G_2} \|\phi\|_{s+1,G_2}. \tag{1.5}
 \end{aligned}$$

Also, by (0.6), (0.10), and the embedding  $H^{s+1}(G) \subset W^{\delta/(\delta+1),1+(1/\delta)}(G)$ ,

$$\begin{aligned}
 (q, \pi_h \omega a \nabla \phi - \omega a \nabla \phi) &\leq K \|q\|_{0,1+\delta;G_2} \|\pi_h \omega a \nabla \phi - \omega a \nabla \phi\|_{0,1+(1/\delta);G_2} \\
 &\leq K \|q\|_{0,1+\delta;G_2} h^{\delta/(\delta+1)} \|\omega a \nabla \phi\|_{\delta/(\delta+1),1+(1/\delta);G_2} \\
 &\leq K h^{\delta/(\delta+1)} \|q\|_{0,1+\delta;G_2} \|\nabla \phi\|_{s+1,G_2} \tag{1.6}
 \end{aligned}$$



and

$$\begin{aligned}
 (\eta, P_h \omega \phi - \omega \phi) &\leq K \|\eta\|_{0,1+\delta;G_2} \|P_h \omega \phi - \omega \phi\|_{0,1+(1/\delta);G_2} \\
 &\leq K \|\eta\|_{0,1+\delta;G_2} \|\omega \phi\|_{\delta/(\delta+1),1+(1/\delta);G_2} h^{\delta/(\delta+1)} \\
 &\leq K \|\eta\|_{0,1+\delta;G_2} \|\phi\|_{s+1,G_2} h^{\delta/(\delta+1)}.
 \end{aligned} \tag{1.7}$$

Next, note that

$$(\underline{f}_0, \omega a \nabla \phi) \leq K \|\underline{f}_0\|_{-s-1,G_2} \|\nabla \phi\|_{s+1,G_2}. \tag{1.8}$$

$$(f_1, \operatorname{div} \omega a \nabla \phi) \leq K \|f_1\|_{-s,G_2} \|\operatorname{div} \omega a \nabla \phi\|_{s,G_2}, \tag{1.9}$$

$$(g_0, \omega \phi) \leq K \|g_0\|_{-s-2,G_2} \|\phi\|_{s+2,G_2}. \tag{1.10}$$

Also, (0.6) and (0.10) imply that

$$\begin{aligned}
 (\underline{f}_0 - \alpha \underline{\zeta} - \underline{\Gamma} z, \pi_h \omega a \nabla \phi - \omega a \nabla \phi) &\leq \\
 &\leq K (\|\underline{f}_0\|_{0,G_2} + \|\underline{\zeta}\|_{0,G_2} + \|z\|_{0,G_2}) \|\pi_h \omega a \nabla \phi - \omega a \nabla \phi\|_{0,G_2} \\
 &\leq K h^{l+1} (\|\underline{f}_0\|_{0,G_2} + \|\underline{\zeta}\|_{0,G_2} + \|z\|_{0,G_2}) \|\nabla \phi\|_{l+1,G_2},
 \end{aligned} \tag{1.11}$$

$$\begin{aligned}
 (g_0 - \operatorname{div} \underline{\zeta} - \gamma z, P_h \omega \phi - \omega \phi) &\leq \\
 &\leq K (\|g_0\|_{0,G_2} + \|\operatorname{div} \underline{\zeta}\|_{0,G_2} + \|z\|_{0,G_2}) \|P_h \omega \phi - \omega \phi\|_{0,G_2} \\
 &\leq K h^{l+2-\delta_{lk}} (\|g_0\|_{0,G_2} + \|\operatorname{div} \underline{\zeta}\|_{0,G_2} + \|z\|_{0,G_2}) \|\phi\|_{l+2,G_2}
 \end{aligned} \tag{1.12}$$

$$\begin{aligned}
 (f_1, \operatorname{div} (\pi_h \omega a \nabla \phi) - \operatorname{div} (\omega a \nabla \phi)) &= (f_1, P_h \operatorname{div} (\omega a \nabla \phi) - \operatorname{div} (\omega a \nabla \phi)) \\
 &\leq K \|f_1\|_{0,G_2} \|P_h \operatorname{div} (\omega a \nabla \phi) - \operatorname{div} (\omega a \nabla \phi)\|_{0,G_2} \\
 &\leq K h^l \|f_1\|_{0,G_2} \|\operatorname{div} (\omega a \nabla \phi)\|_{l,G_2}.
 \end{aligned} \tag{1.13}$$

Finally,

$$(\underline{\zeta}, (\nabla \omega) \phi) \leq K \|\underline{\zeta}\|_{-s-2,G_2} \|\phi\|_{s+2,G_2}, \tag{1.14}$$

$$(z, \nabla \omega a \nabla \phi) \leq K \|z\|_{-s-1,G_2} \|a \nabla \phi\|_{s+1,G_2}. \tag{1.15}$$

The lemma now follows trivially from the substitution of (1.4)-(1.15) into (1.3).

We can now derive some preliminary local error estimates from this lemma.

1.2 : If  $h$  is sufficiently small,  $0 \leq r \leq k+1$ ,  $0 \leq s \leq k$ , and  $p \in H^r(G)$ , then

$$\begin{aligned}
 (i) \quad \|\xi\|_{-s,G'} &\leq K [h^{s+1} (\|\xi\|_{0,G} + \|\underline{\zeta}\|_{0,G}) + h^{s+2-\delta_{sk}} \|\operatorname{div} \underline{\zeta}\|_{0,G} \\
 &\quad + h^{r+s} \|p\|_{r,G} + \|\xi^2\|_{0,1+\delta;G} + \|\xi \underline{\zeta}\|_{0,1+\delta;G} \\
 &\quad + \|\xi\|_{-k-1,G} + \|\underline{\zeta}\|_{-k-1,G}],
 \end{aligned}$$

$$(ii) \quad \|\zeta\|_{-s,G'} \leq K[h^s(\|\xi\|_{0,G} + \|\zeta\|_{0,G}) + h^{s+1}\|\operatorname{div} \zeta\|_{0,G} + h^{r+s}\|p\|_{r,G} + \|\xi^2\|_{0,1+\delta;G} + \|\xi\zeta\|_{0,1+\delta;G} + \|\xi\|_{-k-1,G} + \|\zeta\|_{-k-1,G}].$$

$$(iii) \quad \|\operatorname{div} \zeta\|_{-s,G'} \leq K[h^s(\|\xi\|_{0,G} + \|\operatorname{div} \zeta\|_{0,G}) + \|\rho\xi^2\|_{0,1+\delta;G} + \|\xi\|_{-s,G}].$$

*Proof:* Rewrite (0.12) in the form

$$\left. \begin{aligned} (\alpha(p)\zeta, v) - (\operatorname{div} v, \tau) + (\Gamma\tau, v) &= ([\kappa\xi + \lambda\zeta]\xi + \Gamma\theta, v), \quad v \in V_h(G), \\ (\operatorname{div} \zeta, w) + (\gamma\tau, w) &= (\rho\xi^2 + \gamma\theta, w), \quad w \in W_h(G). \end{aligned} \right\} \quad (1.16)$$

We can now apply Lemma 1.1 to (1.15), by setting  $g = (\kappa\xi + \lambda\zeta)\xi$ ,  $\eta = \rho\xi^2$ ,  $f_0 = \Gamma\theta$ ,  $f_1 = 0$ , and  $g_0 = \gamma\theta$  in (1.2). Thus,

$$\begin{aligned} \|\tau\|_{-s,G'} &\leq K[h^{s+1}\|\zeta\|_{0,G_2} + h^{s+2-\delta_{sk}}\|\operatorname{div} \zeta\|_{0,G_2} + \|(\kappa\xi + \lambda\zeta)\xi\|_{0,1+\delta;G_2} + \|\rho\xi^2\|_{0,1+\delta;G_2} \\ &\quad + h^{s+1}\|\Gamma\theta\|_{0,G_2} + h^{s+2-\delta_{sk}}\|\gamma\theta\|_{0,G_2} + h^{s+1}\|\tau\|_{0,G_2} + \|\Gamma\theta\|_{-s-1,G_2} + \|\gamma\theta\|_{-s-2,G_2} \\ &\quad + \|\zeta\|_{-s-2,G_2} + \|\tau\|_{-s-1,G_2}]. \end{aligned} \quad (1.17)$$

For  $s = 0$ , (1.17) gives the estimate

$$\begin{aligned} \|\tau\|_{0,G'} &\leq K[h\|\zeta\|_{0,G_2} + h^{2-\delta_{0k}}\|\operatorname{div} \zeta\|_{0,G_2} + \|(\kappa\xi + \lambda\zeta)\xi\|_{0,1+\delta;G_2} + \|\rho\xi^2\|_{0,1+\delta;G_2} + h\|\Gamma\theta\|_{0,G_2} + h^{2-\delta_{0k}}\|\gamma\theta\|_{0,G_2} \\ &\quad + h\|\tau\|_{0,G_2} + \|\Gamma\theta\|_{-1,G_2} + \|\gamma\theta\|_{-2,G_2} + \|\zeta\|_{-2,G_2} + \|\tau\|_{-1,G_2}] \\ &\leq K[h\|\zeta\|_{0,G_2} + h^{2-\delta_{0k}}\|\operatorname{div} \zeta\|_{0,G_2} + \|(\kappa\xi + \lambda\zeta)\xi\|_{0,1+\delta;G_2} + \|\rho\xi^2\|_{0,1+\delta;G_2} + h\|\Gamma\theta\|_{0,G_2} + h^{2-\delta_{0k}}\|\gamma\theta\|_{0,G_2} \\ &\quad + h\|\tau\|_{0,G_2} + h^{-s}\|\Gamma\theta\|_{-s-1,G_2} + h^{-s/2}\|\gamma\theta\|_{-s-2,G_2} + h^{-s/2}\|\zeta\|_{-s-2,G_2} + h^{-s}\|\tau\|_{-s-1,G_2}], \end{aligned} \quad (1.18)$$

where we have used interpolation between Sobolev spaces, [4].

If we now substitute (1.18) recursively into itself we obtain the bound

$$\begin{aligned} \|\tau\|_{0,G'} &\leq K[h\|\zeta\|_{0,G_{2k}} + h^{2-\delta_{0k}}\|\operatorname{div}\zeta\|_{0,G_{2k}} + \|(\kappa\xi + \lambda\zeta)\xi\|_{0,1+\delta;G_{2k}} \\ &\quad + \|\rho\xi^2\|_{0,1+\delta;G_{2k}} + h\|\tilde{\Gamma}\theta\|_{0,G_{2k}} + h^{2-\delta_{0k}}\|\gamma\theta\|_{0,G_{2k}} \\ &\quad + h^k\|\tau\|_{0,G_{2k}} + h^{-s}\|\tilde{\Gamma}\theta\|_{-s-1,G_{2k}} + h^{-s/2}\|\gamma\theta\|_{-s-2,G_{2k}} \\ &\quad + h^{-s/2}\|\zeta\|_{-s-2,G_{2k}} + h^{-s}\|\tau\|_{-s-1,G_{2k}}], \end{aligned}$$

which when substituted into (1.17) gives the estimate

$$\begin{aligned} \|\tau\|_{-s,G'} &\leq K[h^{s+1}\|\zeta\|_{0,G_1} + h^{s+2-\delta_{sk}}\|\operatorname{div}\zeta\|_{0,G_1} \\ &\quad + \|(\kappa\xi + \lambda\zeta)\xi\|_{0,1+\delta;G_1} + \|\rho\xi^2\|_{0,1+\delta;G_1} \\ &\quad + h^{s+1}\|\tilde{\Gamma}\theta\|_{0,G_1} + h^{s+2-\delta_{sk}}\|\gamma\theta\|_{0,G_1} \\ &\quad + h^{s+k+1}\|\tau\|_{0,G_1} + \|\tilde{\Gamma}\theta\|_{-s-1,G_1} + \|\gamma\theta\|_{-s-2,G_1} \\ &\quad + \|\zeta\|_{-s-2,G_1} + \|\tau\|_{-s-1,G_1}], \end{aligned} \quad (1.19)$$

(where  $G'_1 = G_{2k}$ ) which in turn implies (using (0.6) and (0.7)) that

$$\begin{aligned} \|\xi\|_{-s,G'} &\leq K[h^{s+1}\|\zeta\|_{0,G_1} + h^{s+2-\delta_{sk}}\|\operatorname{div}\zeta\|_{0,G_1} \\ &\quad + \|\zeta\xi\|_{0,1+\delta;G_1} + \|\xi^2\|_{0,1+\delta;G_1} \\ &\quad + h^{r+s}\|p\|_{r,G_{2k+1}} + h^{k+s+1}\|\xi\|_{0,G_1} \\ &\quad + \|\zeta\|_{-s-2,G_1} + \|\xi\|_{-s-1,G_1}], \end{aligned} \quad (1.20)$$

Let us consider the divergence of  $\zeta$ . Let  $\phi \in H_0^s(G_1)$ . Then, we see from (1.16) that

$$\begin{aligned} (\omega \operatorname{div} \zeta, \phi) &= (\operatorname{div} \zeta, P_h(\omega\phi)) + (\operatorname{div} \zeta, \omega\phi - P_h(\omega\phi)) \\ &= -(\gamma\xi, \omega\phi) + (\operatorname{div} \zeta + \gamma\xi, \omega\phi - P_h(\omega\phi)) + (\rho\xi^2, P_h(\omega\phi)), \end{aligned}$$

which implies the bound

$$\|\operatorname{div} \zeta\|_{-s,G'} \leq K[\|\xi\|_{-s,G_1} + h^s(\|\xi\|_{0,G_1} + \|\operatorname{div} \zeta\|_{0,G_1}) + \|\rho\xi^2\|_{0,G_1}] \quad (1.21)$$

for  $0 \leq s \leq k+1$ . This gives (iii) of the lemma.

Finally, let us obtain a bound for  $\zeta$  in  $H^{-s}(G')^2$ . Let  $\psi \in H_0^s(G_1)^2$  and take  $\phi \in H^{s+1}(G_1) \cap H_0^1(G_1)$  as the solution of  $-\nabla \cdot (a(p)\nabla\phi) = \operatorname{div} \psi$  in  $G_1$ . Set  $\tilde{\delta} = \psi + a(p)\nabla\phi$ . Then,

$$\operatorname{div} \tilde{\delta} = 0 \quad (1.22)$$

and

$$\|\tilde{\omega}\|_{s,G_1} + \|\phi\|_{s+1,G_1} \leq Q \|\psi\|_{s,G_1}, \quad s \geq 0. \quad (1.23)$$

Also, integrating by parts, we obtain the relation

$$(\omega\alpha(p)\tilde{\zeta}, \psi) = (\operatorname{div} \tilde{\zeta}, \omega\phi) + (\tilde{\zeta}, \phi\nabla\omega) + (\alpha(p)\tilde{\zeta}, \omega\tilde{\omega}). \quad (1.24)$$

Using the relation that led to (1.21) with  $(\rho\xi^2, P_h(\omega\phi))$  replaced by  $(\rho\xi^2, \omega\phi) + (\rho\xi^2, P_h(\omega\phi) - \omega\phi)$ , we obtain the estimate

$$\begin{aligned} |(\operatorname{div} \tilde{\zeta}, \omega\phi) + (\tilde{\zeta}, \phi\nabla\omega)| &= |-(\gamma\xi, \omega\phi) + (\operatorname{div} \tilde{\zeta} + \gamma\xi, \omega\phi - P_h(\omega\phi)) \\ &\quad + (\rho\xi^2, \omega\phi) + (\rho\xi^2, P_h(\omega\phi) - \omega\phi) + (\tilde{\zeta}, \phi\nabla\omega)| \\ &\leq K[\|\xi\|_{-s-1,G_1} + \|\tilde{\zeta}\|_{-s-1,G_1} + \|\rho\xi^2\|_{0,1+\delta;G_1} \\ &\quad + h^{s+1}(\|\xi\|_{0,G_1} + \|\operatorname{div} \tilde{\zeta}\|_{0,G_1})] \|\phi\|_{s+1,G_1}. \end{aligned} \quad (1.25)$$

Next, it follows from (0.12) that

$$\begin{aligned} (\alpha(p)\tilde{\zeta}, \omega\tilde{\omega}) &= (\alpha(p)\tilde{\zeta}, \pi_h\omega\tilde{\omega}) + (\alpha(p)\tilde{\zeta}, \omega\tilde{\omega} - \pi_h\omega\tilde{\omega}) \\ &= (\operatorname{div} \pi_h\omega\tilde{\omega}, \xi) - (\Gamma\xi, \omega\tilde{\omega}) + (\alpha(p)\tilde{\zeta} + \Gamma\xi, \omega\tilde{\omega} - \pi_h\omega\tilde{\omega}) \\ &\quad + ([\kappa\xi + \lambda\tilde{\zeta}]\xi, \omega\tilde{\omega}) + ([\kappa\xi + \lambda\tilde{\zeta}]\xi, \pi_h\omega\tilde{\omega} - \omega\tilde{\omega}). \end{aligned} \quad (1.26)$$

Now, (0.8), (0.9), and (1.22) imply (using integration by parts) that

$$\begin{aligned} (\operatorname{div} \pi_h\omega\tilde{\omega}, \xi) &= (\operatorname{div} [\pi_h\omega\tilde{\omega} - \omega\tilde{\omega}], \tau) + (\operatorname{div} \omega\tilde{\omega}, \tau) \\ &= (\operatorname{div} \omega\tilde{\omega}, \tau) \\ &= (\tilde{\omega}, \nabla\omega, \tau), \end{aligned} \quad (1.27)$$

since  $\tau = \xi + \theta \in W_h$ ; it now follows from (1.26) and (1.27) that

$$\begin{aligned} |(\alpha(p)\tilde{\zeta}, \omega\tilde{\omega})| &\leq K[\|\tau\|_{-s,G_1} + \|\xi\|_{-s,G_1} + h^s(\|\xi\|_{0,G_1} + \|\tilde{\zeta}\|_{0,G_1}) \\ &\quad + \|[\kappa\xi + \lambda\tilde{\zeta}]\xi\|_{0,1+\delta;G_1}] \|\tilde{\omega}\|_{s,G_1}, \quad 1 \leq s \leq k+1. \end{aligned} \quad (1.28)$$

Using (0.6), (1.23), (1.25), and (1.28) in (1.24) we obtain the bound

$$\begin{aligned} \|\tilde{\zeta}\|_{-s,G'} &\leq K[\|\xi\|_{-s,G_1} + h^{r+s}\|p\|_{r,G_2} + h^s(\|\xi\|_{0,G_1} + \|\tilde{\zeta}\|_{0,G_1}) \\ &\quad + \|\xi^2\|_{0,1+\delta;G_1} + \|\tilde{\zeta}\xi\|_{0,1+\delta;G_1} + h^{s+1}\|\operatorname{div} \tilde{\zeta}\|_{0,G_1} + \|\tilde{\zeta}\|_{-s-1,G_1}] \end{aligned} \quad (1.29)$$

for  $1 \leq s \leq k$ , and since the relation holds trivially for  $s = 0$ , using interpolation, [4], we see that it holds for  $0 \leq s \leq k$ .

Recursive substitution of (1.20) into (1.29) gives the inequality

$$\begin{aligned} \|\zeta\|_{-s, G'} &\leq K[h^s(\|\xi\|_{0, G_{k+1}} + \|\zeta\|_{0, G_{k+1}}) + h^{s+1} \|\operatorname{div} \zeta\|_{0, G_{k+1}} \\ &\quad + \|\xi^2\|_{0, 1+\delta; G_{k+1}} + \|\zeta\xi\|_{0, 1+\delta; G_{k+1}} + h^{r+s} \|p\|_{r, G_{k+2}} \\ &\quad + \|\zeta\|_{-s-1, G_{k+1}} + \|\xi\|_{-k-1, G_{k+1}}]. \end{aligned} \quad (1.30)$$

Now substitute (1.30) recursively into itself to see that

$$\begin{aligned} \|\zeta\|_{-s, G'} &\leq K[h^s(\|\xi\|_{0, G_{k(k+1)}} + \|\zeta\|_{0, G_{k(k+1)}}) + h^{s+1} \|\operatorname{div} \zeta\|_{0, G_{k(k+1)}} \\ &\quad + \|\xi^2\|_{0, 1+\delta; G_{k(k+1)}} + \|\zeta\xi\|_{0, 1+\delta; G_{k(k+1)}} + h^{r+s} \|p\|_{r, G_{k(k+2)}} \\ &\quad + \|\xi\|_{-k-1, G_{k(k+1)}} + \|\zeta\|_{-k-1, G_{k(k+1)}}], \end{aligned}$$

which gives part (ii) of this lemma.

We now substitute (ii) into (1.20) (with  $G'_1$  shifted to  $G_1$ ) to obtain the estimate

$$\begin{aligned} \|\xi\|_{-s, G'} &\leq K[h^{s+1}(\|\xi\|_{0, G_2} + \|\zeta\|_{0, G_2}) + h^{s+2-\delta_{sk}} \|\operatorname{div} \zeta\|_{0, G_2} \\ &\quad + \|\xi^2\|_{0, 1+\delta; G_2} + \|\zeta\xi\|_{0, 1+\delta; G_2} + h^{r+s} \|p\|_{r, G_2} \\ &\quad + \|\xi\|_{-s-1, G_2} + \|\zeta\|_{-k-1, G_2}], \end{aligned}$$

which recursively substituted into itself yields (i) of the lemma.

Observe that we have a quadratic error term with no powers of  $h$  in front of it (just in the nonlinear case). This will produce eventually bounds with a quite atypical term in them.

## 2. THE LOCAL ESTIMATES FOR THE LINEAR CASE

Observe that  $s = 0$  in Lemma 1.2(i) gives the bound

$$\begin{aligned} \|\xi\|_{0, G_0} &\leq K[h(\|\xi\|_{0, G_1} + \|\zeta\|_{0, G_1}) + h^{2-\delta_{0k}} \|\operatorname{div} \zeta\|_{0, G_1} \\ &\quad + h^r \|p\|_{r, G_1} + \|\xi^2\|_{0, 1+\delta; G_1} + \|\zeta\xi\|_{0, 1+\delta; G_1} \\ &\quad + \|\xi\|_{-k-1, G_1} + \|\zeta\|_{-k-1, G_1}], \end{aligned} \quad (2.1)$$

for  $2 \leq r \leq k+1$ .

Also,  $\kappa = 0$ ,  $\lambda = 0$ , and  $\rho = 0$  in the linear problem, and the bound

$$\|\xi\|_{-k-1, \Omega} + \|\zeta\|_{-k-1, \Omega} + \|\operatorname{div} \zeta\|_{-k-1, \Omega} \leq K \|p\|_{j, \Omega} h^{j+k-1} \quad (2.2)$$

for  $2 \leq j \leq k+3$  if  $p \in H^j(\Omega)$  is given in [3].

Thus, (2.1) can be rewritten in the form

$$\| \xi \|_{0,G_0} \leq K[h(\| \xi \|_{0,G_1} + \| \zeta \|_{0,G_1}) + h^{2-\delta_{0k}} \| \operatorname{div} \zeta \|_{0,G_1} + h^r \| p \|_{r,G_1} + h^{k+1} \| p \|_{2,\Omega}], \quad (2.3)$$

$2 \leq r \leq k+1$ , if  $p \in H^2(\Omega) \cap H^r(G_1)$ . We do not have similar estimates for  $\zeta$  and  $\operatorname{div} \zeta$ .

We shall need to have error estimates for a local solution of the homogeneous equations.

LEMMA 2.1 : Let  $(v_h, q_h) \in V_h \times W_h$  be a solution of the system

$$\left. \begin{aligned} (\alpha v_h, v) - (\operatorname{div} v, q_h) + (\beta q_h, v) &= 0, \quad v \in V_h(G), \\ (\operatorname{div} v_h, w) + (c q_h, w) &= 0, \quad w \in W_h(G), \end{aligned} \right\} \quad (2.4)$$

where  $G$  is a ball of sufficiently small diameter. Then, for fixed  $\varepsilon > 0$ , any  $m \geq 0$ , and  $h$  sufficiently small,

$$\begin{aligned} h^{-\varepsilon}(\| q_h \|_{0,G'} + \| v_h \|_{0,G'}) + \| \operatorname{div} v_h \|_{0,G'} &\leq \\ &\leq Q[h^m(\| v_h \|_{0,G} + \| \operatorname{div} v_h \|_{0,G} + \| q_h \|_{0,G}) + \\ &\quad + (\| v_h \|_{-k-1,G} + \| q_h \|_{-k-1,G}) h^{-\varepsilon}]. \end{aligned}$$

Proof : Lemma 1.1 implies directly the bound

$$\| q_h \|_{0,G_0} \leq K[h(\| q_h \|_{0,G_1} + \| v_h \|_{0,G_1}) + h^{2-\delta_{0k}} \| \operatorname{div} v_h \|_{0,G_1} + \| q_h \|_{-1,G_1} + \| v_h \|_{-2,G_1}]. \quad (2.5)$$

We should now like to take  $w = \operatorname{div} v_h$  in the second equation of (2.4), but there is no reason to believe that  $\operatorname{div} v_h \in W_h(G)$ . However, there is  $\chi \in W_h(G_1)$  such that

$$\| \chi - \omega \operatorname{div} v_h \|_0 \leq Qh \| \operatorname{div} v_h \|_{0,G_1}. \quad (2.6)$$

Thus, (2.4)-(2.6) imply that

$$\begin{aligned} \| \operatorname{div} v_h \|_{0,G'}^2 &\leq (\omega \operatorname{div} v_h, \operatorname{div} v_h) \\ &= (\omega \operatorname{div} v_h - \chi, \operatorname{div} v_h + c q_h) - (\omega \operatorname{div} v_h, c q_h) \\ &\leq Q \| \operatorname{div} v_h \|_{0,G_1} (h \| \operatorname{div} v_h \|_{0,G_1} + \| q_h \|_{0,G_1}) \\ &\leq Q[h(\| q_h \|_{0,G_2} + \| v_h \|_{0,G_2} + \| \operatorname{div} v_h \|_{0,G_1}) + \| q_h \|_{-1,G_2} \\ &\quad + \| v_h \|_{-2,G_2}] \| \operatorname{div} v_h \|_{0,G_1}. \end{aligned} \quad (2.7)$$

Let us try to estimate  $\| \underline{v}_h \|_{0,G_0}$ . Again, there are  $\bar{\chi} \in \underline{V}_h(G_1)$  and  $\psi \in W_h(G_1)$  such that

$$\left. \begin{aligned} \| \omega \underline{v}_h - \bar{\chi} \|_0 &\leq Qh \| \underline{v}_h \|_{0,G_1}, \\ \| \operatorname{div} (\omega \underline{v}_h - \bar{\chi}) \|_0 &\leq Qh \| \operatorname{div} \underline{v}_h \|_{0,G_1}, \\ \| \omega q_h - \psi \|_0 &\leq Qh \| q_h \|_{0,G_1}. \end{aligned} \right\} \quad (2.8)$$

Then, it follows from (2.4) and (2.8) that

$$\begin{aligned} \| \underline{v}_h \|_{0,G'}^2 &\leq Q(\omega \alpha \underline{v}_h, \underline{v}_h) \\ &= Q[(\alpha \underline{v}_h, \omega \underline{v}_h - \bar{\chi}) + (\operatorname{div} [\bar{\chi} - \omega \underline{v}_h], q_h) - (\beta q_h, \bar{\chi} - \omega \underline{v}_h) \\ &\quad + (\operatorname{div} \omega \underline{v}_h, q_h) - (\omega \beta q_h, \underline{v}_h)] \\ &\leq Q[h(\| \underline{v}_h \|_{0,G_1}^2 + \| q_h \|_{0,G_1} \| \operatorname{div} \underline{v}_h \|_{0,G_1}) \\ &\quad + \| q_h \|_{0,G_1} \| \underline{v}_h \|_{0,G_1} + (\omega \operatorname{div} \underline{v}_h, q_h) + (\underline{v}_h, \nabla \omega, q_h)] \\ &\leq Q[h(\| \underline{v}_h \|_{0,G_1}^2 + \| q_h \|_{0,G_1} \| \operatorname{div} \underline{v}_h \|_{0,G_1}) \\ &\quad + \| q_h \|_{0,G_1} \| \underline{v}_h \|_{0,G_1} - (\omega c q_h, q_h) + O(h \| q_h \|_{0,G_1} [\| q_h \|_{0,G_1} \\ &\quad + \| \operatorname{div} \underline{v}_h \|_{0,G_1}])] \\ &\leq Q[h(\| \underline{v}_h \|_{0,G_1}^2 + \| q_h \|_{0,G_1} \| \operatorname{div} \underline{v}_h \|_{0,G_1}) \\ &\quad + \| q_h \|_{0,G_1} (\| q_h \|_{0,G_1} + \| \underline{v}_h \|_{0,G_1})]. \end{aligned} \quad (2.9)$$

Lemma 1.2, (i) and (ii) imply that, for  $0 \leq s \leq k$ ,

$$\begin{aligned} \| q_h \|_{-s,G'} &\leq Q[h^{s+1}(\| q_h \|_{0,G_1} + \| \underline{v}_h \|_{0,G_1}) + h^{s+2-\delta_{sk}} \| \operatorname{div} \underline{v}_h \|_{0,G_1} \\ &\quad + \| q_h \|_{-k-1,G_1} + \| \underline{v}_h \|_{-k-1,G_1}], \\ \| \underline{v}_h \|_{-s,G'} &\leq Q[h^s(\| q_h \|_{0,G_1} + \| \underline{v}_h \|_{0,G_1}) + h^{s+1} \| \operatorname{div} \underline{v}_h \|_{0,G_1} \\ &\quad + \| q_h \|_{-k-1,G_1} + \| \underline{v}_h \|_{-k-1,G_1}], \end{aligned}$$

where we have omitted the term  $h^{r+s} \| p \|_{r,G_1}$  as it came from the projection errors. Substitution of these relations into (2.5) and (2.7) will replace the terms  $\| q_h \|_{-1,G_1}$  and  $\| \underline{v}_h \|_{-2,G_1}$  by  $\| q_h \|_{-k-1,G_2}$  and  $\| \underline{v}_h \|_{-k-1,G_2}$ , respectively, giving the bounds

$$\begin{aligned} \| q_h \|_{0,G'} &\leq K[h(\| q \|_{0,G_3} + \| \underline{v}_h \|_{0,G_3}) + h^{2-\delta_{0k}} \| \operatorname{div} \underline{v}_h \|_{0,G_3} \\ &\quad + \| q_h \|_{-k-1,G_3} + \| \underline{v}_h \|_{-k-1,G_3}], \end{aligned} \quad (2.10)$$

$$\begin{aligned} \| \operatorname{div} \underline{v}_h \|_{0,G'}^2 &\leq K[h(\| q_h \|_{0,G_3} + \| \underline{v}_h \|_{0,G_3} + \| \operatorname{div} \underline{v}_h \|_{0,G_3}) + \\ &\quad + \| q_h \|_{-k-1,G_3} + \| \underline{v}_h \|_{-k-1,G_3}] \| \operatorname{div} \underline{v}_h \|_{0,G_3}. \end{aligned} \quad (2.11)$$

Recursive substitution of (2.10) into itself (and relabeling the  $G_i$ 's) will give

the estimate

$$\|q_h\|_{0,G'} \leq K[h^t \|q_h\|_{0,G_1} + h \|\underline{v}_h\|_{0,G_1} + h^{2-\delta_{0k}} \|\operatorname{div} \underline{v}_h\|_{0,G_1} + \|q_h\|_{-k-1,G_1} + \|\underline{v}_h\|_{-k-1,G_1}], \quad (2.12)$$

for any fixed integer  $t > 0$ .

Substituting (2.12) into (2.9), we see that

$$\begin{aligned} \|\underline{v}_h\|_{0,G'}^2 &\leq Q[h \|\underline{v}_h\|_{0,G_1}^2 + h^{t+1} \|q_h\|_{0,G_1} \|\operatorname{div} \underline{v}_h\|_{0,G_1} \\ &\quad + h^2 \|\underline{v}_h\|_{0,G_1} \|\operatorname{div} \underline{v}_h\|_{0,G_1} + h^{3-\delta_{0k}} \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 \\ &\quad + h(\|q_h\|_{-k-1,G_1} + \|\underline{v}_h\|_{-k-1,G_1}) \|\operatorname{div} \underline{v}_h\|_{0,G_1} \\ &\quad + h^{2t} \|q_h\|_{0,G_1}^2 + \|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2] \\ &\leq Q[h^{2T} \|q_h\|_{0,G_1}^2 + h \|\underline{v}_h\|_{0,G_1}^2 + h^2 \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 \\ &\quad + \|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2], \end{aligned}$$

where  $T$  is any fixed positive integer. By repeated substitution of this relation into itself (and relabeling of the  $G_i$ 's) we obtain the bound

$$\|\underline{v}_h\|_{0,G'}^2 \leq Q[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2) + h^2 \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 + \|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2]. \quad (2.13)$$

It follows from (2.11), (2.12), and (2.13) that

$$\begin{aligned} \|\operatorname{div} \underline{v}_h\|_{0,G'}^2 &\leq Q[h(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2 + \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2) \\ &\quad + (\|q_h\|_{-k-1,G_1} + \|\underline{v}_h\|_{-k-1,G_1}) \|\operatorname{div} \underline{v}_h\|_{0,G_1}] \\ &\leq Q[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2) + h \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 \\ &\quad + h(\|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2) + (\|q_h\|_{-k-1,G_1} \\ &\quad + \|\underline{v}_h\|_{-k-1,G_1}) \|\operatorname{div} \underline{v}_h\|_{0,G_1}]. \end{aligned} \quad (2.14)$$

If we now substitute repeatedly (2.14) into (2.13) we see that

$$\|\underline{v}_h\|_{0,G'}^2 \leq Q[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2 + \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2) + \|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2]. \quad (2.15)$$

Also, (2.12) implies, using (2.14) repeatedly and (2.15), that

$$\|q_h\|_{0,G'}^2 \leq Q[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2 + \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2) + \|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2], \quad (2.16)$$

for any fixed positive integer  $T$ .



Finally, (2.14) implies, using (2.15) and (2.16), that

$$\begin{aligned} \|\operatorname{div} \underline{v}_h\|_{0,G'}^2 &\leq Q[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2) + h^\varepsilon \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 \\ &\quad + h^{-\varepsilon}(\|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2)] \end{aligned}$$

for any fixed  $\varepsilon > 0$ , and this in turn implies, by repeated substitution into itself, the bound

$$\begin{aligned} \|\operatorname{div} \underline{v}_h\|_{0,G'}^2 &\leq Q[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2 + \|\operatorname{div} \underline{v}_h\|_{0,G_1}^2) \\ &\quad + h^{-\varepsilon}(\|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2)], \end{aligned}$$

which, with (2.15) and (2.16), proves the lemma.

We are now ready to derive the local  $L^2$ -error estimates.

**THEOREM 2.1 :** *If  $p \in H^2(\Omega) \cap H^r(\Omega_1)$  and  $h$  is sufficiently small then*

- (i)  $\|\xi\|_{0,\Omega_0} \leq Q[\|p\|_{r,\Omega_1} h^{r-\delta_{0k}} + \|p\|_{2,\Omega} h^{k+1}]$ ,  $2 \leq r \leq k+1 + \delta_{0k}$
- (ii)  $\|\zeta\|_{0,\Omega_0} \leq Q[\|p\|_{r,\Omega_1} h^{r-1} + \|p\|_{2,\Omega} h^{k+1}]$ ,  $2 \leq r \leq k+2$ ,
- (iii)  $\|\operatorname{div} \zeta\|_{0,\Omega_0} \leq Q[\|p\|_{r,\Omega_1} h^{r-2} + \|p\|_{2,\Omega} h^{k+1-\varepsilon}]$ ,  $2 \leq r \leq k+3$ ,

for any fixed  $\varepsilon > 0$  and for  $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$ .

*Proof :* Let  $(\bar{u}, \bar{p}) = (\omega u, \omega p)$  where  $\omega \in C_0^\infty(G_2)$  and  $\omega = 1$  on  $G_1$ . There is no reason to assume that  $\{E \cap G_1\}_{E \in \mathcal{T}_h}$  is a quasi-regular polygonalization of  $G_1$ , but, if  $h < \operatorname{dist}(G_1, G_2)$ , we can take  $\tilde{G}_1, G_1 \subset \tilde{G}_1 \subset G_2$ , such that  $\tilde{G}_1$  is a union of polygons in  $\mathcal{T}_h$ . Therefore, it follows from [3] that there exists a unique  $(\bar{v}_h, \bar{q}_h) \in \underline{V}_h(G_2) \times W_h(G_2)$  such that

$$\left. \begin{aligned} (\alpha[\bar{u} - \bar{v}_h], \underline{v}) - (\operatorname{div} \underline{v}, \bar{p} - \bar{q}_h) + (\beta[\bar{p} - \bar{q}_h], \underline{v}) &= 0, \quad \underline{v} \in \underline{V}_h(G_2), \\ (\operatorname{div} [\bar{u} - \bar{v}_h], w) + (c[\bar{p} - \bar{q}_h], w) &= 0, \quad w \in W_h(G_2), \end{aligned} \right\} \quad (2.17)$$

and, furthermore, if  $p \in H^r(G_2)$ , we have the bounds

$$\left. \begin{aligned} \|\bar{p} - \bar{q}_h\|_{-s,G'} &\leq Q\|\bar{p}\|_{r,G_1} h^{r+s-\delta_{sk}} \leq Q\|p\|_{r,G_2} h^{r+s-\delta_{sk}}, \\ &\quad 2 \leq r \leq k+1 + \delta_{sk'} \\ \|\bar{u} - \bar{v}_h\|_{-s,G'} &\leq Q\|p\|_{r,G_2} h^{r+s-1}, \quad 2 \leq r \leq k+2, \\ \|\operatorname{div} (\bar{u} - \bar{v}_h)\|_{-s,G'} &\leq Q\|p\|_{r,G_2} h^{r+s-2}, \quad 2 \leq r \leq k+3, \end{aligned} \right\} \quad (2.18)$$

for  $0 \leq s \leq k$ .

Now, if we restrict  $(\underline{v}, w)$  to the space  $\underline{V}_h(G_1) \times W_h(G_1)$  in (2.17), then  $(\bar{u}, \bar{p})$

can be replaced by  $(\underline{u}, p)$ ; then, using (0.12) with  $(\underline{v}, w) \in \underline{V}_h(G_1) \times W_h(G_1)$  we can replace  $(\underline{u}, p)$  by  $(\underline{u}_h, p_h)$  so that

$$\left. \begin{aligned} (\alpha[\underline{u}_h - \bar{v}_h], \underline{v}) - (\operatorname{div} \underline{v}, p_h - \bar{q}_h) + (\beta[p_h - \bar{q}_h], \underline{v}) &= 0, \quad \underline{v} \in \underline{V}_h(G_1), \\ (\operatorname{div} [\underline{u}_h - \bar{v}_h], w) + (c[p_h - \bar{q}_h], w) &= 0, \quad w \in W_h(G_1). \end{aligned} \right\}$$

Since  $(\underline{u}_h - \bar{v}_h, p_h - \bar{q}_h)$  is a local solution of the mixed method equations, Lemma 2.1 implies (taking  $\underline{v}_h = \underline{u}_h - \bar{v}_h$ ,  $q_h = p_h - \bar{q}_h$  in (2.4) and using (2.18)) that

$$\begin{aligned} & (\|p_h - \bar{q}_h\|_{0,G'} + \|\underline{u}_h - \bar{v}_h\|_{0,G'}) h^{-\varepsilon} + \|\operatorname{div}(\underline{u}_h - \bar{v}_h)\|_{0,G'} \\ & \leq Q[h^m(\|p_h - \bar{q}_h\|_{0,G_1} + \|\underline{u}_h - \bar{v}_h\|_{0,G_1} + \|\operatorname{div}(\underline{u}_h - \bar{v}_h)\|_{0,G_1}) \\ & \quad + h^{-\varepsilon}(\|p_h - \bar{q}_h\|_{-k-1,G_1} + \|\underline{u}_h - \bar{v}_h\|_{-k-1,G_1})] \\ & \leq Q[h^m(\|\xi\|_{0,G_1} + \|\zeta\|_{0,G_1} + \|\operatorname{div} \zeta\|_{0,G_1}) \\ & \quad + h^{k+r-1-\varepsilon} \|p\|_{r,G_1} + h^{-\varepsilon}(\|\xi\|_{-k-1,G_1} + \|\zeta\|_{-k-1,G_1})], \end{aligned} \quad (2.19)$$

for  $2 \leq r \leq k+3$ .

If we now take  $s = 0$  in (2.18),  $r = 2$ , and  $\varepsilon + m = k+1$  in (2.19), we see that

$$\begin{aligned} \|\xi\|_{0,G'} & \leq \|\bar{p} - \bar{q}_h\|_{0,G'} + \|p_h - \bar{q}_h\|_{0,G'} \\ & \leq Q[\|p\|_{r,G}(h^{r-\delta_{0k}} + h^{k+1}) \\ & \quad + h^{k+1}(\|\xi\|_{0,G} + \|\zeta\|_{0,G} + \|\operatorname{div} \zeta\|_{0,G}) \\ & \quad + \|\xi\|_{-k-1,G} + \|\zeta\|_{-k-1,G}], \quad 2 \leq r \leq k+1 + \delta_{0k}, \end{aligned} \quad (2.20)$$

$$\begin{aligned} \|\zeta\|_{0,G'} & \leq Q[\|p\|_{r,G}(h^{r-1} + h^{k+1}) + h^{k+1}(\|\xi\|_{0,G} + \|\zeta\|_{0,G} + \|\operatorname{div} \zeta\|_{0,G}) \\ & \quad + \|\xi\|_{-k-1,G} + \|\zeta\|_{-k-1,G}], \quad 2 \leq r \leq k+2, \end{aligned} \quad (2.21)$$

$$\begin{aligned} \|\operatorname{div} \zeta\|_{0,G'} & \leq Q[\|p\|_{r,G}(h^{r-2} + h^{k+1-\varepsilon}) + h^{k+1}(\|\xi\|_{0,G} + \|\zeta\|_{0,G} + \|\operatorname{div} \zeta\|_{0,G}) \\ & \quad + h^{-\varepsilon}(\|\xi\|_{-k-1,G} + \|\zeta\|_{-k-1,G})], \quad 2 \leq r \leq k+3. \end{aligned} \quad (2.22)$$

If we now use a finite cover of  $\Omega_1$  by a collection of  $G$ 's such that the corresponding  $(G')$ 's will cover  $\Omega_0$ , we obtain again the bounds (2.20)-(2.22) with  $G'$  replaced by  $\Omega_0$  and  $G$  by  $\Omega_1$ .

Finally, [3] implies that for  $p \in H^2(\Omega)$ ,

$$\left. \begin{aligned} \|\xi\|_{0,\Omega_1} + \|\zeta\|_{0,\Omega_1} + \|\operatorname{div} \zeta\|_{0,\Omega_1} & \leq Q \|p\|_{2,\Omega} \\ \|\xi\|_{-k-1,\Omega_1} + \|\zeta\|_{-k-1,\Omega_1} & \leq Q h^{k+1} \|p\|_{2,\Omega}, \end{aligned} \right\}$$

which completes the proof of the theorem.

We have also obtained (by combining Lemma 1.2, (2.2), and Theorem 2.1) the following local negative norm error estimates.

**THEOREM 2.2 :** *If  $p \in H^j(\Omega) \cap H^r(\Omega_1)$ ,  $2 \leq j \leq k+3$ , and  $h$  is sufficiently small, then, for  $0 \leq s \leq k+1$ ,*

- (i)  $\|\xi\|_{-s, \Omega_0} \leq K[\|p\|_{r, \Omega_1} h^{r+s-(s-k+1)^+} + \|p\|_{j, \Omega} h^{j+k-1}],$   
 $2 \leq r \leq k+1+(s-k+1)^+,$
- (ii)  $\|\zeta\|_{-s, \Omega_0} \leq K[\|p\|_{r, \Omega_1} h^{r+s-1-(s-k)^+} + \|p\|_{j, \Omega} h^{j+k-1}],$   
 $2 \leq r \leq k+2+(s-k)^+,$
- (iii)  $\|\operatorname{div} \zeta\|_{-s, \Omega_0} \leq K[\|p\|_{r, \Omega_1} h^{r+s-2} + \|p\|_{j, \Omega} h^{k-1+\min\{j, s+2-\varepsilon\}}],$   
 $2 \leq r \leq k+3.$

Theorem 2.2 for  $s = 0$  restates the results of Theorem 2.1, and (i) is, of course, what was expected from (2.3).

### 3. THE LOCAL ESTIMATES FOR THE NONLINEAR CASE

Let  $\omega_i \in C_0^\infty(G_{i+2})$ ,  $\omega_i = 1$  on  $G_{i+1}$ , just as before.

Set

$$\begin{aligned} (\bar{u}_i, \bar{p}_i) &= (\omega_i u, \omega_i p), \\ \bar{\Gamma} &= \int_0^1 [\alpha_p(p_h + t\xi) dt] u_h + \int_0^1 \beta_p(p_h + t\xi) dt \in L^\infty(\bar{\Omega})^2, \\ \bar{\gamma} &= \int_0^1 c_p(p_h + t) dt \in L^\infty(\bar{\Omega}). \end{aligned} \quad (3.1)$$

We can now rewrite the error equations (0.12) in the form (see [5])

$$\left. \begin{aligned} (\alpha(p) \zeta, v) - (\operatorname{div} v, \xi) + (\bar{\Gamma} \xi, v) &= 0, \quad v \in V_h, \\ (\operatorname{div} \zeta, w) + (\bar{\gamma} \xi, w) &= 0, \quad w \in W_h, \end{aligned} \right\} \quad (3.2)$$

which are the mixed method equations corresponding to the operator  $\bar{M}: H^2(\Omega) \rightarrow L^2(\Omega)$  given by

$$\bar{M}w = -\nabla \cdot (a(p) \nabla w + a(p) \bar{\Gamma} w) + \bar{\gamma} w.$$

It was shown in [5] that  $\bar{M}^*$  has a bounded inverse  $L^2(\Omega) \rightarrow H^2(\Omega) \cap H_0^1(\Omega)$ . However, it does not follow that  $(\bar{M}^*)^{-1}$  maps  $H^s(\Omega)$  into  $H^{s+2}(\Omega) \cap H_0^1(\Omega)$  boundedly if  $s > 0$ . Therefore, the duality argument developed in Section 2 for the linear problem (more specifically, the use of Lemma 1.1 to prove Lemma

2.1, and the bounds (2.18)) does not apply to the nonlinear problem. We can obtain a result analogous to Lemma 2.1 for the equations (3.2) with  $(\zeta, \xi)$  replaced by a local solution  $(v_h, q_h)$  as in (2.4), which we shall do for the sake of completeness.

**LEMMA 3.1 :** *Let  $(v_h, q_h) \in V_h \times W_h$  be a local solution of the homogeneous mixed method equations for the operator  $\overline{M}$  on  $G$ ; that is,*

$$\left. \begin{aligned} (\alpha(p) v_h, v) - (\operatorname{div} v, q_h) + (\overline{\Gamma} q_h, v) &= 0, \quad v \in V_h(G), \\ (\operatorname{div} v_h, w) + (\overline{\gamma} q_h, w) &= 0, \quad w \in W_h(G). \end{aligned} \right\} \quad (3.3)$$

Let  $\varepsilon > 0$  and  $m \geq 0$ . Then, if  $h$  is sufficiently small,

$$\|q_h\|_{0,G} + \|v_h\|_{0,G} + \|\operatorname{div} v_h\|_{0,G} \leq K[h^{-\varepsilon}(\|q_h\|_{-k-1,G_1} + \|v_h\|_{-k-1,G_1}) + h^m(\|q_h\|_{0,G_1} + \|v_h\|_{0,G_1} + \|\operatorname{div} v_h\|_{0,G_1})].$$

*Proof:* The existence of a bounded inverse  $(\overline{M}^*)^{-1} : L^2(G) \rightarrow H^2(G) \cap H_0^1(G)$  implies that Lemma 1.1 is valid on (3.3) with  $s = 0$ . Thus,

$$\|q_h\|_{0,G} \leq K[h\|v_h\|_{0,G_1} + h^{2-\delta_0k}\|\operatorname{div} v_h\|_{0,G_1} + h\|q_h\|_{0,G_1} + \|v_h\|_{-2,G_1} + \|q_h\|_{-1,G_1}]. \quad (3.4)$$

Using again interpolation between Sobolev spaces (see [4]), we obtain from (3.4) the bound

$$\|q_h\|_{0,G} \leq K[h^{\varepsilon/2k}(\|q_h\|_{0,G_1} + \|v_h\|_{0,G_1}) + h\|\operatorname{div} v_h\|_{0,G_1} + h^{-\varepsilon/2}(\|q_h\|_{-k-1,G_1} + \|v_h\|_{-k-1,G_1})], \quad k > 0. \quad (3.5)$$

Next,  $\overline{\gamma} \in L^\infty(G)$  implies that (2.7) holds again, so that

$$\begin{aligned} \|\operatorname{div} v_h\|_{0,G}^2 &\leq Q\|\operatorname{div} v_h\|_{0,G_1}(h\|\operatorname{div} v_h\|_{0,G_1} + \|q_h\|_{0,G_1}) \\ &\leq Q[h^{\varepsilon/2k}\|\operatorname{div} v_h\|_{0,G_1}^2 + h^{-\varepsilon/2k}\|q_h\|_{0,G_1}^2]. \end{aligned} \quad (3.6)$$

Also,  $\overline{\Gamma} \in L^\infty(G)^2$  implies that (2.9) still holds, so that

$$\begin{aligned} \|v_h\|_{0,G}^2 &\leq Q[h(\|q_h\|_{0,G_1}^2 + \|v_h\|_{0,G_1}^2 + \|\operatorname{div} v_h\|_{0,G_1}^2) \\ &\quad + \|q_h\|_{0,G_1}(\|q_h\|_{0,G_1} + \|\operatorname{div} v_h\|_{0,G_1})] \\ &\leq K[h^{\varepsilon/2k}\|\operatorname{div} v_h\|_{0,G_1}^2 + h\|v_h\|_{0,G_1}^2 + h^{-\varepsilon/2k}\|q_h\|_{0,G_1}^2]. \end{aligned} \quad (3.7)$$

Also, we can obtain from (3.5) the estimate

$$\|q_h\|_{0,G}^2 \leq K[h^{\varepsilon/k}(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2) + h\|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 + h^{-\varepsilon}(\|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2)],$$

which when recursively substituted into itself will give (after relabeling the  $G_i$ 's) the bound

$$\|q_h\|_{0,G}^2 \leq K[h^t\|q_h\|_{0,G_1}^2 + h^{\varepsilon/k}\|\underline{v}_h\|_{0,G_1}^2 + h\|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 + h^{-\varepsilon}(\|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2)], \quad (3.8)$$

where  $t$  is any fixed positive number.

Substituting (3.8) into (3.7), we see that

$$\|\underline{v}_h\|_{0,G'}^2 \leq K[h^{2T}\|q_h\|_{0,G_1}^2 + h^{\varepsilon/2k}\|\underline{v}_h\|_{0,G_1}^2 + h^{\varepsilon/2k}\|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 + h^{-3\varepsilon/2}(\|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2)],$$

for any fixed  $T > 0$ , and recursive substitution of this relation into itself (and relabeling of the  $G_i$ 's) will show that

$$\|\underline{v}_h\|_{0,G'}^2 \leq K[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2) + h^{\varepsilon/2k}\|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 + h^{-3\varepsilon/2}(\|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2)]. \quad (3.9)$$

We now substitute (3.9) into (3.8) to obtain the estimate

$$\|q_h\|_{0,G'}^2 \leq K[h^{2T}(\|q_h\|_{0,G_2}^2 + \|\underline{v}_h\|_{0,G_2}^2) + h^{3\varepsilon/2k}\|\operatorname{div} \underline{v}_h\|_{0,G_2}^2 + h^{-3\varepsilon/2}(\|q_h\|_{-k-1,G_2}^2 + \|\underline{v}_h\|_{-k-1,G_2}^2)]. \quad (3.10)$$

If we now substitute (3.10) into (3.6), and the resulting relation repeatedly into itself, we see (after relabeling the  $G_i$ 's) that, for any fixed positive number  $T$ ,

$$\|\operatorname{div} \underline{v}_h\|_{0,G'}^2 \leq K[h^{2T}(\|\operatorname{div} \underline{v}_h\|_{0,G_1}^2 + \|\underline{v}_h\|_{0,G_1}^2 + \|q_h\|_{0,G_1}^2) + h^{-2\varepsilon}(\|q_h\|_{-k-1,G_1}^2 + \|\underline{v}_h\|_{-k-1,G_1}^2)]. \quad (3.11)$$

Substitution of (3.11) into (3.9) and (3.10) finishes the proof of the lemma when  $k > 0$ . The case  $k = 0$  is covered by the proof of Lemma 2.1 starting from (3.4), (2.7), and (2.9) without making use of Lemma 1.2, which is obviously unnecessary anyway if  $k = 0$ .

This result, though, will not be sufficient to allow us to reproduce the proof of Theorem 2.1 for the nonlinear problem, since, although the boundedness of

$(\overline{M}^*)^{-1} : L^2(\Omega) \rightarrow H^2(\Omega) \cap H_0^1(\Omega)$  still guarantees the existence and uniqueness of the solution  $(\overline{v}_h, \overline{q}_h)$  of the analogue of (2.17) and also the error bounds (2.18) for  $s = 0$  (only), we do not have any estimates to replace (2.18) for  $s = k$ .

Let now  $\omega \in C_0^\infty(G_2)$ ,  $\omega = 1$  on  $G_1$ , and set  $(\underline{u}, \overline{p}) = (\omega \underline{u}, \omega p)$ . Let

$$t(j, k) = 2j - 2 - \varepsilon - 2(j - k - 2)^+$$

for  $\varepsilon > 0$  and  $2 + \varepsilon \leq j \leq k + 3$ .

**THEOREM 3.1 :** *Let  $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$  be open subdomains of  $\Omega$ . If  $p \in H^j(\Omega) \cap H^r(\Omega_1)$  and  $k \geq 1$ , then*

- (i)  $\|\xi\|_{0, \Omega_0} \leq K[h^r \|p\|_{r, \Omega_1} + h^{t(j, k)} \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1)],$   
 $2 \leq r \leq k + 1,$
- (ii)  $\|\zeta\|_{0, \Omega_0} \leq K[h^{r-1} \|p\|_{r, \Omega_1} + h^{t(j, k)-\varepsilon} \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1)],$   
 $2 \leq r \leq k + 2 - \varepsilon,$
- (iii)  $\|\operatorname{div} \zeta\|_{0, \Omega_0} \leq K[h^{r-2} \|p\|_{r, \Omega_1} + h^{t(j, k)-\varepsilon} \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1)],$   
 $2 \leq r \leq k + 3 - \varepsilon.$

*Proof :* Throughout this proof, the subindices will be shifted so that the smaller ball on any norm bound will be called  $G$ , and the larger one  $G_1$ .

Let  $(\underline{v}_h, \overline{q}_h) \in \underline{V}_h(G_1) \times W_h(G_1)$  satisfy

$$\left. \begin{aligned} (\alpha(p) [\underline{u} - \underline{v}_h], v) - (\operatorname{div} v, \overline{p} - \overline{q}_h) + (\overline{\Gamma}[\overline{p} - \overline{q}_h], v) &= 0, \quad v \in \underline{V}_h(G_1), \\ (\operatorname{div} [\underline{u} - \underline{v}_h], w) + (\overline{\gamma}[\overline{p} - \overline{q}_h], w) &= 0, \quad w \in W_h(G_1). \end{aligned} \right\} \quad (3.12)$$

As it was pointed out earlier,  $(\underline{v}_h, \overline{q}_h)$  exists and is unique. Furthermore, if  $p \in H^r(G_1)$ ,

$$\left. \begin{aligned} \|\overline{p} - \overline{q}_h\|_{0, G} &\leq Q \|p\|_{r, G_1} h^r, & 2 \leq r \leq k + 1, \\ \|\underline{u} - \underline{v}_h\|_{0, G} &\leq Q \|p\|_{r, G_1} h^{r-1}, & 2 \leq r \leq k + 2, \\ \|\operatorname{div} (\underline{u} - \underline{v}_h)\|_{0, G} &\leq Q \|p\|_{r, G_1} h^{r-2}, & 2 \leq r \leq k + 3, \end{aligned} \right\} \quad (3.13)$$

just as in (2.18) for  $s = 0$ , where the constant  $Q$  depends on  $\|p\|_{2+\varepsilon}$ .

It follows from [5] that, if  $k \geq s$ , then

$$\|\xi\|_{-k-1} + \|\zeta\|_{-k-1} + \|\operatorname{div} \zeta\|_{-k-1} \leq Ch^{t(j, k)} \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1), \quad (3.14)$$

and also that

$$\begin{aligned} \|(\kappa\zeta + \lambda\zeta)\xi\|_0 &\leq K(\|\zeta\|_0 + \|\xi\|_0)\|\xi\|_{0,\infty} \\ &\leq Ch^{t(j,k)}\|p\|_{j,\Omega}(\|p\|_{j,\Omega} + 1). \end{aligned} \quad (3.15)$$

It now follows from (2.1), (3.14), and (3.15) that

$$\begin{aligned} \|\xi\|_{0,G} &\leq K[h(\|\xi\|_{0,G_1} + \|\zeta\|_{0,G_1}) + h^2\|\operatorname{div}\zeta\|_{0,G_1} + \\ &\quad + h^r\|p\|_{r,G_1} + h^{t(j,k)}\|p\|_{j,\Omega}(\|p\|_{j,\Omega} + 1)], \\ 2 &\leq r \leq k+1. \end{aligned} \quad (3.16)$$

As in the proof of Theorem 2.1, we can replace  $(\bar{u}, \bar{p})$  in (3.12) by  $(y_h, p_h)$  using (3.2). What we have to do next is to take  $(v_h, q_h) = (y_h - \bar{v}_h, p_h - \bar{q}_h)$  in (3.3) and to change Lemma 3.1 slightly. It follows from (3.13) and (3.16) that

$$\begin{aligned} \|q_h\|_{0,G} &\leq K[h(\|q_h\|_{0,G_1} + \|v_h\|_{0,G_1}) + h^2\|\operatorname{div}v_h\|_{0,G_1} \\ &\quad + h^r\|p\|_{r,G_1} + h^{t(j,k)}\|p\|_{j,\Omega}(\|p\|_{j,\Omega} + 1)], \quad 2 \leq r \leq k+1, \end{aligned}$$

so that, by nesting and relabeling the  $G_i$ 's, we can see that

$$\begin{aligned} \|q_h\|_{0,G}^2 &\leq K[h^{2T}\|q_h\|_{0,G_1}^2 + h^2\|v_h\|_{0,G_1}^2 + h^4\|\operatorname{div}v_h\|_{0,G_1}^2 + \\ &\quad + h^{2r}\|p\|_{r,G_1}^2 + h^{2t(j,k)}(\|p\|_{j,\Omega}^2 + \|p\|_{j,\Omega}^4)], \\ 2 &\leq r \leq k+1, \end{aligned} \quad (3.17)$$

where  $T$  is any fixed positive number. Next, (3.6) and (3.17) imply that

$$\begin{aligned} \|\operatorname{div}v_h\|_{0,G}^2 &\leq K[h^{2T}\|q_h\|_{0,G_1}^2 + h^{2-2\epsilon}\|v_h\|_{0,G_1}^2 + h^{2\epsilon}\|\operatorname{div}v_h\|_{0,G_1}^2 + \\ &\quad + h^{2r-2\epsilon}\|p\|_{r,G_1}^2 + h^{2t(j,k)-2\epsilon}(\|p\|_{j,\Omega}^2 + \|p\|_{j,\Omega}^4)], \\ 2 &\leq r \leq k+1, \end{aligned}$$

which repeatedly substituted into itself will give (after renaming the  $G_i$ 's) the bound

$$\begin{aligned} \|\operatorname{div}v_h\|_{0,G}^2 &\leq K[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\operatorname{div}v_h\|_{0,G_1}^2) + h^{2-2\epsilon}\|v_h\|_{0,G_1}^2 \\ &\quad + h^{2r-2\epsilon}\|p\|_{r,G_1}^2 + h^{2t(j,k)-2\epsilon}(\|p\|_{j,\Omega}^2 + \|p\|_{j,\Omega}^4)], \\ 2 &\leq r \leq k+1. \end{aligned} \quad (3.18)$$

We see from (3.17) and (3.18) that

$$\begin{aligned} \|q_h\|_{0,G}^2 &\leq K[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\operatorname{div}v_h\|_{0,G_1}^2) + h^2\|v_h\|_{0,G_1}^2 \\ &\quad + h^{2r}\|p\|_{r,G_1}^2 + h^{2t(j,k)}(\|p\|_{j,\Omega}^2 + \|p\|_{j,\Omega}^4)], \\ 2 &\leq r \leq k+1. \end{aligned} \quad (3.19)$$

Also, (3.7), (3.18), and (3.19) imply that

$$\begin{aligned} \|v_h\|_{0,G}^2 &\leq K[h^{2T}(\|q_h\|_{0,G_1}^2 + \|\operatorname{div} v_h\|_{0,G_1}^2) + h\|v_h\|_{0,G_1}^2 \\ &\quad + h^{2r-2\epsilon}\|p\|_{r,G_1}^2 + h^{2t(j,k)-2\epsilon}(\|p\|_{j,\Omega}^2 + \|p\|_{j,\Omega}^4)], \\ 2 &\leq r \leq k+1, \end{aligned}$$

and substituting this bound recursively into itself leads to the bound

$$\begin{aligned} \|v_h\|_{0,G}^2 &\leq K[h^{2T}(\|q_h\|_{0,G_1}^2 + \|v_h\|_{0,G_1}^2 + \|\operatorname{div} v_h\|_{0,G_1}^2) \\ &\quad + h^{2r-2\epsilon}\|p\|_{r,G_1}^2 + h^{2t(j,k)-2\epsilon}(\|p\|_{j,\Omega}^2 + \|p\|_{j,\Omega}^4)], \quad (3.20) \\ 2 &\leq r \leq k+1. \end{aligned}$$

Substituting (3.20) into (3.18) and (3.19) finally gives

$$\begin{aligned} h^{-\epsilon}\|q_h\|_{0,G} + \|v_h\|_{0,G} + \|\operatorname{div} v_h\|_{0,G} &\leq K[h^T(\|q_h\|_{0,G_1} + \|v_h\|_{0,G_1} \\ &\quad + \|\operatorname{div} v_h\|_{0,G_1}) + h^{r-\epsilon}\|p\|_{r,G_1} + h^{t(j,k)-\epsilon}\|p\|_{j,\Omega}(\|p\|_{j,\Omega} + 1)] \quad (3.21) \end{aligned}$$

for  $2 \leq r \leq k+1$  and any fixed  $T > 0$ .

It follows from [5] that

$$\|\xi\|_0 + \|\zeta\|_0 + \|\operatorname{div} \zeta\|_0 \leq K\|p\|_{2+\epsilon,\Omega}. \quad (3.22)$$

Now, (3.13), (3.21), and (3.22) imply that

$$\begin{aligned} \|q_h\|_{0,G} &\leq K[h^T(\|\bar{p} - \bar{q}_h\|_{0,G_1} + \|\xi\|_{0,G_1} + \|\bar{u} - \bar{v}_h\|_{0,G_1} + \|\zeta\|_{0,G_1} \\ &\quad + \|\operatorname{div}(\bar{u} - \bar{v}_h)\|_{0,G_1} + \|\operatorname{div} \zeta\|_{0,G_1}) + h^r\|p\|_{r,G_1} \\ &\quad + h^{t(j,k)}\|p\|_{j,\Omega}(\|p\|_{j,\Omega} + 1)] \\ &\leq K[h^r\|p\|_{r,G_1} + h^{t(j,k)}\|p\|_{j,\Omega}(\|p\|_{j,\Omega} + 1)], \quad 2 \leq r \leq k+1, \end{aligned}$$

and

$$\begin{aligned} \|v_h\|_{0,G} + \|\operatorname{div} v_h\|_{0,G} &\leq K[h^{r-\epsilon}\|p\|_{r,G_1} + h^{t(j,k)-\epsilon}\|p\|_{j,\Omega}(\|p\|_{j,\Omega} + 1)], \\ 2 &\leq r \leq k+1. \end{aligned}$$

These estimates together with (3.13) and the finite covering process used in Theorem 2.1 complete the proof of the theorem.

We have also obtained (by combining Lemma 1.2, (3.14), (3.15), and Theorem 3.1) the following local negative norm error estimates.

**THEOREM 3.2 :** *If  $p \in H^j(\Omega) \cap H^r(\Omega_1)$ ,  $2 + \epsilon \leq j \leq k + 3$ ,  $k > 0$ , and  $h$  is sufficiently small, then, for  $0 \leq s \leq k + 1$ ,*

$$\begin{aligned} (i) \quad \|\xi\|_{-s,\Omega_0} &\leq K[\|p\|_{r,\Omega_1} h^{r+s-(s-k+1)^+} + \|p\|_{j,\Omega}(\|p\|_{j,\Omega} + 1) h^{t(j,k)}], \\ 2 &\leq r \leq k+1 + (s-k+1)^+, \end{aligned}$$



- (ii)  $\|\zeta\|_{-s, \Omega_0} \leq K[\|p\|_{r, \Omega_1} h^{r+s-1-(s-k)^+} + \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1) h^{t(j,k)-(s-k)^+}], \quad 2 \leq r \leq k+2-\varepsilon+(s-k)^+,$
- (iii)  $\|\operatorname{div} \zeta\|_{-s, \Omega_0} \leq K[\|p\|_{r, \Omega_1} h^{r+s-2} + \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1) h^{t(j,k)-(s-k)^+}], \quad 2 \leq r \leq k+3-\varepsilon.$

#### 4. CONVERGENCE OF DIFFERENCE QUOTIENTS

Assume a *uniform* polygonalization in the interior of  $\Omega$  for the remainder of this paper.

Let  $\mu = (\mu_1, \mu_2) \in Z^2$ , and let

$$T_h^\mu v(x) = v(x + \mu h).$$

Set

$$\begin{aligned} \partial_{h,j} v(x) &= h^{-1}(T_h^{e_j} - I)v(x), \\ &= \frac{v(x + he_j) - v(x)}{h}, \quad j = 1, 2; \\ \partial^\alpha u &= \partial_h^\alpha u = \partial_{h,1}^{\alpha_1} \partial_{h,2}^{\alpha_2} u. \end{aligned}$$

DÉFINITION 4.1 : An operator  $Q_h$  of the form

$$Q_h w = \sum_{\mu, \nu} c_{\mu, \nu} T_h^\mu \partial^\nu w,$$

where only finitely many  $c_{\mu, \nu}$ 's are nontrivial, is called a difference operator. Observe that for some  $M \geq 0$ ,

$$Q_h w = \sum_{|\delta| \leq M} \tilde{c}_\delta T_h^\delta w.$$

DÉFINITION 4.2 : We say that the difference operator  $Q_h$  is of order  $m$  if

$$\inf_{\delta} \{ l_\delta / \tilde{c}_\delta = O(h^{l_\delta}) \} = -m.$$

We shall make the assumption throughout the rest of the paper that, for any fixed  $\mu$  and  $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$ , there exists  $h_0$  such that, for  $h \in (0, h_0]$ ,

$$(T_h^\mu \tilde{v}, T_h^\mu w) \in \tilde{V}_h(\Omega_1) \times W_h(\Omega_1) \quad \text{if} \quad (\tilde{v}, w) \in \tilde{V}_h(\Omega_0) \times W_h(\Omega_0),$$

where  $T_h^\mu \tilde{v} = T_h^\mu(v_1, v_2) = (T_h^\mu v_1, T_h^\mu v_2)$ .

We shall also make use of the discrete Leibnitz formula :

$$\partial_h^\phi(uv) = \sum_{\psi \leq \phi} \binom{\phi}{\psi} T_h^\psi \partial_h^\phi u \partial_h^\psi v, \quad (4.1)$$

where  $\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix}$ . Note that  $Q_h$  commutes with any differential operator with constant coefficients, as well as with  $P_h$  and  $\pi_h$ .

We wish to obtain, following the ideas developed in [7], estimates for  $Q_h \xi$  and  $Q_h \operatorname{div} \zeta$  in  $H^s(\Omega_0)$ , and for  $Q_h \zeta$  in  $H^s(\Omega_0)^2$ , with  $s = -k - 1$  or  $s = 0$ . It is clearly sufficient to derive estimates for the case  $Q_h = T_h^\mu \partial^\nu$ . Moreover, since

$$\|T_h^\mu v\|_{0,G} \leq \|v\|_{0,G_1}$$

for sufficiently small  $h$ , it suffices to consider  $Q_h = \partial_h^\nu$ .

We need to find equations satisfied by  $\partial^\nu \xi$ ,  $\partial^\nu \zeta$  and  $\partial^\nu \operatorname{div} \zeta$ . We see that (4.1) and (0.12) imply that, for  $|\nu| > 0$ ,

$$\begin{aligned} & (\alpha(p) \partial^\nu \zeta, v) - (\operatorname{div} v, \partial^\nu \xi) + (\tilde{\Gamma} \partial^\nu \xi, v) \\ &= (\alpha(p) \zeta, \partial^{-\nu} v) - (\partial^{-\nu} \operatorname{div} v, \xi) + (\tilde{\Gamma} \xi, \partial^{-\nu} v) \\ &\quad - \sum_{\mu < \nu} \binom{\nu}{\mu} \{ ([T_h^\mu \partial^{\nu-\mu} \alpha(p)] \partial^\mu \zeta, v) + ([T_h^\mu \partial^{\nu-\mu} \tilde{\Gamma}] \partial^\mu \xi, v) \} \\ &= (\partial^\nu [\kappa \xi^2 + \lambda \zeta \xi], v) - \sum_{\mu < \nu} \binom{\nu}{\mu} ([T_h^\mu \partial^{\nu-\mu} \alpha(p)] \partial^\mu \zeta \\ &\quad + [T_h^\mu \partial^{\nu-\mu} \tilde{\Gamma}] \partial^\mu \xi, v), \quad v \in V_h(G), \end{aligned} \quad (4.2)$$

since  $\partial^{-\nu} v \in V_h(G)$  if  $v \in V_h(G_1)$ . Also,

$$\begin{aligned} & (\operatorname{div} \partial^\nu \zeta, w) + (\gamma \partial^\nu \xi, w) = (\partial^\nu [\rho \xi^2], w) - \sum_{\mu < \nu} \binom{\nu}{\mu} ([T_h^\mu \partial^{\nu-\mu} \gamma] \partial^\mu \xi, w), \\ & w \in W_h(G). \end{aligned} \quad (4.3)$$

We are now ready to derive estimates for the linear problem.

**THEOREM 4.1 :** *If  $p \in H^j(\Omega) \cap H^r(\Omega_1)$ ,  $2 \leq j \leq k + 3$ , the operator  $M$  of (1.1) is linear, and  $h$  is sufficiently small, then for  $0 \leq s \leq k + 1$ ,*

- (i)  $\|Q_h \xi\|_{-s, \Omega_0} \leq K[\|p\|_{r, \Omega_1} h^{r+s-(s-k+1)^+-m} + \|p\|_{j, \Omega} h^{j+k-1}],$   
 $m + 2 \leq r \leq m + k + 1 + (s - k + 1)^+,$
- (ii)  $\|Q_h \zeta\|_{-s, \Omega_0} \leq K[\|p\|_{r, \Omega_1} h^{r+s-1-(s-k)^+-m} + \|p\|_{j, \Omega} h^{j+k-1}],$   
 $m + 2 \leq r \leq m + k + 2 + (s - k)^+,$

$$(iii) \quad \| Q_h \operatorname{div} \zeta \|_{-s, \Omega_0} \leq K [\| p \|_{r, \Omega_1} h^{r+s-2-m} + \| p \|_{j, \Omega} h^{k-1+\min\{j; s+2-\varepsilon\}}],$$

$$m+2 \leq r \leq m+k+3,$$

where  $m \geq 0$  is the order of  $Q_h$ .

*Proof* : The proof will proceed by induction. As we pointed out earlier, it suffices to consider  $Q_h = \partial^v$ , with  $m = |v|$ . The case  $|v| = 0$  is just Theorem 2.2, so that we shall assume that  $|v| \geq 1$  and that (i), (ii), and (iii) hold for  $\partial^\mu$ , with  $|\mu| \leq |v| - 1$ .

Let us recall that in the linear problem  $\alpha(p) = \alpha$ ,  $\Gamma = \beta$ ,  $\gamma = c$ ,  $\kappa = \varrho$ ,  $\lambda = \rho = 0$ . Therefore, since  $\xi = \tau - \theta$ , we can rewrite (4.2) and (4.3) as follows :

$$\left. \begin{aligned} (\alpha \partial^v \zeta, v) - (\operatorname{div} v, \partial^v \tau) + (\beta \partial^v \tau, v) &= (\beta \partial^v \theta, v) \\ &- \sum_{\mu < v} \binom{v}{\mu} ([T_h^\mu \partial^{v-\mu} \alpha] \partial^\mu \zeta + [T_h^\mu \partial^{v-\mu} \beta] \partial^\mu \xi, v), \quad v \in V_h(G), \\ (\operatorname{div} \partial^v \zeta, w) + (c \partial^v \xi, w) &= (c \partial^v \theta, w) - \sum_{\mu < v} \binom{v}{\mu} ([T_h^\mu \partial^{v-\mu} c] \partial^\mu \xi, w), \\ &w \in W_h(G). \end{aligned} \right\} \quad (4.4)$$

Since equations (4.4) are of the form (1.1), it is clear that the same argument that led from Lemma 1.1 to Theorems 2.1 and 2.2 shows that, for  $0 \leq s \leq k+1$  and  $0 \leq \bar{r} \leq k+1 + (s-k+1)^+$ ,

$$\begin{aligned} \|\partial^v \xi\|_{-s, G} &\leq K \left[ \|\partial^v p\|_{r, G_1} h^{\bar{r}+s-(s-k+1)^+} + h^{s+1} \left( \sum_{\mu < v} \|\partial^\mu \xi\|_{0, G_1} \right. \right. \\ &\quad \left. \left. + \sum_{\mu < v} \|\partial^\mu \zeta\|_{0, G_1} \right) + \sum_{\mu \leq v} (\|\partial^\mu \xi\|_{-k-1-|\mu|} + \|\partial^\mu \zeta\|_{-k-1-|\mu|}) \right] \\ &\leq K [\|p\|_{r, G_1} h^{\bar{r}+s-(s-k+1)^+-|v|} + \|p\|_{j, \Omega} h^{j+k} \\ &\quad + \|\xi\|_{-k-1} + \|\zeta\|_{-k-1}] \end{aligned} \quad (4.5)$$

$$\begin{aligned} &\leq K [\|p\|_{r, G_1} h^{\bar{r}+s-(s-k+1)^+-|v|} + \|p\|_{j, \Omega} h^{j+k-1}], \\ &|v| + 2 \leq r \leq |v| + k + 1 + (s-k+1)^+, \end{aligned}$$

$$\|\partial^v \zeta\|_{-s, G} \leq K [\|p\|_{r, G_1} h^{\bar{r}+s-1-(s-k)^+-|v|} + \|p\|_{j, \Omega} h^{j+k-1}], \quad (4.6)$$

$$|v| + 2 \leq r \leq |v| + k + 2 + (s - k)^+,$$

$$\|\partial^v \operatorname{div} \zeta\|_{-s, G} \leq K[\|p\|_{r, G_1} h^{r+s-2-|v|} + \|p\|_{j', \Omega} h^{j+k-1}], \quad (4.7)$$

$$|v| + 2 \leq r \leq |v| + k + 3,$$

since

$$\|\partial^v \theta\|_{0, G} = \|P_h(\partial^v p) - \partial^v p\|_{0, G} \leq K \|p\|_{r+|v|, G_1} h^r, \quad 0 \leq r \leq k + 1,$$

and

$$\|\partial^v \theta\|_{-k-1-|v|, G} \leq K \|\theta\|_{-k-1, G_1} \leq K \|p\|_{r, G_1} h^{r+k+1}, \quad 0 \leq r \leq k + 1.$$

Finally, the finite covering process used in the previous sections, together with (4.5), (4.6), and (4.7), finishes the proof of the theorem.

Note that, in particular, we have shown that

$$\begin{aligned} \|Q_h \xi\|_{-k-1, \Omega_0} + \|Q_h \zeta\|_{-k-1, \Omega_0} + \|Q_h \operatorname{div} \zeta\|_{-k-1, \Omega_0} \\ \leq K h^{2k+2} (\|p\|_{k+3, \Omega} + \|p\|_{k+3+m, \Omega_1}). \end{aligned} \quad (4.8)$$

We can also derive the estimates for the nonlinear problem. Let  $m \leq k + 1$  be the order of  $Q_h$ , and let

$$l(j, k) = 2(j - 1) + s - m - \varepsilon - 2(j - k - 2)^+$$

so that  $l(j, k) \geq t(j, k)$  if  $s \geq m$ .

**THEOREM 4.2 :** Assume  $k \geq 1$ ,  $p \in H^j(\Omega) \cap H^r(\Omega_1)$ ,  $2 + \varepsilon \leq j \leq k + 3$ , the operator  $M$  of (1.1) is nonlinear, and  $h$  is sufficiently small; then, for  $0 \leq s \leq k + 1$ ,

$$\begin{aligned} \text{(i)} \quad \|Q_h \xi\|_{-s, \Omega_0} &\leq K[\|p\|_{r, \Omega_1} h^{r+s-(s-k+1)^+-m} \\ &\quad + \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1) h^{l(j, k)}], \\ &\quad m + 2 \leq r \leq m + k + 1 + (s - k + 1)^+, \\ \text{(ii)} \quad \|Q_h \zeta\|_{-s, \Omega_0} &\leq K[\|p\|_{r, \Omega_1} h^{r+s-1-(s-k)^+-m} + \\ &\quad + \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1) h^{l(j, k) - (\varepsilon - s)^+}], \\ &\quad m + 2 \leq r \leq m + k + 2 + (s - k)^+ - \varepsilon, \\ \text{(iii)} \quad \|Q_h \operatorname{div} \zeta\|_{-s, \Omega_0} &\leq \\ &\leq K[\|p\|_{r, \Omega_1} h^{r+s-2-m} \\ &\quad + \|p\|_{j, \Omega} (\|p\|_{j, \Omega} + 1) h^{l(j, k) - (\varepsilon - s)^+}], \\ &\quad m + 2 \leq r \leq m + k + 3 - \varepsilon. \end{aligned}$$

*Proof:* Let us rewrite (4.2) and (4.3) in the form

$$\left. \begin{aligned}
 &(\alpha(p) \partial^\nu \zeta, \underline{v}) - (\operatorname{div} \underline{v}, \partial^\nu \tau) + (\Gamma \partial^\nu \tau, \underline{v}) = (\Gamma \partial^\nu \theta, \underline{v}) \\
 &+ (\partial^\nu [\kappa \xi^2 + \lambda \xi \zeta], \underline{v}) - \sum_{\mu < \nu} \binom{\nu}{\mu} ([T_h^\mu \partial^{\nu-\mu} \alpha(p)] \partial^\mu \zeta \\
 &+ [T_h^\mu \partial^{\nu-\mu} \Gamma] \partial^\mu \xi, \underline{v}), \quad \underline{v} \in \underline{V}_h(G), \\
 &(\operatorname{div} \partial^\nu \zeta, w) + (\gamma \partial^\nu \xi, w) = (\gamma \partial^\nu \theta, w) + (\partial^\nu [\rho \xi^2], w) \\
 &\quad - \sum_{\mu < \nu} \binom{\nu}{\mu} ([T_h^\mu \partial^{\nu-\mu} \gamma] \partial^\mu \xi, w), \\
 &\quad \quad \quad w \in W_h(G).
 \end{aligned} \right\} \quad (4.9)$$

Just as in the linear case (Theorem 4.1) it suffices to consider  $Q_h = \partial^\nu$  and to proceed by induction. Again,  $|\nu| = 0$  is just Theorem 3.2, and so, we assume that  $|\nu| > 0$  and that (i), (ii), and (iii) are valid for  $m \leq |\nu| - 1$ . The proof will be the same as in Theorem 4.1, but before we can apply Lemma 1.1 to (4.9) we need estimates for  $\|\partial^\nu(\kappa \xi^2 + \lambda \xi \zeta)\|_{0,G_1}$  and  $\|\partial^\nu[\rho \xi^2]\|_{0,G_1}$ . Note that, by definition,

$$\partial^\nu w = h^{-|\nu|} (T_h^{e_2} - I)^{\nu_2} (T_h^{e_1} - I)^{\nu_1} w,$$

which implies that

$$\|\partial^\nu w\|_{0,\Omega_0} \leq Q h^{-|\nu|} \|w\|_{0,\Omega_1}.$$

Therefore, this estimate and (3.15) imply that

$$\begin{aligned}
 \|\partial^\nu(\kappa \xi^2 + \lambda \xi \zeta)\|_{0,\Omega_0} &\leq Q h^{-|\nu|} \|\kappa \xi^2 + \lambda \xi \zeta\|_{0,\Omega_1} \\
 &\leq Q h^{t(j,k)-|\nu|} \|p\|_{j,\Omega}^2 (\|p\|_{j,\Omega} + 1), \quad |\nu| \leq k+1.
 \end{aligned} \quad (4.10)$$

Similarly,

$$\|\partial^\nu(\rho \xi^2)\|_{0,\Omega_0} \leq Q h^{t(j,k)-|\nu|} \|p\|_{j,\Omega}^2 (\|p\|_{j,\Omega} + 1), \quad |\nu| \leq k+1. \quad (4.11)$$

We can now apply Lemma 1.1, then the same argument that led to Theorems 4.1 and 4.2 will show (in view of (3.14), (4.10), and (4.11)) that

$$\begin{aligned} \|\partial^v \xi\|_{-s,G} &\leq K \left[ \|\partial^v p\|_{\bar{r},G_1} h^{r+s-(s-k+1)^+} + h^{s+1} \left( \sum_{\mu < v} \|\partial^\mu \xi\|_{0,G_1} \right. \right. \\ &\quad + \sum_{\mu < v} \|\partial^\mu \zeta\|_{0,G_1} + \|\partial^v(\kappa \xi^2 + \lambda \xi \zeta)\|_{0,G_1} + \|\partial^v(\rho \xi^2)\|_{0,G_1} \Big) \\ &\quad + \sum_{\mu \leq v} (\|\partial^\mu \xi\|_{-k-1-|\mu|} + \|\partial^\mu \zeta\|_{-k-1-|\mu|}) \\ &\quad \left. + \|\partial^v(\kappa \xi^2) + \lambda \xi \zeta\|_{-k-1-|v|} + \|\partial^v(\rho \xi^2)\|_{-k-1-|v|} \right] \quad (4.12) \end{aligned}$$

$$\begin{aligned} &\leq K [\|p\|_{r,G_1} h^{r+s-(s-k+1)^+-|v|} + \|p\|_{j,\Omega} (\|p\|_{j,\Omega} + 1) h^{l(j,k)} \\ &\quad + \|\xi\|_{-k-1} + \|\zeta\|_{-k-1}] \\ &\leq K [\|p\|_{r,G_1} h^{r+s-(s-k+1)^+-|v|} + \|p\|_{j,\Omega} (\|p\|_{j,\Omega} + 1) h^{l(j,k)}], \\ &\quad |v| + 2 \leq r \leq |v| + k + 1 + (s - k + 1)^+, \end{aligned}$$

$$\begin{aligned} \|\partial^v \zeta\|_{-s,G} &\leq K [\|p\|_{r,G_1} h^{r+s-1-(s-k)^+-|v|} + \|p\|_{j,\Omega} (\|p\|_{j,\Omega} + 1) \\ &\quad \times h^{l(j,k)-(e-s)^+}], \quad |v| + 2 \leq r \leq |v| + k + 2 + (s - k)^+ - \varepsilon \quad (4.13) \end{aligned}$$

$$\begin{aligned} \|\partial^v \operatorname{div} \zeta\|_{0,G} &\leq K [\|p\|_{r,G_1} h^{r+s-2-|v|} + \|p\|_{j,\Omega} (\|p\|_{j,\Omega} + 1) \\ &\quad \times h^{l(j,k)-(e-s)^+}], \quad |v| + 2 \leq r \leq |v| + k + 3 - \varepsilon. \quad (4.14) \end{aligned}$$

We now finish the proof using (4.12), (4.13), and (4.14) with the finite covering process described before.

Note that, in particular, Theorem 4.2 shows that (4.8) holds also for the nonlinear problem, with the exponent  $2k + 2$  replaced by  $2k + 2 - \varepsilon$ .

## 5. THE SUPERCONVERGENCE ESTIMATES

We shall introduce in this section a kernel  $K_h$  as defined in [1, 2], which will produce by convolution with  $\underline{u}_h$  and  $p_h$  a new approximation to  $(\underline{u}, p)$  in  $\tilde{V} \times W$ , the convergence of which will be (for sufficiently smooth  $p$ ) twice as fast as that of  $(\underline{u}_h, p_h)$  to  $(\underline{u}, p)$ . Recall that [3] and [5] show that if  $p \in H^{k+3}(\Omega)$ , then

$$\begin{aligned} \|\xi\|_{0,\Omega_0} + \|\zeta\|_{0,\Omega_0} + \|\operatorname{div} \zeta\|_{0,\Omega_0} &\leq \|\xi\|_0 + \|\zeta\|_0 + \|\operatorname{div} \zeta\|_0 \\ &\leq C \|p\|_{k+3} h^{k+1} \quad (5.1) \end{aligned}$$

We shall obtain in a moment the following estimates :

$$\begin{aligned} & \|p - K_h * p_h\|_{0,\Omega_0} + \|\underline{u} - K_h * \underline{u}_h\|_{0,\Omega} + \|\operatorname{div} \underline{u} - K_h * \operatorname{div} \underline{u}_h\|_{0,\Omega_0} \\ & \leq C(\|p\|_{2k+4-\varepsilon,\Omega_1} + \|p\|_{k+3,\Omega}) h^{2k+2-\varepsilon}, \\ & \text{if } p \in H^{k+3}(\Omega) \cap H^{2k+4-\varepsilon}(\Omega_1), \end{aligned} \quad (5.2)$$

where the exponent  $-\varepsilon$  does not appear in the linear case. Comparing (5.1) and (5.2) we see that we have indeed doubled the rate of convergence.

Let

$$\chi(t) = \begin{cases} 1, & |t| \leq \frac{1}{2}, \\ 0, & |t| > \frac{1}{2}, \end{cases}$$

and for  $l$  a positive integer set

$$\psi_1^{(l)}(t) = \chi * \chi * \cdots * \chi, \quad \text{convolution } l - 1 \text{ times.}$$

Let  $c_i$ ,  $0 \leq i \leq k$ , be determined as the unique solution of the linear system of algebraic equations (see [2])

$$\sum_{i=0}^k c_i \int_{\mathbb{R}} \psi_1^{(k+2)}(y) (y+i)^{2m} dy = \delta_{0m}, \quad \text{for } 0 \leq m \leq k,$$

and define  $K_h$  by

$$K_h(x) = K_{h,k+2}^{2k+2}(x) = \prod_{m=1}^2 \left( \sum_{i=-k}^k h^{-1} c'_i \psi_1^{(k+2)}(h^{-1} x_m - i) \right), \quad x \in \mathbb{R}^2, \quad (5.3)$$

where the constants  $c'_j$  are given by  $c'_{-i} = c'_i = c_i/2$ ,  $1 \leq i \leq k$ , and  $c'_0 = c_0$ .

It follows from [2] that

$$\|K_h * w - w\|_{0,\Omega_0} \leq C \|w\|_{r,\Omega_1} h^r, \quad \text{if } w \in H^r(\Omega_1), \quad 0 \leq r \leq 2k+2, \quad (5.4)$$

$$\|D^\nu(K_h * w)\|_{s,\Omega_0} \leq C \|\partial^\nu w\|_{s,\Omega_1}, \quad \text{if } w \in H^s(\Omega_1). \quad (5.5)$$

Also, it is easily shown by induction (see [2] for a detailed proof) that

$$\|w\|_{0,\Omega_0} \leq C \sum_{|v| \leq s} \|D^\nu w\|_{-s,\Omega_1}, \quad s \in \mathbb{Z}, \quad s \geq 0, \quad \text{if } w \in L^2(\Omega). \quad (5.6)$$

Next, note that

$$\underline{u} - K_h * \underline{u}_h = (\underline{u} - K_h * u) + K_h * \zeta, \quad (5.7)$$

$$p - K_h * p_h = (p - K_h * p) + K_h * \xi, \quad (5.8)$$

$$\operatorname{div} \underline{u} - K_h * \operatorname{div} \underline{u}_h = (\operatorname{div} \underline{u} - K_h * \operatorname{div} u) + K_h * \operatorname{div} \zeta. \quad (5.9)$$

We are now ready to demonstrate the superconvergence estimates announced in (5.2).

**THEOREM 5.1 :** *Assume the operator  $N$  is linear (i.e.,  $N = L$ ),  $p \in H^j(\Omega) \cap H^r(\Omega_1)$ ,  $2 \leq j \leq k + 3$ , and  $k + 1 \leq r \leq 2k + 4$ ; then, for sufficiently small  $h$ ,*

- (i)  $\|p - K_h * p_h\|_{0,\Omega_0} \leq C[\|p\|_{r,\Omega_1} h^{r-(r/2-k)^+} + \|p\|_{j,\Omega} h^{j+k-1}],$
- (ii)  $\|\underline{u} - K_h * \underline{u}_h\|_{0,\Omega_0} \leq C[\|p\|_{r,\Omega_1} h^{r-1-(r/2-k-1)^+} + \|p\|_{j,\Omega} h^{j+k-1}];$
- (iii)  $\|\operatorname{div} \underline{u} - K_h * \operatorname{div} \underline{u}_h\|_{0,\Omega_0} \leq C[\|p\|_{r,\Omega_1} h^{r-2} + \|p\|_{j,\Omega} h^{j+k-1-(j-k-3+\varepsilon)^+}].$

**THEOREM 5.2 :** *Assume the operator  $N$  is nonlinear,  $k \geq 1$ ,  $p \in H^j(\Omega) \cap H^r(\Omega_1)$ ,  $2 + \varepsilon \leq j \leq k + 3$ , and  $k + 1 \leq r \leq 2k + 4 - \varepsilon$ ; then, for sufficiently small  $h$ ,*

- (i)  $\|p - K_h * p_h\|_{0,\Omega_0} \leq C[\|p\|_{r,\Omega_1} h^{r-(r/2-k)^+} + \|p\|_{j,\Omega} (\|p\|_{j,\Omega} + 1) h^{t(j,k)}],$
- (ii)  $\|\underline{u} - K_h * \underline{u}_h\|_{0,\Omega_0} \leq C[\|p\|_{r,\Omega_1} h^{r-1-(r/2-k-1)^+} + \|p\|_{j,\Omega} (\|p\|_{j,\Omega} + 1) h^{t(j,k)}],$
- (iii)  $\|\operatorname{div} \underline{u} - K_h * \operatorname{div} \underline{u}_h\|_{0,\Omega_0} \leq C[\|p\|_{r,\Omega_1} h^{r-2} + \|p\|_{j,\Omega} (\|p\|_{j,\Omega} + 1) h^{t(j,k)}],$

where  $t(j, k) = 2(j - 1) - \varepsilon - 2(j - k - 2)^+.$

*Proof of Theorems 5.1 and 5.2.* Use the triangle inequality on (5.7), (5.8), and (5.9), and apply then (5.4), (5.6), (5.5), and Theorems 4.1 and 4.2, respectively.



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